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دفتـر

ابلايد

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Lec-1-

28-1/2020

Applied

13.1 → complex Numbers and their Geometric Representation

$x^2 = -1 \rightarrow x = \pm\sqrt{-1} = \pm \hat{i} = \pm \hat{j}$

* any complex Number can be written In the form :—

$z = x + \hat{i} y$
$\begin{cases} x = Re\{z\} \\ y = IM\{z\} \end{cases}$

ex:- $z_1 = 1 + \hat{i} = 1 + \hat{i}(1)$

$x = 1$

$y = 1$

$z_2 = -3 - 4\hat{i}$

$x = -3$

$y = -4$

$z_3 = 5 = 5 + \hat{i}(0)$

$x = 5$

$y = 0$

$z_4 = -4\hat{i} = 0 - 4\hat{i}$

$x = 0$

$y = -4$

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Rules :- Two complex number are equal if Real part are equal IMaginary part are equal

ex:- $z_1 = 2 + 4\hat{i}$

$z_2 = x + 2y\hat{i}$

find x, y if $z_1 = z_2$?

* نساوي الكلية مع بعض

$2 = x$

$4 = 2y \longrightarrow y = 2$

* Addition of complex number :-

$$z_1 + z_2 = (x_1 + y_1\hat{i}) + (x_2 + y_2\hat{i})$$

$$= (x_1 + x_2) + \hat{i}(y_1 + y_2)$$

ex:- Let $z_1 = 2 + 4\hat{i}$

$z_2 = -3 + 2\hat{i}$

Find $z_1 + z_2 = ??$

$$z_1 + z_2 = -1 + 6\hat{i}$$

* subtraction of complex number :-

$$z_1 - z_2 = (x_1 + y_1\hat{i}) - (x_2 + y_2\hat{i})$$

$$= x_1 + y_1\hat{i} - x_2 - y_2\hat{i}$$

$$= \underbrace{(x_1 - x_2)}_{x} + \hat{i}\underbrace{(y_1 - y_2)}_{y}$$

$$= x + \hat{i}y$$

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ex:- $z_1 = 3 - 3\hat{i}$

$z_2 = -2 - 2\hat{i}$

find $z_1 - z_2$?

$z_1 - z_2 = 5 - 1\hat{i}$

* Multiplication of complex number :-

$z_1 z_2 = (x_1 + y_1 \hat{i})(x_2 + y_2 \hat{i})$

يعوض بالها (-1)

$= x_1 x_2 + x_1 y_2 \hat{i} + x_2 y_1 \hat{i} + y_1 y_2 \hat{i}^2$

$\hat{i} = \sqrt{-1}$

$\hat{i}^2 = -1$

$\hat{i}^3 = -\hat{i}$

* $= (x_1 x_2 - y_1 y_2) + \hat{i}(x_1 y_2 + x_2 y_1)$

* Ex :- $z_1 = 1 - \hat{i}$

$z_2 = 2 + 2\hat{i}$

find $z_1 z_2$??

$z_1 z_2 = (1 - \hat{i})(2 + 2\hat{i}) = 2 \cancel{+ 2\hat{i}} \cancel{- 2\hat{i}} - 2\hat{i}^2$

$= 2 + 2 = 4$

$= 4 + \hat{i}(0)$

* scalar Multiplication of complex number :-

$cz = c(x + \hat{i}y)$

$= cx + cy\hat{i}$

Ex:- $-5(-2 + 4\hat{i}) = 10 - 20\hat{i}$

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* complex conjegate of complex number :- $(\bar{z})$

$$z = x + \hat{i}y \longrightarrow \bar{z} = x - \hat{i}y$$

ex:- Let : $z_1 = 2 + 4\hat{i}$

$$\bar{z} = 2 - 4\hat{i}$$

$$z_2 = -4\hat{i} + 2 = 2 - 4\hat{i}$$

$$\bar{z} = 2 + 4\hat{i}$$

ex:- $(2+4\hat{i})(2-4\hat{i})$ * فحالة الضرب بتخلص من I Mag

$$= 4 - \cancel{8\hat{i}} + \cancel{8\hat{i}} - 16\hat{i}^2$$

$$= 4 + 16 = 20$$

* Division of complex number :-

$$\frac{z_1}{z_2} = \frac{x_1 + y_1\hat{i}}{x_2 + y_2\hat{i}}$$ ⇐ بضرب بمرافق للمقام

$$= \frac{x_1 + y_1\hat{i}}{x_2 + y_2\hat{i}} * \frac{x_2 - y_2\hat{i}}{x_2 - y_2\hat{i}}$$

$$= \frac{x_1x_2 - x_1y_2\hat{i} + x_2y_1\hat{i} - y_1y_2\overset{-1}{\hat{i}^2}}{x_2^2 - \cancel{x_2y_2\hat{i}} + \cancel{x_2y_2\hat{i}} - y_2^2\underset{-1}{\hat{i}^2}}$$

$$= \frac{(x_1x_2 + y_1y_2) + \hat{i}(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

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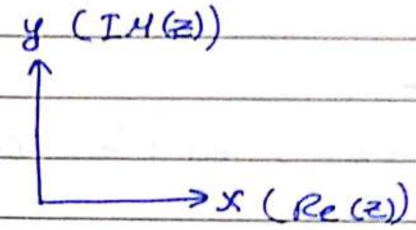
ex:- find $\frac{1+\hat{i}}{2-\hat{i}}$

$$\frac{1+\hat{i}}{2-\hat{i}} * \frac{2+\hat{i}}{2+\hat{i}} = \frac{2+\hat{i}+2\hat{i}+\hat{i}^2}{4+1} = \frac{1+3\hat{i}}{5} = \frac{1}{5} + \frac{3}{5}\hat{i}$$

* complex plane :-



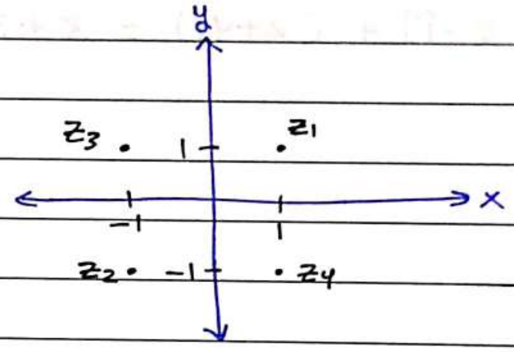
ex:-

$z_1 = 1+\hat{i}$

$z_2 = -1-\hat{i}$

$z_3 = -1+\hat{i}$

$z_4 = 1-\hat{i}$



* Rule :-

① $z\bar{z} = x^2 + y^2$

② $Re(z) = x = \frac{z+\bar{z}}{2}$

③ $IM(z) = y = \frac{z-\bar{z}}{2\hat{i}}$

④ $\overline{(z_1+z_2)} = \bar{z}_1 + \bar{z}_2$

⑤ $\overline{(z_1-z_2)} = \bar{z}_1 - \bar{z}_2$

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⑥ $\overline{(z_1 z_2)} = \overline{z_1}\,\overline{z_2}$

⑦ $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$

ex:- $z_1 = 1 + \hat{i}$

$z_2 = 2 - 4\hat{i}$

find $\overline{(z_1 + z_2)}$, $\overline{z_1} + \overline{z_2}$??

$\overline{(z_1 + z_2)} = \overline{(3 - 3\hat{i})} = 3 + 3\hat{i}$

$\overline{z_1} + \overline{z_2} = (1 - \hat{i}) + (2 + 4\hat{i}) = 3 + 3\hat{i}$

Lec - 2-

30/1/2020

13.2 → polar from of complex Numbers powers and Roots :-

$$z = x + \hat{i}y = re^{\hat{i}\theta}$$

$|z| = r = \sqrt{x^2+y^2}$ → absolute value
→ modulus
→ Magnituds

$\theta = \tan^{-1} \frac{y}{x}$ $y \to$ IM, $x \to$ Re

$x = r\cos\theta$

$y = r\sin\theta$

ex:- Let $z = 1+\hat{i}$ find the polar form ?

$x = 1 \quad y = 1$

$r = \sqrt{x^2+y^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$z = \sqrt{2}\, e^{\hat{i}\frac{\pi}{4}}$

ex:- Let $z = \sqrt{2}\, e^{\hat{i}\frac{\pi}{4}}$

$r = \sqrt{2}$

$\theta = \frac{\pi}{4}$

$x = r\cos\theta = \sqrt{2}\cos\frac{\pi}{4} = 1$

$y = r\sin\theta = \sqrt{2}\sin\frac{\pi}{4} = 1$

$z = 1 + \hat{i}$

Lec 2

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ex:- find θ for

① $z = 1 + \hat{i}$ → الربع الأول
② $z = -1 - \hat{i}$ → الربع الثالث
③ $z = -1 + \hat{i}$ → الربع الثاني
④ $z = 1 - \hat{i}$ → الربع الرابع

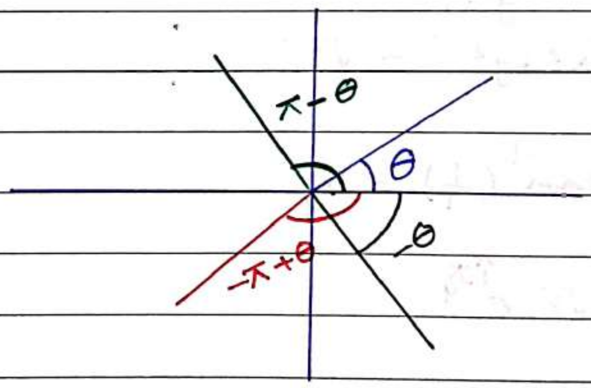
① $\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

② $\theta = \tan^{-1}\left(\frac{-1}{-1}\right) = \frac{-4}{4}\pi + \frac{\pi}{4} = \frac{-3\pi}{4}$

③ $\theta = \tan^{-1}\left(\frac{+1}{-1}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

④ $\theta = \tan^{-1}\left(\frac{-1}{1}\right) = \frac{-\pi}{4}$

* $-\pi \leq \theta \leq \pi$



* Eular formula :-

$$e^{\hat{i}\theta} = \underbrace{\cos\theta}_{x} + \hat{i}\underbrace{\sin\theta}_{y}$$

Lec 2

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* principle value = $Ag(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$

* $\arg(z) = Arg(z) \pm 2\pi n$

ex:- Let $z = 1 + \hat{i}$ find $Arg(z)$, $\arg(z)$??

$x = 1 \quad y = 1$

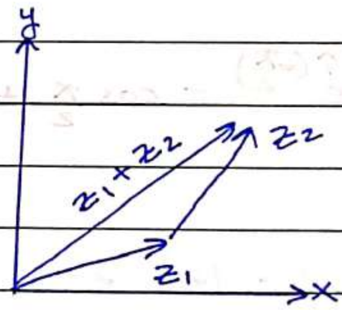
$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$Arg(z) = \theta = \frac{\pi}{4}$

$\arg(z) = \frac{\pi}{4} \pm 2\pi n$

* ~~Pitagutra~~ Trianglue Ingquality :-

$|z_1 + z_2| \leq |z_1| + |z_2|$



* Multiplication and division of complex number :-

① Let $z_1 = r_1 e^{\hat{i}\theta_1}$, $z_2 = r_2 e^{\hat{i}\theta_2}$

$z_1 z_2 = (r_1 e^{\hat{i}\theta_1})(r_2 e^{\hat{i}\theta_2}) = r_1 r_2 e^{\hat{i}(\theta_1+\theta_2)}$

$z = r_1 r_2 \left[\cos(\theta_1+\theta_2) + \hat{i}\sin(\theta_1+\theta_2)\right]$

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* division :-

$$\frac{z_1}{z_2} = \frac{r_1 e^{\hat{i}\theta_1}}{r_2 e^{\hat{i}\theta_2}} = \frac{r_1}{r_2} e^{\hat{i}(\theta_1 - \theta_2)}$$

$$z = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + \hat{i}\sin(\theta_1 - \theta_2)\right]$$

ex:— $z_1 = 2e^{\hat{i}\frac{\pi}{4}}$ $\quad z_2 = 4e^{\hat{i}\frac{3\pi}{4}}$

① $z_1 z_2 = 8e^{\hat{i}\left(\frac{\pi}{4}+\frac{3\pi}{4}\right)} = 8e^{\hat{i}\pi} = -8$

② $\frac{z_1}{z_2} = \frac{1}{2}e^{\hat{i}\left(\frac{\pi}{4}-\frac{3\pi}{4}\right)} = \frac{1}{2}e^{\hat{i}\left(-\frac{\pi}{2}\right)} = -\frac{1}{2}\hat{i}$

$$e^{\hat{i}\pi} = \cos\pi + \hat{i}\sin\pi = -1$$

$$e^{\hat{i}\left(-\frac{\pi}{2}\right)} = \cos\frac{\pi}{2} + \hat{i}\sin\frac{\pi}{2} = \hat{i}$$

* $|z_1 z_2| = |z_1|\,|z_2|$
* $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

* Power of complex number :-

$$(1+\hat{i})^2 = 1 + 2\hat{i} + \hat{i}^2 = 1 + 2\hat{i} - 1 = 2\hat{i}$$

$$\Rightarrow z^n = (x+\hat{i}y)^n = (re^{\hat{i}\theta})^n = r^n e^{\hat{i}\theta n}$$

$$= r^n\left[\cos(\theta n) + \hat{i}\sin(\theta n)\right]$$ but $\rightarrow n =$ integer value

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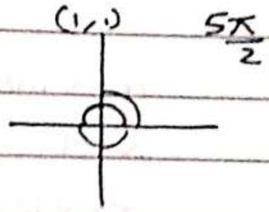
Lec 2

ex:- find $(1+\hat{i})^{10} = ?$



$n = 10$ integer

$x = 1 \quad y = 1$

$r = \sqrt{2} \qquad \theta = \tan^{-1} 1 = \frac{\pi}{4}$

$$(1+\hat{i})^{10} = \left(\sqrt{2}\, e^{\hat{i}\frac{\pi}{4}}\right)^{10} = 2^5\, e^{\hat{i}\frac{\pi}{4}\times 10} = 32\, e^{\hat{i}\frac{5\pi}{2}}$$

$$= 32\left[\cos\frac{5\pi}{2} + \hat{i}\sin\frac{5\pi}{2}\right]$$

$$= 32\left[0 + \hat{i}\right] = 32\hat{i}$$

* Demoiver formula :-

$$e^{\hat{i}\theta n} = e^{\hat{i}\theta n}$$

$$\left[e^{\hat{i}\theta}\right]^n = e^{\hat{i}[\theta n]}$$

$$\left[\cos\theta + \hat{i}\sin\theta\right]^n = \cos(\theta n) + \hat{i}\sin(\theta n)$$

* Root of complex number :-

$$\sqrt[n]{z} = \sqrt[n]{x+\hat{i}y}$$

$$= \sqrt[n]{r}\left[\cos\frac{\theta + 2\pi k}{n} + \hat{i}\sin\frac{\theta + 2\pi k}{n}\right]$$

$\Rightarrow \; r = \sqrt{x^2+y^2} \qquad \theta = \tan^{-1}\frac{y}{x}$

$n =$ integer $\qquad k = 0, 1, 2 \text{ ---- } (n-1)$

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Lec - 2 -

ex:— find $\sqrt[3]{1} = \sqrt[3]{1+\hat{i}0}$

$n = 3$ integer

$x = 1 \quad y = 0$

$r = \sqrt{1^2+0^2} = 1$

$\theta = \tan^{-1}(\frac{0}{1}) = 0$

$k = 0, 1, 2 \leftarrow$ نستدل عليهم (n) = 3 تعني هناك

٣ حلول للمعادلة (الجذر)

① the 1^{st} Root at $k=0$

$$= \sqrt[3]{1}\left[\cos\frac{0+2\pi.0}{3} + \hat{i}\sin\theta\ \frac{0+2\pi\times 0}{3}\right]$$

$$= 1\left[1+0\right] = 1$$

② the 2^{nd} Root at $k=1$

$$\sqrt[3]{1}\left[\cos\frac{0+2\pi\,1}{3} + \hat{i}\sin\frac{0+2\pi}{3}\right]$$

$$= 1\left[\cos\frac{2\pi}{3} + \hat{i}\sin\frac{2\pi}{3}\right]$$

③ the 3^{rd} Root at $k=2$

$$\sqrt[3]{1}\left[\cos\frac{0+2\pi\times 2}{3} + \hat{i}\sin\frac{0+2\pi\times 2}{3}\right]$$

$$= 1\left[\cos\frac{4\pi}{3} + \hat{i}\sin\frac{4\pi}{3}\right.$$

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Lec-2

H.W :—

① $\sqrt[5]{1}$ ⇒ $x=1$ $y=0$ $r=\sqrt{1}=1$ $\theta=\tan 0=0$ $n=5$

$k=0,1,2,3,4$

① first root at $k=0$ → $\sqrt[5]{1}\left[\cos\frac{0+2\pi\times 0}{\cancel{8}5}+\hat{i}\sin\frac{0+2\pi\times 0}{5}\right]=1[1+0]=1$

② ~~s~~second root at $k=1$ → $\sqrt[5]{1}\left[\cos\frac{2\pi}{5}+\hat{i}\sin\frac{2\pi}{5}\right]$

③ ~~third~~ 3^{ed} root at $k=2$ → $\sqrt[5]{1}\left[\cos\frac{4\pi}{5}+\hat{i}\sin\frac{4\pi}{5}\right]$

④ 4^{ed} root at $k=3$ → $\sqrt[5]{1}\left[\cos\frac{6\pi}{5}+\hat{i}\sin\frac{6\pi}{5}\right]$

⑤ 5^{th} root at $k=4$ → $\sqrt[5]{1}\left[\cos\frac{8\pi}{5}+\hat{i}\sin\frac{8\pi}{5}\right]$

② $\sqrt[6]{1}$ ⇒ $x=1$ $y=0$ $r=1$ $\theta=0$ $n=6$ $k=0,1,2,3,4,5$

① 1^{st} root at $k=0$ → 1

② 2^{nd} root at $k=1$ → $\sqrt[6]{1}\left[\cos\frac{2\pi}{6}+\hat{i}\sin\frac{2\pi}{6}\right]=\cos\frac{\pi}{3}+\hat{i}\sin\frac{\pi}{3}$

③ 3^{rd} root at $k=2$ → $\sqrt[6]{1}\left[\cos\frac{4\pi}{6}+\hat{i}\sin\frac{4\pi}{6}\right]=\cos\frac{2\pi}{3}+\hat{i}\sin\frac{2\pi}{3}$

④ 4^{th} root at $k=3$ → $\sqrt[6]{1}\left[\cos\frac{6\pi}{6}+\hat{i}\sin\frac{6\pi}{6}\right]=\cos\pi+\hat{i}\sin\pi=-1$

⑤ 5^{th} root at $k=4$ → $\sqrt[6]{1}\left[\cos\frac{8\pi}{6}+\hat{i}\sin\frac{8\pi}{6}\right]=\cos\frac{4\pi}{3}+\hat{i}\sin\frac{4\pi}{3}$

⑥ 6^{th} root at $k=5$ → $\sqrt[6]{1}\left[\cos\frac{10\pi}{6}+\hat{i}\sin\frac{10\pi}{6}\right]=\cos\frac{5\pi}{3}+\hat{i}\sin\frac{5\pi}{3}$

2/2/2020

Lec -8-

13.5 → exponetial function of complex number :-

$$z = x + \hat{i}y$$

$$e^{z} = e^{x+\hat{i}y} = e^{x}\, e^{\hat{i}y}$$

$$= e^{x}\left[\cos y + \hat{i}\sin y\right]$$

$$= \underbrace{e^{x}\cos y}_{x} + \hat{i}\,\underbrace{\sin y\, e^{x}}_{y}$$

$$|e^{z}| = r = \sqrt{x^2 + y^2} = \sqrt{(e^{x}\cos y)^2 + (e^{x}\sin y)^2}$$

$$= e^{x}\underbrace{\sqrt{\cos^2 y + \sin^2 y}}_{1} = e^{x}$$

* To find $\theta = Arg(z)$ ⇒

$$\theta = \tan^{-1}\frac{y}{x}$$

$$= \tan^{-1}\left[\frac{\cancel{e^{x}}\sin y}{\cancel{e^{x}}\cos y}\right]$$

$$= \tan^{-1}[\tan y]$$

$$= y$$

* find $arg(z)$ ⇒

$$arg(z) = Arg(z) \pm 2\pi n$$

$$= y \pm 2\pi n$$

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Lec - 3 -

ex:- find r, θ $e^{1.4-0.6\hat{i}} = ?$

$$e^{1.4-0.6\hat{i}} = e^{1.4} e^{-0.6\hat{i}}$$

$$= e^{1.4} \left[\cos(-0.6) + \hat{i}\sin(-0.6)\right]$$

$$= \underbrace{e^{1.4}\cos(0.6)}_{Re} - \underbrace{\hat{i}e^{1.4}\sin(0.6)}_{IM}$$

$r = e^{x} = e^{1.4}$

$\theta = -0.6$

$\arg(z) = -0.6 \pm 2\pi n$

or $r = \sqrt{x^2+y^2} = \sqrt{(e^{1.4}\cos(0.6))^2 + (e^{1.4}\sin(0.6))^2} = e^{1.4}$

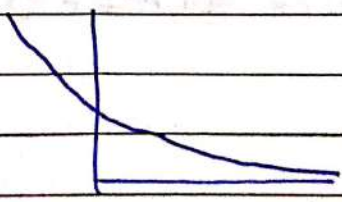
$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left[\frac{-e^{1.4}\sin(0.6)}{e^{1.4}\cos(0.6)}\right] = -0.6$$

$\arg(z) = -0.6 \pm 2\pi n$

* $e^{\hat{i}\pi} = \cos\pi + \hat{i}\sin\pi = -1$

* $e^{\hat{i}2\pi} = \cos 2\pi + \hat{i}\sin 2\pi = 1$

* Note $\Rightarrow$ $e^{z} \neq 0$ so e^{z} is entire function



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Lec-3-

3.6 → Trigonometric and hyperbolic function:-

$e^{\hat{i}x} = \cos x + \hat{i}\sin x$ — — ①

$e^{\hat{i}(-x)} = \cos(-x) + \hat{i}\sin(-x)$ $\qquad \cos(-x) \longrightarrow \cos x$

$= \cos x - \hat{i}\sin x$ ——— ② $\qquad \sin(-x) \longrightarrow -\sin x$

① $1+2 \longrightarrow e^{\hat{i}x} + e^{\hat{i}(-x)} = 2\cos x$

$$\boxed{\cos x = \frac{e^{\hat{i}x} + e^{-\hat{i}x}}{2}}$$

② $1-2 \longrightarrow e^{\hat{i}x} - e^{\hat{i}(-x)} = 2\hat{i}\sin x$

$$\boxed{\sin x = \frac{e^{\hat{i}x} - e^{-\hat{i}x}}{2\hat{i}}}$$

* $\cos(z) = \dfrac{e^{\hat{i}z} + e^{-\hat{i}z}}{2}$

* $\sin(z) = \dfrac{e^{\hat{i}z} - e^{-\hat{i}z}}{2\hat{i}}$

* $\tan(z) = \dfrac{\sin(z)}{\cos(z)} = \dfrac{\frac{e^{\hat{i}z} - e^{-\hat{i}z}}{2\hat{i}}}{\frac{e^{\hat{i}z} + e^{-\hat{i}z}}{2}} = \dfrac{e^{\hat{i}z} - e^{-\hat{i}z}}{\hat{i}(e^{\hat{i}z} + e^{-\hat{i}z})}$

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Lec -3-

* $\cot(z) = \frac{1}{\tan(z)} = \frac{\cos z}{\sin z} = \frac{\hat{i}(e^{\hat{i}z} + e^{-\hat{i}z})}{e^{\hat{i}z} - e^{-\hat{i}z}}$

* $\sec(z) = \frac{1}{\cos(z)} = \frac{2}{e^{\hat{i}z} + e^{-\hat{i}z}}$

* $\csc(z) = \frac{1}{\sin(z)} = \frac{2\hat{i}}{e^{\hat{i}z} - e^{-\hat{i}z}}$

* $[\cos(z)]' = -\sin(z)$

* $[\sin(z)]' = \cos(z)$

* $[\tan(z)]' = \sec^2(z)$

* $(e^z)' = e^z$

* $e^{z_1+z_2} = e^{z_1} e^{z_2}$

* $e^{z_1-z_2} = \frac{e^{z_1}}{e^{z_2}}$

* $\cos(z) = \cos(x+\hat{i}y)$
$= \cos x \cosh y - \hat{i} \sin x \sinh(y)$

* $\sin(z) = \sin(x+\hat{i}y)$
$= \sin x \cosh(y) + \hat{i} \cos x \sinh(y)$

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Lec-3-

ex:- find $\cos(\pi + \hat{i}5)$?

$= \cos\pi \cosh(5) + \hat{i}\sin\pi \sinh(5)$

$= -\cosh(5)$

ex:- find $\sin(2\pi + 4\hat{i})$??

$= \sin(2\pi)\cosh(4) + \hat{i}\cos(2\pi)\sinh(4)$

$= \hat{i}\sinh(4)$

$\Rightarrow \sin^2(z) + \cos^2(z) = 1$

$\Rightarrow \cosh^2(\cancel{x}) = 1 + \sinh^2(\cancel{x})$

$\Rightarrow |\cos(z)|^2 = \cos^2 x + \sinh^2(y)$

$\Rightarrow |\sin(z)|^2 = \sin^2(x) + \sinh^2(y)$

$\Rightarrow \sin(z_1)\cos(z_2) = \frac{1}{2}[\sin(z_1+z_2) + \sin(z_1-z_2)]$

$\Rightarrow \cos(z_1 \pm z_2) = \cos(z_1)\cos(z_2) \mp \sin(z_1)\sin(z_2)$

$\Rightarrow \sin(z_1 \pm z_2) = \sin(z_1)\cos(z_2) \pm \sin(z_2)\cos(z_1)$

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Lec - 4 -

* hyperbolic function :-

① $\cosh(z) = \frac{e^{z} + e^{-z}}{2}$

② $\sinh(z) = \frac{e^{z} - e^{-z}}{2}$

③ $[\cosh(z)]' = \sinh(z)$

④ $[\sinh(z)]' = \cosh(z)$

⑤ $\tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$

⑥ $\coth(z) = \frac{\cosh(z)}{\sinh(z)} = \frac{e^{z} + e^{-z}}{e^{z} - e^{-z}}$

⑦ $\operatorname{sech}(z) = \frac{1}{\cosh(z)} = \frac{2}{e^{z} + e^{-z}}$

⑧ $\operatorname{csch}(z) = \frac{1}{\sinh(z)} = \frac{2}{e^{z} - e^{-z}}$

* complex number trigonometric and hyperbolic function are related

* $\cosh(iz) = \cos(z)$

* $\sinh(iz) = i\sin(z)$

* $\cos(iz) = \cosh(z)$

* $\sin(iz) = i\sinh(z)$

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Lec - 4 -

* $\hat{i}z = \hat{i}\,[x + \hat{i}y] = \hat{i}x + \hat{i}^2 y$

$= \underbrace{-y}_{Rel} + \hat{i}\underbrace{x}_{IM}$

* $\frac{z}{\hat{i}} = \frac{x + \hat{i}y}{\hat{i}} = \frac{x + \hat{i}y}{\hat{i}} * \frac{\hat{i}}{\hat{i}}$

$= \frac{\hat{i}x + \hat{i}^2 y}{\hat{i}^2} = \frac{\hat{i}x - y}{-1}$

$= \underbrace{y}_{Rel} - \hat{i}\underbrace{x}_{IM}$

* $\cosh(z) = \cosh(x + \hat{i}y)$

$= \cosh(x)\cos y + \hat{i}\sinh x \sin y$

* $\sinh(z) = \sinh(x + \hat{i}y)$

$= \sinh(x)\cos y + \hat{i}\cosh x \sin y$

ex:- find $\overset{\cosh}{\hat{i}}(\underbrace{5}_{x} + \underbrace{2\pi}_{y}\hat{i})$

$= \cosh 5 \cos 2\pi + \hat{i}\sinh 5 \sin 2\pi$

$= \cosh 5 + \hat{i}(0)$

ex:- find $\sinh(2 + \hat{i}\pi)$

$= \sinh 2 \cos\pi + \hat{i}\cosh 2 \sinh\pi$

$= -\sinh 2$

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Lec - 4 -

* $\cosh(z_1+z_2) = \cosh(z_1)\cosh(z_2) + \sinh(z_1)\sinh(z_2)$

* $\sinh(z_1+z_2) = \sinh(z_1)\cosh(z_2) + \cosh z_1 \sinh z_2$

* $\cosh^2(z) = \sinh^2(z) = 1$

* $\cosh^2(z) + \sinh^2(z) = \cosh(2z)$

13.7 → logarithem, General power, principle value :-

$$z = x + \hat{i}y = re^{\hat{i}\theta}$$

$$\begin{aligned} \operatorname{Ln} z = \operatorname{Ln} x+\hat{i}y &= \operatorname{Ln} re^{\hat{i}\theta} \\ &= \operatorname{Ln} r + \operatorname{Ln} e^{\hat{i}\theta} \\ &= \operatorname{Ln} r + \hat{i}\theta \operatorname{Ln} e \\ &= \underbrace{\operatorname{Ln} r}_{x} + \hat{i}\underbrace{\theta}_{y} \end{aligned}$$

$\operatorname{Ln} e = 1$

$\operatorname{Log} 10 = 1$

$$r = \sqrt{x^2+y^2}$$

$$\theta = \tan^{-1}\frac{y}{x} \pm 2\pi n$$

*ex:- find $\operatorname{Ln}(1+\hat{i}) = ??$

$x = 1$

$y = 1$

$\theta = \frac{\pi}{4}$

$r = \sqrt{2}$

$$\begin{aligned} \operatorname{Ln}(1+\hat{i}) &= \operatorname{Ln} r + \hat{i}\theta \\ &= \operatorname{Ln}\sqrt{2} + \hat{i}\left[\frac{\pi}{4} \pm 2\pi n\right] \end{aligned}$$

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Lec - 4 -

* find $\ln(1) = ??$

$\ln(1) = \ln(1 + \hat{i}(0))$ $\quad x = 1 \quad r = 1$

$y = 0 \quad \theta = 0 \leftarrow$ الربع الأول

$= \ln r + \hat{i}\theta$

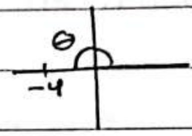
$= \ln 1 + \hat{i}[0 \pm 2\pi n]$

$= \hat{i}[2\pi n]$

* ex:- find $\ln(-4) = ??$

$\ln[-4 + \hat{i}(0)]$ $\quad x = -4 \quad y = 0$

$r = 4 \quad \theta = \pi \leftarrow$



$\ln[-4 + \hat{i}(0)] = \ln 4 + \hat{i}[\pi \pm 2\pi n]$

* $\ln(z_1 z_2) = \ln z_1 + \ln z_2$

* $\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$

* principle value of $\ln(z)$ is called $\operatorname{Ln}(z)$

$\operatorname{Ln}(z) = \ln(z) + \hat{i}\operatorname{Arg}(z)$

$\ln(z) = \operatorname{Ln}(z) \pm 2\pi n \hat{i}$

* ex:- find $\operatorname{Ln}(2\hat{i})$??

$\ln[0 + \hat{i}(2)] = \ln(r) + \hat{i}\theta$

$x = 0 \quad r = 2$

$y = 2 \quad \theta = \tan^{-1}\frac{2}{0} = \frac{\pi}{2}$

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lec - 4 -

$$\ln[0+\hat{i}2] = \ln 2 + \hat{i}\left[\frac{\pi}{2} \pm 2\pi n\right]$$

$$\mathrm{Ln}[0+\hat{i}2] = \ln 2 + \hat{i}\frac{\pi}{2}$$

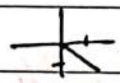
* ex:— find $\ln(1-\hat{i})$, $\mathrm{Ln}(1-\hat{i})$??

$x = 1$

$y = -1$

$r = \sqrt{2}$

$\theta = \tan^{-1}\frac{-1}{1} = \frac{-\pi}{4}$



$$\ln(+1-\hat{i}) = \ln r + \hat{i}\theta$$
$$= \ln\sqrt{2} + \hat{i}\left[\frac{-\pi}{4} \pm 2\pi n\right]$$

$$\mathrm{Ln}(1-\hat{i}) = \ln\sqrt{2} + \hat{i}\left[\frac{-\pi}{4}\right]$$

* $[\ln z]' = \frac{1}{z}$

* Logarithem of complex number :-

$\ln(x+\hat{i}y) = \ln r + \hat{i}\left[\tan^{-1}\frac{y}{x} \pm 2\pi n\right]$

ex:- find $\ln(-1-\hat{i})$, $Ln(-1-\hat{i})$??

$\ln(-1-\hat{i}) = \ln r + \hat{i}\theta$

$r = \sqrt{-1^2 + -1^2} = \sqrt{2}$

$\theta = \tan^{-1}\frac{-1}{-1} = -\pi + \frac{\pi}{4} = \frac{-3\pi}{4}$

$\ln(-1-\hat{i}) = \ln\sqrt{2} + \hat{i}\left[\frac{-3\pi}{4} \pm 2\pi n\right]$

$Ln(-1-\hat{i}) = \ln\sqrt{2} + \hat{i}\left[-\frac{3\pi}{4}\right]$

* General power of complex number:-

$z^c \Rightarrow z = x + \hat{i}y$

$c = 1, 2, 3 \text{ ---}$

$c = -1, -2, -3 \text{ ---}$

$c = \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \text{ --- } \frac{1}{n}$

$c = \frac{n_1}{n_2} \cdots \frac{2}{3} \text{ --- } \frac{3}{4}$

c = complex number

$z^n = e^{\ln(z)^c} = e^{c\ln z}$

ex:- $(\hat{i})^{\hat{i}} = e^{\ln(\hat{i})^{\hat{i}}} = e^{\hat{i}\ln\hat{i}}$

$= \ln\hat{i} = \ln r + \hat{i}\theta$

$= \ln(0+\hat{i}) = \ln 1 + \hat{i}\left[\frac{\pi}{2} \pm 2\pi n\right]$

$= \hat{i}\left[\frac{\pi}{2} \pm 2\pi n\right.$

$e^{\hat{i}[\hat{i}][\frac{\pi}{2} \pm 2\pi n]} = e^{\frac{-\pi}{2} \pm 2\pi n}$

ex:- find the principle value :-

$(\hat{i})^{\hat{i}} = e^{-\frac{\pi}{2} \mp 2\pi n}$

principle value happened at $n=0$

$= e^{-\frac{\pi}{2}}$

ex:- Find $(1+\hat{i})^{\hat{i}} = ??$

$(1+\hat{i})^{\hat{i}} = e^{\ln(1+\hat{i})^{\hat{i}}} = e^{\hat{i}\ln(1+\hat{i})}$

$\ln(1+\hat{i}) = \ln r + \hat{i}\theta$

$= \ln\sqrt{2} + \hat{i}\left[\frac{\pi}{4} \pm 2\pi n\right]$

$e^{\hat{i}[\ln\sqrt{2} + \hat{i}[\frac{\pi}{4} \pm 2\pi n]}$

$e^{\hat{i}\ln\sqrt{2}} \; e^{-\frac{\pi}{4} \pm 2\pi n} = e^{\hat{i}\ln\sqrt{2}} \; e^{-[\frac{\pi}{4} \pm 2\pi n]}$

$= e^{-[\frac{\pi}{4} \pm 2\pi n]} \left[\cos(\ln\sqrt{2}) + \hat{i}\sin(\ln\sqrt{2})\right]$

ex:— find $(1+\hat{i})^{2-\hat{i}} = ??$

$$(1+\hat{i})^{2-\hat{i}} = e^{\ln(1+\hat{i})^{2-\hat{i}}} = e^{(2-\hat{i})\ln(1+\hat{i})}$$

$$\ln(1+\hat{i}) = \ln r + \hat{i}\theta = \ln\sqrt{2} + \hat{i}\frac{\pi}{4} \pm 2\pi n$$

$$e^{(2-\hat{i})\left[\ln\sqrt{2} + \hat{i}\frac{\pi}{4} \pm 2\pi n\right]} = x_3 + \hat{i}y_3$$

$$= (2-\hat{i})\left[\ln\sqrt{2} + \hat{i}\left[\frac{\pi}{4} \pm 2\pi n\right]\right]$$

$$e^{x_3+\hat{i}y_3} = e^{x_3}\, e^{\hat{i}y_3}$$

$$= e^{x_3}\left[\cos y_3 + \hat{i}\sin y_3\right]$$

$$= e^{x_3}\cos y_3 + e^{x_3}\sin y_3\,\hat{i}$$

* principle value for $(1+\hat{i})^{2-\hat{i}}$ → happend at $n=0$

$$= e^{(2-\hat{i})(\ln\sqrt{2} + \hat{i}\frac{\pi}{4})}$$

$$= e^{2\ln\sqrt{2} + 2\hat{i}\frac{\pi}{4} - \hat{i}\ln\sqrt{2} + \frac{\pi}{4}}$$

$$= e^{\left(2\ln\sqrt{2} + \frac{\pi}{4}\right] + \hat{i}\left[\frac{\pi}{2} - \ln\sqrt{2}\right)}$$

6/2/2020

Lec - 5 —

ex:— $[x_1 + \hat{i} y_1]^{x_2+\hat{i}y_2}$

$$[x_1+\hat{i}y_1]^{x_2+\hat{i}y_2} = e^{\ln[x_1+\hat{i}y_1]^{x_2+\hat{i}y_2}}$$

$$= e^{[x_2+\hat{i}y_2]\ln[x_1+\hat{i}y_1]}$$

$$\ln[x_1+\hat{i}y_1] = \ln r + \hat{i}\theta$$

$$= \ln\sqrt{x_1^2+y_1^2} + \hat{i}\left[\tan^{-1}\frac{y_1}{x_1} \pm 2\pi n\right]$$

$$x_3 + \hat{i}y_3 = [x_2+\hat{i}y_2]\left[[\ln\sqrt{x_1^2+y_1^2}] + \hat{i}\{\tan^{-1}\frac{y_1}{x_1} \pm 2\pi n\}\right]$$

$$e^{x_3+\hat{i}y_3} = e^{x_3}e^{\hat{i}y_3}$$

$$= e^{x_3}[\cos y_3 + \hat{i}\sin y_3]$$

$$= e^{x_3}\cos y_3 + e^{x_3}\sin y_3\,\hat{i}$$

* $\left|e^{x_3+\hat{i}y_3}\right| = e^{x_3}$

* $\theta = \tan^{-1}\left(\frac{y}{x}\right) = y_3$

9/2/2020

Lec-6-

7.1 → Matrix, vector, addition and scalar multiplication:-

* matrix → is arectangular ariay of numbers or function which we will enclose in brakets.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} x & y \\ \sin x & \cos y \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

⇒ size of matrix $= \underbrace{m}_{\text{row}} \times \underbrace{n}_{\text{column}}$

⇒ vector is a matrix with one row of one column.

$$\Rightarrow V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \Rightarrow \text{column vector}$$

3x1

$$\Rightarrow b = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \Rightarrow \text{Row vector}$$

1x3

9/2/2020
Lec - 6 -

* linear system of equation ⇒ $\boxed{Ax = b}$

$$4x_1 + 6x_2 + 9x_3 = 6$$
$$6x_1 + 0x_2 - 2x_3 = 20$$
$$5x_1 - 8x_2 + x_3 = 10$$

↓

matrix ⇒ $\begin{bmatrix} 4 & 6 & 9 \\ 6 & 0 & -2 \\ 5 & -8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 10 \end{bmatrix}$

$A \quad x = b$

* Augmenting Matrix of $\boxed{A = \tilde{A}}$

$$\tilde{A} = [A \vdots b] = \left[\begin{array}{ccc:c} 4 & 6 & 9 & 6 \\ 6 & 0 & -2 & 20 \\ 5 & -8 & 1 & 10 \end{array}\right]$$

$$x = \begin{bmatrix} 3 \\ 1/2 \\ -1 \end{bmatrix}$$

* Square matrix : amatrix that has same Row and column

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

⇒ a_{11} , a_{22} , a_{33} ⇒ main diagonal element

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Lec -6-

Def:- equality of matrix

Two matrix $A=[a_{jk}]$ and $B=[b_{jk}]$ are equal iff

① have the same size $m \times n$

② The cross ponding element are equal

if $A \neq B$ then A, B, are different

① do not have same size

② the cross ponding element are not equal.

ex:- Let :- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

size = 2X2 size = 2X2

cross poinding element are equal A, B

so $A = B$

ex:- Let :- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ size = ~~2X3~~ 3X2

$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ size = ~~3X2~~ 2X3

so $A \neq B$

9/2/2020

Lec-6-

ex:- $A = \begin{bmatrix} 2x & 1 \\ 3y & z \end{bmatrix}$ $B = \begin{bmatrix} 8 & 1 \\ 6 & 2 \end{bmatrix}$

find x, y, z if A=B ??

$\Rightarrow$ $2x = 8 \rightarrow \boxed{x=4}$

$3y = 6 \longrightarrow \boxed{y=2}$

$\boxed{z=2}$

ex:- Let:- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$ $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}_{2\times 2}$

both of them are same size but not cross ponding

So $A \neq B$

Def :- addition of matrix

The sum of two matrix $A = [a_{jk}]$ and $B = [b_{jk}]$

of $\boxed{\text{same size}}$ is written $A + B = [a_{jk} + b_{jk}]$

and obtained by addition the cross ponding element

Note $\rightarrow$ matrix with different size can not be added.

9/1/2020

Lec - 6 -

ex:- Let:- $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times3}$

$B = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}_{2\times3}$

find A+B , A-B ??

$A+B = \begin{bmatrix} 1 & 3 & 3 \\ 6 & 5 & 7 \end{bmatrix}_{2\times3}$

$A-B = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 5 & 5 \end{bmatrix}_{2\times3}$

Def :- scalar multiplication :-

the product of any $m \times n$ matrix $A = [a_{jk}]$ and any scalar $\underline{\underline{C}}$ written CA and is an $m \times n$ matrix $CA = [ca_{jk}]$ obtained by multipling each element of A by c

* the size will be stay the same

ex:- let $A = \begin{bmatrix} 1, & 2 \\ 3 & 4 \end{bmatrix}$ find 3A
-A ??

① $3A = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$

② $-A = \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}$

9/9/2020

Lec - 8 -

ex:- let $A = \begin{bmatrix} 2.7 & -1.8 \\ 0 & 0.9 \\ 9 & -4.5 \end{bmatrix}$

find $-A$, $\frac{10}{9}A$??

① $-A = \begin{bmatrix} -2.7 & 1.8 \\ 0 & -0.9 \\ -9 & +4.5 \end{bmatrix}$

② $\frac{10}{9}A = \begin{bmatrix} 3 & -2 \\ 0 & 1 \\ 10 & -5 \end{bmatrix}$

* Rules ⇒
① $A + B = B + A$
② $(A+B) + C = A + (B+C) = A + B + C$
③ $A + 0 = A$
④ $A + (-A) = 0$

* Rules ⇒
① $c(A+B) = cA + cB$
② $(c+k)A = cA + kA$
③ $c(kA) = ck(A)$
④ $1A = A$

11/5/2020

Lec - 7 -

7.2 → matrix multiplication ⇒ الترتيب مهم جداً ⇐

Def → the product $C = AB$ of an $m \times n$ matrix $A = [a_{jk}]$ times are $r \times p$ matrix $B = [b_{jk}]$ is defined iff $\boxed{n = r}$ ⇒ the $m \times p$ matrix $C = [c_{jk}]$

$$c_{jk} = \sum_{i=1}^{n} a_{jk} b_{jk} = a_{j1} b_{j1} + \cdots + a_{jn} b_{jn}$$

↳ $j = 1, 2, \ldots, m$

$k = 1, 2, \ldots, p$

* $AB = C$

$\underbrace{m \times n \quad r \times p}_{\text{equal}} = m \times p$

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ex:- Let $A_{3\times2}$, $B_{2\times3}$ find size $C = AB = ??$

$\underbrace{A_{3\times2} \; B_{2\times3}}_{\text{equal}} = C_{3\times3}$

11/2/2020

Lec -7-

ex:- Let $A = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix}$

find AB ??

$A_{3\times\underbrace{3 \quad B_{3}}_{equal}\times 4} = C = 3\times4$

$C_{3\times4} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \end{bmatrix}$

* [كل الصف الاول * كل الاعمدة]

توضيح ← الاول بالصف * الاول بالعمود + الثاني بالصف * الثاني بالعمود

$C_{11} = (3\times2) + (5\times5) + (-1\times9) = 22$

$C_{12} = (3\times-2) + (5\times0) + (-1\times-4) = -2$

$C_{21} = (4\times2) + (0\times5) + (2\times9) = 26$

$\vdots$

$C_{34} = (-6\times1) + (-3\times8) + (2\times1) = -28$

$C_{3\times4} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & 14 & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$

ex:- Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

Find C= AB ?? $A_{2\times2} \; B_{2\times2} = C_{2\times2}$

$c = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) \end{bmatrix}$

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Lec -7-

ex:- $\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 22 \\ 43 \end{bmatrix}$

$\underbrace{2\times2 \quad 2\times1}_{\text{equal}} = 2\times1$

ex:- $\begin{bmatrix} 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 19 \end{bmatrix}$

$1\times3 \quad 3\times1 = 1\times1$

ex:- $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 1 \\ 6 & 12 & 2 \\ 12 & 24 & 4 \end{bmatrix}$

$3\times1 \quad 1\times3 = 3\times3$

* Note ⇒ $AB \neq BA$

① $A_{2\times2} \; B_{2\times1} = C_{2\times1}$

② $B_{2\times1} \; A_{2\times2}$ = can not be multiplication

* Rules :-
① $(KA)B = K(AB)$ K = scalar
② $A(BC) = (AB)C$
③ $(A+B)C = AC + BC$] الترتيب مهم جداً
④ $C(A+B) = CA + CB$

↳ order of multiplication is important ⇄

11/2/2020

Lec - 7 -

* product in terms of row and column vectors.

Let $A_{3\times3} = \begin{bmatrix} \text{——} & a_1 & \text{——} \\ \text{——} & a_2 & \text{——} \\ \text{——} & a_3 & \text{——} \end{bmatrix}$

$B_{3\times4} = \begin{bmatrix} | & | & | & | \\ b_1 & b_2 & b_3 & b_4 \\ | & | & | & | \end{bmatrix}$

$\underbrace{A_{3\times3} \; B_{3\times4}}_{\text{equal}} = C_{3\times4} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 \end{bmatrix}$

* Parallel processing of product :-

$AB = A \begin{bmatrix} | & | & | & & | \\ b_1 & b_2 & b_3 & \cdots & b_n \\ | & | & | & & | \end{bmatrix}$

$= \begin{bmatrix} | & | & | & & | \\ Ab_1 & Ab_2 & Ab_3 & \cdots & Ab_n \\ | & | & | & & | \end{bmatrix}$

ex :- $AB = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 7 \\ -1 & 4 & 6 \end{bmatrix}$

$2\times2 \quad 2\times3 = 2\times3$

$= \begin{bmatrix} 11 & 4 & 34 \\ -17 & 8 & -23 \end{bmatrix}$

or $\Longrightarrow$ سيع

Five Apple

11/2/2020

Lec -7-

$$\Rightarrow AB = \begin{bmatrix} | & | & | \\ Ab_1 & Ab_2 & Ab_3 \\ | & | & | \end{bmatrix}$$

$$Ab_1 = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ -17 \end{bmatrix}$$

$$Ab_2 = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$Ab_3 = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 34 \\ -23 \end{bmatrix}$$

$$AB = \begin{bmatrix} 11 & 4 & 34 \\ -17 & 8 & -23 \end{bmatrix}$$

13/2/2020

Lec -8-

* transposition :- the transpose of $m \times n$ matrix $A = [a_{jk}]$ is $n \times m$ matrix A^T (A transpose) that has first raw of A as it's first column and second raw of A as it's seconds column and raw.

ex:- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$ $A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}_{3\times3}$

$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 2 \end{bmatrix}_{2\times3}$ $B^T = \begin{bmatrix} 1 & 4 \\ 2 & 0 \\ 3 & 2 \end{bmatrix}_{3\times2}$

⇒ الصف يصبح عامود ⇐

ex:- Let

$A = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}$ $A^T = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}$

$B = [\,6 \;\; 2 \;\; 3\,]$ raw vector $B^T = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$ column vector

* Rules :-

① $(A^T)^T = A$

② $(A+B)^T = A^T + B^T$

③ $(cA)^T = cA^T$ c = scalar

④ $(AB)^T = B^T A^T$ ⇐ AB الأسماء للقلب

⑤ $(ABC)^T = C^T B^T A^T$ c = matrix

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Lec - 8 -

* squar matrix → matrix that has same column and raw.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{\underline{\underline{2}}\times\underline{\underline{2}}}$$

* symmetric matrix → iff $A^T = A$

ex :- $A = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix}_{3\times3}$

$$A^T = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix} = A$$

so $A = A^T$ ⇒ matrix is symmetric

* skew - symmetric iff $A^T = -A$

ex :- Let $A = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$

① $A^T = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$

② $-A = \begin{bmatrix} 0 & -1 & 3 \\ +1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$

$A^T = -A$ ⇒ then A is skew - symmetric.

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Lec-8-

* upper trangular matrix :- [المثلث الفوقي]

⇐ هو مثلث نهاية 90° وتحته كل القيم = صفر يبدأ من يمين ~~ت~~ matrix

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$$

Not upper trangular

* Lower triangular matrix :-

⇐ مثلث قائم الزاوية 90° يبدأ من يسار matrix في الأسفل، كل القيم فوقه = صفر

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 0 \\ 8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}$$

* ~~diagonal~~ diagonal matrix :-

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

⇒ هنا عبارة عن قيم تبدأ من ~~يمين~~ أقصى اليسار إلى نهاية أقصى اليمين فوقه أصفار وتحته أصفار (غير مهمة القيم ما قيمتها)

* scalar matrix

$$S = \begin{bmatrix} \pm C & 0 & 0 \\ 0 & \pm C & 0 \\ 0 & 0 & \pm C \end{bmatrix}$$

* قيم C كلها متساوية

13/2/2020

Lec -8-

* unit matrix or Identity matrix :-

$$I_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_{3\times3} = \begin{bmatrix} \underline{1} & 0 & 0 \\ 0 & \underline{1} & 0 \\ 0 & 0 & \underline{1} \end{bmatrix}$$

ـه دائماً يكون [1]

⇐ هي حالة خاصة من scalar & diagonal

$$\boxed{AI = IA = A}$$

ex:-

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}_{2\times2} \quad \text{find } IA,\ AI\ ??$$

① $$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2} \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}_{2\times2} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} = A$$

② $$AI = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} = A$$

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Lec - 9 -

7.3 → Linear system of equation Gauss - elimination

* Let

Linear system with m-equation and n-unknown

$$
\begin{aligned}
a_{11}x_1 + \text{-----} &+ a_{1n}x_n = b_1 \\
a_{21}x_1 + \text{-----} &+ a_{2n}x_n = b_2 \\
&\vdots \\
a_{m1}x_1 + \text{-----} &+ a_{mn}x_n = b_n
\end{aligned}
$$

① system is linear, $x_1, x_2 \text{---}, x_n$ of order 1

② for Linear system with all (b_j) are zero
⇒ we call it Homogenouse linear system

③ for linear system with at lest one (b_j) is not zero
⇒ call it Nonhomogenouse linear system.

⇒ for Linear system $Ax = b$ ⇒ تحويل المعادلة إلى matrix

$$
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
$$

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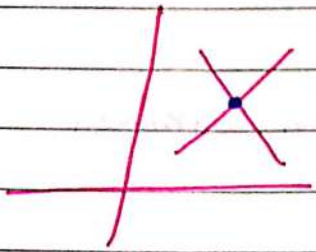
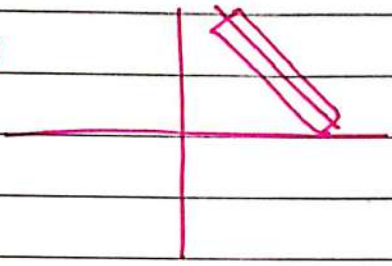
Lec- 9 -

$\Rightarrow$ for $Ax=b$ has three possible cases :-

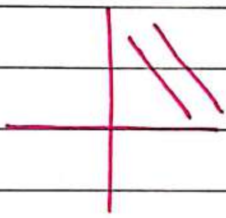
① 

$\Rightarrow$ precisly one solution if lines intersect

تقاطعها

② 

$\Rightarrow$ infinity many solution if lines concide

③ 

$\Rightarrow$ no - solution if the line parallel

* Gauss - elemination + back substitution :-

$$Ax=b \quad \begin{cases} 2x_1 + 5x_2 = 2 \\ 0x_1 + 13x_2 = -26 \end{cases}$$

$$x_2 = \frac{-26}{13} = -2$$

$$x_1 = 6$$

$$Ax=b \Rightarrow \begin{bmatrix} 2 & 5 \\ 0 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -26 \end{bmatrix}$$

upper triangular بيسمّونها

16/2/2020

Lec - 9 -

ex:- $2x_1 + 5x_2 = 2$ Find the x_1 و x_2 ??

$-4x_1 + 3x_2 = -30$

2 eq① + eq②

$\cancel{4x_1} + 10x_2 = 4$ $\Rightarrow 13x_2 = -26$

$\cancel{-4x_1} + 3x_2 = -30$ $x_2 = \frac{-26}{13} = -2$

$x_1 = 6$

ex:- $2x_1 + 5x_2 = 2$

$-4x_1 + 3x_2 = -30$

solution

① $Ax = b$ → matrix

$$\begin{bmatrix} 2 & 5 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -30 \end{bmatrix}$$

② $\tilde{A} = [A \vdots b]$

$$= \left[\begin{array}{cc|c} 2 & 5 & 2 \\ -4 & 3 & -30 \end{array}\right] \begin{array}{l} \rightarrow \text{Pivot} \\ \rightarrow R_2 + 2R_1 \end{array}$$

Pivot يعني يبقى الصف ثابت

$$= \left[\begin{array}{cc|c} 2 & 5 & 2 \\ 0 & 13 & -26 \end{array}\right] \Leftarrow R_2 + 2R_1$$

لازم أحصل عليها ←

[upper triangular]

$2x_1 + 5x_2 = 2$

$0x_1 + 13x_2 = -26$

لازم أحصل ع هذه المعادلات

بحيث ف كل معادلة ينقص متغير

16/2/2020

Lec -9-

$$x_2 = \frac{-26}{13} = -2 \rightarrow \text{back substitution}$$

$$x_1 = 6$$

ex:- with one solotion

$$x_1 - x_2 + x_3 = 0$$
$$-x_1 + x_2 - x_3 = 0$$
$$10x_2 + 25x_3 = 90$$
$$20x_1 + 10x_2 = 80$$

Fined $x_1, x_2, x_3 = ??$

① $Ax = b$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 10 & 25 \\ 20 & 10 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 90 \\ 80 \end{bmatrix}$$

② $\tilde{A} = [A : b]$

$$= \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right] \begin{array}{l} \rightarrow \text{Pivot} \\ \\ \\ \\ \end{array}$$

$$= \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right] \begin{array}{l} \\ \rightarrow R_2 + R_1 \\ \\ \rightarrow R_4 - 20R_1 \end{array}$$

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Lec -9-

$$= \left[\begin{array}{ccc|c} +1 & -1 & +1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \rightarrow \text{Pivot} \\ \\ \\ \end{array}$$

* عمم تنزيل لصف الذي فيه أصفار
لأن الترتيب للقيم لا يتغير

$$= \left[\begin{array}{ccc|c} +1 & -1 & +1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \rightarrow R_3 - 3R_2 \\ \\ \end{array}$$

$$x_1 - x_2 + x_3 = 0$$

$$10x_2 + 25x_3 = 90$$

$$-95x_3 = -190$$

$x_3 = 2$ $\qquad$ $x_2 = +4$ $\qquad$ $x_1 = 2$

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Lec- 10-

* elemintary Row opration for matrix :-

① Interchange of two rows

② addition of constant multiple of one Row to another Row.

③ Multiplication of one Row by non-zero constant $\underline{c}$

⇒ has no effected of matrix A or the solution

ex:-

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

↳

$$\begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_3 \\ b_1 \end{bmatrix}$$

⇒ both system has same solution.

* over deterimined linear system :-

if it has more equation than unknowns.

* determined linear system

if number of equation equal number of unknowns

* under determind linear system

if it has fewer equation than unkowns

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Lec -10-

* consistant linear system :-
if it has at lest one solution

* In-consistant linear system
if it has no- solution

ex:- with infinitly many solution

$$3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$$
$$0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 = 2.7$$
$$1.2x_1 - 0.3x_2 - 0.3x_3 + 2.4x_4 = 2.1$$

Find x_1, x_2, x_3, x_4 ??

هنا يوجد ٣ معادلات ← ٤ مجاهيل
* under detemined linear system.

(1) $Ax = b$

$$\begin{bmatrix} 3 & 2 & 2 & -5 \\ 0.6 & 1.5 & 1.5 & -5.4 \\ 1.2 & -0.3 & -0.3 & 2.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 2.7 \\ 2.1 \end{bmatrix}.$$

(2) $\hat{A} = [A : b]$

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right] \rightarrow \text{Pivot}$$

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Lec-10-

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right] \begin{array}{l} \\ \to R_2 - 0.2R_1 \\ \to R_3 - 0.4R_1 \end{array}$$

$$\begin{array}{r} \\ \text{Pivot} \to \\ \\ \end{array} \left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \to R_3 + R_2 \end{array}$$

→ infinitly many solution يشيرالى

$3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$

$+1.1x_2 + 1.1x_3 - 4.4x_4 = 1.1$

$x_2 + x_3 - 4x_4 = 1$

$x_2 + x_3 = 1 + 4x_4$

$3x_1 + 2[x_2 + x_3] - 5x_4 = 8$

$3x_1 + 2[1 + 4x_4] - 5x_4 = 8$

$3x_1 + 2 + 8x_4 - 5x_4 = 8$

$3x_1 + 3x_4 = 6$

$\boxed{x_1 = 2 - x_4}$

$\boxed{x_2 = 1 - x_3 + 4x_4}$

$x_3 = \text{arbitrary}$

غير معروفة القيمة نفرض من عندنا

$x_4 = \text{arbitary}$

$x_1 = 1$

$x_2 = 4$

$x_3 = 1$

$x_4 = 1$

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Lec - 10 -

ex:- with no solution ?

$$3x_1 + 2x_2 + x_3 = 3$$
$$2x_1 + x_2 + x_3 = 0$$
$$6x_1 + 2x_2 + 4x_3 = 6$$

Find x_1 , x_2 , x_3 ??

① $Ax = b$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 6 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$$

② $\check{A} = [A : b] = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right] \rightarrow \text{Pivot}$

$$= \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & -2 & 2 & 0 \end{array}\right] \begin{array}{l} \\ \rightarrow R_2 - \frac{2}{3}R_1 \\ \rightarrow R_3 - 2R_1 \end{array}$$

Pivot →

$$= \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right] \begin{array}{l} \\ \\ \rightarrow R_3 - 6R_2 \end{array}$$

*عند ظهور إليهم جميعا صفر، تساوي رقم

لايوجد حل ~~[illegible]~~ للمسألة

[false statment]

$$3x_1 + 2x_2 + x_3 = 3$$
$$-\frac{1}{3}x_2 + \frac{1}{3}x_3 = -2$$

$0 = 12 \Rightarrow$ مستحيلة

false statement system has no solution

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Lec - 11 -

* Row - echelon form and information form it

For Linear system $Ax=b$ augmented matrix $\tilde{A}=[A:b]$ after doing the Gauss - elemination $[R:F]$

$$[R:F] = \left[\begin{array}{ccccc|c} r_{11} & r_{12} & \cdots & & r_{1m} & F_1 \\ & r_{22} & \cdots & & r_{2m} & F_2 \\ & & \cdots & & \vdots & \vdots \\ & & & r_{rr} & \cdots\ r_{rn} & F_r \\ & & & & & F_{r+1} \\ & & & & & F_{r+2} \\ & & & & & \vdots \\ & & & & & F_m \end{array}\right]$$

① $r_{11} \neq 0$

② $r \leq m$

③ all element in triangular and triangle must be zero

$Ax=b$

① No - solution : if $r<m$ and at least one number $F_{r+1}, F_{r+2} \cdots F_m$ is not zero.

ex :-

$$[R:F] = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & 1/3 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

* اصفار بجانبهم رقم يعني ← No - solution

① $r_{11} = 3 \neq 0$

② $r_{rn} = \frac{1}{3} \longrightarrow r_{23} = 1/3 \longrightarrow \boxed{r=2}$ $\boxed{n=3}$

③ $F_m = 12 \longrightarrow F_3 = 12 \longrightarrow \boxed{m=3}$

② One solution : if $r = n$ and all element $F_{r+1}, F_{r+2}, \dots F_m$ are zero

ex:-

$$[R:F] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \to F_r \\ \to F_{r+1} \end{array}$$

F_{r+2}

F_m → are zero So one solution

① $r_{11} = 1 \neq 0$

② $r_{rn} = -95 \longrightarrow r_{33} = -95 \longrightarrow \boxed{r=3}\ \boxed{n=3}$

③ $F_m = 0 \longrightarrow F_4 = 0 \longrightarrow \boxed{m=4}$

③ infinitly many solution : if $r < m$ and all element $F_{r+1} \text{ --- } F_m$ are zero

ex:-

$$[R:F] = \left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

عندما تكون السطر بالكامل zero
يدل على infinity solution

① $r_{11} = 3 \neq 0$

② $r_{rn} = -4.4 \longrightarrow r_{24} = -4.4 \longrightarrow \boxed{r=2}\ \boxed{n=4}$

③ $F_m = 0 \longrightarrow F_3 = 0 \longrightarrow \boxed{m=3}$

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Lec-11-

7.4 → linear independent, Rank of matrix, vector space

Def :-
$a_1, a_2, a_3, \ldots a_m$ → are m = vector
$C_1, C_2, C_3, \ldots C_m$ → are m = scalar

then $C_1a_1 + C_2a_2 + C_3a_3 \ldots C_ma_m = 0$ → eq(1)

① if eq(1) holds for all C → C_1, C_2, C_m are zero
then vector a_1, a_2, a_m are linearly independent

② if eq(1) holds for at lest one C → $C_1, C_2, \ldots C_m$
are not zero then the vector a_1, a_2, a_m
are linearly dependent

Let :-
$a_1 = [3 \quad 0 \quad 2 \quad 3]$
$a_2 = [-6 \quad 42 \quad 24 \quad 54]$
$a_3 = [21 \quad -21 \quad 0 \quad -15]$
are a_1, a_2, a_3 linear dependent?

$C_1a_1 + C_2a_2 + C_3a_3 = 0$
$C_1[3 \quad 0 \quad 2 \quad 3] + C_2[-6 \quad 42 \quad 24 \quad 54] + C_3[21 \quad -21 \quad 0 \quad -15] = 0$

$3C_1 - 6C_2 + 21C_3 = 0$
$42C_2 - 21C_3 = 0$ $\qquad C_1 = 6 \qquad C_2 = \frac{1}{2} \qquad C_3 = -1$
$2C_1 + 24C_2 = 0$
$3C_1 + 54C_2 - 15C_3 = 0$ $\qquad a_1, a_2, a_3$ → linearly depentent

20/12/2020

Lec-11-

* Rank of matrix A :-

is the max. number of linearly indepent Row vector of A

* Rank دايمًا +ve مش ممكن تكون سالبة

ex:-

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & 15 \end{bmatrix} \begin{array}{l} \rightarrow a_1 \\ \rightarrow a_2 \\ \rightarrow a_3 \end{array}$$

Find the Rank of A ??

* باخذ كل vector مع الآخر

* باخذهم كلهم مع بعض

① a_1, a_2

$$c_1 a_1 + c_2 a_2 = 0$$

$a_1, a_2 \longrightarrow$ linearly independent ✓

$$c_1 = c_2 = 0$$

② a_1, a_3

$$c_1 a_1 + c_3 a_3 = 0$$

$$c_1 \begin{bmatrix} 3 \\ 0 \\ 2 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 21 \\ -21 \\ 0 \\ 15 \end{bmatrix} = 0$$

$a_1, a_3 \longrightarrow$ linearly independent ✓

$$c_1 = c_3 = 0$$

③ a_2, a_3

$$c_2 a_2 + c_3 a_3 = 0$$

$a_2, a_3 \longrightarrow$ linearly independent ✓

$$c_2 = c_3 = 0$$

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Lec -11-

(4) a_1, a_2, a_3.

$c_1 a_1 + c_2 a_2 + c_3 a_3 = 0$

$a_1, a_2, a_3 \rightarrow$ linearly dependent X

Max. number $\{2, 2, 2\} = \{2\}$

* يوجد طريقة (3) لإيجاد Rank هو أني أعمل matrix وأشوف عدد الصفوف التي لم تصفر = Rank

ex:-

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -51 \end{bmatrix} \begin{array}{l} \rightarrow \text{Pivot} \\ \\ \\ \end{array}$$

$$\begin{array}{r} \\ \text{Pivot} \rightarrow \\ \\ \end{array} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix} \begin{array}{l} \\ \rightarrow R_2 + 2R_1 \\ \rightarrow R_3 - 7R_1 \end{array}$$

$$= \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \\ \rightarrow R_3 + \frac{1}{2} R_2 \end{array}$$

Rank = 2

20/2/2020

Lec-11-

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{Rank} = 1$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{Rank} = 2$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix} \longrightarrow \text{Rank} = 3$$

23/2/2020

Lec-12-

* Row-equalent matrix

① Matrix A_1 is a row equa to matrix A_2 if A_1 can obtained from A_2 by elementary Row operation

② Row-equi lent matrix have the same Rank.

* Theorem:-

consider p-vectors that each have n-component then these vectors are linearly independent

if the matrix formed with these vectors as Row vector has Rank P. however these vector linearly dependent

if matrix has rank less than p

* Theorem:-

Rank of A is the Max. number of lienerly indep. column vector of A

$$\boxed{Rank(A) = Rank(A^T)}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

23/2/2020

Lec -12-

* Theorem:—

consider p-vectors each have n- com

if $n<p$ then these vector are $\Rightarrow$ linearly dependent.

ex:—

$P_1 = [1 \quad 2]$

$P_2 = [3 \quad 4]$

$P_3 = [5 \quad 6]$

$P_4 = [7 \quad 8]$

$P_5 = [9 \quad 10]$

are these vector linearly dependent ??

$n = 2 \longrightarrow$ p الـ element الـ

$p = 5$

$n < p$

$2 < 5$ ✓ $\Rightarrow$ linearly dep.

* vector pace :—

Let $\rightarrow$ $V = \begin{bmatrix} | & | & | & & | \\ v_1 & v_2 & v_3 & \cdots & v_n \\ | & | & | & & | \end{bmatrix}$

① dimention of V :—

→ is the max. number of linearly indep. vector of V.

② basis of V = subset of V with Max. possible number of linearly indep. vector in V

③ span of V = Subset of V with Max. possible number of linearly dep. vector in V

23/2/2020

Lec - 12 -

7.5 → Solution of linear system : existance and uniqueness

theorem :-

For linear system $Ax=b$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \tilde{A} = [A : b]$$

m - equation

n - unknowns

① existence :- linear system is consistant , has solution

iff ⇒ $Rank(A) = Rank(\tilde{A})$

② Uniqueness :- linear system has one solution

iff ⇒ $Rank(A) = Rank(\tilde{A}) = n$

③ infinitly ~~many~~ many solution

iff ⇒ $Rank(A) < n$

④ No - solution :- linear system is in - consistant

iff ⇒ $Rank(A) \neq Rank(\tilde{A})$

25/2/2020

Lec-13-

7.6 → Secound and third order determinats :-

Let :- $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

↳ $D = \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

ex:- $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \longrightarrow D = (2\times2) - (1\times1) = 4 - 1 = 3$

$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \longrightarrow D = (0\times1) - (1\times2) = 0 - 2 = -2$

* Cramers Rules :-

For secound order linear system $Ax = b$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Find x_1, x_2 ? $x_1 = \frac{D_1}{D}$ $x_2 = \frac{D_2}{D}$

$D = \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = b_1 a_{22} - a_{12} b_2$

$D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = a_{11} b_2 - b_1 a_{21}$

25/2/2020

Lec| 13|

ex:— $4x_1 + 3x_2 = 12$

$2x_1 + 5x_2 = -8$

Find the x_1, x_2 ??

$$Ax = b \qquad \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \end{bmatrix}$$

$$x_1 = \frac{D_1}{D} = \frac{\begin{vmatrix} 12 & 3 \\ -8 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \frac{(12\times5) - (3\times-8)}{(4\times5) - (3\times2)} = 6$$

$$x_2 = \frac{D_2}{D} = \frac{\begin{vmatrix} 4 & 12 \\ 2 & -8 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \frac{(4\times-8) - (12\times2)}{(4\times5) - 3\times2} = -4$$

* third order determination :—

For linear system $Ax = b$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x_1 = \frac{D_1}{D}$$

$$x_2 = \frac{D_2}{D}$$

$$x_3 = \frac{D_3}{D}$$

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Lec- 13 -

$$D = \det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= +a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11} [a_{22} a_{33} - a_{23} a_{32}] - a_{12} [a_{21} a_{33} - a_{23} a_{31}] + a_{13} [a_{21} a_{32} - a_{22} a_{31}]$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$= b_1 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} b_2 & a_{23} \\ b_3 & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} b_2 & a_{22} \\ b_3 & a_{32} \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} b_2 & a_{23} \\ b_3 & a_{33} \end{vmatrix} - b_1 \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & b_1 \\ a_{31} & b_2 \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_{11} & a_{21} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & b_2 \\ a_{32} & b_3 \end{vmatrix} - a_{21} \begin{vmatrix} a_{21} & b_2 \\ a_{31} & b_3 \end{vmatrix} + b_1 \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{23} \end{vmatrix}$$

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Lec-14-

7.7 → determinat + crumer Rules :-

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$D = \det A = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{12} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

* The minor matrix of A called M :-

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$$M_1 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

$$M_2 = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$$

$$M_3 = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}$$

* the cofactor matrix of A is called C ??

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$C_{jk} = (-1)^{j+k} M_{jk} \Lleftarrow \text{bèp}$$

$$C = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}$$

$$A \to M \to C$$

Theorm :-

① Inter change of two rows ~~[illegible]~~ Multiplies the value of determinat by (-1)

② addition of Multiple of row to another row does not olter the value of determinate.

③ Multiplication of raw by Non zero constant (c) = Multiply the value of determinit by (c) ex:- $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$A = 1 - 4 = -3$ $\quad$ $B = 4 - 1 = 3$

ex:- $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \rightarrow A = 1 - 4 = -3$

$= \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} \begin{matrix} \rightarrow \text{Pivot} \\ \rightarrow R_2 - 2R_1 \end{matrix}$

$= -3 - 0 = -3$

ex:- $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \rightarrow A = 1 - 4 = -3$

$B = \begin{bmatrix} 2 & 4 \\ 2 & 1 \end{bmatrix} \rightarrow B = 2 - 8 = -6 = 2[A]$

* Rule :-

$$\det[cA] = c^n \det[A]$$

↳ A = Matrix

c = Scalar

n = number of Rows in A

ex :- $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \rightarrow A = 1 - 4 = -3$

$2A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \rightarrow 2A = 4 - 16 = -12$

$|2A| = 2^2 \, |A| = 4 \, [-3] = -12$

\# evalution of determinat by reducation to trigler from

$\Rightarrow$ "Gauss elimination" $\Leftarrow$

$$D = \begin{bmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{bmatrix} \begin{matrix} \rightarrow \text{Pivot} \\ \\ \\ \\ \end{matrix}$$

$$= \begin{bmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 2 & 6 & -1 \\ 0 & 8 & 3 & 10 \end{bmatrix} \begin{matrix} \\ \rightarrow R_2 - 2R_1 \rightarrow \text{Pivot} \\ \\ \rightarrow R_4 + 1.5R_1 \end{matrix}$$

$$= \begin{bmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & -11.4 & 29.2 \end{bmatrix} \begin{matrix} \\ \rightarrow \text{Pivot} \\ \rightarrow R_3 - 0.4R_1 \\ \rightarrow R_4 - 1.6R_2 \end{matrix}$$

$$= \begin{bmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & 0 & 27.25 \end{bmatrix} \begin{matrix} \\ \\ \rightarrow \text{Pivot} \\ \rightarrow R_4 + 4.75R_3 \end{matrix}$$

$D = 2 \times 5 \times 2.4 \times 27.25 = 1134$

* Theorm :- continous

* نفس لهقواعد السابقة

تكن بنذل Row ← column

(4) Transpose leve the value of determinat unchang :-

$$\det(A) = \det(A)^T$$

(5) Zero Row or zero column render the value of determinat to zero

(6) propertional rows or column render the value of determinat to zero

ex:- $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \Rightarrow A = 2 - 2 = 0$

$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \quad |B| = 0$

* $A = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$

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$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 3$$

$$A^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 3$$

* Cramer Rules :-

$Ax = b$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$x_1 = \frac{D_1}{D} \qquad x_2 = \frac{D_2}{D}$$

$$x_n = \frac{D_n}{D}$$

$D = \det(A)$

D_n = is the determinat obtaind form D by replacing in D the rxr colum by column $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

5/3/2020

Lec — 15 —

7.8 → Inverese of matrix , Gauss - Jordan elimination

⇒ For squar matrix $n \times n$ A it's inverse is A^{-1}

$$\boxed{A A^{-1} = A^{-1} A = I}$$

Note :—

① if A has inverse than A is called non - singular matrix

② if A has no - inverse than A is called singular matrix

* for system $Ax = b$ we can Find x using A^{-1}

$$Ax = b$$
$$A^{-1} A x = A^{-1} b \quad \leftarrow \text{لازم الضرب من اليسار}$$
$$Ix = A^{-1} b$$
$$x = A^{-1} b$$

* Theorm :—

existance of inverse.

- the inverse A^{-1} of an $n \times n$ matrix A exist

iff

① Rank $A = n$

② $\det A \neq 0$

③ A is non - singular

if rank $A < n$

than ① A is singular

② A has no inverse

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Lec - 15

⇒ Finding invers of matrix by Gauss - Jordan elimination

ex:- let $A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$

solution ⇒

$$\tilde{A} = [A : I] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \to \text{Pivot}$$

$$= \begin{array}{c} \\ \text{Pivot} \to \\ \\ \end{array} \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \to R_2 + 3R_1 \\ \to R_3 + R_1 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] \begin{array}{l} \\ \\ \to R_3 - R_2 \end{array}$$

$$= \begin{array}{c} \\ \\ \text{Pivot} \to \end{array} \left[\begin{array}{ccc|ccc} +1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 3.5 & 1.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \begin{array}{l} \to -R_1 \\ \to \frac{1}{2}R_2 \\ \to -0.2\,R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0.6 & 0.4 & -0.4 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \begin{array}{l} \to R_1 + 2R_3 \\ \to R_2 - 3.5R_3 \\ \\ \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \rightarrow R_1+R_2$$

$$\tilde{A} = [I : A^{-1}]$$

$$A^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

* Theorm :—

the inverse of non singular $n \times n$ matrix $A = [a_{jk}]$ is given by

$$\boxed{A^{-1} = \frac{1}{|A|} C^T}$$

↳ $|A|$ = determinent of A

↳ C^T = cofactor matrix Transpose

* For $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} C^T = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$|A| = a_{11} a_{22} - a_{12} a_{21}$$

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Lec - 15 -

ex:- let $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ Find A^{-1} ??

$$A^{-1} = \frac{1}{|A|} C^T = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$$

ex:- Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} C^T$$

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C^T \Rightarrow A \Rightarrow M \rightarrow C \longrightarrow C^T$$

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

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Lec-15-

$$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \qquad M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$C = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}$$

$$C^T = \begin{bmatrix} M_{11} & -M_{21} & M_{31} \\ -M_{12} & M_{22} & M_{32} \\ M_{13} & -M_{32} & M_{33} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} C^T$$

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Lec - 16 -

* inverse of matrix A $\Rightarrow A^{-1} = \frac{1}{|A|} C^T$

ex:- let $A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$ Find A^{-1} ??

$$A^{-1} = \frac{1}{|A|} C^T$$

$$|A| = -1 \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} + 2 \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 10$$

$$A \rightarrow M \rightarrow C \rightarrow C^T$$

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7 \qquad -M_{21} = - \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2$$

$$-M_{12} = - \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = \cancel{2} -13 \qquad M_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2$$

$$M_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8 \qquad -M_{23} = - \begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2$$

$M_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3$ $\quad -M_{32} = -\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = 7$

$M_{33} = \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = -2$

$$C = \begin{bmatrix} -7 & -13 & 8 \\ 2 & -2 & 2 \\ 3 & 7 & -2 \end{bmatrix}$$

$$C^T = \begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{10} C^T = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

* for diagonal Matrix A

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & 0 \\ 0 & \frac{1}{a_{22}} & 0 \\ 0 & 0 & \frac{1}{a_{33}} \end{bmatrix}$$

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Lec-16-

ex:- let $A = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

* Rules :-

(1) $(AC)^{-1} = C^{-1}A^{-1}$

(2) $(A^{-1})^{-1} = A$

(3) $(A^{T})^{-1} = (A^{-1})^{T}$

* unusual properties of matrix multiplication.

(1) $AB \neq BA$

(2) $AB = 0$ it dose not generaly imply that $A = 0$, $B = 0$

ex:- $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}_{2\times2}$ $B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}_{2\times2}$

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}$$

* نلاحظ ان $A \neq B$ بس ضربهم = zero

(3) $AC = AD$ it dose not generly imply $C = D$

Five Apple

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Lec -16-

* theorm :-

cancolation lows :-

Let A, B, C be $n \times n$ matrix then :-

① if Rank $A = n$ and $AB = AC$, than $B = C$

② if Rank $A = n$ and $AB = 0$, then $B = 0$

③ if $AB = 0$ and $A \neq 0$, $B \neq 0$ then $\text{rank}(A) < n$

$\text{rank}(B) < n$

④ if A is singular then BA, AB are singular.

* Theorm :-

determent of prodect of matrix :-

$$\det(AB) = \det(BA) = \det|A| \det(B)$$

ex :- $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$

Find $|AB|$??

① $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix}$

$|AB| = 12 - 6 = 6$

or $\Rightarrow$ $|AB| = |A||B| = (1-4)(0-2) = 6$

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Lec - 16 -

* adjoint Matrix :-

① $\text{adj}(A) = [\text{cof}(A)]^T = C^T$

② $A[\text{adj}(A)] = [\text{adj}(A)]A = \det(A)\, I$

$$A^{-1} = \frac{1}{|A|} C^T = \frac{1}{|A|}[\text{adj}(A)]$$

10/3/2020

Lec-17-

8.1 → the matrix eigen value problem.

let $A = [a_{jk}]$ is an $n \times n$ square matrix then the matrix eigen value ~~pr~~ problem is

$$Ax = \lambda x$$

when → A = squar matrix

λ = unknown scalar called ((eigen value))

x = unknown vector ((eigen vector))

Note:-

that $x=0$ is solution with no intrest so we are looking for solution for x

* spectrum of A :-

the set of all eigen value of A which consist of at least one eigen value and at most n-different eigen value

* How to Find eigen value and eigen vector:-

$$Ax = \lambda x$$

$$Ax - \lambda x = 0$$

$$[A - \lambda I] x = 0$$

(1) Find charecteristics equation

$$\det [A - \lambda I] = 0$$

we can find λ From the characteristic equation.

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② by Find λ ⇒ eigen value we substitue in $(A-\lambda I)x=0$ to find eign vector x

ex:-

let $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$ Find eigen value and eigen vector?

① det $(A-\lambda I)=0$

$$\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} = 0$$

$$(-5-\lambda)(-2-\lambda) - 4 = 0$$

$$10 + 5\lambda + 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 + 7\lambda + 6 = 0$$

$$(\lambda+6)(\lambda+1) = 0$$

$$\boxed{\lambda=-1} \quad \boxed{\lambda=-6}$$

For $\lambda=-1$ ⇒ the cross poinding eigen vector is
$x = (A-\lambda I)x = 0$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-4x_1 + 2x_2 = 0$

$x_1 = 1$

$x_2 = 2$

$\hat{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

for $\lambda = -6 \Rightarrow$ the cross bending eigen vector x is $(A - \lambda I)x = 0$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 = 0$

$x_1 = 2$

$x_2 = -1$

$\hat{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

* normaliz of x:-

$$\text{Normaliz of } x = \frac{x}{||x||}$$

$$x = [x_1, x_2, x_3, x_n]$$

$$||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2 - - - + x_n^2}$$

ex:- let $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ Find the normalized of x ??

$$\text{norm}(x) = \frac{x}{||x||} = \frac{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}{\sqrt{5}} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

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ex :- Find eigen value and eigen vector ? for

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

① $\det(A-\lambda I) = 0$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} = 0$$

$$-2-\lambda \begin{vmatrix} 1-\lambda & -6 \\ -2 & -\lambda \end{vmatrix} - 2 \begin{vmatrix} 2 & -6 \\ -1 & -\lambda \end{vmatrix} - 3 \begin{vmatrix} 2 & 1-\lambda \\ -1 & -2 \end{vmatrix} = 0$$

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

$$\lambda = 5, -3, \cancel{+3}$$
-3

* For $\lambda = 5$ the cross ponding vector x is $(A-\lambda I)x = 0$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

using Gauss - elimination :-

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & -24/7 & -48/7 \\ 0 & -16/7 & -32/7 \end{bmatrix} \begin{matrix} \rightarrow \text{Pivot} \\ \rightarrow R_2 + \frac{2}{7} R_1 \\ \rightarrow R_3 - \frac{1}{7} R_1 \end{matrix}$$

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$$= \begin{bmatrix} -7 & 2 & -3 \\ 0 & -24/7 & -48/7 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \to R_1 - \frac{2}{3}R_2 \end{matrix}$$

$$-7x_1 + 2x_2 - 3x_3 = 0$$

$$-\frac{24}{7}x_2 - \frac{48}{7}x_3 = 0$$

let $x_3 = -1$ $\quad x_2 = 2$ $\quad x_1 = 1$ $\quad \hat{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

* for $\lambda = -3$ the x is :- $(A - \lambda I)x = 0$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

using Gauss elimination

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \to \text{Pivot} \\ \to R_2 - 2R_1 \\ \to R_3 + R_1 \end{matrix}$$

$x_1 + 2x_2 - 3x_3 = 0$ $\quad \hat{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

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* For $\lambda = -3$ the x is $(A - \lambda I)x = 0$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Gauss elimination

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \rightarrow \text{Pivot} \\ \rightarrow R_2 - 2R_1 \\ \rightarrow R_3 + R_1 \end{array}$$

$$x_1 + 2x_2 - 3x_3 = 0 \qquad \hat{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

ex:- show that $[1 \quad 1 \quad 1]^T$ is eigen vector For eigen value 2 For matrix A

$$A = \begin{bmatrix} 4 & -1 & -1 \\ 0 & -1 & 3 \\ 1 & 3 & -2 \end{bmatrix}$$

$$Ax = \lambda x$$

$$\begin{bmatrix} 4 & -1 & -1 \\ 0 & -1 & 3 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \overset{?}{=} 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \checkmark$$

ex:- Find λ, x for $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

① $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 0-\lambda & 1 \\ -1 & 0-\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$
$$\lambda = \pm\sqrt{-1}$$
$$= \pm i$$

* For $\lambda = 1$ the x is $(A - \lambda I)x = 0$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

using G-E

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{matrix} \rightarrow \text{Pivot} \\ \rightarrow R_2 + 1R_1 \end{matrix}$$

$$-1x_1 + x_2 = 0 \qquad \hat{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = -i$ the x is $(A - \lambda I)x = 6$

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

using G-E

$$\begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix} \rightarrow R_2 - iR_1$$

$ix_1 + x_2 = 0 \qquad \hat{x} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

* theorem:-

the transpose A^T of squar matrix A has the same eigen ~~value~~ value as A

$$A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(1) $\det(A^T - \lambda I) = 0$

$\begin{vmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = 0 \qquad \lambda^2 + 1 = 0$

$\lambda = \pm i$

8.3 → symmatric, skew-symmatric, orthogonal matrix

Def :- areal squar matrix $A = [a_{jk}]$ is called :-

① symmatric matrix $\Rightarrow A^T = A$
② skew-symmetric $\Rightarrow A^T = -A$
③ orthogonal matrix $\Rightarrow A^T = A^{-1}$

* example symatric

$$A = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix} \qquad A = A^T$$

* example skew-symmatric :-

$$A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{bmatrix} \qquad A^T = -A$$

* example orthogonal :-

$$A = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{-2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \end{bmatrix} \longrightarrow$$

$A^T = A^{-1}$

* متى أعرف انو هاي المصفوفة orthogonal أضرب المصفوفة بـ A^T

$A^T = A^{-1} \rightarrow AA^{-1} = I$

$AA^T = I$ *

$$\overset{A}{\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{-2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \end{bmatrix}} \times \overset{A^T}{\begin{bmatrix} \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

A is orthogonal

⇒ For any real squar matrix:-

① symmetric $(R) = \frac{1}{2}(A + A^T)$

② skew-symmetric $(S) = \frac{1}{2}(A - A^T)$

* example :-

Find R, S ??

$$A = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix} \qquad A^T = \begin{bmatrix} 9 & 2 & 5 \\ 5 & 3 & 4 \\ 2 & -8 & 3 \end{bmatrix}$$

$$R = \frac{1}{2}(A + A^T) = \begin{bmatrix} 9 & 3.5 & 3.5 \\ 3.5 & 3 & -2 \\ 3.5 & -2 & 3 \end{bmatrix}$$

$$S = \frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & 1.5 & -1.5 \\ -1.5 & 0 & -6 \\ 1.5 & 6 & 0 \end{bmatrix}$$

* Theorem :-

① eigen value of symmetric are real

② eigen value of skew-symmetric are pure imaginary or zero

* Theorem

$AA^{-1} = I$, $AA^T = I$ orthogonal

* Theorem :-

① Determinat of othognal matrix have value +1 or −1

② eigen value of othognal matrix are real or complex conject with absolute value (1)

* Diagonalization of matrix :-

– symilar matrix :- matrix $\hat{A}$ is called similar to n×n

$$\hat{A} = P^{-1} A P$$

Is Non-singular matrix

theorem :-

① $\hat{A}$ is similar to A then $\hat{A}$ has eigen value

② if x is eigen vector for A then $\hat{A}$ has eigen vector $y = P^{-1} x$ cross ponding.

* example :-

Find the eigen value and eigen vector for $A, \hat{A}$??

$$A = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$\hat{A} = P^{-1} A P$$

$$P^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \times \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

eigen value

$\lambda = 3, 2$

$$\det(\hat{A} - I\lambda) = 0$$

$$\begin{vmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

For $\lambda = 2 \rightarrow (\hat{A} - 2\lambda) y = 0$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad y' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For $\lambda = 3 \longrightarrow (\hat{A} - 3I)\, y = 0$

$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \hat{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

* الحل القصير ⊏ أنه نخرج أول شي القيم التي تخص A على النظريات التي لدينا نوجد

$$y_1 = P^{-1} x_1 = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y_2 = P^{-1} x_2 = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \#$$

* Diagonalization of matrix if an $n \times n$ matrix A has basis of eigen vector.

diagonal matrix $\leftarrow D = X^{-1} A X$

$\hookrightarrow$ eigen vector of A with column vector

* example :-

Find D for $A = \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix} = \begin{bmatrix} 7.3-\lambda & 0.2 & -3.7 \\ -11.5 & 1-\lambda & 5.5 \\ 17.7 & 1.8 & -9.3-\lambda \end{bmatrix}$

① $\det(A - \lambda I) = 0$

$$7.3-\lambda \begin{vmatrix} 1-\lambda & 5.5 \\ 1.8 & -9.3-\lambda \end{vmatrix} - 0.2 \begin{vmatrix} -11.5 & 5.5 \\ 17.7 & -9.3-\lambda \end{vmatrix} + (-3.7) \begin{vmatrix} -11.5 & 1-\lambda \\ 17.7 & 1.8 \end{vmatrix}$$

$\lambda = 0, 3, -4$

②

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

③ For $\lambda = 0$ $\hat{x} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ For $\lambda = 3$ $\hat{x} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$ For $\lambda = -4$ $\hat{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

④ $x = \begin{bmatrix} \text{x بنقسم} \\ \text{هنا عالدير} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$

$$x^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

$D = x^{-1} A x$ الشيت مهم

$$= \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix} * \begin{bmatrix} 7.3 & 0.2 & 3.7 \\ -11.5 & 1 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix} * \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

* ch. 9

9.1 → vector :-

* scalar → determint by its magnitude
(voltage, tempreture, length, speed)

* vector → has both magnitude + direction
(Force, displacement, velocity)

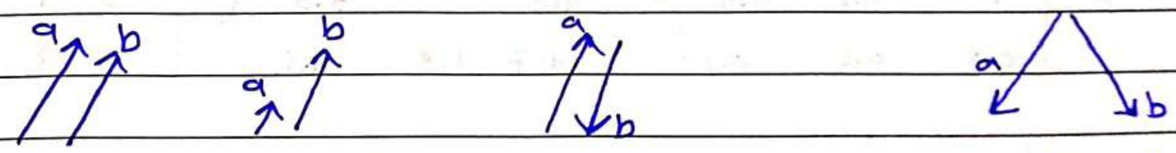
Note :-

① Avector is drawing by arrow $a = \vec{a}$ يمثل الـ vector بواسطة سهم

② $|\vec{a}|$ = length = magnitude → $|a| = \sqrt{(\,)^2 + (\,)^2}$

③ trail is initial point , Tip is a termial point
الذيل نقطة أولية , والرأس نقطة طرفية

④ $|\vec{a}| = 1$ $\vec{a}$ = unit vector

⑤


a = b — different length — differant direction — different length and direction

⑤ componant of vector $P(x_1, y_1, z_1)$ → initial point
$Q(x_2, y_2, z_2)$ → terminal point

then $\vec{a} = [a_1, a_2, a_3]$ $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$a_1 = x_2 - x_1$ $a_2 = y_2 - y_1$ $a_3 = z_2 - z_1$

* example :-

$P = [4, 0, 2]$ $Q = [6, -1, 2]$

Find component and length of vector $\vec{a}$??

Sol

$\vec{a} = [2, -1, 0]$

$|\vec{a}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

* Position vector $\vec{v}$ of a point $A(x_1, y_1, z_1)$ is a vector with origin $(0, 0, 0)$ initial point and A is termind point

* vector addition + scalar multiplication :-

① $\vec{a} = [a_1, a_2, a_3]$
$\vec{b} = [b_1, b_2, b_3]$

then → $\vec{a} + \vec{b} = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$
$\vec{a} - \vec{b} = [a_1 - b_1, a_2 - b_2, a_3 - b_3]$

* Rule :-

* $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
* $[\vec{u} + \vec{v}] + \vec{w} = \vec{u} + [\vec{v} + \vec{w}] = \vec{u} + \vec{v} + \vec{w}$
* $\vec{a} + 0 = \vec{a}$
vector
* $\vec{a} + (-\vec{a}) = zero$

0 scalar = 0 / 0 vector = 0, 0, 0

* scalar multiplication :-

$\vec{a} = (a_1, a_2, a_3)$

$\downarrow$

$c\vec{a} = [ca_1, ca_2, ca_3]$

* Rule :-

* $c[\vec{a}+\vec{b}] = c\vec{a} + c\vec{b}$
* $[c+k]\vec{a} = c\vec{a} + k\vec{a}$
* $c[k\vec{a}] = (c\vec{a})k = (ck)\vec{a}$
* $1\vec{a} = \vec{a}$
* $0\vec{a} = \vec{0}$
* $-1\vec{a} = -\vec{a}$

* example :-

$\vec{a} = [4, 0, 1]$ $\quad \vec{b} = [2, -5, \frac{1}{3}]$

① $-\vec{a} = [-4, 0, -1]$

② $7\vec{a} = [28, 0, 7]$

③ $\vec{a}+\vec{b} = [6, -5, \frac{4}{3}]$

④ $2[\vec{a}+\vec{b}] = [12, -10, \frac{8}{3}]$

⑤ $2[\vec{a}-\vec{b}] = 2[2, 5, \frac{+2}{3}] = [4, 10, \frac{4}{3}]$

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* **unite vector** :— $\hat{i}, \hat{j}, \hat{k}$

$$\hat{i} = [1, 0, 0]$$
$$\hat{j} = [0, 1, 0]$$
$$\hat{k} = [0, 0, 1]$$

example :—

$$\vec{a} = [a_1, a_2, a_3] \quad = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

* **inner product** [dot product]

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$\vec{a}\cdot\vec{b} = \text{zero}$ if $a = 0$ or $b = \text{zero}$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 \cdots}$$
$$|\vec{b}| = \sqrt{b_1^2 + b_2^2 \cdots}$$

* $\vec{a} = [a_1, a_2, a_3] \quad \vec{b} = [b_1, b_2, b_3]$

$$\vec{a}\cdot\vec{b} = [a_1b_1 + a_2b_2 + a_3b_3] = \_\_\_$$

* $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$ منه اوجد الزاوية. ـــ

* example :—

$\vec{a} = [1, 2, 0] \quad \vec{b} = [3, -2, 1]$ Find θ between $\vec{a}$ and $\vec{b}$??

$$\cos\theta = \frac{3 - 4 + 0}{\sqrt{5}\sqrt{14}} = \frac{-1}{\sqrt{70}} \qquad \theta = 96.86^\circ$$

* unit vector $\hat{i}, \hat{j}, \hat{k}$

① $\hat{i} \cdot \hat{i} = 1$ ④ $\hat{i} \cdot \hat{j} = 0$

② $\hat{j} \cdot \hat{j} = 1$ ⑤ $\hat{j} \cdot k = 0$

③ $\hat{k} \cdot \hat{k} = 1$ ⑥ $k \cdot \hat{i} = 0$

* Rule :-

* $(q_1 \vec{a} + q_2 \vec{b}) \cdot \vec{c} = q_1 \vec{a} \cdot \vec{c} + q_2 \vec{b} \cdot \vec{c}$ q_1, q_2 = scalar

* $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
* $\vec{a} \cdot \vec{a} \geq 0$
* $\vec{a} \cdot \vec{a} = 0$ iff $\vec{a} = 0$
* $(\vec{a} + \vec{b}) \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$
* $|\vec{a} \cdot \vec{b}| \leq |\vec{a}| |\vec{b}|$
* $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$

$|\vec{a} + \vec{b}|^2 + |\vec{a} + \vec{b}|^2 = 2 \left[|\vec{a}|^2 + |\vec{b}|^2 \right]$

* vector product [cross product]

$\vec{v} = \vec{a} \times \vec{b}$

① if $\vec{a} = 0$ or $\vec{b} = 0$ then $\vec{v} = \vec{a} \times \vec{b} = 0$

② if $\vec{a} \neq 0$, $\vec{b} \neq 0$ then $|\vec{v}| = |\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin\theta$

③ if θ between $\vec{a}, \vec{b}$ are zero or 180° $|\vec{v}| = |\vec{a} \times \vec{b}| = \text{zero}$

* $\vec{a} = [a_1, a_2, a_3]$
  $\vec{b} = [b_1, b_2, b_3]$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \rightarrow \hat{i}(a_2b_3 - a_3b_2) - \hat{j}(a_1b_3 - a_3b_1) + \hat{k}(a_1b_2 - b_1a_2)$$

example :—

$\vec{a} = [1, 1, 0]$ $\quad \vec{b} = [3, 0, 0]$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix} = \hat{i}(0) - \hat{j}(0) + \hat{k}(-3) = -3\hat{k}$$
$$= [0, 0, -3]$$

* Note :—

$\hat{i} \times \hat{j} = \hat{k}$ $\qquad \hat{j} \times \hat{i} = -\hat{k}$

$\hat{j} \times \hat{k} = \hat{i}$ $\qquad \hat{k} \times \hat{j} = -\hat{i}$

$\hat{i} \times \hat{k} = \hat{j}$ $\qquad \hat{k} \times \hat{i} = -\hat{j}$

* Rule :—

① $(L\vec{a}) \times \vec{b} = L(\vec{a} \times \vec{b}) = \vec{a} \times L\vec{b}$

② $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

③ $(\vec{a} \times \vec{b}) * \vec{c} = \vec{a} \times c + \vec{b} \times \vec{c}$

④ $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

⑤ $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times [\vec{b} \times \vec{c}]$

* Scalar Triple product :-

for vector , $\vec{a} = [a_1, a_2, a_3]$
$\vec{b} = [b_1, b_2, b_3]$
$\vec{c} = [c_1, c_2, c_3]$

$$(\vec{a}\,\vec{b}\,\vec{c}) = \vec{a} \cdot [\vec{b} \times \vec{c}] \quad \text{or} = [\vec{a} \times \vec{b}] \cdot \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

* $|\vec{a}\,\vec{b}\,\vec{c}|$ → volum of parallel eipied

* volum of tetrahedion = $\frac{1}{6}$ volum of parrallel eiped

* في cross product ناتجه ← vector
* في dot product ناتجه ← scalar

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* example :-

$\vec{a} = [2, 0, 3]$
$\vec{b} = [0, 4, 1]$
$\vec{c} = [5, 6, 0]$

Find , ① $(\vec{a}\,\vec{b}\,\vec{c})$
② volum of box
③ volum of tetrahelron
④ are $\vec{a}, \vec{b}, \vec{c}$ linear indepent

① $(\vec{a}, \vec{b}\,\vec{c}) = \begin{vmatrix} 2 & 0 & 3 \\ 0 & 4 & 1 \\ 5 & 6 & 0 \end{vmatrix} = 2(0-6) - 0(0-5) + 3(0-20)$
$= -12 + 0 - 60 = -72$

② volum $= |-72| = 72$

③ volum $= \frac{1}{6} \times 72 = 12$

④ are $\vec{a}, \vec{b}, \vec{c}$ independent $\rightarrow -72 \neq 0$ then it's independent

* vector + scalar function "derivation"

* vector Function $(\vec{v})$ → depend on point $p(x, y, z)$

$$\vec{v}(x,y,z) = [v_1(x,y,z), v_2(x,y,z), v_3(x,y,z)]$$

example → tangent vector
normal vector

* scalar Function $f = f(x,y,z)$

example → temprature
pressure

* example $\vec{v}(x,y,z) = 5xy\,\hat{i} + x^2z\,\hat{j} + y^2z^2\,\hat{k}$ [vector function]

$f(x,y,z) = x^2y + xyz + z^2yx$ [scalar function]

* drevative function

$\vec{v}(t) = [v_1(t), v_2(t), v_3(t)]$
$= v_1(t)\hat{i} + v_2(t)\hat{j} + v_3(t)\hat{k}$

$\vec{v}(t)' = v_1'(t) + v_2(t)' + v_3'(t)$

* example

$\vec{v}(t) = [t, t^2, 0]$

$\vec{v}(t)'\Big|_{t=1} = [1 + 2t + 0]$

$= [1, 2, 0]$

* Rule :-

① $(c\vec{v})' = c\vec{v}'$
② $(\vec{v}+\vec{u})' = \vec{v}' + \vec{u}'$

③ $(\vec{u}.\vec{v})' = \vec{u}'\vec{v} + \vec{u}\vec{v}'$

④ $(\vec{u}\times\vec{v})' = \vec{u}'\vec{v} + \vec{u}\vec{v}'$

⑤ $(\vec{u}\,\vec{v}\,\vec{w})' = \vec{u}'\vec{v}\vec{w} + \vec{u}\vec{v}'\vec{w} + \vec{u}\vec{v}\vec{w}'$

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* Partial derivative :-

$$\vec{v}(t) = [v_1(t), v_2(t), v_3(t)]$$

① $$\frac{d\vec{v}(t)}{dt_m} = \frac{dv_1}{dt_m}\hat{i} + \frac{dv_2}{dt_m}\hat{j} + \frac{dv_3}{dt_m}\hat{k}$$

② second partial drivitive

$$\frac{d\vec{v}^2(t)}{dt_m^2} = \left[\frac{d^2v_1}{dt_m}, \frac{d^2v_2}{dt_m}, \frac{d^2v_3}{dt_m}\right]$$

* example :-

let $\vec{r}(t_1, t_2) = a\cos t_1\,\hat{i} + a\sin t_1\,\hat{j} + t_2\,\hat{k}$

Find ① $\frac{dr}{dt_1}$ ② $\frac{dr}{dt_2}$ ③ $\frac{d^2r}{dt_1^2}$ ④ $\frac{d^2r}{dt_2^2}$

Sol ① $$\frac{dr}{dt_1} = -a\sin t_1\,\hat{i} + a\cos t_1\,\hat{j} + 0\hat{k}$$

② $$\frac{dr}{dt_2} = 0\hat{i} + 0\hat{j} + \hat{k}$$

③ $$\frac{d^2r}{dt_1^2} = -a\cos t_1\,\hat{i} - a\sin t_1\,\hat{j}$$

④ $$\frac{d^2r}{dt_2^2} = 0\hat{k}$$

example :-

let $r(t_1, t_2) = a \cos t_1 \hat{i} + a \sin t_1 \hat{j} + t_1 t_2 \hat{k}$

Find ① $\frac{dr}{dt_1}$ ② $\frac{dr}{dt_2}$ ③ $\frac{dr^2}{dt_1 dt_2}$ ④ $\frac{dr^2}{dt_2 dt_1}$

Sol

① $\frac{dr}{dt_1} = -a \sin t_1 \hat{i} + a \cos t_1 \hat{j} + t_2 \hat{k}$

② $\frac{dr}{dt_2} = 0\hat{i} + 0\hat{j} + t_1 \hat{k}$

③ $\frac{dr^2}{dt_1 dt_2} = \frac{dr}{dt_1}\left(\frac{dr}{dt_2}\right) = \frac{dr}{dt_1}\left(t_1 \hat{k}\right) = \hat{k}$

④ $\frac{dr^2}{dt_2 (dt_1)} = \frac{dr}{dt_2}\left(\frac{dr}{dt_1}\right) = \frac{dr}{dt_2}\left(-a \sin t_1 \hat{i} + a \cos t_1 \hat{j} + t_2 \hat{k}\right)$

$= \hat{k}$

* Gradiant of scalar field

$$\text{grad}[F] = \nabla F = \left[\frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}\right]$$

$$= \frac{df}{dx}\hat{i} + \frac{df}{dy}\hat{j} + \frac{df}{dz}\hat{k}$$

* ∇F = Nabla f

* $\nabla = \frac{d}{dx}\hat{i} + \frac{d}{dy}\hat{j} + \frac{d}{dz}\hat{k}$

* scalar function $\xrightarrow{\nabla F}$ vector function

* example :-

let $f(x,y,z) = 2y^{3z} + 4xz + 3x$ find ∇F ??

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sol $\nabla F = \frac{df}{dx}\hat{i} + \frac{df}{dy}\hat{j} + \frac{df}{dz}\hat{k}$

$= (4z+3)\hat{i} + (6y^2)\hat{j} + 4z\hat{k}$

$\nabla F\Big|_{(1,1,1)} = 7\hat{i} + 6\hat{j} + 4\hat{k} = (7,6,4)$

* directional derivative :-

$D_F = \frac{\vec{a}}{|a|} \cdot \text{grad } f$

$\vec{a} = [a_1, a_2, a_3]$

$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$\text{grad } f = \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz}$

$\cdot$ = dot product

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* example :-

Find the direction derivitive of $f(x,y,z) = 2z^2 + 3y^2 + z^2$

$p = (2, 1, 3)$ in a direction $\vec{a} = [1, 0, -2]$

sol

$$D_F = \frac{[1, 0, -2]}{\sqrt{5}} \cdot [4x, 6y, 2z]$$

$$= \frac{[1, 0, -2]}{\sqrt{5}} \cdot [8, 6, 6]$$

$$= \frac{1}{\sqrt{5}} [8 + 0 - 12] = \frac{-4}{\sqrt{5}}$$

* Gradient of surface normal vector :-

* unit of normal vector $= \vec{n} = \frac{\nabla F}{|\nabla F|}$

* example :-

Find unit normal vector $\vec{n}$ if $f(x,y,z) = 4(x^2+y^2) - z^2$

$= 4x^2 + 4y^2 - z^2$

at point $P = (1, 0, 2)$

$$\vec{n} = \frac{\nabla F}{|\nabla F|} = \frac{[8x, 8y, -2z]}{|\nabla F|} = \frac{[8, 0, -4]}{\sqrt{80}}$$

$$= \left[\frac{8}{\sqrt{80}}, 0, \frac{-4}{\sqrt{80}}\right]$$

* Laplace equation :-

$$\nabla^2 F = \frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} + \frac{d^2 f}{dz^2}$$

∇^2 = called nable ~~see~~ squar

* example :-

let $f(x,y,z) = 4(x^2+y^2)-z^2$ find $\nabla^2 f$??

$$\nabla^2 f = [8x, 8y, -2z]$$

$$= [8, 8, -2] = 8+8-2 = 14$$

* divergence of vector field :-

let $\vec{V}(x,y,z) = (v_1, v_2, v_3)$
$= v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$$\operatorname{div}(\vec{V}) = \nabla \cdot \vec{V}$$

$$= \left[\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right] \cdot [v_1, v_2, v_3]$$

$$= \frac{dv_1}{dx} + \frac{dv_2}{dy} + \frac{dv_3}{dz}$$

* vector function $\xrightarrow{\text{div}}$ scalar function

example:-

let $\vec{v} = [3xz, 2xy, -yz^2]$ find $div(\vec{v})$

$$div(\vec{v}) = \frac{dv_1}{dx} + \frac{dv_2}{dy} + \frac{dv_3}{dz}$$

$$= 3z + 2x + (-2yz)$$

$$div(\vec{v})\Big|_{(1,1,1)} = 3 + 2 - 2 = 3$$

* $div(Grad f) = \nabla^2 f$

* Curl of vector field:-

let $\vec{v}(x,y,z) = [v_1, v_2, v_3]$

$$curl(\vec{v}) = \nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \hat{i}\left[\frac{dv_3}{dy} - \frac{dv_2}{dz}\right] - \hat{j}\left[\frac{dv_3}{dx} - \frac{dv_1}{dz}\right] + \hat{k}\left[\frac{dv_2}{dx} - \frac{dv_1}{dy}\right]$$

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example:

let $\vec{v} = [yz, 3zx, z]$ find $\text{curl}(\vec{v})\Big|_{(1,1,1)} = ??$

$$\text{curl}(\vec{v}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ yz & 3zx & z \end{vmatrix}$$

$$= \hat{i}\left[0 - 3x\right] - \hat{j}\left[0 - y\right] + \hat{k}\left[3z - z\right]$$

$$= -3\hat{i} + \hat{j} + 2\hat{k}$$

* $\text{curl}(\nabla f) = \vec{0}$
* $\text{div}(\text{curl}\,\vec{v}) = 0$

Rule $\longrightarrow$ ① $\nabla(fg) = f\nabla g + g\nabla f$

② $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

③ $\text{div}(f\vec{v}) = f\,\text{div}(\vec{v}) + \vec{v}\cdot\nabla f$

④ $\text{div}(f\nabla g) = f\nabla^2 g + \nabla f\cdot\nabla g$

⑤ $\text{div}[\text{grad}(f)] = \nabla^2 f$

⑥ $\nabla^2(fg) = g\nabla^2 f + 2\nabla f\cdot\nabla g + f\nabla^2 g$

⑦ $\text{curl}(f\vec{v}) = \nabla f \times \vec{v} + f\,\text{curl}\,\vec{v}$

⑧ $\text{div}(\vec{u}\times\vec{v}) = \vec{v}\cdot\text{curl}(\vec{u}) - \vec{u}\cdot\text{curl}(\vec{v})$

* Scalar Field $\xrightarrow{\nabla F}$ vector field

* Scalar Field $\xrightarrow{\nabla^2 F}$ Scalar field

* vector Field $\xrightarrow{div\ \vec{v}}$ Scalar Field

* vector Field $\xrightarrow{curl\ \vec{v}}$ vector field

* vector field $\xrightarrow{d/dt}$ vector field

* vector field $\xrightarrow{d/dt_m}$ vector field