Find eign value + eigen value for

$$A = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$$

1) det $[A - \lambda I] = 0$

$$\begin{vmatrix} 0-\lambda & 1 \\ -1 & 0-\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm\sqrt{-1} = \pm i$$

For $\lambda = i$ the X is :

$[A - \lambda I] X = 0$

$$\begin{vmatrix} -i & 1 \\ -1 & -i \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$$X = \begin{vmatrix} 1 \\ i \end{vmatrix} \qquad i x_1 + 1 x_2 = 0$$

For $\lambda = -i$ the X is :-

$[A - \lambda I] X = 0$

$$\begin{vmatrix} i & 1 \\ -1 & i \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \qquad i x_1 + 1 x_2 = 0$$

$$X = \begin{vmatrix} 1 \\ i \end{vmatrix}$$

Theorem:

The transpose A^T of square matrix A has the same eigen value as A.

$A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad \det[A^T - \lambda I] = 0$

$$\begin{vmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = 0$$

$\lambda^2 + 1 = 0 \quad \lambda = \pm i$

Symmetric, skew-symmetric and orthogonal Matrix:

Def: a real square matrix $A = [a_{jk}]$ is called:

1) symmetric if $A^T = A$
2) skew-symmetric if $A^T = -A$
3) orthogonal if $A^T = A^{-1}$

EX: 1) symmetric

$$A = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

2) skew-symmetric

$$A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

3) orthogonal

$$A = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$$

* for any real square matrix A
  1) symmetric Matrix $(R) = 1/2\,[A + A^T]$
  2) skew-symmetric Matrix $(S) = 1/2\,[A - A^T]$

EX: let $A = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix}$, find R, S?

$$A^T = \begin{bmatrix} 9 & 2 & 5 \\ 5 & 3 & 4 \\ 2 & -8 & 3 \end{bmatrix}$$

$$R = 1/2\,[A + A^T] = \begin{bmatrix} 9 & 3.5 & 3.5 \\ 3.5 & 3 & -2 \\ 3.5 & -2 & 3 \end{bmatrix}$$

$$S = 1/2\,[A - A^T] = \begin{bmatrix} 0 & 1.5 & -1.5 \\ -1.5 & 0 & -6 \\ 1.5 & 6 & 0 \end{bmatrix}$$

Ex: Find det, eigenvalue of

$$A = \begin{vmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{vmatrix} \qquad |A| = -1$$

$$\det |A - \lambda I| = 0$$

$$\begin{vmatrix} 2/3 - \lambda & 1/3 & 2/3 \\ -2/3 & 2/3 - \lambda & 1/3 \\ 1/3 & 2/3 & -2/3 - \lambda \end{vmatrix} = 0$$

$$= -\lambda^3 + 2/3\,\lambda^2 + 2/3\,\lambda - 1 = 0$$

$$\lambda = -1,\ \frac{5 + i\sqrt{11}}{6},\ \frac{5 - i\sqrt{11}}{6}$$

$$|Z| = \sqrt{x^2 + y^2} = \sqrt{(5/6)^2 + (\sqrt{11}/6)^2} = 1$$

orthogonal Matrix

Ex: let $A = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

$$A^T = A^{-1}$$

$$A^{-1} = 1/|A|\, C^T$$

$$A^T = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = 1/1 \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$$

Theorem:

1) The determinant of orthogonal Matrix has value of 1 or -1

2) The eigen value of orthogonal Matrix A are Real or complex conjugate and has absolute value of 1.

Theorem:-

1) The eigen value of symmetric Matrix are Real

2) The eigen value of skew-symmetric matrix are pure Imaginary or zero.

EX: let $A = \begin{vmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{vmatrix}$ Find eigen v...

$\det(A-\lambda I)=0 \quad \begin{vmatrix} 0-\lambda & 9 & -12 \\ -9 & 0-\lambda & 20 \\ 12 & -20 & 0-\lambda \end{vmatrix} = 0$

$+(-\lambda)\begin{vmatrix} -\lambda & 20 \\ -20 & -\lambda \end{vmatrix} - 9\begin{vmatrix} -9 & 20 \\ 12 & -\lambda \end{vmatrix} + (-12)\begin{vmatrix} -9 & -\lambda \\ 12 & -20 \end{vmatrix} = 0$

$-\lambda^3 - 625\lambda = 0 \quad -\lambda[\lambda^2 + 625] = 0 \quad \lambda = 0, \pm 25i$

Theorem:

if an $n \times n$ Matrix A has n-distent eigen value then A has a basis of eigen vector $X_1, X_2, \ldots X_n$ for R^n

EX: Find eigen value + eigen vector for $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

1) $\det[A-\lambda I) = 0$

$\begin{vmatrix} 5 & 3 \\ 3 & 5 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} =$

$= \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0$

$(5-\lambda)(5-\lambda) - 9 = 0$

$\lambda^2 - 10\lambda + 16 = 0$

$(\lambda-2)(\lambda-8) = 0 \quad \lambda = 2, 8$

for $\lambda = 2$ the X is:

$[A - \lambda I]X = 0$

$\begin{vmatrix} 3 & 3 \\ 3 & 3 \end{vmatrix} \begin{vmatrix} X_1 \\ X_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \quad 3X_1 + 3X_2 = 0 \quad X = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

* for $\lambda = 8$ the X is:

$[A - \lambda I]X = 0$

$\begin{vmatrix} -3 & 3 \\ 3 & -3 \end{vmatrix} \begin{vmatrix} X_1 \\ X_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$

$-3x_1 + 3x_2 = 0. \qquad X = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$

Similar Matrix + similarity transformation

let $n \times n$ matrix $\hat{A}$ is called similar to $n \times n$ matrix A iff

$$\hat{A} = P^{-1} A P$$

where P = is non-singular

Theorem:

1) if $\hat{A}$ is similar to A, then $\hat{A}$ has the same eigen value as A.

2) if X is an eigen vector of A, then $y = P^{-1} X$ is the eigen vector of $\hat{A}$ corresponding to same eigen value.

$$\begin{array}{ccc} A & \xrightarrow{\hat{A} = P^{-1} A P} & \hat{A} \\ \downarrow & & \downarrow \\ \lambda & \xrightarrow{=} & \lambda \\ \downarrow & & \downarrow \\ X & \xrightarrow{y = P^{-1} X} & y \end{array}$$

Ex: let $A = \begin{vmatrix} 6 & -3 \\ 4 & -1 \end{vmatrix}$ and $P = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}$

find eigen value + eigen vector of A and $\hat{A}$?

for A:

$$\det [A - \lambda I] = 0$$

$$\begin{vmatrix} 6-\lambda & -3 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$(6-\lambda)(-1-\lambda) + 12 = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0 \qquad \lambda = 2, 3$

- for $\lambda = 2$ the X is:

$[A - \lambda I] X = 0$

$\begin{vmatrix} 4 & -3 \\ 4 & -3 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \qquad 4x_1 - 3x_2 = 0$

$X = \begin{vmatrix} 3 \\ 4 \end{vmatrix}$

- for $\lambda = 3$ the X is:

$[A - \lambda I] X = 0$

$\begin{vmatrix} 3 & -3 \\ 4 & -4 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \qquad 3x_1 - 3x_2 = 0$

$X = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$

$$\hat{A} = P^{-1}AP = \frac{1}{1}\begin{vmatrix} 4 & -3 \\ -1 & 1 \end{vmatrix}\begin{vmatrix} 6 & -3 \\ 4 & -1 \end{vmatrix}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix}$$

$\det[\hat{A} - 2I| = 0$

$$\begin{vmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0 \qquad (3-\lambda)(2-\lambda) - 0 = 0$$

$$\lambda = 3, 2$$

For $\lambda = 2$ they is:

$(\hat{A} - 2I)\, y = 0 \qquad \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\begin{vmatrix} y_1 \\ y_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$

$y = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$

For $\lambda = 3$ they is:

$[\hat{A} - 2I]\; y = 0$

$$\begin{vmatrix} 0 & 0 \\ 0 & -1 \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$$y = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$$

$$y_1 = P^{-1} X_1 = 1/1 \begin{vmatrix} 4 & -3 \\ -1 & 1 \end{vmatrix} \begin{vmatrix} 3 \\ 4 \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$$

$$y_2 = P^{-1} X_2 = 1/1 \begin{vmatrix} 4 & -3 \\ -1 & 1 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$$