* Maximum and Minimum value

• Defn: for a function $f(x)$ defined on a set "S" and $c \in S$.

[1] $f(c)$ is absolute Max for $f(x)$ if:

$f(c) \geq f(x)$ for all $x \in S$.

[2] $f(c)$ is absolute Min for $f(x)$ if:

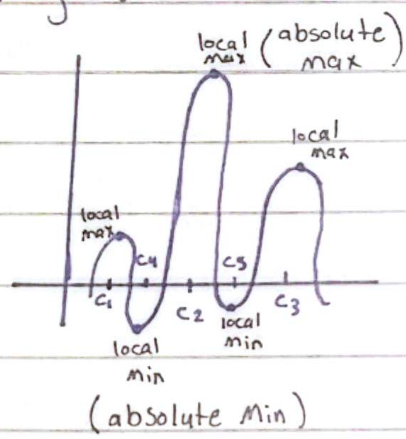
$f(c) \leq f(x)$ for all $x \in S$.

• Defn: [1] $f(c)$ is a local Max for $f(x)$ if:

$f(c) \geq f(x)$ for all x in some open interval containg (c)

[2] $f(c)$ is a local Min for $f(x)$ if:

$f(c) \leq f(x)$ for all x in some open interval containg (c).



• Thrm: if $f(x)$ is cont on $[a,b]$, then (x) attains Max & Min.

• Defn: A number $c \in$ Domain $f(x)$ is called a critical number of $f(x)$ if: $f'(c) = 0$ or $f'(c)$ d.n.e

• Thrm: if $f(c)$ is a local externum "Min or Max" then c must be a critical number for $f(x)$, but the converse is not true.

• Defn: A function $f(x)$ is strictly increasing for all $x_1 < x_2 \in$ open interval (I) then $f(x_1) < f(x_2)$.

• A function $f(x)$ is strictly decreasing for all $x_1 < x_2 \in$ open interval (I) then $f(x_1) > f(x_2)$.

• Thrm: suppose $f(x)$ is differentiable on I, then:

[1] if $f'(x) > 0$ for all $x \in I$ then $f(x)$ increasing on I.

[2] if $f'(x) < 0$ for all $x \in I$ then $f(x)$ decreasing on I

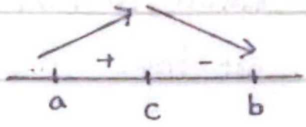
[3] if $f'(x) = 0$ for all $x \in I$ then $f(x)$ is Constant.

• First Derivative test:

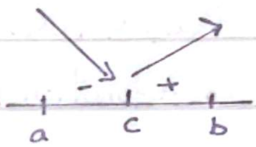
① let $f(x)$ is cont on $[a,b]$ and

② let $c \in [a,b]$ a critical number then

(1) if $f'(x) > 0$ for all $x \in (a,c)$ & $f'(x) < 0$ for all $x \in (c,b)$ then $(c, f(c))$ local max.



(2) if $f'(x) < 0$ for all $x \in (a,c)$ & $f'(x) > 0$ for all $x \in (c,b)$ then $(c, f(c))$ local min



• Defn: for a function $f(x)$ that is diffble on I then:

مقعر للأعلى

① $f(x)$ is concave up on the interval I if $f'(x)$ increasing on I " $f''(x) > 0$ for all $x \in I$ "

مقعر للأسفل

② $f(x)$ is concave down on the interval I if $f'(x)$ decreasing on I " $f''(x) < 0$ for all $x \in I$ "

• Defn: suppose that $f(x)$ is cont on (a,b) and that graph changes concavity at point $(c, f(c))$, $c \in (a,b)$ is an inflection point.

نقطة انعطاف

e.g) let $f(x) = x^3 - 12x$, $[0,4]$ Domain

1. find the increasing and decreasing domain.
2. find the extrema point. (max / min)
3. find x such that $f(x)$ has critical points.
4. find the domains of concavity. (المشتقة الثانية)
5. find the inflection point.

① Dom: $[0,4]$

② $f'(x) = 3x^2 - 12$ هل ينتمي للـ Dom

critical $f'(x) = 0$ ↳ $3x^2 - 12 = 0 \Rightarrow x^2 = 4 \Rightarrow \boxed{x = \pm 2}$

critical no. ← $2 \in Dom$

$-2 \notin Dom$

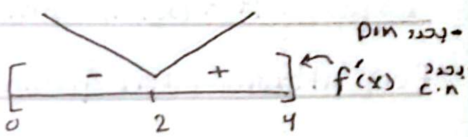
أطراف الفترة

critical number $x \in \{2, 0, 4\}$

* دائماً أطراف الفترة نقاط حرجة *

فرع

③ critical point $(2, f(2))$ $(4, f(4))$ $(0, f(0))$ -18, 16, 0



تزداد

increasing $(2,4)$

دالة متزايدة

↳ $(2, f(2))$ absolute Min point

decreasing $(0,2)$

يتبع

* اطراف الفترة نعوض absolute

$f(0) = 0$

$f(4) = 16$

$\boxed{(4,16) \text{ absolute max}}$

$\rightarrow f''(x) = 6x$

$6x = 0 \Rightarrow \boxed{x=0} \in \text{Dom}$

f'': $[0 \quad + \quad + \quad + \quad + \quad 4]$

$\therefore$ Concave up $(0,4)$

$\therefore$ there is no inflection point.

$f''(x) = 6x - 12$

$6x - 12 = 0$

$\boxed{x=2}$

$f''(x)$: $\cap \; - \; 2 \; + \; \cup$

$\rightarrow$ concave down $(-\infty, 2)$

$\rightarrow$ concave up $(2, \infty)$

$\rightarrow$ inflection point $(2, \overset{f(2)}{-15})$

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e.g) $f(x) = x^3 - 6x^2 + 1 \rightarrow D \Rightarrow R$

احسب كل شيء من السؤال

compute all of the above.

$f'(x) = 3x^2 - 12x$

$3x^2 - 12x = 0 \Rightarrow x(3x - 12) = 0$

$\boxed{x=0}$ ✓ $\boxed{x=4}$ ✓

⊛ critical point : $(0, \overset{f(0)}{1})$ / $(4, \overset{f(4)}{-31})$

$f'(x)$: $+ \; 0 \; - \; 4 \; +$

$\rightarrow$ increasing $(-\infty, 0) \cup (4, \infty)$

$\rightarrow$ decreasing $(0,4)$

$\rightarrow$ local min point $(4, -31)$

$\rightarrow$ local Max point $(0, 1)$

---

e.g) let $f(x) = x e^{-x}$ $\therefore$ Dom: R

a) find the intervals of increasing and decreasing

b) find the externa points.

c) find x such that $f(x)$ has critical points.

$f'(x) = -x e^{-x} + e^{-x}$

$-x e^{-x} + e^{-x} = 0 \Rightarrow e^{-x}(-x+1) = 0$

$e^{-x} \neq 0 \quad \boxed{x=1} \in R$

$\rightarrow$ critical points $(1, f(1))$

$f'(x)$: $+ \; 1 \; -$

$\rightarrow$ increasing $(-\infty, 1)$

$\rightarrow$ decreasing $(1, \infty)$

$\rightarrow$ absolute Max point $(1, f(1))$

d) find the interval of concavity, inflection point.

$f''(x) = x e^{-x} + -e^{-x} + -e^{-x} \Rightarrow f''(x) = x e^{-x} - 2e^{-x}$

$x e^{-x} - 2e^{-x} = 0 \Rightarrow e^{-x}(x-2) = 0$

$e^{-x} \neq 0 \quad \boxed{x=2}$

$f''(x)$: $\cap \; - \; 2 \; + \; \cup$

$\rightarrow$ concave up $(2, \infty)$

$\rightarrow$ concave down $(-\infty, 2)$

inflection point $(2, f(2))$

e.g) 1. find the interval of increasing and decreasing.

2. find the extrema points.

3. find x such that $f(x)$ has critical points

4. find the intervals of concavity inflection point for the following :-

① $f(x) = \dfrac{x^2}{x^2-1}$ $\quad\therefore$ Dom $\mathbb{R}-\{\pm 1\}$

$2x^3-2x-2x^3$

$$f'(x) = \frac{(x^2-1)(2x) - x^2(2x)}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2}$$

البسط

$\dfrac{-2x}{(x^2-1)^2} = 0 \Rightarrow -2x = 0$ $\boxed{x=0}$ $\in$ Dom

$f'(x) = 0$

المقام

$f'(x) = d.n.e \Rightarrow (x^2-1)^2 = 0 \Rightarrow x = \pm 1 \notin$ Dom

→ critical point $(0, f(0))$

$f'(x)$: $+$ -1 $+$ 0 $-$ 1 $-$

→ increasing $(-\infty, 0)$

→ decreasing $(0, \infty)$

→ absolute max point $(0, f(0))$ and local max

$(x^2-1)(-2x^2+2+2x(4x))$ → $(-2x^2+2+8x^2)$

$$f''(x) = \frac{-2(x^2-1)^2 + (2x)(2)(x^2-1)(2x)}{(x^2-1)^{4}}$$

$$= \frac{\cancel{(x^2-1)}(6x^2+2)}{(x^2-1)^{\cancel{4}3}} = \frac{6x^2+2}{(x^2-1)^3}$$

$\dfrac{6x^2+2}{(x^2-1)^3} = 0 \qquad 6x^2+2 \neq 0$

$f''(x)$: $\cup$ $+$ -1 $\cap$ $-$ 1 $\cup$ $+$

→ concave up $(-\infty,-1)\cup(1,\infty)$

→ concave down $(-1,1)$

→ inflection point ← there is no $\mathbb{R}-\{\pm 1\}$

$\boxed{2}$ $f(x) = x^{\frac{2}{3}} - x$ $\qquad$ Dom: $\mathbb{R}$

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} - 1 = \frac{2}{3x^{\frac{1}{3}}} - \frac{1}{1} \times \frac{3x^{\frac{1}{3}}}{3x^{\frac{1}{3}}}$$

$$f'(x) = \frac{2-3x^{\frac{1}{3}}}{3x^{\frac{1}{3}}}$$

$f'(x) = 0 \Rightarrow 2-3x^{\frac{1}{3}} = 0 \Rightarrow 3x^{\frac{1}{3}} = 2 \Rightarrow x^{\frac{1}{3}} = \frac{2}{3}$ $\boxed{x = \frac{8}{27}}$

$f'(x) = d.n.e \Rightarrow 3x^{\frac{1}{3}} = 0$ $\boxed{x=0}$

→ critical point : $(\frac{8}{27}, f(\frac{8}{27}))$ / $(0, f(0))$

$-$ 0 $+$ $\frac{8}{27}$ $-$

→ increasing $(0, \frac{8}{27})$

→ dicreasing $(-\infty, 0)\cup(\frac{8}{27}, \infty)$

→ local min point $(0, f(0))$

→ local max point $(\frac{8}{27}, f(\frac{8}{27}))$

$$f''(x) = \frac{-2}{9}x^{-4/3} = \frac{-2}{9x^{\frac{4}{3}}} \neq 0$$

→ concave down $(-\infty, \infty)$ f''

→ there is no inflection point

[3] $f(x) = \sin x - \cos x$, Dom $[0, \pi]$

$f'(x) = \cos x + \sin x$

$\cos x + \sin x = 0 \Rightarrow \cos x = -\sin x$

$1 = -\tan x$

$\boxed{x = \frac{3\pi}{4}} \in \text{Dom}$

→ critical point $\left(\frac{3\pi}{4}, f\left(\frac{3\pi}{4}\right)\right)$

$+ \quad - \quad f'(x)$

$0 \quad \frac{3\pi}{4} \quad \pi$

→ increasing $\left(0, \frac{3\pi}{4}\right)$

→ decreasing $\left(\frac{3\pi}{4}, \pi\right)$

→ absolute max point $\left(\frac{3\pi}{4}, f\left(\frac{3\pi}{4}\right)\right)$

$f(0) = -1 \Rightarrow$ absolute Min point $(0, -1)$

$f(\pi) = 1$ X

$f''(x) = -\sin x + \cos x = 0 \Rightarrow \tan x = 1$

$\boxed{x = \frac{\pi}{4}}$

$\cup \quad \cap$

$+ \quad -$

$0 \quad \frac{\pi}{4} \quad \pi$

→ concave up $\left(0, \frac{\pi}{4}\right)$

→ concave down $\left(\frac{\pi}{4}, \pi\right)$

→ inflection point $\left(\frac{\pi}{4}, f\left(\frac{\pi}{4}\right)\right)$

[4] $f(x) = \tan^{-1}(2x)$ Dom: $\mathbb{R}$

$f'(x) = \frac{2}{1+4x^2}$ $\neq 0$, $\neq$ d.n.e

$+ \quad + \quad +$

→ there is no critical point

→ increasing $(-\infty, \infty)$

→ there is no decreasing

→ No exterma

$f''(x) = \frac{-2(8x)}{(1+4x^2)^2} = \frac{-16x}{(1+4x^2)^2} = 0$

$\boxed{x = 0}$

$\cup \quad \cap$

$+ \quad -$

0

→ conave up $(-\infty, 0)$

→ concave down $(0, -\infty)$

→ inflection point $(0, f(0))$

- Integration

① $\int a\,dx = ax + c$

② $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

③ $\int f(x) \pm g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$

e.g) $\int 3x - x^2 + 5\,dx$

$\frac{3x^2}{2} - \frac{x^3}{3} + 5x + c$

e.g) $\int 2x(3x-1)^2\,dx$

$\int 2x\,(9x^2 - 6x + 1)\,dx$

$\int 18x^3 - 12x^2 + 2x\,dx$

$\frac{18x^4}{4} - \frac{12x^3}{3} + x^2 + c$

e.g) $\int \frac{5x^3 + 4}{\sqrt{x}}\,dx$

$\int \frac{5x^3}{\sqrt{x}} + \frac{4}{\sqrt{x}} = 5x^3 \cdot x^{-\frac{1}{2}} + 4x^{-\frac{1}{2}}$

$= \int 5x^{\frac{5}{2}} + 4x^{-\frac{1}{2}}\,dx$

$5x^{\frac{7}{2}}\left(\frac{2}{7}\right) + 4x^{\frac{1}{2}}(2) + c$

$= \frac{10}{7}x^{\frac{7}{2}} + 8x^{\frac{1}{2}} + c$

- Remark:

1. $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$

2. $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

3. $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c$

4. $\int \csc^2(ax+b) = -\frac{1}{a}\cot(ax+b) + c$

5. $\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + c$

6. $\int \csc(ax+b)\cot(ax+b)\,dx = \frac{-1}{a}\csc(ax+b) + c$

e.g) $\int \cos(2x-3)\,dx = \frac{1}{2}\sin(2x-3) + c$

e.g) $\int \frac{\sin 2x}{\cos x}\,dx = \int \frac{2\sin x \cancel{\cos x}}{\cancel{\cos x}}\,dx$

$\int 2\sin x\,dx = -2\cos x + c$

e.g) $\int 1 + \sin^2\theta \csc\theta\,d\theta =$

$\int 1 + \sin^{\cancel{2}}\theta \cdot \frac{1}{\cancel{\sin\theta}}\,d\theta = \int 1 + \sin\theta\,d\theta$

$= \theta + -\cos\theta + c$

e.g) $\int \sec(2x)\left(\sec(2x)+\tan(2x)\right)\, dx$

$= \int \sec^2(2x) + \sec(2x)\tan(2x)\, dx$

$= \frac{1}{2}\tan(2x) + \frac{1}{2}\sec(2x) + C$

e.g) $\int \frac{\sin 4x}{\cos^2 4x}\, dx = \int \frac{\sin 4x}{\cos 4x \cdot \cos 4x}\, dx$

$= \int \tan 4x \sec 4x\, dx = \frac{1}{4}\sec 4x + C$

e.g) $\int \sin(5x)\cot(5x)\, dx$

$= \int \cancel{\sin(5x)} \cdot \frac{\cos(5x)}{\cancel{\sin(5x)}}\, dx$

$= \int \cos(5x)\, dx = \frac{1}{5}\sin(5x) + C$

e.g) $\int (\sin x + \cos x)^2\, dx$

$\int \underbrace{\sin^2 x + \cos^2 x}_{1} + 2\sin x \cos x\, dx$

$\int 1 + \sin 2x\, dx$

$x + -\frac{1}{2}\cos 2x + C$

e.g) $\int \cot^2(4x) + \tan^2(5x)\, dx$

$\int \frac{1}{2} + \frac{1}{2}\cos(8x) + \sec^2(5x) - 1\, dx$

$\frac{1}{2}x + \frac{1}{16}\sin(8x) + \frac{1}{5}\tan(5x) - x + C$

• Remark :

① $\int \cosh(ax+b)\, dx = \frac{1}{a}\sinh$

② $\int \sinh(ax+b)\, dx = \frac{1}{a}\cosh(ax+b) + C$

③ $\int \operatorname{sech}^2(ax+b)\, dx = \frac{1}{a}\tanh(ax+b) + C$

④ $\int \operatorname{csch}^2(ax+b)\, dx = \frac{-1}{a}\coth(ax+b) + C$

⑤ $\int \operatorname{sech}(ax+b)\tanh(ax+b)\, dx = \frac{-1}{a}\operatorname{sech}(ax+b) + C$

⑥ $\int \operatorname{csch}(ax+b)\coth(ax+b)\, dx = \frac{-1}{a}\operatorname{csch}(ax+b) + C$

e.g) $\int \cosh(3x-5)\, dx$

$= \frac{1}{3}\sinh(3x-5) + C$

e.g) $\int \text{sech}^2(2x)\,dx$

$\frac{1}{2}\tanh(2x) + c$

e.g) $\int \text{sech}(3x)\tanh(3x)\,dx$

$-\frac{1}{3}\text{sech}(3x) + c$

e.g) $\int \frac{3e^x}{\cosh x + \sinh x}\,dx = \int \frac{3e^x}{e^x}\,dx$

$= 3x + C$

• Remark :

① $\int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + C$

② $\int a^{bx+c}\,dx = \frac{a^{bx+c}}{b \cdot \ln a} + C$

e.g) $\int e^{2x-1}\,dx$

$\frac{e^{2x-1}}{2} + C$

e.g) $\int 2^{3x+5}\,dx$

$\frac{2^{3x+5}}{3\ln 2} + c$

e.g) $\int \frac{e^x + 3}{e^x}\,dx$

$\int \frac{e^x}{e^x} + \frac{3}{e^x}\,dx = \int 1 + 3e^{-x}\,dx$

$x + \frac{3e^{-x}}{-1} + c = x - 3e^{-x} + c$

e.g) $\int 3^{3-2\log_3 x}\,dx$

$\int 3^3 \cdot 3^{-2\log_3 x}$

$\int 27\ \cancel{3}^{\log_{\cancel{3}} x^{-2}}\,dx$

$\int 27 x^{-2}\,dx$

$= 27\frac{x^{-1}}{-1} + C = \frac{-27}{x} + C$