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تفاضل وتكامل 2

Calculus 2

إعداد : ليلى عمايرة



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Calculus 2

$$- \int (f(x))^n f'(x) dx \rightarrow n \neq -1 = \frac{f(x)^{n+1}}{n+1} + C$$

$$\text{ex: } \int \sin^{100} x \cos x dx = \frac{\sin^{101} x}{101} + C$$

$$- \int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C$$

$$\text{ex: } \int \frac{\sin x}{\cos x} dx = -\ln |\cos x| + C$$

or

$$u = \cos x, du = -\sin x dx \rightarrow \int \frac{du}{u} = \ln |u| + C$$

$$\rightarrow \ln |\cos x| + C$$

$$\therefore \textcircled{1} \int x^n dx = \frac{x^{n+1}}{n+1} (n \neq -1) \quad \textcircled{2} \int \frac{1}{x} dx = \ln |x|$$

$$\textcircled{3} \int e^x dx = e^x$$

$$\textcircled{4} \int a^x dx = \frac{a^x}{\ln a}$$

$$\textcircled{5} \int \sin x dx = -\cos x$$

$$\textcircled{6} \int \cos x dx = \sin x$$

$$\textcircled{7} \int \sec^2 x dx = \tan x$$

$$\textcircled{8} \int \csc^2 x dx = -\cot x$$

$$\textcircled{9} \int \sec x \tan x dx = \sec x$$

$$\textcircled{10} \int \csc x \cot x dx = -\csc x$$

$$\textcircled{11} \int \sec x dx = \ln |\sec x + \tan x|$$

$$\textcircled{12} \int \csc x dx = \ln |\csc x - \cot x|$$

$$\textcircled{13} \int \tan x dx = \ln |\sec x|$$

$$\textcircled{14} \int \cot x dx = \ln |\sin x|$$

$$\textcircled{15} \int \sinh x dx = \cosh x$$

$$\textcircled{16} \int \cosh x dx = \sinh x$$

$$\textcircled{17} \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$$

$$\textcircled{18} \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right), a > 0$$

$$\textcircled{19} \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right|$$

$$\textcircled{20} \int \frac{dx}{\sqrt{x^2 + a^2}} = \ln \left| x + \sqrt{x^2 + a^2} \right|$$

*Integration by parts

بشكل آخر،

$$\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + g(x)f'(x) \rightarrow \text{الجزء الثاني}$$

$$\int \frac{d}{dx}(f(x) \cdot g(x)) = \int f(x)g'(x)dx + \int g(x)f'(x)dx$$

بشكل آخر،

$$f(x) \cdot g(x) = \int f(x)g'(x)dx + \int g(x)f'(x)dx$$

بشكل آخر،

$$\int f(x)g'(x)dx = f(x) \cdot g(x) - \int g(x)f'(x)dx \rightarrow \text{هذا القاعده}$$

$$\left. \begin{array}{l} u = f(x) \\ du = f'(x)dx \end{array} \right\} \text{ (الجزء الثاني)}$$

بشكل آخر،

$$\int u dv = u \cdot v - \int v du$$

$$\text{Ex: } \int x \sin x dx =$$

u → الجزء الثاني

$$\left. \begin{array}{l} u = x \\ du = 1 dx \end{array} \right\} \text{ الجزء الثاني}$$

dv → الجزء الثاني

$$\int dv = \int \sin x dx$$

$$v = -\cos x$$

$$u = x \rightarrow du = dx$$

$$dv = \sin x dx \rightarrow v = -\cos x$$

$$u \cdot v - \int v du \rightarrow -x \cos x - \int -\cos x dx$$

$$-x \cos x + \sin x + C$$

$$\text{Ex: } \int \ln x dx =$$

$$u = \ln x \rightarrow du = \frac{1}{x} dx \quad \left\{ \begin{array}{l} u \cdot v - \int v du \\ dv = \frac{1}{x} dx \rightarrow v = \ln x \end{array} \right.$$

$$x \ln x - \int x \cdot \frac{1}{x} dx \rightarrow x \ln x - \int 1 dx$$

$$x \ln x - x + C$$

$$\text{Ex: } \int x^2 e^x dx =$$

$$u = x^2 \rightarrow du = 2x dx \quad \left\{ \begin{array}{l} u \cdot v - \int v du \\ dv = e^x dx \rightarrow v = e^x \end{array} \right.$$

$$x^2 e^x - \int 2x e^x dx$$

الآن نستخدم قاعدة لايبنز
بالجزء

$$x^2 e^x - 2 \int x e^x dx$$

$$\int x e^x dx$$

$$u = x \rightarrow du = dx$$

$$dv = e^x dx \rightarrow v = e^x$$

$$x \cdot e^x - \int e^x dx$$

$$x e^x - e^x$$

$$x^2 e^x - 2(x e^x - e^x)$$

$$x^2 e^x - 2x e^x + 2e^x + C$$

$$e^x (x^2 - 2x + 2) + C$$

* Tabular method for repeated integration by parts

طريقة ٢) تسمى طريقة التكامل بالجزء المتكرر

* $\int x^2 e^x dx$

u	dv
x^2	e^x
$2x$	$+ e^x$
2	$- x e^x$
0	$+ e^x$

$x^2 e^x - 2x e^x + 2e^x + C$

u	dv
Polynomial	e^{ax+b}
	$\cos(ax+b)$
	$\sin(ax+b)$
	$\sqrt{ax+b}$

← هذه هي الطريقة ٢

Ex: $\int x^4 \sin 2x dx$ → طريقة ٢

u	dv
x^4	$\sin 2x$
$4x^3$	$- \frac{\cos 2x}{2}$
$12x^2$	$- \frac{\sin 2x}{4}$
$24x$	$+ \frac{\cos 2x}{8}$
24	$+ \frac{\sin 2x}{16}$
0	$- \frac{\cos 2x}{32}$

$$-\frac{x^4 \cos 2x}{2} + \frac{x^3 \sin 2x}{4} + \frac{\frac{3}{2} x^2 \cos 2x}{8} - \frac{\frac{3}{2} x \sin 2x}{16}$$

$$-\frac{\frac{3}{2} x \cos 2x}{32} + C$$

$$(x^3 - \frac{3}{2}x) \sin 2x - (\frac{x^4}{2} - \frac{3x^3}{2} + \frac{3}{4}) \cos 2x + C$$

Ex: $\int e^x \cdot \sin x \, dx$

$$u = e^x \rightarrow du = e^x \, dx$$

$$dv = \sin x \rightarrow u = -\cos x$$

$$\int e^x \sin x \, dx = e^x \cos x - \int -\cos x e^x \, dx$$

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int \sin x e^x \, dx$$

$$\frac{2}{2} \int e^x \sin x \, dx = \frac{-e^x (\cos x + \sin x)}{2}$$

$$\int e^x \sin x \, dx = \frac{-e^x (\cos x + \sin x)}{2} + C$$

$$u = e^x \rightarrow du = e^x \, dx$$

$$dv = \cos x \rightarrow v = \sin x$$

$$e^x \sin x = \int \sin x e^x \, dx$$

$$\int \sin x e^x \, dx$$

$$\int \sin x e^x \, dx$$

Ex: $\int_0^{\pi} e^{\cos x} \sin 2x \, dx \rightarrow$

$$y = \cos x \rightarrow dy = -\sin x \, dx$$

$$dx = \frac{-dy}{\sin x}$$

$$x = 0 \rightarrow y = 1$$

$$x = \pi \rightarrow y = -1$$

$$\int_{-1}^1 e^y \sin 2x \cdot \frac{dy}{-\sin x}$$

→ Note → $\sin 2x = 2 \sin x \cos x$

$$\int_{-1}^1 2e^y \cancel{\sin x} \cos x \frac{dy}{-\cancel{\sin x}} \rightarrow -2 \int_{-1}^1 e^y y dy$$

by parts

$$\Rightarrow u = y \xrightarrow{'} du = dy$$

$$dv = e^y dy \xrightarrow{'} v = e^y$$

$$ye^y - \int e^y dy = ye^y - e^y$$

$$-2(ye^y - e^y) \Big|_{-1}^1 = -2(-e^1 - e^{-1} - (-e^{-1} + e^1))$$

$$-2(-2e^{-1}) = \boxed{\frac{4}{e}}$$

② $\int \frac{x e^{2x}}{(1+2x)^2} dx$

$$u = x e^{2x} \xrightarrow{'} 1 \cdot 2x e^{2x} + e^{2x}$$

$$dv = (1+2x)^{-2} \xrightarrow{'} v = \frac{(1+2x)^{-1}}{-2}$$

$$\frac{x e^{2x} \cdot (1+2x)^{-1}}{-2} + \int \frac{(1+2x)^{-1}}{-2} \cdot (2x e^{2x} + e^{2x}) dx$$

$$-\frac{x e^{2x} (1+2x)^{-1}}{2} + \frac{1}{2} \int \frac{e^{2x} (2x+1)}{(1+2x)} dx$$

$$-\frac{x e^{2x}}{2} (1+2x)^{-1} + \frac{1}{2} \frac{e^{2x}}{2} + C$$

$$\textcircled{3} \int x^4 (\ln x)^2 dx$$

$$y = \ln x \rightarrow dy = \frac{1}{x} dx \quad e^{y^4} = x$$

$$dx = x dy$$

$$\int x^4 y^2 x dy \rightarrow \int \underline{x^5} y^2 dy$$

$$\rightarrow \int e^{5y} y^2 dy$$

u	dv
y^2	e^{5y}
$2y$	$\frac{e^{5y}}{5}$
2	$-\frac{e^{5y}}{25}$
0	$\frac{e^{5y}}{125}$

$$\frac{y^2}{5} e^{5y} - \frac{2y e^{5y}}{25} + \frac{2 e^{5y}}{125} + C$$

$$e^{5y} \left(\frac{y^2}{5} - \frac{2y}{25} + \frac{2}{125} \right) + C$$

$$e^{5 \ln x} \left(\frac{(\ln x)^2}{5} - \frac{2 \ln x}{25} + \frac{2}{125} \right) + C$$

$$x^5 \left(\frac{(\ln x)^2}{5} - \frac{2 \ln x}{25} + \frac{2}{125} \right) + C$$

$$\text{H.w } \int 2^x \sin x dx =$$

$$u = 2^x$$

$$dv = \sin x dx$$

$$\int (\sin^{-1} x)^2 dx =$$

~~*~~

$$\rightarrow -2^x \cos x + \ln 2 \cdot 2^x \sin x \quad \checkmark$$

$$(\ln 2)^2 + 1$$

$\hookrightarrow$

$$u = 2^x$$

$$= i g'$$

$$dv = \sin x$$

$$\int \ln x \, dx = x \ln x - x + C$$

$$u = \ln x \rightarrow du = \frac{dx}{x}$$

$$dv = dx \rightarrow v = x$$

$$\int \ln x \, dx = x \ln x - \int \frac{x}{x} \, dx$$

$$= x \ln x - x + C$$

Integrating power of sine & cosine

$$\textcircled{1} \int \sin x \, dx = -\cos x + C$$

$$\textcircled{2} \int \sin^2 x \, dx = \int \frac{1}{2} (1 - \cos 2x) \, dx = \frac{1}{2} (x - \frac{\sin 2x}{2}) + C$$

$$\sin^2 x = \frac{1}{2} (1 - \cos 2x)$$

$$\textcircled{3} \int \sin^3 x \, dx = \int \sin x \cdot \sin^2 x \, dx$$

$$= \int \sin x (1 - \cos^2 x) \, dx$$

$$\sin^2 x + \cos^2 x = 1$$

$$= u = \cos x \rightarrow du = -\sin x \, dx$$

$$\rightarrow \int \sin x (1 - u^2) \frac{du}{-\sin x} = \int -1 + u^2 \, du = -\frac{u^3}{3} + u + C$$

$$= \frac{\cos^3 x}{3} - \cos x + C$$

$$\textcircled{4} \int \sin^4 x \, dx = \int (\sin^2 x)^2 \, dx$$

$$\rightarrow \sin^2 x = \frac{1}{2} (1 - \cos 2x)$$

$$= \int \left(\frac{1}{2} (1 - \cos 2x) \right)^2 \, dx$$

$$= \frac{1}{4} \int (1 + 2 \cos 2x + \cos^2 2x) \, dx$$

$$\cos^2 x = \frac{1}{2} (1 + \cos 2x)$$

$$= \frac{1}{4} \int \left(1 + 2 \cos 2x + \frac{1 + \cos 4x}{2} \right) \, dx$$

$$= \frac{1}{4} \int \left(\frac{3}{2} + 2 \cos 2x + \frac{\cos 4x}{2} \right) \, dx$$

$$= \frac{1}{4} \left(\frac{3}{2} x - \frac{\sin 2x}{1} + \frac{1}{2} \frac{\sin 4x}{4} \right) + C$$

$$\int \sin 4x \sin 5x \, dx$$

$$① \int \cos x \, dx = \sin x + c$$

$$② \int \cos^2 x \, dx = \int \frac{1}{2} (1 + \cos 2x) \, dx = \frac{1}{2} (x + \frac{\sin 2x}{2}) + c$$

$$③ \int \cos^3 x \, dx = \int \cos x \cdot \cos^2 x \, dx = \int \cos x (1 - \sin^2 x) \, dx$$

$$u = \sin x \rightarrow du = \cos x \, dx \rightarrow dx = \frac{du}{\cos x}$$

$$\int \cos x (1 - u^2) \frac{du}{\cos x} = u - \frac{u^3}{3} + c \rightarrow \sin x - \frac{\sin^3 x}{3} + c$$

$$④ \int \cos^4 x \, dx = \int (\cos^2 x)^2 \, dx$$

$$= \int \left(\frac{1}{2} (1 + \cos 2x) \right)^2 \, dx = \frac{1}{4} \int (1 + 2\cos 2x + \cos^2 2x) \, dx$$

$$= \frac{1}{4} \int (1 + 2\cos 2x + \frac{1}{2} + \frac{\cos 4x}{2}) \, dx$$

$$\frac{1}{2} (1 + \cos 4x)$$

$$= \frac{1}{4} \int \left(\frac{3}{2} + 2\cos 2x + \frac{\cos 4x}{2} \right) \, dx$$

$$= \frac{1}{4} \left(\frac{3}{2}x + \frac{2}{2}\sin 2x + \frac{\sin 4x}{8} \right) + c$$

$$* \int \sin^5 x \, dx \rightarrow \int \sin x \cdot \sin^4 x \, dx = \int \sin x (1 - \cos^2 x)^2 \, dx$$

$$\int \sin x (1 - 2\cos^2 x + \cos^4 x) \, dx$$

$$y = \cos x \rightarrow dy = -\sin x \, dx \rightarrow \int \sin x (1 - 2y^2 + y^4) \frac{dy}{-1} = \int (1 - 2y^2 + y^4) \, dy$$

$$\rightarrow \int (1 - 2y^2 + y^4) \, dy = y - \frac{2y^3}{3} + \frac{y^5}{5} + c = \cos x - \frac{2\cos^3 x}{3} + \frac{\cos^5 x}{5} + c$$

$$* \int \cos^6 x \, dx = \int (\cos^2 x)^3 = \int \left(\frac{1}{2} (1 + \cos 2x) \right)^3 \, dx$$

W

* Reduction Formulas *

$$\int \sin^n x dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx \quad \left. \begin{array}{l} n=2,3,4,\dots \end{array} \right\}$$

$$\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx$$

مثال:

Ex: $\int \sin^5 x dx = -\frac{1}{5} \sin^4 x \cos x + \frac{4}{5} \int \sin^3 x dx$ (reduction formulas) أو بالخطوات

$$= -\frac{1}{5} \sin^4 x \cos x + \frac{4}{5} \left(-\frac{1}{3} \sin^2 x \cos x + \frac{2}{3} \int \sin x dx \right)$$

$$= -\frac{1}{5} \sin^4 x \cos x + \frac{4}{5} \left(-\frac{1}{3} \sin^2 x \cos x + \frac{2}{3} \cos x \right) + C$$

$$\int \cos^6 2x dx$$

→ reduction formulas حاجب الزاوية خفيفة لتبسيط ما قبلها

$$u = 2x \rightarrow du = 2 dx$$

بدلية → تعويض

$$\int \cos^6 u \frac{du}{2} = \frac{1}{2} \int \cos^6 u du =$$

$$= \frac{1}{6} \cos^5 u \sin u + \frac{5}{6} \int \cos^4 u du$$

$$= \frac{1}{6} \cos^5 u \sin u + \frac{5}{6} \left(\frac{1}{4} \cos^3 u \sin u + \frac{3}{4} \int \cos^2 u du \right)$$

$$= \frac{1}{6} \cos^5 u \sin u + \frac{5}{24} \cos^3 u \sin u + \frac{15}{24} \left(\frac{1}{2} (u + \frac{\sin 2u}{2}) \right) + C$$

$$= \frac{1}{2} \left(\frac{1}{6} \cos^5 2x \sin 2x + \frac{5}{24} \cos^3 2x \sin 2x + \frac{15}{48} 2x + \frac{15}{84} \sin 4x \right) + C$$

ملاحظة:

$\int \sin^n x dx$	$\rightarrow$ even	$\rightarrow (\int \sin^2 x dx)$	$\rightarrow$ زوج (n)
$\int \sin^n x dx$	$\rightarrow$ odd	$\rightarrow (\int \sin x dx)$	$\rightarrow$ فردي (n)
$\int \cos^n x dx$	$\rightarrow$ odd	$\rightarrow (\int \cos x dx)$	$\rightarrow$ فردي
$\int \cos^n x dx$	$\rightarrow$ even	$\rightarrow (\int \cos^2 x dx)$	$\rightarrow$ زوج

$$* \sin A \cos B = \frac{1}{2} [\sin(A-B) + \sin(A+B)]$$

$$* \sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$$

$$* \cos A \cos B = \frac{1}{2} [\cos(A-B) + \cos(A+B)]$$

المثلثات
جيب

$$\rightarrow \int \sin mx \cos nx dx$$

$$\rightarrow \int \sin mx \sin nx dx$$

$$\rightarrow \int \cos mx \cos nx dx$$

المثلثات
جيب

$$\text{EX: ① } \int \sin^a x \cos^b x dx$$

$$\rightarrow \int \frac{1}{2} (\sin(-x) + \sin(9x)) dx$$

$$\cos(-x) = \cos x \leftarrow = \frac{1}{2} \left(\frac{-\cos(-x)}{-1} - \frac{\cos 9x}{9} \right) + C$$

$$= \frac{1}{2} \left(\cos x - \frac{1}{9} \cos 9x \right) + C$$

$$\text{② } \int x \sin 3x \sin 5x dx$$

$$u = x \rightarrow du = dx$$

$$dv = \sin 3x \sin 5x \rightarrow \int \sin 3x \sin 5x dx = v$$

$$v = \frac{1}{2} \int \cos(-2x) - \cos(8x) dx$$

$$\frac{1}{2} \left(\frac{\sin 2x}{2} - \frac{\sin 8x}{8} \right)$$

$$v = \frac{\sin 2x}{4} - \frac{\sin 8x}{16}$$

$$u \cdot v = \int u dv$$

$$x \left(\frac{\sin 2x}{4} - \frac{\sin 8x}{16} \right) - \int \frac{\sin 2x}{4} - \frac{\sin 8x}{16} dx$$

$$x \left(\frac{\sin 2x}{4} - \frac{\sin 8x}{16} \right) - \left(-\frac{\cos 2x}{8} + \frac{\cos 8x}{128} \right) + C$$

$$\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$$

sec

$$\int \sec^n x dx = \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x dx$$

$$\int \tan^n x \sec^n x dx \rightarrow (1) \text{ rule}$$

$$① \rightarrow \sec^2 x = 1 + \tan^2 x \text{ or } \tan^2 x = \sec^2 x - 1$$

$$② \rightarrow \int \tan x dx ? \rightarrow \text{بالقوى}$$

$$\int \frac{\sin x}{\cos x} dx \xrightarrow{\text{بالتعويض}} y = \cos x \rightarrow dx = \frac{dy}{-\sin x}$$

$$\int \frac{\sin x}{y} \cdot \frac{dy}{-\sin x} \rightarrow - \int \frac{1}{y} dy = -\ln|y| + C$$

$$\int \tan x dx = -\ln|\cos x| + C = \ln|\sec x| + C$$

$$③ \int \tan^2 x dx \rightarrow \int \sec^2 x - 1 dx \rightarrow \tan x - x + C$$

$$④ \int \sec x dx \rightarrow \int \sec x \cdot \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$$

$$\Rightarrow \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \rightarrow \text{هذا مقلد } \frac{f'(x)}{f(x)} \text{ بالتعويض } u = \sec x + \tan x$$

$$= \ln|\sec x + \tan x| + C$$

$$\int \frac{g(x)}{g(x)} dx \quad ②$$

$$\rightarrow ① \int \tan^n x \sec^n x dx = \ln|g(x)|$$

$\therefore$ even $\rightarrow$ the power of secant is even.

$$\text{Ex: } \int \tan^4 x \sec^6 x dx \xrightarrow{\text{even}}$$

$\rightarrow \sec^2 x, \sec^2 x$

$$\rightarrow \int \tan^4 x \sec^2 x \sec^2 x dx \rightarrow \sec^2 x = 1 + \tan^2 x$$

$$\rightarrow \int \tan^4 x \sec^2 x (1 + \tan^2 x) dx \rightarrow \text{نكتب بالقوى}$$

$$y = \tan x \rightarrow dy = \sec^2 x dx$$

$$\rightarrow \int \frac{y^4 \sec^2 x (1+y^2)^2 dy}{\sec x} \rightarrow \int y^4 (1+y^2)^2 dy$$

$$\rightarrow \int y^4 + 2y^6 + y^8 dy \rightarrow \frac{y^5}{5} + 2\frac{y^7}{7} + \frac{y^9}{9} + C$$

$$\rightarrow \frac{\tan^5 x}{5} + 2\frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C$$

② $\int \tan^{\text{odd}} x \sec^n x dx$ (2) rule

$\therefore$ odd $\rightarrow$ The power of tangent is odd

EX: $\int \tan^{\text{odd}} x \sec^7 x dx \rightarrow$ sec & tan is odd

$$\rightarrow \int \tan x \sec x \tan^4 x \sec^6 x dx \rightarrow \tan x \sec x - 1$$

$$\rightarrow \int \tan x \sec x (\sec^2 x - 1)^3 \sec^6 x dx \rightarrow$$

$$y = \sec x \rightarrow dy = \sec x \tan x dx$$

$$\rightarrow \int \frac{\sec x \tan x (y^2 - 1)^3 y^6 dy}{\sec x \tan x}$$

$$\rightarrow \int y^6 - 2y^8 + y^{10} dy \rightarrow \frac{y^7}{7} - \frac{2y^9}{9} + \frac{y^{11}}{11} + C$$

$\sec x = y$ (المعروف)

$$\textcircled{1} \int \tan^m x \, dx$$

$$\rightarrow \text{Ex: } \int \tan^3 x \, dx \rightarrow \int \tan x \tan^2 x \, dx$$

$$\rightarrow \int \tan x (\sec^2 x - 1) \, dx \rightarrow \int \tan x \sec^2 x \, dx - \int \tan x \, dx$$

$$\int \tan x \sec^2 x \, dx - \int \tan x \, dx$$

$$\frac{\tan^2 x}{2} - \ln|\sec x| + C$$

$$\textcircled{2} \int \tan^4 x \, dx$$

$$= \int \tan^2 x \tan^2 x \, dx$$

$$\rightarrow \int \tan^2 x (\sec^2 x - 1) \, dx$$

$$\rightarrow \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx$$

$$\textcircled{A} \, y = \tan x$$

$$dx = \frac{dy}{\sec^2 x}$$

$$\int y^2 \sec^2 x \frac{dy}{\sec^2 x}$$

$$\int y^2 \, dy$$

$$\frac{y^3}{3} = \frac{\tan^3 x}{3}$$

$$y = \tan x$$

$$dx = \frac{dy}{\sec^2 x}$$

$$\int y \sec^2 x \frac{dy}{\sec^2 x}$$

$$\int y \, dy = \frac{y^2}{2}$$

$$\textcircled{B} \int \sec^2 x - 1 \, dx$$

$$\tan x - x$$

$$\downarrow$$

$$\frac{\tan^3 x}{3} - \tan x + x + C$$

① $\int \sec^3 x dx$

$\rightarrow \int \sec x \sec^2 x dx \rightarrow$ بتقسيم الأس على 1

$u = \sec x \rightarrow du = \sec x \tan x dx$
 $du = \sec^2 x dx \rightarrow v = \tan x$

$\sec x \tan x - \int \sec x \tan^2 x dx$ قاعدة

$\int \sec^3 x dx = \sec x \tan x - \int \sec^3 x + \int \sec x dx$

$\int \sec^3 x dx = \frac{\sec x \tan x + \ln |\sec x + \tan x|}{2} + C$

② $\int \sec^5 x dx$

$\rightarrow \int \sec^3 x \sec^2 x dx \rightarrow \int \sec^3 x (\tan^2 x + 1) dx$

$\rightarrow \int \sec^3 x \tan^2 x dx + \int \sec^3 x dx$

$y = \tan x$
 $dy = \frac{dy}{\sec^2 x}$

$\rightarrow \int \sec^3 x \frac{y^2 dy}{\sec^2 x} + \tan x$

$\frac{y^3}{3} + \tan x + C \rightarrow \frac{\tan^3 x}{3} + \tan x + C$

Trigonometric substitution
جاء من الجذور التي هي عبارة عن دالة
عن دالة، لاقتراحات

$$\sqrt{a^2 - x^2}, \sqrt{x^2 - a^2}, \sqrt{x^2 + a^2} \rightarrow a \in \mathbb{R}$$

① $\sqrt{a^2 - x^2}$ $\xrightarrow[\frac{a}{\sin \theta}]{\frac{a \cos \theta}{\sin^2 \theta}}$ $x = a \sin \theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ $D \sin^{-1} x$

② $\sqrt{a^2 + x^2}$, $\rightarrow x = a \tan \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $D_{\tan} x$

③ $\sqrt{x^2 - a^2}$, $\longrightarrow x = a \sec \theta$, $0 < \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$

EX: ① $\int \frac{\sqrt{a^2 - x^2}}{x^2} dx \rightarrow x = a \sin \theta \rightarrow x = 3 \sin \theta$
 $dx = 3 \cos \theta d\theta$ $\int \dots$

$$\sqrt{9^2 - x^2} \rightarrow \sqrt{9 - 9\sin^2\theta} \rightarrow \sqrt{9(1 - \sin^2\theta)} \rightarrow 3\sqrt{\cos^2\theta}$$

$$\begin{aligned} \rightarrow 3 |\cos \theta| &\rightarrow -\frac{\pi}{2} < \theta < \frac{\pi}{2} \quad \frac{1}{\sqrt{2}} \cos \theta \quad \oplus \\ \rightarrow 3 \cos \theta \end{aligned}$$

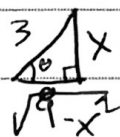
$$\therefore \sqrt{9-x^2} = 3\cos\theta \rightarrow \text{عوضا في السؤال} \rightarrow \int \frac{3\cos\theta \cdot 3\cos\theta d\theta}{\cancel{3\sin^2\theta}}$$

$$\int \frac{\cos \theta}{\sin \theta} d\theta \rightarrow \int \cot \theta d\theta \rightarrow \text{siehe 10.1}$$

$$\rightarrow \int \cos 2\theta - 1 \, d\theta \rightarrow -\cot \theta - \theta + C \rightarrow$$

الحل: بالدرجة ② يجب أن يكون بالدرجة ① ←

$$x = 3 \sin \theta \rightarrow \frac{x}{3} = \sin \theta \rightarrow \theta = \sin^{-1}\left(\frac{x}{3}\right)$$



المثلث

$$= -\cot \theta - \theta + C \rightarrow \frac{\sqrt{9-x^2}}{x} - \sin^{-1} \frac{x}{3} + C$$

Ex ② $\int \frac{1}{x^2 \sqrt{x^2+4}} dx$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\rightarrow X = 2 \tan \theta \rightarrow dx = 2 \sec^2 \theta d\theta$

$\rightarrow \sqrt{x^2+4} = \sqrt{4 \tan^2 \theta + 4} = 2 \sqrt{\tan^2 \theta + 1} = 2 \sec \theta$

$\therefore \sqrt{x^2+4} = 2 \sec \theta$

$\rightarrow \int \frac{1}{4 \tan^2 \theta \cdot 2 \sec \theta} \cdot 2 \sec^2 \theta d\theta \rightarrow \frac{1}{4} \int \frac{\sec \theta}{\tan^2 \theta} d\theta$

$\rightarrow \frac{1}{4} \int \frac{1}{\cos \theta} \cdot \frac{\cos^2 \theta}{\sin^2 \theta} d\theta \rightarrow \frac{1}{4} \int \frac{\cos \theta}{\sin^2 \theta} d\theta$

تحويل القوسين العادي

$u = \sin \theta$
 $du = \cos \theta d\theta$

$\rightarrow \frac{1}{4} \int \frac{\cos \theta}{u^2} \cdot \frac{du}{\cos \theta}$

$\rightarrow \frac{1}{4} \int u^{-2} du \rightarrow \frac{1}{4} \cdot \frac{u^{-1}}{-1} + C \rightarrow -\frac{1}{4u} + C \rightarrow -\frac{1}{4 \sin \theta} + C$

$\rightarrow -\frac{1}{4 \sin \theta} + C \rightarrow -\frac{1}{4} \csc \theta + C$

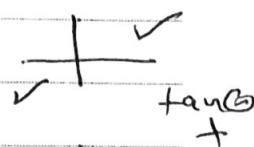
$X = 2 \tan \theta \rightarrow \tan \theta = \frac{x}{2}$

$-\frac{1}{4} \frac{\sqrt{x^2+4}}{x} + C$

Ex ③ $\int \frac{\sqrt{x^2-9}}{x} dx$

$$x = a \sec \theta \rightarrow x = 3 \sec \theta \quad \left\{ \begin{array}{l} \sqrt{x^2-9} = \sqrt{9 \sec^2 \theta - 9} = 3 \sqrt{\tan^2 \theta} \\ dx = 3 \sec \theta \tan \theta \end{array} \right. \rightarrow \begin{array}{l} 3 |\tan \theta| \\ 3 \tan \theta \end{array} \rightarrow \theta \rightarrow \begin{array}{l} \pi < \theta < \frac{3\pi}{2} \\ 0 < \theta < \frac{\pi}{2} \end{array}$$

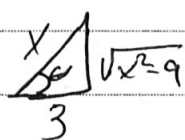
$$\rightarrow \int \frac{3 \tan \theta}{3 \sec \theta} \cdot 3 \sec \theta \tan \theta d\theta$$



$$\rightarrow 3 \int \tan^2 \theta d\theta \rightarrow 3 \int \sec^2 \theta - 1 d\theta$$

$$3 (\tan \theta - \theta) + C \rightarrow x = 3 \sec \theta \rightarrow \sec \theta = \frac{x}{3} \rightarrow \theta = \sec^{-1} \frac{x}{3}$$

$$3 \left(\frac{\sqrt{x^2-9}}{3} - \sec^{-1} \left(\frac{x}{3} \right) \right) + C$$



Ex: ④ $\int e^x \sqrt{1-e^{2x}} dx$

اولاً نبدأ بتحديد المتغير المناسب، لنفرض $u = e^x$

$$u = e^x \rightarrow du = e^x dx$$

$$\rightarrow \int \frac{u \sqrt{1-u^2}}{u} du \rightarrow \int \sqrt{1-u^2} du \rightarrow u = \sin \theta$$

$$du = \cos \theta d\theta$$

$$\int \sqrt{1-\sin^2 \theta} \cos \theta d\theta \rightarrow \int \cos \theta \cos \theta d\theta$$

$$\rightarrow \int \cos^2 \theta d\theta \rightarrow \int \frac{1}{2} (1 + \cos 2\theta) d\theta$$

$$= \frac{1}{2} (\theta + \sin 2\theta) + C \rightarrow \frac{1}{2} \theta + \frac{1}{2} \sin \theta \cos \theta + C$$



$$= \frac{1}{2} \sin^{-1} u + \frac{1}{2} \frac{u}{1} \cdot \frac{\sqrt{1-u^2}}{1} + C \rightarrow u = e^x$$

$$= \frac{1}{2} \sin^{-1} e^x + \frac{1}{2} e^x \sqrt{1-e^{2x}} + C$$

$$\text{ex 6) } \int \frac{x}{\sqrt{3-2x-x^2}} dx \rightarrow$$

الكمال مربع

يحتاج يكون معاد $1 = (x)^2$

$$1 = \left(\frac{-2}{2}\right)^2 = \left(\frac{b}{2}\right)^2$$

$$(3-2x-x^2) : 1 \rightarrow -(x^2+2x-3)$$

$$\rightarrow -(\underbrace{x^2+2x+1}_{\text{مربع كامل}} - 1 - 3) \rightarrow -(x+1)^2 - 4$$

$$\rightarrow (4 - (x+1)^2) \rightarrow \int \frac{x}{\sqrt{4-(x+1)^2}}$$

$$u = x+1 \rightarrow du = dx \rightarrow \int \frac{u-1}{\sqrt{4-u^2}} du \rightarrow u = 2 \sin \theta$$

$$x = u-1$$

$$\rightarrow \int \frac{2 \sin \theta - 1}{2 \cos \theta} 2 \cos \theta d\theta \quad \text{بالتعويض } du = 2 \cos \theta d\theta$$

$$\leftarrow \sqrt{4-4 \sin^2 \theta} = 2 \cos \theta$$

$$\rightarrow \int 2 \sin \theta - 1 d\theta \rightarrow -2 \cos \theta - \theta + C$$

$$u = \frac{2}{2} \sin \theta$$

$$\frac{\frac{2}{2} \sqrt{4-u^2}}{\frac{2}{2} \sqrt{4-u^2}} \rightarrow$$

$$-2 \frac{\sqrt{4-u^2}}{2} - \sin^{-1}\left(\frac{u}{2}\right) + C$$

$$\leftarrow u = x+1 \rightarrow$$

$$-\sqrt{4-(x+1)^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

يمكن استخدام هذه الطريقة مع الدالة التي لا تكون جيب

$$\text{Ex 7) } \int \frac{dx}{x^4+2x^2+1} \rightarrow \int \frac{dx}{(x^2+1)^2} \rightarrow x = a + \tan \theta$$

$$x = \tan \theta$$

$$dx = \sec^2 \theta d\theta$$

$$\int \frac{\sec^2 \theta d\theta}{(\tan^2 \theta + 1)^2}$$

$$\rightarrow \int \frac{\sec^2 \theta d\theta}{(\sec^2 \theta)^2} \rightarrow \int \frac{1}{\sec^2 \theta} d\theta \rightarrow \int \cos^2 \theta d\theta$$

$$\rightarrow \frac{1}{2} \int 1 + \cos 2\theta d\theta \rightarrow \frac{1}{2} \left(\theta + \frac{\sin 2\theta}{2} \right) + C \rightarrow x = \tan \theta$$

$$\frac{\sqrt{1+x^2}}{2} + \frac{x}{1+x^2}$$

$$\rightarrow \frac{1}{2} \left(\tan^{-1}(x) + \frac{x}{1+x^2} \right) + C$$

$$\rightarrow \frac{1}{2} \left(\tan^{-1} x + \frac{x}{1+x^2} \right) + C$$

$$\frac{(1 - \sin^2 \theta)^{3/2}}{(\sqrt{\cos^2 \theta})^3} \cos^3 \theta$$

$$\textcircled{2} \int \frac{dx}{(1-x^2)^{3/2}} \rightarrow x = \sin \theta \quad dx = \cos \theta d\theta$$

$$\rightarrow \int \frac{\cos \theta d\theta}{(1 - \sin^2 \theta)^{3/2}} \rightarrow \int \frac{\cos \theta}{(\cos^2 \theta)^{3/2}} d\theta$$

$$\rightarrow \int \frac{\cos \theta}{\cos^3 \theta} d\theta \rightarrow \int \frac{1}{\cos^2 \theta} d\theta = \int \sec^2 \theta d\theta$$

$$= \tan \theta + C \rightarrow x = \sin \theta \quad \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{x}{\sqrt{1-x^2}} + C$$

Integration of Rational Functions by Partial Fractions.

تكامل الدوال النسبية

$$\int f(x) dx = \int \frac{P(x)}{Q(x)} dx \quad ?$$

الاستزانات النسبية تكامل بالكسور الجزئية

عدد - $\frac{\text{Poly}}{\text{Poly}}$

~~عدد~~

$$f(x) = \frac{P(x)}{Q(x)}$$

$$\deg P(x) \geq \deg Q(x)$$

↓
نقسم اقسم قسمة طويلة

$$\deg P(x) < \deg Q(x)$$

↓
مباشرة (كسور جزئية)

$$\begin{array}{l} \text{باقى القسمة} \\ S(x) + \frac{R(x)}{Q(x)} \\ \text{مقوم عليه} \end{array}$$

Ex: $\int \frac{x^3 + x}{x-1} dx$

$x^3 + x > x-1$

$$\begin{array}{r} x^2 + x + 2 \\ x-1 \overline{) x^3 + x} \\ \underline{-(x^3 + x^2)} \\ x^2 + x \\ \underline{-(x^2 + x)} \\ 2x \\ \underline{-(2x + 2)} \\ 2 \end{array}$$

$\rightarrow \int x^2 + x + 2 + \frac{2}{x-1}$

$\frac{x^3}{3} + \frac{x^2}{2} + 2x + 2 \ln|x-1| + C$

* Factor the denominator

تجزيه المقام

$\rightarrow (ax+b)$ or (ax^2+bx+c)

Linear

irr quadratic

تريع غير قابل

للخيل

تريع الاصل مكرر

Case 1	Case 2	Case 3	Case 4
Linear factors	repeated Linear factors	irreducible quadratic factor	repeated irr quadratic factor
$x(x-1)(x+5)$	$(x-3)^2(2x+1)^3$	$(x^2+1)(x^2+x+1)$	$(x^2+4)^2(x^2+1)^3$
$\frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x+5)}$	$\frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(2x+1)} + \frac{D}{(2x+1)^2} + \frac{E}{(2x+1)^3}$	$\frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+x+1}$	$\frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$
	$\frac{1}{x^2(x+5)^4}$		
	$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+5)} + \frac{D}{(x+5)^2} + \frac{E}{(x+5)^3} + \frac{F}{(x+5)^4}$		

PRENKO

ثم اجب الثوابع واتحد

write out the form of the partial fraction decomposition of the function?

form 1.2.1

1) $\frac{1-3x^4}{(x-2)(x^2+1)^2} \rightarrow$ partial fraction

$$\frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

2) $\frac{x^4-2x^3+x^2+2x-1}{x^2-2x+1} \rightarrow$ partial fraction

$$\begin{array}{r} x^2-2x+1 \overline{) x^4-2x^3+x^2+2x-1} \\ \underline{x^4-2x^3+x^2} \\ 2x-1 \end{array}$$

$$x^2 + \frac{2x-1}{x^2-2x+1}$$

$$x^2 + \frac{2x-1}{(x+1)(x-1)} \rightarrow x^2 + \frac{2x-1}{(x-1)^2}$$

$$x^2 + \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$\text{Ex ①} \quad \int \frac{dx}{x^2+x-2}$$

$$\frac{1}{x^2+x-2} = \frac{1}{(x+2)(x-1)} \rightarrow \frac{A(x-1)}{(x+2)(x-1)} + \frac{B(x+2)}{(x-1)(x+2)}$$

$$\rightarrow \frac{A(x-1) + B(x+2)}{(x-1)(x+2)} = \frac{1}{(x+2)(x-1)} \rightarrow 1 = \underbrace{A(x-1)}_{x=1} + \underbrace{B(x+2)}_{x=-2}$$

$$\boxed{x=1} \quad 1 = A(0) + B(3)$$

$$\boxed{B = \frac{1}{3}}$$

$$\boxed{x=-2} \quad 1 = A(-3) + B(0) \quad \left. \begin{array}{l} B = \frac{1}{3} \\ A = -\frac{1}{3} \end{array} \right\} \rightarrow \frac{-\frac{1}{3}}{x+2} + \frac{\frac{1}{3}}{x-1} = \frac{1}{x^2+x-2}$$

$$\rightarrow \int \frac{-\frac{1}{3}}{x+2} + \frac{\frac{1}{3}}{x-1} dx = -\frac{1}{3} \ln|x+2| + \frac{1}{3} (\ln|x-1|) + C$$

$$= \frac{1}{3} \left(\ln \left| \frac{x-1}{x+2} \right| \right) + C$$

$$\text{②} \quad \int \frac{1}{x^3+x} dx$$

$$\frac{1}{x^3+x} = \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \rightarrow \frac{(x^2+1)A + x(Bx+C)}{x^2+1(x)} = \frac{1}{x(x^2+1)}$$

$$1 = \underbrace{A(x^2+1)}_{x=0} + \underbrace{(Bx+C)}_{x=0} x$$

$$\boxed{x=0} \rightarrow 1 = A + 0 = \boxed{A=1}$$

وأيضا افترضنا B و C في المعادلة

$$1 = 1(x^2+1) + Bx^2 + Cx \rightarrow 1 = (1+B)x^2 + Cx + 1$$

$$0 = (1+B)x^2 + Cx$$

$$0 = (1+B)x^2 + Cx$$

$$0x^2 + 0x = (1+B)x^2 + Cx \rightarrow 0 = 1+B \quad \& \quad \boxed{C=0}$$

$$\boxed{B=-1}$$

$$\rightarrow \frac{A}{x} + \frac{Bx+C}{x^2+1} \rightarrow \frac{1}{x} + \frac{-1x+0}{x^2+1}$$

$$\rightarrow \int \frac{1}{x} - \frac{x}{x^2+1} dx \rightarrow \ln|x| - \frac{\ln|x^2+1|}{2} + C$$

$$\textcircled{3} \int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx$$

$$\begin{array}{r} x+1 \\ x^3 - x^2 - x + 1 \overline{) x^4 - 2x^3 + 4x + 1} \\ \underline{-x^4 + x^3 + x^2 + x} \\ x^3 - x^2 + 3x + 1 \\ \underline{-x^3 + x^2 - x + 1} \\ 4x \end{array}$$

$$\boxed{4x}$$

$$\rightarrow \int x+1 dx + \int \frac{4x}{x^3 - x^2 - x + 1} dx$$

$$\frac{x^2}{2} + x$$

$$x^3 - x^2 - x + 1 \rightarrow \text{معاملة الحلق}$$

$$x=1 \rightarrow 20 \checkmark$$

$$(x-1)(x^2+1) \rightarrow \text{متعددة الحدود}$$

$$x^2 - 1 \rightarrow \text{الباقي}$$

$$\begin{array}{r} x-1 \overline{) x^3 - x^2 - x + 1} \\ \underline{x^3 - x^2} \\ -x + 1 \\ \underline{-x + 1} \\ 0 \end{array}$$

$$\frac{4x}{(x-1)(x^2-1)} \rightarrow \frac{4x}{(x-1)(x-1)(x+1)} \rightarrow \frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$\frac{(x-1) \cdot A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{x+1} (x-1)^2$$

$$\rightarrow \frac{A(x-1)(x+1) + B(x+1) + C(x-1)^2}{(x-1)^2(x+1)} = \frac{4x}{(x-1)^2(x+1)}$$

$$A(x-1)(x+1) + \underbrace{B(x+1)}_{x=-1} + \underbrace{C(x-1)^2}_{x=1} = 4x$$

$$\left. \begin{array}{l} \boxed{x=1} \rightarrow 4 = B(2) \rightarrow \boxed{B=2} \\ \boxed{x=-1} \rightarrow -4 = C(-2) \rightarrow \boxed{C=-2} \end{array} \right\} \begin{array}{l} \uparrow \\ (A) \end{array}$$

$$\rightarrow A(x-1)(x+1) + 2(x+1) - 2(x-1)^2 = 4x$$

$$Ax^2 - A + 2x + 2 + 2 - 4x + 2 = 4x$$

$$(A-1)x^2 + 4x - A + 1 = 4x$$

$$0 = (A-1)x^2 - A + 1 \rightarrow 0x^2 + 0 = (A-1)x^2 - A + 1$$

$$0 = A - 1$$

$$\int \frac{1}{x-1} + \frac{2(x-1)^{-2}}{(x-1)^2} - \frac{1}{x+1} dx$$

$$\ln|x-1| + \frac{2(x-1)^{-1}}{-1} - \ln|x+1|$$

$$\ln \left| \frac{x-1}{x+1} \right| - \frac{2}{x-1} + \frac{x^2}{2} + x + C$$

Ex ① $\int \frac{3x^2+2}{x^3+2x-8} dx$

ليذكرنا كوكور حربية
مقام

$= \ln|x^3+2x-8|$

② $\int \frac{2x-1}{x^2+1} dx = -\ln|x^2+1|$

$\rightarrow \int \frac{2x}{x^2+1} - \frac{1}{x^2+1} dx$

$\ln|x^2+1| - \tan^{-1}x + c$

* Strategy of integration. استراتيجيات التكامل

- Basic Rules of integration $\rightarrow$ القواعد الأساسية للتكامل

- The substitution Rule $\rightarrow$ التكامل بالقويعة

- Integration by part $\rightarrow$ بالاجزاء

- Trigonometric substitution $\rightarrow$ ~~التكامل بالتريغونوميترى~~
 $\rightarrow$ القويعة المثلثية

- Trigonometric integrals $\rightarrow$ المتكاملات التريغونوميترية

- Partial fractions $\rightarrow$ الكسور الجزئية

Ex: ③ $\int \frac{\tan^3 x}{\cos^3 x} dx$ 0-1 $\rightarrow$ odd $\int \tan^3 x \sec^2 x dx \rightarrow y = \sec x$

② $\int \frac{\sin^3 x}{\cos^6 x} dx \rightarrow y = \cos x$

$$\textcircled{2} \int \frac{dx}{1-\cos x} \rightarrow \int \frac{dx}{1-\cos x} \cdot \frac{1+\cos x}{1+\cos x}$$

$$\rightarrow \int \frac{1+\cos x}{1-\cos^2 x} dx = \int \frac{1+\cos x}{\sin^2 x} dx$$

$$\rightarrow \int \csc^2 x dx + \int \frac{\cos x}{\sin^2 x} dx \xrightarrow{\text{let } u = \sin x}$$

$$\textcircled{3} \int \frac{x^3}{(4x^2+9)^{\frac{3}{2}}} dx \rightarrow \text{[1]} \quad 2x = 3 \tan \theta \quad \text{--- right}$$

$$\int \frac{x^3}{u^{\frac{3}{2}}} \frac{du}{8x} \quad \text{[2]} \quad u = 4x^2 + 9 \rightarrow \frac{du}{8x} = dx$$

$$\rightarrow \frac{1}{8} \int \frac{x^2}{u^{\frac{3}{2}}} du \rightarrow \frac{1}{8} \int \frac{u-9}{u^{\frac{3}{2}}} du \quad \leftarrow x^2 = \frac{u-9}{4}$$

$$\rightarrow \frac{1}{32} \int \frac{u-9}{u^{\frac{3}{2}}} du \rightarrow \frac{1}{32} \int u^{-\frac{3}{2}} - 9u^{-\frac{3}{2}} du$$

$$\rightarrow \frac{1}{32} \left(-2u^{\frac{1}{2}} + 18u^{-\frac{1}{2}} \right) \rightarrow \frac{1}{32} \left(-2\sqrt{u} + \frac{18}{\sqrt{u}} \right) \rightarrow \frac{1}{32} \left(-2\sqrt{4x^2+9} + \frac{18}{\sqrt{4x^2+9}} \right)$$

$$\textcircled{4} \int \sqrt{1-\sin x} dx \rightarrow \text{multiply by } \frac{\sqrt{1+\sin x}}{\sqrt{1+\sin x}}$$

$$\rightarrow \int \frac{\sqrt{1-\sin x} \cdot \sqrt{1+\sin x}}{\sqrt{1+\sin x}} dx \rightarrow \int \frac{\sqrt{1-\sin^2 x}}{\sqrt{1+\sin x}} dx$$

$$\rightarrow \int \frac{\sqrt{\cos^2 x}}{\sqrt{1+\sin x}} dx \rightarrow y = 1 + \sin x \rightarrow dy = \cos x dx$$

$$\rightarrow \int \frac{\cos x}{\sqrt{y}} \cdot \frac{dy}{\cos x} \rightarrow \int \frac{dy}{\sqrt{y}} \rightarrow \int y^{-\frac{1}{2}} dy$$

$$2y^{\frac{1}{2}} + c \rightarrow 2\sqrt{1+\sin x} + c$$

$$\textcircled{5} \int e^{\sqrt{x}} dx \rightarrow \text{let } y = \sqrt{x} \rightarrow dy = \frac{1}{2\sqrt{x}} dx \rightarrow dx = 2\sqrt{x} dy$$

$$dx = 2y dy$$

$$\int e^y 2y dy \rightarrow 2 \int e^y \cdot y dy$$

↓
Py Parts

$$u = y$$

$$du = dy$$

$$\textcircled{6} \int \frac{\sqrt{1-x}}{\sqrt{1+x}} dx \rightarrow \int \frac{\sqrt{1-x}}{\sqrt{1+x}} \cdot \frac{\sqrt{1-x}}{\sqrt{1-x}} dx$$

نفسه هو افادت المقام

ادع اربع بجزء $x = \sin \theta$ من
ما افكده / يبين

$$\rightarrow \int \frac{1-x}{\sqrt{1-x^2}} dx \rightarrow \int \frac{1}{\sqrt{1-x^2}} + \frac{-x}{\sqrt{1-x^2}} dx$$

$$\sin^{-1} x$$

$$x = \sin \theta \dots$$

$$\textcircled{7} \int \frac{1}{\sqrt{x} - \sqrt[3]{x}} dx$$

المقام نفسه بفرمته x قوة الجذر
نوع الجذر $\rightarrow 3 \cdot 2 = 6 \rightarrow$

$$dx = 6u^5 du$$

$$\rightarrow \int \frac{6u^5}{\sqrt{u^6} - \sqrt[3]{u^6}} du \rightarrow (u^6)^{\frac{1}{2}} = u^3 \text{ \& } (u^6)^{\frac{1}{3}} = u^2$$

$$\rightarrow \int \frac{6u^5}{u^3 - u^2} du \rightarrow$$

السطح المقام

$$\begin{array}{r}
 6u^2 + 6u + 6 \\
 u^3 - u^2 \overline{) 6u^5} \\
 \underline{-6u^5 + 6u^4} \\
 6u^4 \\
 \underline{-6u^4 + 6u^3} \\
 6u^3 \\
 \underline{-6u^3 + 6u^2} \\
 6u^2
 \end{array}$$

$$\rightarrow \int 6u^2 + 6u + 6 + \frac{6u^2}{u^3 - u^2} du$$

$$\int \frac{6u}{u^2(u-1)} du$$

$$\int \frac{6}{u-1} du \rightarrow 6 \cdot \ln|u-1|$$

$$\textcircled{8} \int \frac{\sqrt{1+\sqrt{x}}}{x} dx$$

$$\begin{aligned}
 u^2 - 1 &= \sqrt{x} \\
 (u^2 - 1)^2 &= x
 \end{aligned}$$

$$u = \sqrt{1+\sqrt{x}} \rightarrow u^2 = 1+\sqrt{x} \rightarrow 2u du = \frac{1}{2\sqrt{x}} dx$$

$$du = \frac{1}{4u\sqrt{x}} dx$$

$$du = \frac{1}{4u(u^2-1)} dx$$

$$\rightarrow \int \frac{u}{x} \cdot 4u(u^2-1) du$$

$$\rightarrow \int \frac{4u^2(u^2-1)}{(u^2-1)^2} du \rightarrow \int \frac{4u^2}{u^2-1} du \rightarrow 4 \int \frac{u^2}{u^2-1} du$$

$$\frac{1}{2} \ln \left| \frac{u-1}{u+1} \right|$$

$$\begin{array}{r}
 u-1 \overline{) u^2} \\
 \underline{-u^2 + 1} \\
 1
 \end{array}$$

$$\rightarrow 4 \int 1 + \frac{1}{u^2-1} du$$

$$\frac{1}{(u-1)(u+1)}$$

جزء

$$\textcircled{9} \int \frac{x e^x}{\sqrt{1+e^x}} dx$$

$$\rightarrow y = \sqrt{1+e^x}$$

$$dy = \frac{e^x}{2\sqrt{1+e^x}} dx \rightarrow dx = \frac{2\sqrt{1+e^x}}{e^x} dy$$

$$dx = \frac{2y}{e^x} dy$$

PRENKO

$$(y) = (y^2 - 1)e^x \rightarrow y^2 - 1 = e^x \rightarrow \ln y^2 - 1 = x$$

برجع للمفرد ←

$$\rightarrow \int \frac{x e^x}{y} \cdot \frac{2y}{e^x} dy \rightarrow 2 \int x dy$$

$$\rightarrow 2 \int \ln(y^2 - 1) dy \rightarrow \text{بالاجزاء}$$

$$u = \ln(y^2 - 1) \rightarrow du = \frac{2y}{y^2 - 1}$$

$$dv = dy \rightarrow v = y$$

$$2 \left[y \ln(y^2 - 1) - \int \frac{2y^2}{y^2 - 1} dy \right]$$

قسمه بواسطة كوك جزئية

$$\textcircled{16} \int \frac{e^{2x}}{e^x + 3e^x + 2} dx \rightarrow u = e^x \rightarrow \frac{du}{dx} = dx$$

$$\leftarrow \frac{du}{u} = dx$$

$$\rightarrow \int \frac{\frac{2}{u}}{u^2 + 3u + 2} \frac{du}{x}$$

$$\rightarrow \int \frac{u}{u^2 + 3u + 2} du \rightarrow \int \frac{u}{(u+1)(u+2)} du$$

كوك جزئية

* Improper Integrals → تكاملات غير صحيحة

هنا ٣ أنواع

Type (1): Infinite Intervals

$$\textcircled{1} \int_a^{+\infty} f(x) dx$$

$$\textcircled{2} \int_{-\infty}^a f(x) dx$$

اول من نستبدل (∞) بتغير t نحو ∞ ونحسب النهاية

$$\rightarrow \textcircled{1} \int_a^{+\infty} f(x) dx \rightarrow \lim_{t \rightarrow +\infty} \int_a^t f(x) dx$$

$$\rightarrow \textcircled{2} \int_{-\infty}^a f(x) dx \rightarrow \lim_{t \rightarrow -\infty} \int_t^a f(x) dx$$

* called convergent if limit exists

* divergent if the limit d.n.e

$$\textcircled{3} \int_{-\infty}^{+\infty} f(x) dx \rightarrow \int_{-\infty}^a f(x) dx + \int_a^{+\infty} f(x) dx$$

EX: $\textcircled{1} \int_1^{+\infty} \frac{1}{x} dx \rightarrow \lim_{t \rightarrow +\infty} \int_1^t \frac{1}{x} dx$

$$\rightarrow \lim_{t \rightarrow +\infty} \ln |x| \Big|_1^t \rightarrow \lim_{t \rightarrow +\infty} \ln |t| - \ln |1|$$

$$\rightarrow \lim_{t \rightarrow +\infty} \ln |t| = \infty \quad \} \rightarrow \int_1^{+\infty} \frac{1}{x} dx \text{ is divergent}$$

لا يوجد نهاية

الحد لا يوجد
 $\lim_{t \rightarrow +\infty} \ln |t|$

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{1}{x} &= 0 \\ \lim_{x \rightarrow \infty} \frac{1}{x^p} &= 0 \quad p > 0 \\ \lim_{x \rightarrow \infty} \frac{1}{x^p} &= \infty \quad p < 0 \end{aligned}$$

$$\textcircled{2} \int_1^{\infty} \frac{1}{x^2} dx \rightarrow \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2} dx \rightarrow \lim_{t \rightarrow \infty} \left[-\frac{1}{x} \right]_1^t = \lim_{t \rightarrow \infty} \left(-\frac{1}{t} + 1 \right) = 1$$

$$\rightarrow \lim_{t \rightarrow \infty} -\frac{1}{t} + 1 = 1$$

$$\rightarrow -\frac{1}{\infty} + 1 = 1$$

$$\rightarrow \int_1^{\infty} \frac{1}{x^p} dx = 1 \rightarrow \text{convergent}$$

$$\text{Then: } \int_1^{\infty} \frac{1}{x^p} dx = \int_1^{\infty} \frac{1}{x^p} dx \rightarrow \text{convergent to } \frac{1}{p-1}$$

$$\int_1^{\infty} \frac{1}{x^3} dx \rightarrow \text{convergent} = \frac{1}{3-1} = \frac{1}{2}$$

$$\textcircled{3} \int_1^{\infty} \frac{1}{x^{10}} dx \rightarrow \text{convergent} = \frac{1}{10-1} = \frac{1}{9}$$

$$\textcircled{4} \int_1^{\infty} \frac{1}{\sqrt{x}} dx \rightarrow \int_1^{\infty} \frac{1}{x^{1/2}} dx \rightarrow \text{divergent}$$

$$\textcircled{5} \int_1^{\infty} x^{-2/3} dx \rightarrow \int_1^{\infty} \frac{1}{x^{2/3}} dx \rightarrow \text{divergent}$$

$$\text{Ex: } \int_{-\infty}^0 x e^x dx \rightarrow \lim_{t \rightarrow -\infty} \int_t^0 x e^x dx$$

$$\rightarrow \int_t^0 x e^x dx \rightarrow u = x \rightarrow \frac{du}{dx} = e^x \rightarrow \frac{du}{u} = dx \rightarrow v = e^x$$

$$x e^x - \int e^x dx \rightarrow x e^x - e^x$$

$$\rightarrow \lim_{t \rightarrow -\infty} x e^x - e^x \Big|_t^0 \rightarrow \lim_{t \rightarrow -\infty} (0 - 1 - (t e^t - e^t))$$

$$\rightarrow \lim_{t \rightarrow -\infty} (-1 - \cancel{t e^t} + \cancel{e^t}) \rightarrow \lim_{t \rightarrow -\infty} e^t = 0$$

$$\rightarrow \lim_{t \rightarrow -\infty} t e^t = \infty \cdot 0$$

$$\lim_{t \rightarrow -\infty} [-1]$$

$$\rightarrow \text{L'H} \rightarrow \lim_{t \rightarrow -\infty} \frac{t}{e^t} \rightarrow \lim_{t \rightarrow -\infty} \frac{1}{-e^t}$$

$$\text{convergent} = [-1]$$

$$\rightarrow \lim_{t \rightarrow -\infty} -e^t = 0$$

$$\text{Ex: } \int_{-\infty}^{+\infty} \frac{1}{1+x^2} dx \rightarrow \int_{-\infty}^a \frac{1}{1+x^2} dx + \int_a^{+\infty} \frac{1}{1+x^2} dx \rightarrow a = \infty \text{ or } -\infty$$

$$\rightarrow \int_{-\infty}^0 \frac{1}{1+x^2} dx + \int_0^{+\infty} \frac{1}{1+x^2} dx$$

$$\textcircled{1} \int_{-\infty}^0 \frac{1}{1+x^2} dx \rightarrow \lim_{t \rightarrow -\infty} \int_t^0 \frac{1}{1+x^2} dx \rightarrow \lim_{t \rightarrow -\infty} \tan^{-1} x \Big|_t^0$$

$$\rightarrow \lim_{t \rightarrow -\infty} (0 - \tan^{-1} t) = (-\tan^{-1}(\infty)) \rightarrow -\left(\frac{\pi}{2}\right) \quad \text{convergent}$$

←

$$\textcircled{1} \int_0^{+\infty} \frac{1}{1+x^2} dx \rightarrow \lim_{t \rightarrow +\infty} \int_0^t \frac{1}{1+x^2} dx$$

$$\rightarrow \lim_{t \rightarrow +\infty} \tan^{-1} x \Big|_0^t \rightarrow \lim_{t \rightarrow +\infty} (\tan^{-1}(t) - 0) \Rightarrow$$

$$\rightarrow \boxed{\frac{\pi}{2}} \text{ convergent}$$

$$\textcircled{1} + \textcircled{2} \rightarrow \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$\rightarrow \int_{-\infty}^{+\infty} \frac{1}{1+x^2} dx \rightarrow \pi, \text{ convergent}$$

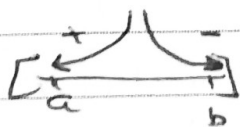
Type 2: Discontinuous integrands

① If f is continuous on $[a, b)$ & is discontinuous at b , then

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

② If f is continuous on $(a, b]$ & is discontinuous at a , then

$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$



③ If $f(x)$ is discontinuous at $[c]$, such that $a < [c] < b$, then

$$\rightarrow \int_a^b f(x) dx$$

$$\left[\begin{array}{ccc} a & c & b \end{array} \right]$$

$$\rightarrow \underbrace{\int_a^c f(x) dx + \int_c^b f(x) dx}$$

Ex: ① $\int_2^5 \frac{1}{\sqrt{x-2}} dx$

و هو improper integral

$\rightarrow \frac{1}{\sqrt{x-2}}$ discts $[2]$ So the function

is Improper Integral

$$\rightarrow \lim_{x \rightarrow 2^+} \int_t^5 (x-2)^{-\frac{1}{2}} dx \rightarrow \lim_{x \rightarrow 2^+} 2(x-2)^{\frac{1}{2}} \Big|_t^5$$

$$\lim_{x \rightarrow 2^+} 2(3)^{\frac{1}{2}} - 2 \sqrt{(4-2)}$$

$$2\sqrt{3} - 2\sqrt{2} = 2(\sqrt{3} - \sqrt{2}) \text{ convergent}$$

Ex ② $\int_0^{\frac{\pi}{2}} \sec x dx$

$\rightarrow$ غير متناهية

$$\sec \frac{\pi}{2} = \frac{1}{\cos \frac{\pi}{2}} = \frac{1}{0} = \infty$$

So $\rightarrow$ improper

integral

$$\rightarrow \lim_{t \rightarrow \frac{\pi}{2}^-} \int_0^t \sec x dx = \frac{\sec x + \tan x}{\sec x + \tan x}$$

$$\rightarrow \lim_{t \rightarrow \frac{\pi}{2}^-} \ln |\sec x + \tan x|$$

$$\rightarrow \lim_{t \rightarrow \frac{\pi}{2}^-} \ln|\sec t + \tan t| - \ln|\sec 0 + \tan 0|$$

$$\rightarrow \lim_{t \rightarrow \frac{\pi}{2}^-} = \text{zero} + \infty \rightarrow \int_0^{\frac{\pi}{2}} \sec x dx \text{ divergent}$$

$$\rightarrow \textcircled{2} \int_0^3 \frac{dx}{x-1} \rightarrow \text{Improper integral } \frac{1}{x-1} \text{ discont at } x=1$$

$$\rightarrow \int_0^1 \frac{1}{x-1} dx + \int_1^3 \frac{1}{x-1} dx$$

$$\lim_{t \rightarrow 1^-} \int_0^t \frac{1}{x-1} dx \rightarrow \int_0^3 \frac{dx}{x-1} = \text{divergent}$$

$$\lim_{t \rightarrow 1^-} \ln|x-1| \Big|_0^t = \text{zero}$$

$$\rightarrow \lim_{t \rightarrow 1^+} \ln|t-1| - \ln|0-1|$$

$$\rightarrow \lim_{t \rightarrow 1^+} \ln|t-1|$$

$$\begin{array}{ccc} \leftarrow & \rightarrow & \\ 1-t & t-1 & \end{array}$$

$$\lim_{t \rightarrow 1^-} \ln|1-t| \rightarrow -\infty$$

divergent

الانطباع وال divergent) انطباع

* Find the values of (P) for which the integral converges (exists) as $\lim_{x \rightarrow \infty}$

$$1 - \int_e^{+\infty} \frac{1}{x (\ln x)^p} dx$$

$$2 - \int_a^b \frac{1}{(x-a)^{p-1}} dx$$

for $a < b$

$$1 - \int_e^{+\infty} \frac{dx}{x (\ln x)^p}$$

$$2 - \int_a^b \frac{dx}{(x-a)^{p-1}}, a < b$$

$$u = \ln x \Rightarrow dx = x du$$

$$\lim_{t \rightarrow a^+} \int_t^b \frac{dx}{(x-a)^{p-1}}$$

$$x=e \rightarrow u = \ln e = 1$$

$$\rightarrow \lim_{t \rightarrow a^+} \int_t^b (x-a)^{1-p} dx$$

$$x=+\infty \rightarrow u = +\infty$$

$$\rightarrow \lim_{t \rightarrow a^+} \left[\frac{(x-a)^{2-p}}{2-p} \right]_t^b$$

$$\int_1^{+\infty} \frac{1}{u^p} du$$

$$\rightarrow \lim_{t \rightarrow a^+} \frac{1}{2-p} \left((b-a)^{2-p} - (t-a)^{2-p} \right)$$

$$\Rightarrow \int_1^{+\infty} \frac{1}{x^p} dx = \begin{cases} p \leq 1 & \text{div} \\ p > 1 & \text{conv} \end{cases}$$

Lim is exists $\leftarrow$ ∞ و ∞

$$\textcircled{1} \frac{1}{2-p} \rightarrow p \neq 2$$

$$\textcircled{2} (b-a)^{2-p} = (+)^{2-p} = +$$

$a < b$ so $(b-a)$ is (+ number).

العدد الموجب يبقى موجب

$$p = \left(\frac{1}{2}, \infty \right)$$

$$\textcircled{3} (t-a)^{2-p} \rightarrow \lim_{t \rightarrow a^+} (t-a)^{2-p}$$

الحوال بغير مرفوع لقوة

$$= 0^{2-p} \rightarrow \begin{cases} 2-p > 0 & 0^+ = 0 \\ 2-p < 0 & 0^- = \frac{1}{0} = \infty \end{cases}$$

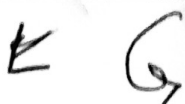
هذه القوة يمكن تكون $+$ or $-$

$$\text{convergen } \frac{1}{2p-1}$$

convergent

divergen

PRENKO

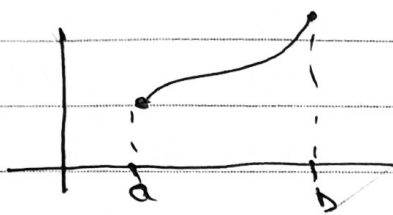


$$2-p > 0 \rightarrow \boxed{p < 2}$$

* So the $\textcircled{1}$ is convergent to $p < 2$

is

~~Are~~ Arc Length . dx is small, dy is small, ds is



* The Arc Length formula

- If f' is continuous on $[a, b]$ then the Length of the curve $y = f(x)$, $a \leq x \leq b$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Ex: find the exact Length of the curve

$f(x) = x^{\frac{3}{2}}$, $1 \leq x \leq 4$

* $f'(x) = \frac{3}{2} x^{\frac{1}{2}} \rightarrow (f'(x))^2 = \frac{9}{4} x$

$$L = \int_1^4 \sqrt{1 + \frac{9}{4} x} dx \rightarrow L = \int_1^4 \left(1 + \frac{9}{4} x\right)^{\frac{1}{2}} dx$$

$$= \left(1 + \frac{9}{4} x\right)^{\frac{3}{2}} \cdot \frac{2}{3} \cdot \frac{4}{9} \Big|_1^4$$

$$= \left[\frac{1}{27} (80\sqrt{10} - 13\sqrt{13}) \right]$$

- curve $x = g(y)$, $c \leq y \leq d$

y axis

$$L = \int_c^d \sqrt{1 + (g'(y))^2} dy$$

or

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Ex: Set up the integral that represents the length of the curve:

$x = y^2$ from $(0,0)$ to $(1,1)$

$$\frac{dx}{dy} = 2y \rightarrow \left(\frac{dx}{dy} \right)^2 = 4y^2$$

$$L = \int_0^1 \sqrt{1 + 4y^2} dy$$

Ex: Find the arc length of the curve $y = \sqrt{4-x^2}$?

Ex: 1)

لم نحدد من السؤال فترة اذن

نأخذ (Domain) $x \in [-2, 2]$

$$4 - x^2 \geq 0 \rightarrow x = \pm 2$$

domain $[-2, 2]$ ✓

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx \quad / \quad y' = \frac{dy}{dx} = \frac{-x}{2\sqrt{4-x^2}} \rightarrow \left(\frac{dy}{dx} \right)^2 = \frac{x^2}{4-x^2}$$

$$L = \int_{-2}^2 \sqrt{1 + \frac{x^2}{4-x^2}} dx \rightarrow \int_{-2}^2 \sqrt{\frac{4-x^2+x^2}{4-x^2}} dx$$

حسابه

$$\int_{-2}^2 \sqrt{\frac{4}{4-x^2}} dx \rightarrow 2 \int_{-2}^2 \sqrt{\frac{1}{4-x^2}} dx = 2 \int_{-2}^2 \frac{1}{\sqrt{4-x^2}} dx$$

$$= 2 \left[\sin^{-1} \frac{x}{2} \right]_{-2}^2 \rightarrow 2 \left[\sin^{-1} \left(\frac{2}{2} \right) - \sin^{-1} \left(\frac{-2}{2} \right) \right]$$

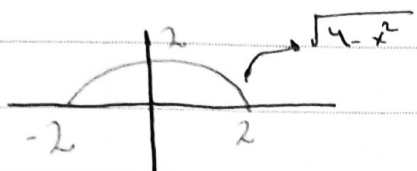
$$2 \left(\sin^{-1}(1) - \sin^{-1}(-1) \right)$$

$$2 \left(\frac{\pi}{2} + \frac{\pi}{2} \right) = 2\pi$$

$$f(x) = \sqrt{4 - x^2} \quad -2 \leq x \leq 2 \quad \text{طريقة (2)}$$

$$f(x) = \sqrt{a^2 - x^2} \quad -a \leq x \leq a$$

هذا الاعتراف عبارة عن نصف دائرة



$$L = 2\pi r \quad \leftarrow \text{محيط الدائرة}$$

$$\frac{1}{2}L = \pi r \quad \leftarrow \text{نصف " " " "}$$

$$\rightarrow L = \pi(2)$$

$$= 2\pi$$

نصف محيط الدائرة

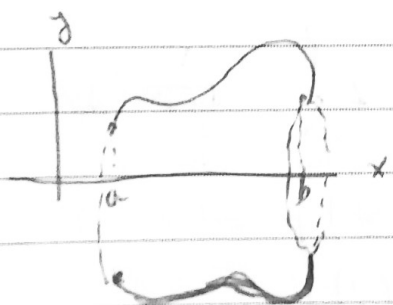
semi circle نصف دائرة

radius = [2]

Area of a surface of Revolution : مساحة سطح دوران
وهذا امر تقريبي ان الشكل

- يكون الشكل اقرب الى محور (x)

- $f(x)$ is positive & has a continuous derivative
we define the surface area obtained by rotating
the curve $y = f(x)$, $a \leq x \leq b$ about the
x-axis



$$S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$$

Arc length

Ex: Find the exact area of the surface obtained by rotating the curve about x-axis

$$f(x) = x^3 \quad x \in [0, 1]$$

$$\rightarrow S = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx$$

$$f'(x) = 3x^2 \rightarrow \int_0^1 x^3 \sqrt{1 + 9x^4} dx$$

$$(f'(x))^2 = 9x^4$$

$$u = 1 + 9x^4 \rightarrow du = 36x^3 dx \rightarrow dx = \frac{du}{36x^3}$$

$$\rightarrow 2\pi \int \frac{x^3 \sqrt{u}}{36x^3} du \rightarrow \frac{2\pi}{36} \int u^{\frac{1}{2}}$$

$$\rightarrow \frac{\pi}{18} \cdot \frac{2}{3} u^{\frac{3}{2}} \rightarrow \frac{\pi}{27} u^{\frac{3}{2}} \rightarrow \frac{\pi}{27} (1 + 9x^4)^{\frac{3}{2}} \Big|_0^1$$

$$\rightarrow \frac{\pi}{27} (10\sqrt{10} - 1) = S$$

Ex: $y = \sqrt{4-x^2} \quad -2 \leq x \leq 2$

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{4-x^2}} \rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{4-x^2}$$

$$\rightarrow S = 2\pi \int_{-2}^2 \sqrt{4-x^2} \sqrt{1 + \frac{x^2}{4-x^2}} dx$$

$$\rightarrow 2\pi \int_{-2}^2 \sqrt{4-x^2} \sqrt{\frac{4}{4-x^2}} dx \rightarrow 4\pi \int_{-2}^2 \sqrt{4-x^2} \cdot \frac{1}{\sqrt{4-x^2}} dx$$

$$4\pi \int_{-2}^2 dx \rightarrow 4\pi (x) \Big|_{-2}^2 \rightarrow 4\pi ((2) - (-2)) = 16\pi$$



مراجعة (2)

لما المنحنى يدور حول محور x سوف يعطينا كرة

$$S = 4r^2\pi$$

اذن مساحة سطح الكرة

$$\rightarrow S = 4(r)^2\pi$$

$$= 16\pi$$

Ex: Set up an Integral for the area of the surface obtained by rotating the curve about the x -axis.

$$y = \tan x \rightarrow 0 \leq x \leq \frac{\pi}{3}$$

$$\rightarrow S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$$

$$\rightarrow f(x) = \tan x \rightarrow f'(x) = \sec^2 x \rightarrow (f'(x))^2 = \sec^4 x$$

$$\rightarrow S = \int_0^{\frac{\pi}{3}} 2\pi \tan x \sqrt{1 + \sec^4 x} dx$$



$x = g(y)$, $c \leq y \leq d$ about x -axis

$$S = \int_c^d 2\pi (y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Ex: Find the exact area of the surface obtained by rotating the curve about the x -axis.

$$x = \frac{1}{3} (y^2 + 2)^{\frac{3}{2}}, \quad 1 \leq y \leq 2$$

$$\frac{dx}{dy} = \frac{1}{3} \cdot \frac{3}{2} (y^2 + 2)^{\frac{1}{2}} (2y) \rightarrow y(y^2 + 2)^{\frac{1}{2}}$$

$$\left(\frac{dx}{dy}\right)^2 = y^2(y^2 + 2) \rightarrow y^4 + 2y^2$$

$$S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$\rightarrow S = \int_1^2 2\pi y \sqrt{1 + y^4 + 2y^2} dy \rightarrow \int_1^2 2\pi y \sqrt{(1 + y^2)^2} dy$$

$$\rightarrow 2\pi \int_1^2 (y + y^3) dy \rightarrow 2\pi \left(\frac{y^2}{2} + \frac{y^4}{4} \right) \Big|_1^2$$

$$2\pi \left(\left(\frac{4}{2} + \frac{16}{4} \right) - \left(\frac{1}{2} + \frac{1}{4} \right) \right) = \frac{21\pi}{4}$$

curves defined by parametric equations:

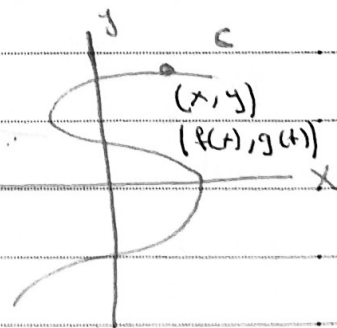
new method for describing curves

t = parameter

$x = f(t)$
 $y = g(t)$ } → parametric equations

Ex: Sketch the curve defined by the parametric equations

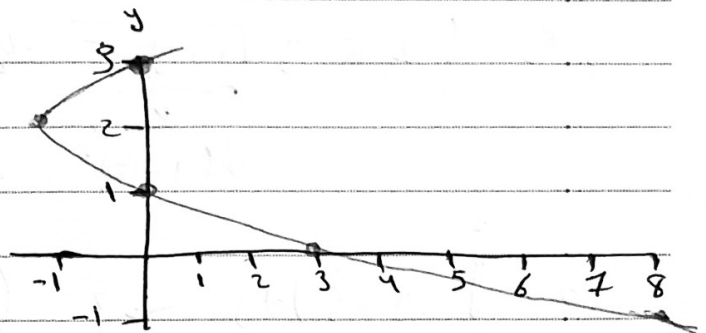
$$x = t^2 - 2t \quad \& \quad y = t + 1$$



اولاً سنرسم (Curve) ١- اخرج قيم t

t	x	y	(x, y)
-2	8	-1	(8, -1)
-1	3	0	(3, 0)
0	0	1	(0, 1)
1	-1	2	(-1, 2)
2	0	3	(0, 3)

$$\begin{aligned} x &= t^2 - 2t \\ y &= t + 1 \end{aligned}$$



اعوضني x
بـ y

Parabola قطع مكافئ
عند $t=0$ نصل الى نقطة البداية
عند $t=2$ نصل الى نقطة النهاية
اذا اخرج اعضاء t كل بدون ما اخرج
بداية $t=0$ ونقطة من t
عوضني x بـ $t = y - 1$

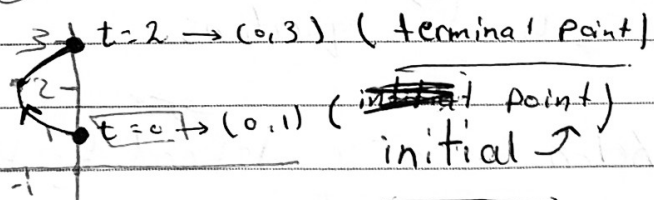
$$\begin{aligned} x &= (y-1)^2 - (y-1) \rightarrow x = y^2 - 2y + 1 - y + 1 \\ x &= y^2 - 3y + 2 \end{aligned}$$

$$x = t^2 - 2t$$

$$y = t + 1$$

$$0 \leq t \leq 2$$

نصل السؤال سنرسم



t	x, y
0	(0, 1)
1	(-1, 2)
2	(0, 3)

Ex: what curve represented by the following parametric equations.

1. $x = \cos t$ & $y = \sin t$, $0 \leq t \leq 2\pi$

معين الحرف نفس خطوط المثلثات

- $\sin^2 t + \cos^2 t = 1$ - بحيث المثلثات

$y^2 + x^2 = 1$
معين الحرف، مثلثات

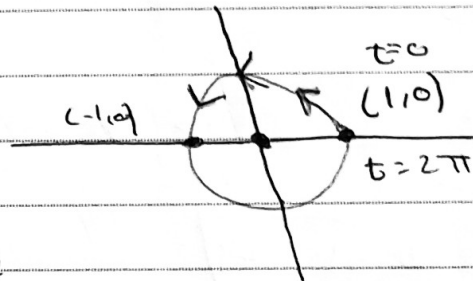
circle center $(0,0)$
radius (1)

انما تكون $(0, 2\pi)$

$t=0 \rightarrow x = \cos 0 = 1$
 $y = \sin 0 = 0 \rightarrow (1,0)$

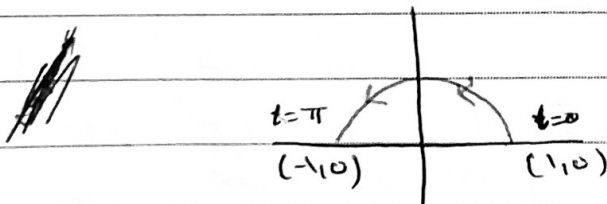
$t=\pi \rightarrow x = \cos \pi = -1$
 $y = \sin \pi = 0 \rightarrow (-1,0)$

$t=2\pi \rightarrow x = \cos 2\pi = 1$
 $y = \sin 2\pi = 0 \rightarrow (1,0)$



فكان $0 \leq t \leq \pi$

انما تقاطع نصف دائرة



Semi-circle $\rightarrow$ center $(0,0)$
radius (1)

Ex: Eliminate the parameter to find cartesian equations of the curve.

اسف (t) لا عا، انا
اكاريت، ارك، ركة

① $x = 3 - 4t$ & $y = 2 - 3t$

↓

$t = \frac{x-3}{-4} \rightarrow y = 2 - 3\left(\frac{3-x}{-4}\right)$ خط (Line)

لا يوصف

② $x = 1 - t^2$ $-2 \leq t \leq 2$
 $y = t - 2$

$t = y + 2 \rightarrow x = 1 - (y+2)^2 \rightarrow$ Parabola

$t = -2 \rightarrow y = -4$
 $t = 2 \rightarrow y = 0$
 $-4 \leq y \leq 0$

دالة (t) جند متغيرية
صفت واطم واول
من اية عديت
مكون (x) لا لا

Parabola, $-4 \leq y \leq 0$ نطاق (y)

③ $x = \sqrt{t}$ $\rightarrow t = x^2$
 $y = 1 - t$ $\rightarrow y = 1 - x^2$
لـ t نختار، بعد ان تكون
اير او = ممر t
نابا x

④ $x = e^t$ $t \in \mathbb{R}$
 $y = e^{-t}$

$t = \ln x \rightarrow y = e^{-\ln x} \rightarrow y = e^{\ln x^{-1}} \rightarrow y = x^{-1} \rightarrow y = \frac{1}{x}$
 $x > 0$

$$\textcircled{5} \begin{cases} x = 3 \cos t \\ y = 4 \sin t \end{cases} \quad 0 \leq t \leq 2\pi$$

$$\cos^2 t + \sin^2 t = 1$$

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

Ellipse

القطع الناقص

$$\textcircled{6} \begin{cases} x = \cosh t \\ y = \sinh t \end{cases} \rightarrow \text{Range } [1, \infty), x \geq 1$$

$$\cosh^2 t - \sinh^2 t = 1$$

$$x^2 - y^2 = 1 \rightarrow \text{hyperbola}$$

القطع الزائد

Ex: Find the parametric equations

$$\textcircled{1} x = y^2 - y$$

$$y = t$$

$$x = t^2 - t$$

parametric

بدي المتغير x و y

بعتبر $y = t$ ثم نعوض في x

$$\textcircled{2} y = 2x + 1$$

$$-1 \leq x \leq 1$$

$$x = t \rightarrow y = 2t + 1$$

$$0 \leq t \leq 1$$

③ line segment from $(-2, 7)$ to $(3, -1)$

المقطع المستقيم بين النقطتين

$$\begin{aligned} P_1(x_1, y_1) \quad x &= x_1 + (x_2 - x_1)t \\ P_2(x_2, y_2) \quad y &= y_1 + (y_2 - y_1)t \end{aligned} \quad 0 \leq t \leq 1$$

$$\begin{aligned} (-2, 7) &\rightarrow (3, -1) \\ x_1 \quad y_1 &\quad x_2 \quad y_2 \end{aligned}$$

$$\begin{aligned} x &= -2 + (3 - (-2))t \rightarrow x = -2 + 5t \\ y &= 7 + (-1 - 7)t \rightarrow y = 7 - 8t \end{aligned} \quad 0 \leq t \leq 1$$

إذا أردت أن أتعرف على متى يكون x و y
 $t=0$

$$t=0 \quad \begin{cases} x = -2 \\ y = 7 \end{cases} \quad (-2, 7)$$

$$t=1 \quad \begin{cases} x = 3 \\ y = -1 \end{cases} \quad (3, -1)$$

④ circle center $(5, 2)$

radius (2)

$$2^2$$

معادلة الدائرة

$$(x - 5)^2 + (y - 2)^2 = 4$$

$$\begin{aligned} \left(\frac{x-5}{2} \right)^2 + \left(\frac{y-2}{2} \right)^2 &= 1 \\ \cos^2 t + \sin^2 t &= 1 \end{aligned}$$

$$\frac{x-5}{2} = \cos t \quad \frac{y-2}{2} = \sin t$$

$$2 \cos t + 5 = x$$

$$2 \sin t + 2 = y$$

$$0 \leq t \leq 2\pi$$

طريقة (2)

circle center (h, k)

radius r

$$x = h + r \cos t$$

$$y = k + r \sin t$$

$$0 \leq t \leq 2\pi$$

center (h, k)

radius r

$$x = 5 + 2 \cos t$$

$$y = 2 + 2 \sin t$$

$$0 \leq t \leq 2\pi$$

* calculus with parametric curves →

$$x = f(t)$$

$$y = g(t)$$

$$\frac{d^2 y}{dx^2}$$

$$\frac{dy}{dx}$$

كيف يمكن نشتق

* $f(t), g(t)$ are differentiable

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \rightarrow 0 \neq \frac{dx}{dt} \text{ or } \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dx} = 0 \text{ or } \frac{dx}{dt} = 0 \text{ or } \frac{dy}{dx} = \frac{0}{0} \text{ is not defined} *$$

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \rightarrow y' = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

✓

$$\frac{dy'}{dt} = \frac{dy'}{dx} \cdot \frac{dx}{dt} \rightarrow u \neq \frac{dx}{dt} \Rightarrow \text{no}$$

$$y'' = \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} \rightarrow \frac{dy'}{dt} \cdot \frac{dt}{dx}$$

* Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$

① $x = t^3 + 1$ $y = t^2 + t$

$$\frac{dx}{dt} = 3t^2$$

$$\frac{dy}{dt} = 2t + 1$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \rightarrow \frac{dy}{dx} = \frac{2t+1}{3t^2} \cdot \frac{1}{3t^2}$$

$$y' = \frac{dy}{dx} = \frac{2t+1}{3t^2} \rightarrow y'' = \frac{2t(2t+1) - (2t+1)^2}{(3t^2)^2}$$

$$\frac{dy'}{dx} = \frac{d}{dx} \left(\frac{dy'}{dt} \cdot \frac{dt}{dx} \right) = \frac{4t - 2t + 2}{4t^2} \cdot \frac{1}{3t^2}$$

$$y'' = \frac{-1}{4t^3}$$

Find $\frac{dy}{dx}$ at $t=1$ $\rightarrow y = \frac{2t+1}{3t^2}$ at $t=1 \rightarrow y' = \frac{3}{2}$

Find $\frac{d^2y}{dx^2}$ at $t=1$ $\rightarrow y'' = \frac{-1}{4t^3}$ at $t=1 \rightarrow y'' = -\frac{1}{4}$

② $y = 2 \sin t$

$t = 3 \cos t$ Find $\frac{dy}{dx}$ & $\frac{dy}{dx^2}$

$\rightarrow \frac{dy}{dt} = 2 \cos t \quad \rightarrow \frac{dx}{dt} = -3 \sin t$

$\rightarrow \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \rightarrow 2 \cos t \cdot \frac{1}{-3 \sin t}$

$y' = -\frac{2}{3} \cot(t)$

$\frac{d^2y}{dx^2} = \frac{dy'}{dt} \cdot \frac{dt}{dx} = \frac{\frac{2}{3} \csc^2(t)}{-3 \sin t}$

$y'' = -\frac{2}{9} \csc^3 t$

equ of

tangent line

معادلة خط المماس

① $y - y_0 = m(t - t_0) \quad m = \text{slope} = \frac{dy}{dx}$
(مماس)

② Curve has horizontal tangent

$m = 0 \rightarrow \frac{dy}{dt} = 0, \frac{dx}{dt} \neq 0$ (نقطة)

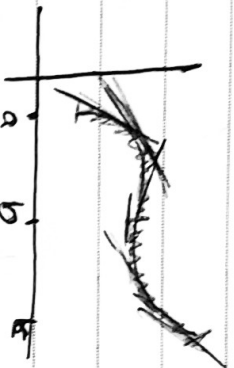
$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \rightarrow$

(3) curve has vertical tangent $\rightarrow$ y is constant

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \rightarrow \frac{dx}{dt} = 0 \quad \text{if } \frac{dy}{dx} \neq 0$$

$$\frac{d^2y}{dx^2} \rightarrow \text{concave up } \frac{d^2y}{dx^2} > 0$$

$$\rightarrow \text{concave down } \frac{d^2y}{dx^2} < 0$$



concave down $\leftarrow$ $\leftarrow$ concave down $\leftarrow$ $\leftarrow$ up $\leftarrow$ $\leftarrow$ up $\leftarrow$ $\leftarrow$

Ex: A curve defined by the parametric equations

$$\begin{aligned} x &= t^2 \\ y &= t^3 - 3t \end{aligned} \quad \text{curve } c$$

① show that c has two tangent at the point $(3,0)$ & Find their equations?

Find the equations?

$$y=0 \rightarrow 0 = t^3 - 3t \rightarrow t(t^2 - 3) = 0$$

$$t=0 \rightarrow x=0 \quad t=\pm\sqrt{3}$$

$$x=3 \rightarrow x=t^2 \quad \text{if } x=3 \text{ then } t=\pm\sqrt{3}$$

$$t=0 \rightarrow x=0$$

$$t=\sqrt{3} \rightarrow x=(\sqrt{3})^2 = 3$$

$$t=-\sqrt{3} \rightarrow x=(-\sqrt{3})^2 = 3$$

$$(3,0) \quad \frac{dx}{dt} = 2t \quad \& \quad \frac{dy}{dt} = 3t^2 - 3$$

$$t = \pm \sqrt{3}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \rightarrow \frac{3t^2 - 3}{2t}$$

t	Slope = $m = \frac{3t^2 - 3}{2t}$	$y - y_0 = m(x - x_0)$ at $(3,0)$
$\sqrt{3}$	$\frac{3(\sqrt{3})^2 - 3}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$	$y = \sqrt{3}(x - 3)$
$-\sqrt{3}$	$\frac{3(-\sqrt{3})^2 - 3}{-2\sqrt{3}} = -\frac{3}{\sqrt{3}} = -\sqrt{3}$	$y = -\sqrt{3}(x - 3)$

② Find the points on C where the tangent is horizontal or vertical

مستقيمات (C) التي يكون ميلها صفر أو غير معرف

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t} \quad \begin{matrix} \text{مستقيم} \\ \text{مقطع} \end{matrix}$$

* Horizontal $\rightarrow \frac{dy}{dx} = 0 \rightarrow 3t^2 - 3 = 0$ Zero = 0

$t = \pm 1 \rightarrow$ نقطتان (t) يعطينان

$t = 1$ Zero $\neq$ غير معرف

$-2t$

$2(1) = 2 \neq 0$ ✓

horizontal at $t = 1$ & $t = -1$ / $t = 1 \rightarrow x = 1$ & $y = -2$

نقطتين

$(1, -2)$

$t = -1 \rightarrow x = 1$ & $y = 2$

* Vertical $\rightarrow \frac{dx}{dt} = 0$ (1, 2)

نقطة واحدة (t) تعطينا Zero = 0

$2t = 0$

Zero $t = 0$ ان التفاضل صفر

$t = 0 \rightarrow 3t^2 - 3 = -3$ ✓

$t = 0 \rightarrow x = 0$ & $y = 0$

نقطة واحدة (t) تعطينا

③ Determine where the curve is concave up &

// down

(2) منحنى المنحنى المعكوف (concave)

$$\frac{d^2y}{dx^2} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\rightarrow \frac{dy}{dx} = \frac{3t^2 - 3}{2t} \quad \frac{dx}{dt} = 2t$$

$$= \frac{6t^2 + 6}{4t^2} \cdot \frac{1}{2t}$$

$$\frac{d^2y}{dx^2} = \frac{2t(6t) - (3t^2 - 3)(2)}{4t^3}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{6t^2 + 6}{8t^3}}$$

$$= \frac{6t^2 + 6}{4t^3}$$

$$0 = \frac{6t^2 + 6}{8t^3}$$

$$6t^2 + 6 = 0$$

$$6t^2 = -6 \quad \times$$

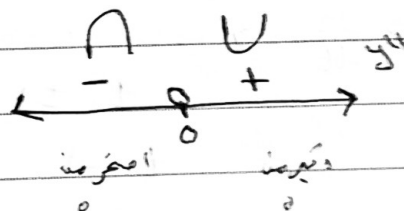
$$8t^3 = 0$$

$$t = 0$$

وذلك اننا نعلم ان $\frac{d^2y}{dx^2}$ ان يكون (0) و

المعكوف (0) يجب ان يكون حرجا لثانية.

$$0 = \frac{d^2y}{dx^2}$$



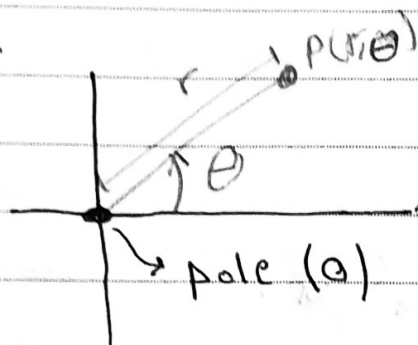
- Concave up $(0, \infty)$

- // down $(-\infty, 0)$

* Polar ~~coordinates~~ coordinates

نقطه مختصات قطبی

$P(x, y) \rightarrow$ polar coordinates $P(r, \theta)$



r the distance from O to P

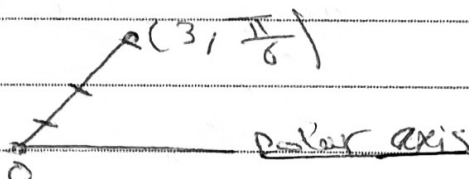
θ the angle between the polar axis & the line OP (radians)

مثال

Ex: Plot the points whose polar coordinates are given:-

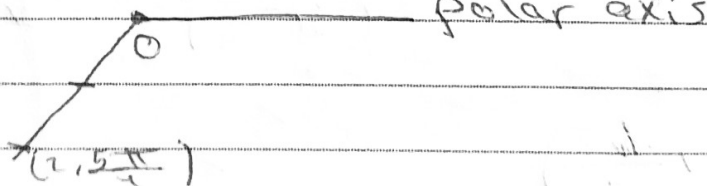
① $(3, \frac{\pi}{6})$

30° or 30°



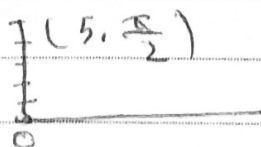
② $(2, \frac{5\pi}{4})$

30° or 125°



③ $(5, \frac{\pi}{2})$

90° or 90°

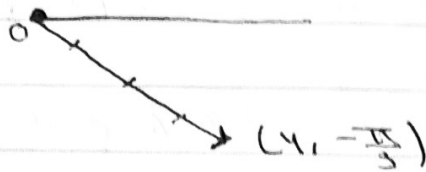


④ $(4, 0)$



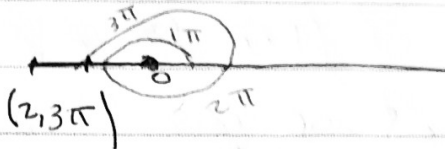
~~5) $(4, \frac{\pi}{3})$~~

5) $(4, -\frac{\pi}{3})$

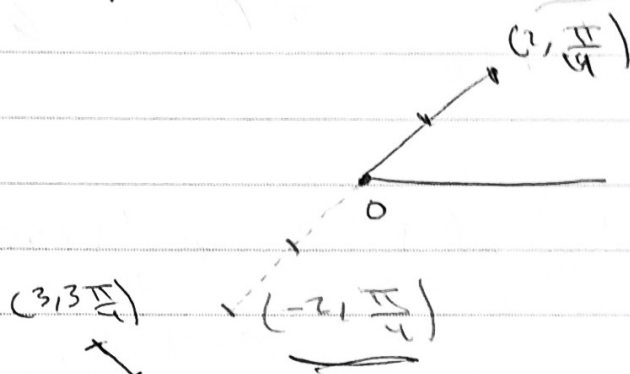


6) $(2, 3\pi)$

540 degrees

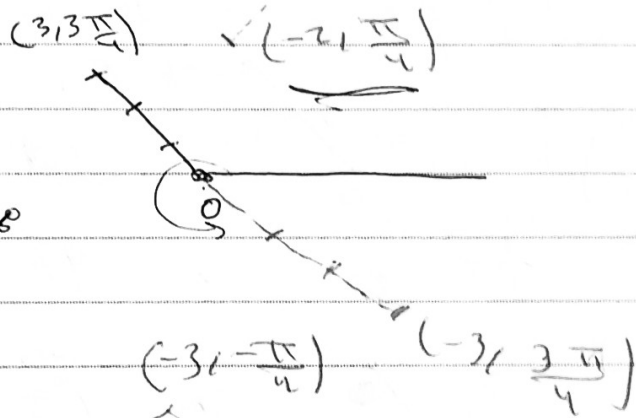


7) $(-2, \frac{\pi}{4})$



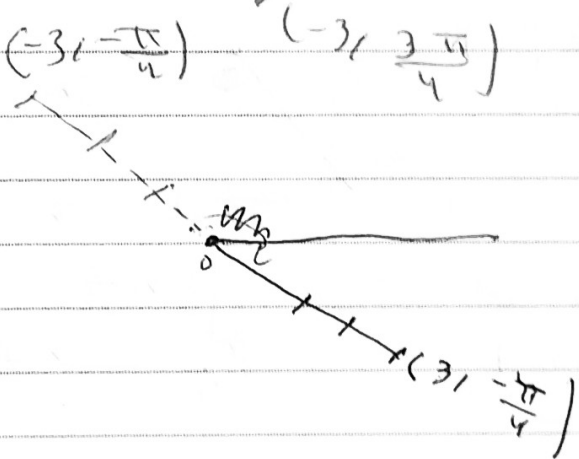
8) $(-3, \frac{3\pi}{4})$

135 degrees



9) $(-3, -\frac{\pi}{4})$

135 degrees



إذا كانت (r) سالبة، فعند النقطة كما هي بدون r ثم سجل
امتداد الخط إلى، سمتو والامتداد، أو، أما تكوني الحضي 180 أو أكثر حتى 180

~~180~~ ←

- If (r, θ) is a point in polar coordinates.

~~Then~~ Then all polar coordinates of (r, θ)

is $(r, \theta + 2n\pi)$ $n \in \mathbb{Z}$ في الدائره

$$\begin{cases} (-r, \theta + \pi + 2n\pi) \\ (-r, \theta + \pi(2n+1)) \end{cases} \quad n \in \mathbb{Z}$$

$$(r, \theta) = (r, \theta + 2n\pi) = (-r, \theta + \pi(2n+1))$$

في المستوى القطبي (0) هي نقطة الأصل (origin) واصلها $(0,0)$

أيضا في الاحداثيات القطبية (0) $\leftarrow$ (polar) واصلها $(0, \theta)$



$(0, \theta)$ any value of θ

$$\text{Ex: } P(0, \frac{\pi}{2}) \rightarrow \bullet$$

$$P(0, \frac{\pi}{3}) \rightarrow \bullet$$

$$P(0, \frac{5\pi}{4}) \rightarrow \bullet$$

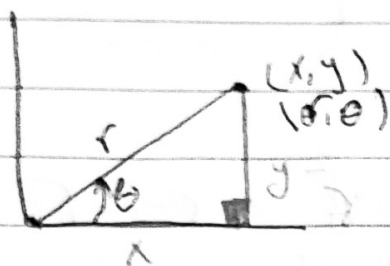
$$P(0, 0) \rightarrow \bullet$$

* Relationship between polar & cartesian coordinates

cartesian & polar are equal

or

Rectangular



$$\cos \theta = \frac{x}{r}$$

$$x = r \cos \theta$$

$$\sin \theta = \frac{y}{r}$$

$$y = r \sin \theta$$

$$r^2 = x^2 + y^2$$

$$\tan \theta = \frac{y}{x}$$

Ex: Convert the following point from polar to cartesian

① $(2, \frac{\pi}{3}) \rightarrow (x, y)$
 (1) $\theta = \frac{\pi}{3}$

$$x = r \cos \theta \rightarrow 2 \cos \frac{\pi}{3} = 2(\frac{1}{2}) = 1$$

$$y = r \sin \theta \rightarrow 2 \sin \frac{\pi}{3} = 2(\frac{\sqrt{3}}{2}) = \sqrt{3}$$

(1) $\theta = \frac{\pi}{3}$, $(1, \sqrt{3})$

② $(-6, \frac{2\pi}{3}) \rightarrow (x, y)$
 (1) $\theta = \frac{2\pi}{3}$

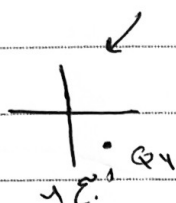
$$x = -6 \cos \frac{2\pi}{3} = -6(-\frac{1}{2}) = 3$$

$$y = -6 \sin \frac{2\pi}{3} = -6(\frac{\sqrt{3}}{2}) = -3\sqrt{3}$$

(1) $\theta = \frac{2\pi}{3}$, $(3, -3\sqrt{3})$

Ex: convert the following points from cartesian to polar coordinates

① $(1, -1) \rightarrow (r, \theta)$



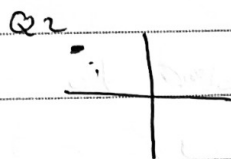
$$r^2 = x^2 + y^2 \rightarrow 1 + 1 = 2 \rightarrow r = +\sqrt{2}$$

$$\tan \theta = \frac{y}{x} \rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\theta = \tan^{-1}\left(\frac{-1}{1}\right) \rightarrow \theta = \tan^{-1}(-1)$$

$$\theta = 2\pi - \frac{\pi}{4}$$

② $(-\sqrt{3}, 1) \rightarrow (r, \theta)$



$$r^2 = x^2 + y^2 \rightarrow r^2 = 3 + 1 = 4 \rightarrow r = 2$$

$$\theta = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right)$$

$$\theta = \left(\pi - \frac{\pi}{6}\right) = \frac{5\pi}{6}$$

$(2, \frac{5\pi}{6})$

Ex: Find The Polar equation for the curve represented by given cartesian equation

① $y = 2 \rightarrow$ horizontal line

$$r \sin \theta = 2$$

$$\rightarrow r = \frac{2}{\sin \theta} = 2 \csc \theta$$

$$r = 2 \csc \theta$$

$$\textcircled{2} \quad xy = 4 \rightarrow (r \cos \theta)(r \sin \theta) = 4$$

$$r^2 \cos \theta \sin \theta = 4$$

المثلث

$$\sin 2\theta = \frac{2 \cos \theta \sin \theta}{\sin \theta}$$

$$r^2 \cdot \frac{1}{2} \sin 2\theta = 4$$

$$r^2 = 8 \csc 2\theta$$

نحوه انعطاف سوال
در x و y دو برابر

EX: Find the cartesian equation for curve represented by given polar equation

$$\textcircled{1} \quad r = 2 \quad \rightarrow \quad r^2 = 4$$

$$x^2 + y^2 = 4 \rightarrow \text{circle with center } (0,0) \text{ radius } (2)$$

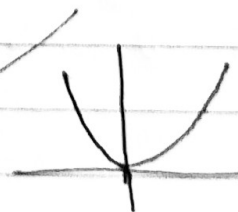
$$\textcircled{2} \quad r = \sec \theta \rightarrow r = \frac{1}{\cos \theta} \rightarrow r \cos \theta = 1$$

Vertical line $\rightarrow \boxed{x = 1} \rightarrow \text{vertical line}$

$$\textcircled{3} \quad r = \sec \theta \tan \theta \rightarrow r = \frac{1}{\cos \theta} \cdot \frac{y}{x}$$

$$r \cos \theta = \frac{y}{x} \rightarrow x = \frac{y}{x} \rightarrow x^2 = y$$

Parabola



$$\textcircled{4} \quad (r = 3 \cos \theta)^2 \rightarrow r^2 = 3r \cos \theta$$

$$x^2 + y^2 = 3x \rightarrow \boxed{x^2 - 3x + y^2 = 0} \rightarrow$$

محل
نقطه

$$\textcircled{5} (\theta = \frac{\pi}{4}) \rightarrow$$

المثلث قائم الزاوية

$$\tan \theta = \tan \frac{\pi}{4} \rightarrow \frac{y}{x} = 1 \quad \boxed{y=x} \text{ Line}$$



polar curves :

منحنيات قطبية

- Polar curves lines

خطوط القطبية

1] horizontal line

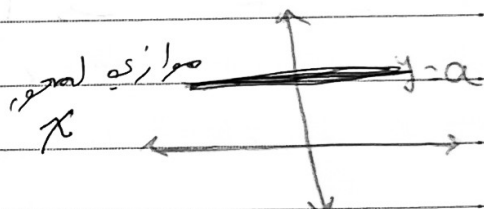
خط أفقي

$$y=a$$

$$r \sin \theta = a \rightarrow r = \frac{a}{\sin \theta}$$

$$\boxed{r = a \csc \theta}$$

horizontal line

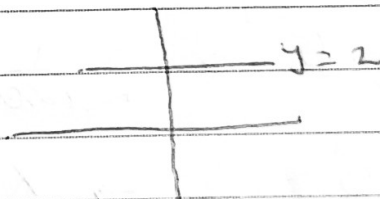


Ex: ① $r = 2 \csc \theta$

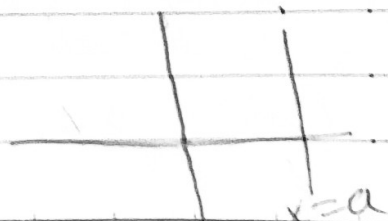
$$y=2$$

② $r = -3 \csc \theta$

$$y=-3$$



2] vertical line



$$x=a \rightarrow r \cos \theta = a \rightarrow r = \frac{a}{\cos \theta}$$

$$\boxed{r = a \sec \theta}$$

vertical line

Ex ① $r = 7 \sec \theta$

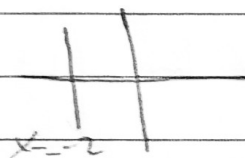
$$\downarrow$$

$$x=7$$

② $r = -2 \sec \theta$

$$\downarrow$$

$$x=-2$$

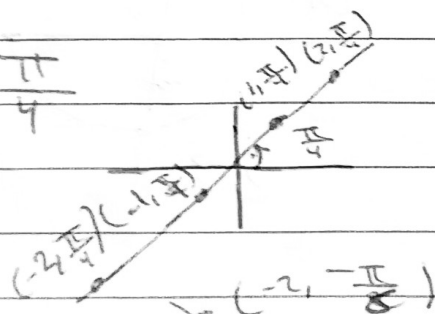


[3] The straight line passes through the Pole

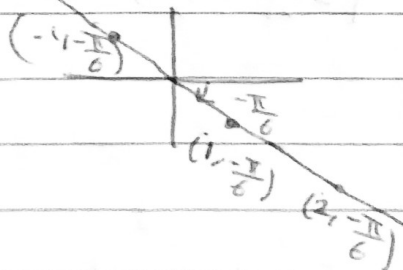
الخط المستقيم الذي يمر بنقطة المركز

$$\Rightarrow \theta = \theta_0$$

Ex: $\theta = \frac{\pi}{4}$



$\theta = -\frac{\pi}{6}$



Polar curves $\rightarrow$ circle

دائرة (4) قطرها

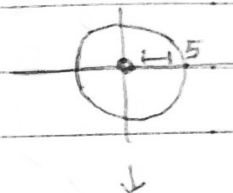
① $r = a$

circle $\rightarrow$ center $(0,0)$

$\rightarrow$ radius $|a|$

Ex: ① $r = 5 \rightarrow$ circle $\rightarrow$ center $(0,0)$

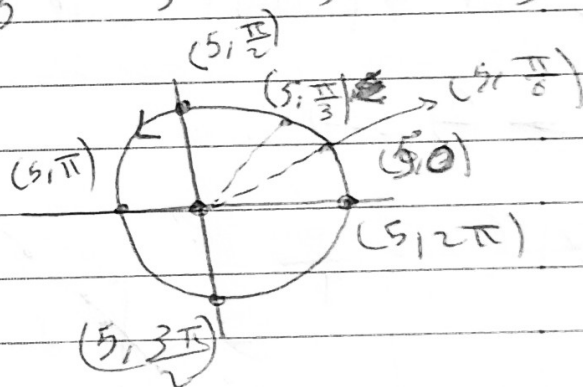
$\rightarrow$ radius $= |5|$



$0 \leq \theta \leq 2\pi$ بين θ دائرة كاملة

θ	0	$\pi/3$	$\pi/6$	π	$3\pi/2$	2π
r	5	5	5	5	5	5

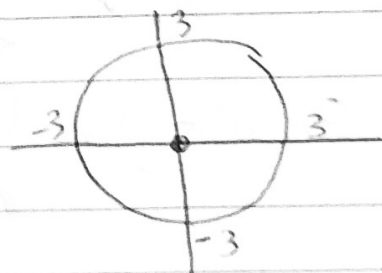
دائرة كاملة بين θ دائرة كاملة



② $r = -3$

circle $\rightarrow$ center $(0,0)$

$\rightarrow$ rad $|-3| = 3$



② $r = a \cos \theta + b \sin \theta$

$0 \leq \theta \leq \pi$

circle $\rightarrow$ center (a,b)

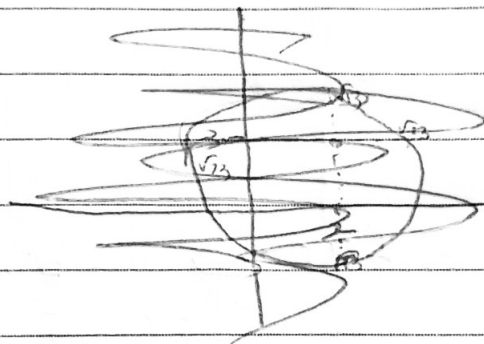
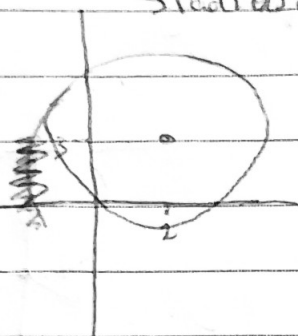
$\rightarrow$ radius $\sqrt{a^2 + b^2}$

Ex: ① $r = 4 \cos \theta + 6 \sin \theta \rightarrow 2(2 \cos \theta + 3 \sin \theta)$

circle $\rightarrow$ center $(\vec{x}, \vec{y}) \rightarrow$ cartesian

$\rightarrow$ radius $\sqrt{4+9} = \sqrt{13}$

circle, center



③ $r = 2a \cos \theta + 2b \sin \theta \quad 0 \leq \theta \leq \pi$

$a = 0$

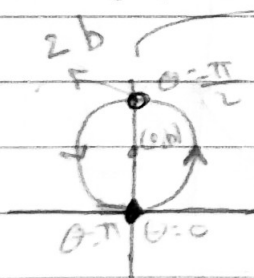
$r = 2b \sin \theta$

$0 \leq \theta \leq \pi$

circle $\rightarrow$ center $(0, b)$

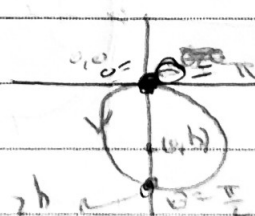
radius $= \sqrt{b^2} = |b| = b$

$2b$



$b > 0$

1 unit



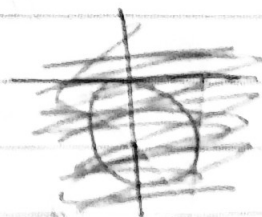
$b < 0$

Ex: ① $r = 8 \sin \theta \rightarrow 2(4 \sin \theta)$

circle $\rightarrow$ center $(0, 4)$

$\rightarrow$ radius $|4| = 4$

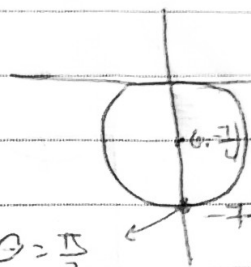
circle $\theta = \pi/2$



$$② r = -7 \sin \theta \rightarrow 2(-\frac{7}{2} \sin \theta)$$

circle $\rightarrow$ center $(0, -\frac{7}{2})$

$\rightarrow$ radiuses $|\frac{-7}{2}| = \frac{7}{2}$



$$④ r = 2a \cos \theta + 2b \sin \theta \quad \theta = \frac{\pi}{2}$$

$$0 < \theta \leq \pi$$

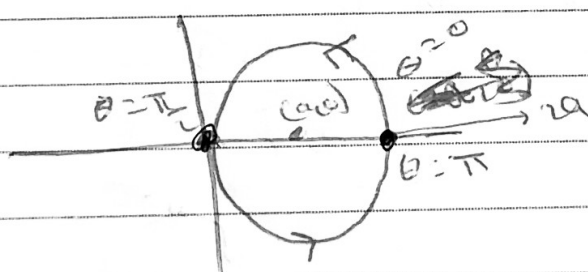
$$b=0 \rightarrow r = 2a \cos \theta \quad 0 < \theta \leq \pi$$

circle $\rightarrow$ center $(a, 0)$

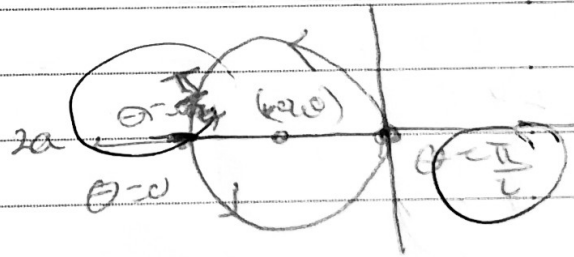
$\rightarrow$ radiuses $\sqrt{a^2} = |a| = a$

$2a \cos \theta$

$1 \cos \theta$



$a > 0$

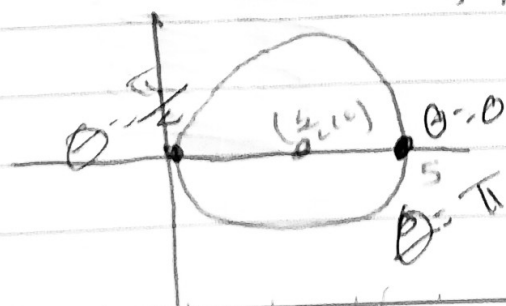


$a < 0$

$$\text{Ex: } ① r = 5 \cos \theta \rightarrow r = 2(\frac{5}{2} \cos \theta)$$

circle $\rightarrow$ center $(\frac{5}{2}, 0)$

$\rightarrow$ radiuses $|\frac{5}{2}| = \frac{5}{2}$

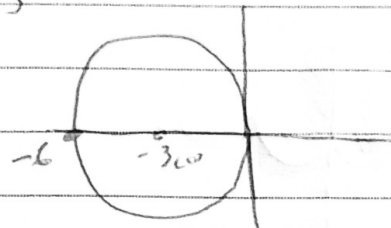


$$\textcircled{1} r = -6 \cos \theta \rightarrow r = 2(-3 \cos \theta)$$

circle $\rightarrow$ center $(-3, 0) \rightarrow$ cartizien

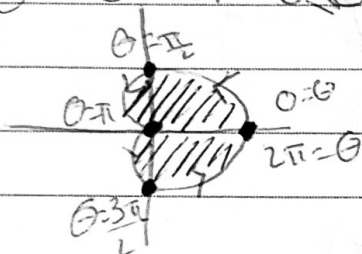
$$\rightarrow \text{rad } |-3| = 3$$

Polar curves cardioid
cardioid



$$\textcircled{1} r = a(1 + \cos \theta)$$

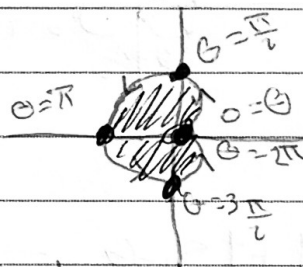
$$r = a(1 - \cos \theta) \quad 0 \leq \theta \leq 2\pi$$



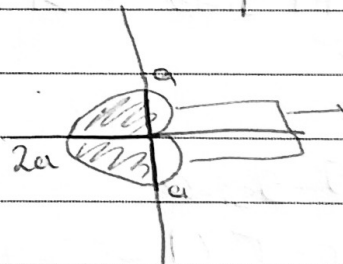
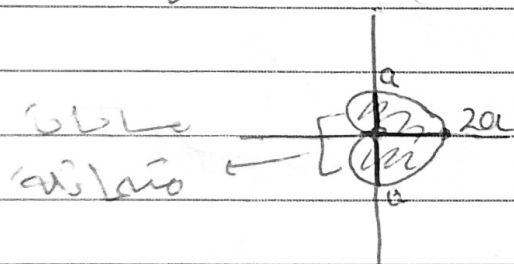
$$\textcircled{2} r = a(1 - \cos \theta)$$

$$r = a(1 + \cos \theta)$$

$$a = +$$



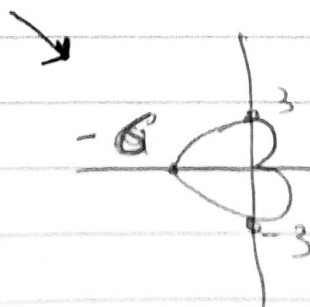
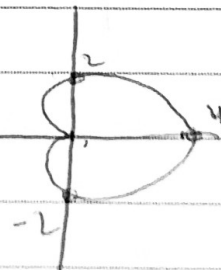
$$0 \leq \theta \leq 2\pi$$



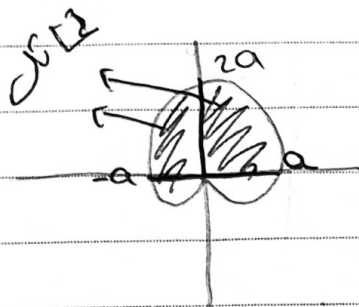
EX: $\textcircled{1} r = 2 + 2 \cos \theta \rightarrow 2(1 + \cos \theta)$

$$\textcircled{2} r = 3 - 3 \cos \theta$$

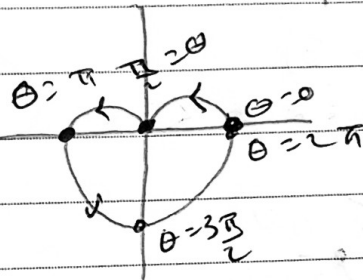
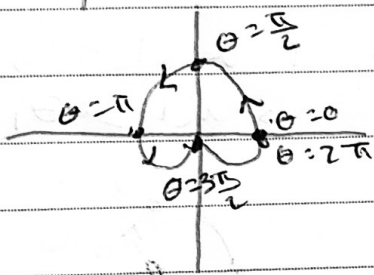
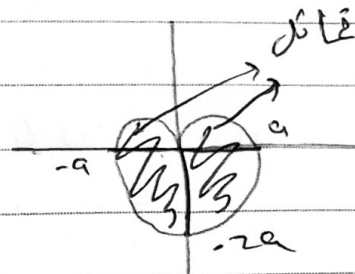
$$r = 3(1 - \cos \theta)$$



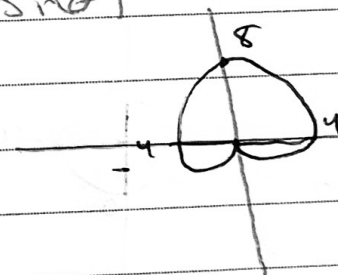
$$\textcircled{3} \begin{aligned} r &= a(1 + \sin\theta) \\ &= a(1 - \sin\theta) \end{aligned}$$



$$\begin{aligned} r &= a(1 - \sin\theta) \\ &= a(1 + \sin\theta) \end{aligned}$$



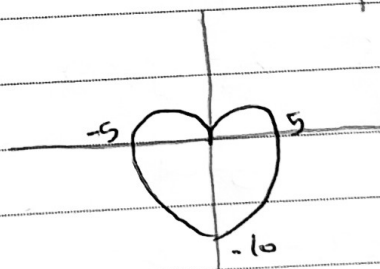
Ex: $r = 4 + 4 \sin\theta \rightarrow r = 4(1 + \sin\theta)$



$$\textcircled{4} \begin{aligned} r &= 5 - 5 \sin\theta \\ &= 5(1 - \sin\theta) \end{aligned}$$

$$r = 5(1 - \sin\theta)$$

$$5 > 0$$



Ex: Find the polar coordinates satisfying the stated conditions for the point whose cartesian coordinates are $(3\sqrt{3}, 3)$

(a) $r > 0$ & $0 \leq \theta < 2\pi$

(b) $r < 0$ & $0 \leq \theta < 2\pi$

(c) $r > 0$ & $-2\pi < \theta \leq 0$

(d) $r < 0$ & $-\pi < \theta \leq \pi$

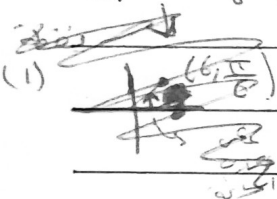
$(3\sqrt{3}, 3)$

$(3\sqrt{3}, 3)$

$r^2 = x^2 + y^2 \rightarrow r^2 = (3\sqrt{3})^2 + (3)^2 = 36 + 9 = 45$

$\tan \theta = \frac{y}{x} \rightarrow \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}} \rightarrow \theta = \frac{\pi}{6}$

(a) $(6, \frac{\pi}{6}) \rightarrow r > 0$ & $0 \leq \theta < 2\pi$



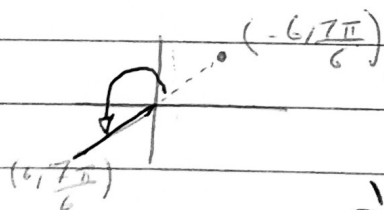
$(r, \theta) \rightarrow (r, \theta + 2\pi n)$

$\rightarrow (-r, \theta + (2n+1)\pi)$

الزاوية (+) اذن بـ π

(b) $r < 0$ $0 \leq \theta < 2\pi$

$\rightarrow (-6, \frac{\pi}{6} + \pi) = (-6, \frac{7\pi}{6})$



نفس
القيمة لـ r

(c) $r > 0$ & $-\pi < \theta \leq 0$

$\rightarrow (6, \frac{\pi}{6}) \rightarrow (6, \frac{\pi}{6} - 2\pi)$

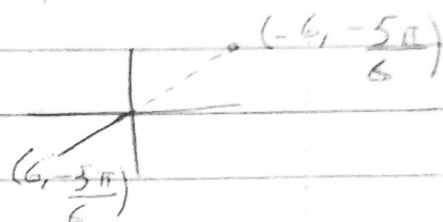


$\rightarrow (6, \frac{-11\pi}{6})$

$\frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$

$$② \quad r < 0 \quad \& \quad -\pi \leq \theta < \pi \rightarrow \left(6, \frac{\pi}{6}\right)$$

$$\left(-6, \frac{\pi}{6} - \pi\right) \rightarrow \left(-6, -\frac{5\pi}{6}\right)$$



* Area In Polar Coordinates ^{المساحة القطبية}

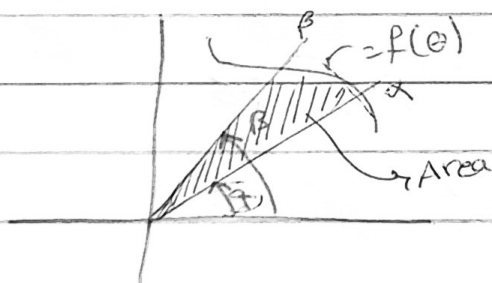
Suppose that α & β are angles that satisfy the condition $0 < \beta - \alpha \leq 2\pi$

① $r = f(\theta)$ is continuous & $r > 0$ for $\alpha \leq \theta \leq \beta$
then the area A of the region R enclosed by the polar curve $r = f(\theta)$ & Lines $\theta = \alpha$ & $\theta = \beta$

is :-

$$A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta = \frac{1}{2} \int_{\alpha}^{\beta} (f(\theta))^2 d\theta$$

المساحة

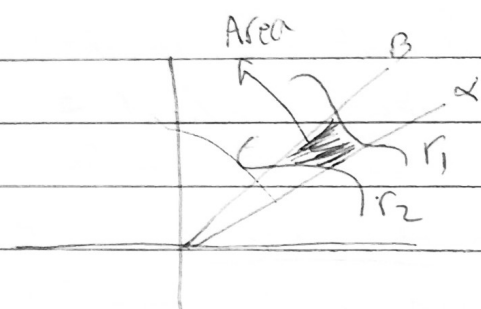


Let $r_1 = f(\theta)$ & $r_2 = g(\theta)$ are continuous & $r_1 > 0$ & $r_2 > 0$.

Then the area A of the region R enclosed by the polar curves r_1 & r_2 & Lines $\theta = \alpha$ & $\theta = \beta$ is

$$A = \int_{\alpha}^{\beta} \frac{1}{2} (f(\theta)^2 - g(\theta)^2) d\theta = \frac{1}{2} \int_{\alpha}^{\beta} r_1^2 - r_2^2 d\theta$$

$$r_1 > r_2$$



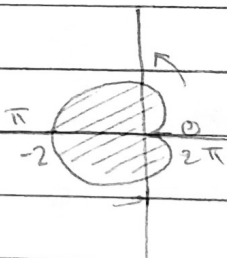
(16)

Ex 1: Find the entire area within the cardioid

$$r = 1 - \cos \theta$$

الرسم التالي

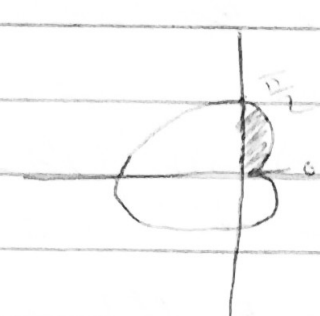
$$A = \frac{1}{2} \int_0^{2\pi} (1 - \cos \theta)^2 d\theta$$



→ المساحة الكلية
الداخلية

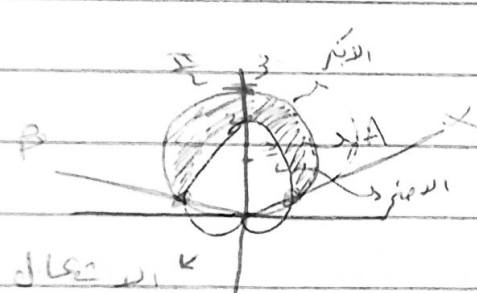
$$A = 2 \cdot \frac{1}{2} \int_0^{\pi} (1 - \cos \theta)^2 d\theta \rightarrow = \frac{3\pi}{2}$$

Ex 2: Find the area of the region in the first quadrant that is within cardioid $r = 1 - \cos \theta$



$$A = \frac{1}{2} \int_0^{\pi/2} (1 - \cos \theta)^2 d\theta = \frac{3}{8} \pi - 1$$

Ex(3): Find the area of the region that lies inside the circle $r = 3 \sin \theta$ & outside the cardioid $r = 1 + \sin \theta$



أوجد نقاط التقاطع
ولكن $r = 3 \sin \theta$ و $r = 1 + \sin \theta$
 $3 \sin \theta = 1 + \sin \theta \rightarrow 2 \sin \theta = 1 \rightarrow \sin \theta = \frac{1}{2}$
عند $\theta = \frac{\pi}{6}$ و $\theta = \frac{5\pi}{6}$

الحدود α و β
 $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

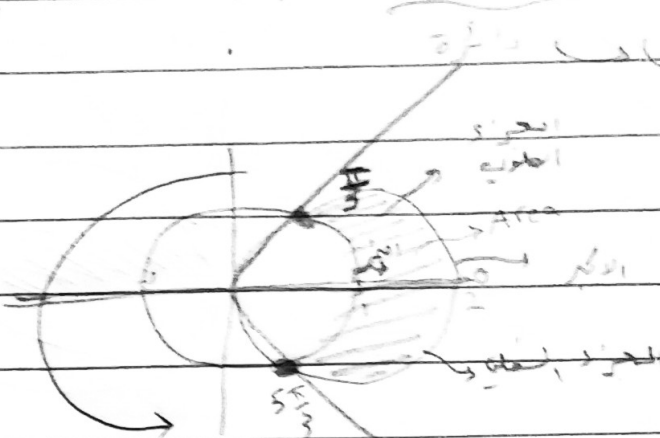
مساحة المنطقة المرغوبة هي $\frac{1}{2} \int_{\pi/6}^{5\pi/6} (3 \sin \theta)^2 - (1 + \sin \theta)^2 d\theta$
 $A = \frac{1}{2} \int_{\pi/6}^{5\pi/6} (9 \sin^2 \theta - 1 - 2 \sin \theta - \sin^2 \theta) d\theta = \pi$

أو $A = 2 \cdot \frac{1}{2} \int_{\pi/6}^{\pi/2} (3 \sin \theta)^2 - (1 + \sin \theta)^2 d\theta = \pi$

المساحة بين المنحنيين
 $A = \frac{1}{2} \int_a^b (f(x))^2 - (g(x))^2 dx$
المساحة بين المنحنيين
 $A = \frac{1}{2} \int_a^b (f(\theta))^2 - (g(\theta))^2 d\theta$
(في القطب)

Ex (4) Find the area of the region that lies outside $r=1$ & inside $r=2\cos\theta$

المركز (0,0) دائرة
نصف القطر (1)
المركز (1,0) دائرة
نصف القطر (1)
المركز (0,0) دائرة
نصف القطر (1)
المركز (1,0) دائرة
نصف القطر (1)



① المثلث نظام الإحداثيات

$$1 = 2\cos\theta$$

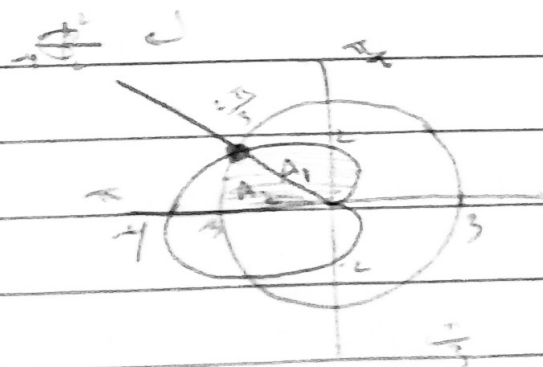
$$\cos\theta = \frac{1}{2} \rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$A = 2 \cdot \frac{1}{2} \int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (2\cos\theta)^2 - (1)^2 d\theta = \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \right)$$

$$A = \frac{1}{2} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (2\cos\theta)^2 - (1)^2 d\theta \rightarrow$$

المساحة المحيطة بالدائرة

Ex (5) Find the Area in the second quadrant that is common to the cardioid $r=2-2\cos\theta$ & circle $r=3$



المساحة المحيطة بالدائرة

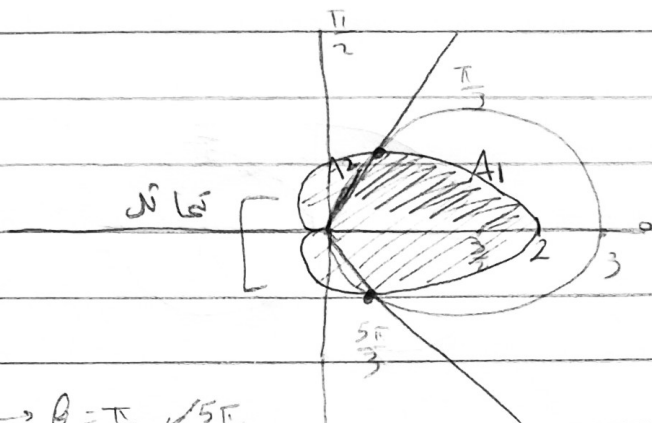
$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \dots = \frac{5\pi}{3} = \frac{\pi}{3}$$

$$3 = 2 - 2\cos\theta \rightarrow \cos\theta = -\frac{1}{2}$$

$$\theta = \frac{2\pi}{3}$$

$$A = A_1 + A_2 = \frac{1}{2} \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (2-2\cos\theta)^2 d\theta + \frac{1}{2} \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} 3^2 d\theta$$

Ex 6: Find the area ~~the~~ of the ~~the~~ region that is common to the circle $r = 3 \cos \theta$ & the cardioid $r = 1 + \cos \theta$?



$$3 \cos \theta = 1 + \cos \theta$$

$$\cos \theta = \frac{1}{2} \rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$A = (A_1 + A_2) \cdot 2 \rightarrow A = 2 \left(\frac{1}{2} \int_0^{\pi/3} (1 + \cos \theta)^2 d\theta + \frac{1}{2} \int_{\pi/3}^{\pi/2} (3 \cos \theta)^2 d\theta \right)$$

Seq

sequence: list of numbers written in a definite order

سلسلة من الأعداد مكتوبة بترتيب محدد

$a_1, a_2, a_3, \dots, a_n$

First Term Second Term general term ~~nth~~ nth term

Ex: 2, 4, 6, 8, ... $\rightarrow 2n$

a_1, a_2, a_3, a_4

$$a_1 \rightarrow n = 1 \rightarrow a_1 = 1 \cdot 2 = 2$$

$$a_2 \rightarrow n = 2 \rightarrow a_2 = 2 \cdot 2 = 4$$

$$a_3 \rightarrow n = 3 \rightarrow a_3 = 3 \cdot 2 = 6$$

$$\textcircled{2} \quad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \boxed{\frac{1}{n}}$$

$\downarrow \quad \downarrow \quad \downarrow$
 $a_1 \quad a_2 \quad a_3$

$$a_1 \Rightarrow n=1 \rightarrow a_1 = \frac{1}{1} = 1 \quad / \quad a_2 \Rightarrow n=2 \rightarrow a_2 = \frac{1}{2}$$

$$a_3 \Rightarrow n=3 \rightarrow a_3 = \frac{1}{3} \quad \dots$$

Def: An infinit seq is a function whose domain is the positive integers

$$f(n) = a_n \quad \rightarrow n = 1, 2, 3, 4, \dots \quad \text{group}$$

notation $\rightarrow \{a_1, a_2, a_3, \dots\}, f(n) = a_n$

$$\{a_n\} \text{ or } \{a_n\}_{n=1}^{+\infty}$$

Ex: Find the first five terms in the following

$$\textcircled{1} a_n = \frac{n}{n+1} \quad \rightarrow n = 1, 2, 3, 4, 5 \rightarrow a_1 = \frac{1}{1+1} = \frac{1}{2} \quad / \quad a_2 = \frac{2}{3}$$

$$a_3 = \frac{3}{4} \quad / \quad a_4 = \frac{4}{5} \quad / \quad a_5 = \frac{5}{6} \quad \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$$

$$\textcircled{2} a_n = \sqrt{n-3} \quad n \geq 3 \quad n = 3, 4, 5, 6, 7$$

$$a_3 = \sqrt{0} = 0 \quad / \quad a_4 = \sqrt{1} = 1$$

$$a_5 = \sqrt{2} \quad / \quad a_6 = \sqrt{3} \quad / \quad a_7 = \sqrt{4} = 2$$

$$0, 1, \sqrt{2}, \sqrt{3}, 2$$

EX: Find the general term of the seq

① $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$ $\rightarrow a_n = \frac{1}{2^n}$ $\rightarrow$ general term

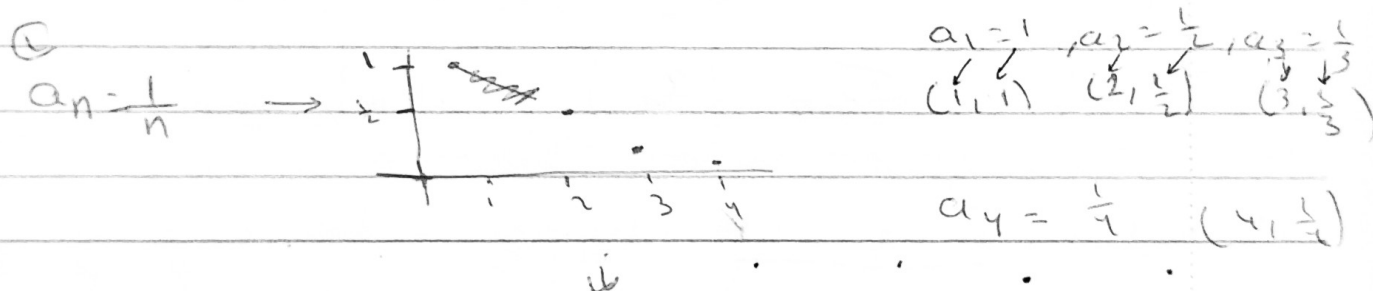
$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$
 a_1 a_2 a_3 a_4
 2^{-1} 2^{-2} 2^{-3} 2^{-4}

② $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$ $\rightarrow a_n = \frac{(-1)^{n+1} n}{n+1}$

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$
 a_1 a_2 a_3 a_4
 $\frac{1}{2}$ $\frac{2}{3}$ $\frac{3}{4}$ $\frac{4}{5}$

$\left\{ \begin{array}{l} \text{odd } n \rightarrow (-1)^{n+1} = 1 \\ \text{even } n \rightarrow (-1)^{n+1} = -1 \end{array} \right.$

- Graph of sequences

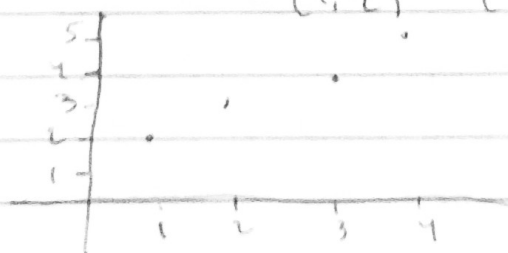


تقارب متناهي
converges

تقارب متناهي
 $n \rightarrow +\infty$
 $\frac{1}{n} \rightarrow 0$ $\rightarrow$ can be 0

② $a_n = n+1 \rightarrow a_1 = 2, a_2 = 3, a_3 = 4, a_4 = 5 \rightarrow$ convergent

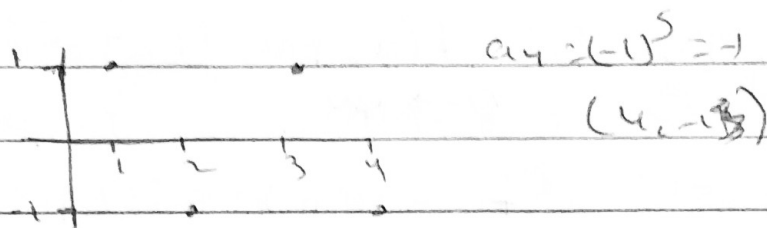
$(1, 2) \quad (2, 3) \quad (3, 4) \quad (4, 5)$



تقارب متناهي
 $n \rightarrow +\infty$
 $n+1 \rightarrow +\infty$ $\rightarrow$ divergent

$$\textcircled{3} a_n = (-1)^{n+1} \rightarrow a_1 = (-1)^2 = 1, a_2 = (-1)^3 = -1, a_3 = (-1)^4 = 1$$

$$(1, 1) \quad (2, -1) \quad (3, 1)$$



$$a_4 = (-1)^5 = -1$$

$$(4, -1)$$

$$n \rightarrow +\infty$$

d.n.e

Divergent

Def: A seq $\{a_n\}$ is said to be convergent to L if $\lim_{n \rightarrow +\infty} a_n = L$ otherwise, we say

the seq divergent.

$$a_n \rightarrow L \text{ as } n \rightarrow +\infty$$

Theorem: If $\{a_n\}, \{b_n\}$ $\left(\begin{array}{l} a_n \rightarrow L \\ b_n \rightarrow M \end{array} \right)$ as $n \rightarrow +\infty$ Then

$$\textcircled{1} \lim_{n \rightarrow +\infty} c = c, c \in \mathbb{R}$$

$$n \rightarrow +\infty$$

$$\textcircled{2} \lim_{n \rightarrow +\infty} c a_n = c \cdot L$$

$$\textcircled{3} \lim_{n \rightarrow +\infty} (a_n \pm b_n) = L \pm M$$

$$\textcircled{4} \lim_{n \rightarrow +\infty} (a_n b_n) = LM$$

$$\textcircled{5} \lim_{n \rightarrow +\infty} \left(\frac{a_n}{b_n} \right) = \frac{L}{M}, M \neq 0$$

~~5/4/07~~

Theorem: If $\lim_{n \rightarrow +\infty} a_n = L$ & the function f is

continuous at L , then $\lim_{n \rightarrow +\infty} f(a_n) = f(L)$

$$\hookrightarrow f(\lim_{n \rightarrow +\infty} a_n) = f(L)$$

Ex:

Determine whether the seq conv or div?

① $\{n^2 + 2n + 1\} \rightarrow \lim_{n \rightarrow \infty} n^2 + 2n + 1 = \lim_{n \rightarrow \infty} n^2 \rightarrow +\infty$ div

② $a_n = \frac{n^2}{n+1} \rightarrow \lim_{n \rightarrow +\infty} \frac{n^2}{n+1} = \lim_{n \rightarrow +\infty} \frac{n^2}{n} \rightarrow \lim_{n \rightarrow +\infty} n = +\infty$ div

③ $\{ne^{-n}\} \rightarrow \lim_{n \rightarrow +\infty} ne^{-n} \rightarrow \infty \cdot 0 \rightarrow L'H$

$\rightarrow \lim_{n \rightarrow +\infty} \frac{n}{e^n} \xrightarrow{L'H} \lim_{n \rightarrow +\infty} \frac{1}{e^n} = \frac{1}{\infty} = 0$ conv to 0

④ $\{\ln(2n+1) - \ln n\} \rightarrow \lim_{n \rightarrow +\infty} \ln\left(\frac{2n+1}{n}\right) = \ln\left(\lim_{n \rightarrow +\infty} \frac{2n+1}{n}\right)$

$\rightarrow \ln\left(\lim_{n \rightarrow +\infty} \frac{2n}{n}\right) \rightarrow \ln(\lim_{n \rightarrow +\infty} 2)$

$\rightarrow \ln 2 \rightarrow \text{conv to } \ln 2$

⑤ $a_n = \sin n \rightarrow \lim_{n \rightarrow +\infty} \sin n \rightarrow d.n.e$

⑥ $a_n = \left(1 + \frac{2}{n}\right)^{4n} \rightarrow \lim_{n \rightarrow +\infty} \left(1 + \frac{2}{n}\right)^{4n} \xrightarrow{\text{div}} \lim_{x \rightarrow +\infty} \left(1 + \frac{2}{x}\right)^{4x} = e^{8}$
 $e^{2.4} = e^8 \rightarrow \text{conv to } e^8$

⑦ ~~enter~~ $a_n = \frac{(2n-1)!}{(2n+1)!} \rightarrow \lim_{n \rightarrow +\infty} \frac{(2n-1)!}{(2n+1)(2n)(2n-1)!}$
 $\xrightarrow{\text{platt abt}}$

$\rightarrow \lim_{n \rightarrow +\infty} \frac{(2n-1)!}{(2n+1)(2n)(2n-1)!} = \lim_{n \rightarrow +\infty} \frac{1}{(2n+1)2n} = \frac{1}{\infty} = 0$
 $\xrightarrow{\text{conv}}$
 $\xrightarrow{\text{div}}$

⑧ $a_n = (3^n + 5^n)^{\frac{1}{n}} \rightarrow \lim_{n \rightarrow +\infty} (3^n + 5^n)^{\frac{1}{n}} \rightarrow \infty^0 \rightarrow L'H$

or

$\lim_{n \rightarrow +\infty} \left(5^n \left(\left(\frac{3}{5}\right)^n + 1\right)\right)^{\frac{1}{n}} \rightarrow \lim_{n \rightarrow +\infty} 5 \left(\left(\frac{3}{5}\right)^n + 1\right)^{\frac{1}{n}}$

$\rightarrow \lim_{n \rightarrow +\infty} (5)(1)^{\frac{1}{n}} \rightarrow 5(1)^0 = 5 \rightarrow \text{Conv to } 5$

Theorem (The Squeezing Thm for sequences)

Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ such that $a_n \leq b_n \leq c_n$

If $\lim_{n \rightarrow +\infty} a_n = \lim_{n \rightarrow +\infty} c_n = L$, then

$$\lim_{n \rightarrow +\infty} b_n = L$$

Ex: determine whether the seq conv or div?

① $a_n = \frac{\sin n}{n} \rightarrow \frac{\text{line } \sin n \text{ bounded } \rightarrow \text{no limit}}{\infty}$

$\lim_{n \rightarrow +\infty} \frac{\sin n}{n}$ $\rightarrow \frac{-1}{n} < \frac{\sin n}{n} \leq \frac{1}{n} \rightarrow \frac{1}{n} \rightarrow 0^+ \rightarrow \text{squeeze thm}$

$$\lim_{n \rightarrow +\infty} \frac{1}{n} = \frac{1}{\infty} = 0$$

$$\lim_{n \rightarrow +\infty} \frac{1}{n} = \frac{1}{\infty} = 0$$

$$\text{So } \lim_{n \rightarrow +\infty} \frac{\sin n}{n} = 0$$

~~seq~~ $a_n = \frac{\sin n}{n} \rightarrow \text{conv to } 0$

Alternating sequences:

$+1, -1, +1, -1, \dots$ $\leftarrow$ $-1, +1, -1, +1, \dots$

$$+ \{(-1)^n a_n\} = \{-a_1, a_2, -a_3, a_4, \dots\}$$

$$\text{EX: } a_n (-1)^n = \{-1, 1, -1, 1, \dots\} \boxed{\text{div}}$$

$$+ a_n = (\cos n\pi) 2^n$$

$$\text{EX: } (\cos \pi) 2^1, (\cos 2\pi) 2^2, (\cos 3\pi) 2^3, \dots$$

$-2, \quad 2, \quad -2^3, \dots$

div

$$+ (-1)^{n+1} / (-1)^{n-1}$$

Thm (1) If $\lim_{n \rightarrow +\infty} |a_n| = 0$ then $\lim_{n \rightarrow +\infty} a_n = 0$

EX: con or div?

$$Q \left\{ \frac{(-1)^n}{n} \right\} = \{a_1, a_2, a_3, a_4, \dots\}$$

$\{-1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

$$\lim_{n \rightarrow +\infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n \rightarrow +\infty} \frac{1}{n} = 0$$

lim $\rightarrow$ 19 (div) diffnly $\rightarrow$ 19 (div)

$\hookrightarrow 0$ conv to 0

Thm 2: A seq converges to L iff the seq of even-numbered terms & odd-numbered terms converges to L .

$$a_1, a_2, a_3, a_4, a_5, \dots$$

ملاحظة: الحدود الفردية تؤول لـ L والحدود الزوجية تؤول لـ L

$$a_1, a_3, a_5, a_7, \dots \rightarrow L \quad a_2, a_4, a_6, a_8, \dots \rightarrow L$$

Ex: Con or div?

$$a_1, a_2, a_3, a_4, a_5, a_6, \dots \xrightarrow{n \rightarrow +\infty} L$$

① $a_n = \frac{(-1)^n n^2 + 1}{2n^2 + 3n + 5}$?

نطبق النظرية الأولى

$$\lim_{n \rightarrow +\infty} \frac{(-1)^n n^2 + 1}{2n^2 + 3n + 5} \neq \lim_{n \rightarrow +\infty} \frac{n^2 + 1}{2n^2 + 3n + 5}$$

↓

زوجيات n

فرديات n

even = a_{2n}

odd = a_{2n+1}

$$n=2, 4, \dots \{ a_{2n} = \frac{(-1)^{2n} n^2 + 1}{2n^2 + 3n + 5}$$

$$n=1, 3, 5, \dots \{ a_{2n+1} = \frac{(-1)^{2n+1} n^2 + 1}{2n^2 + 3n + 5}$$

$= \frac{1}{2} \rightarrow (0)$ ليس

أي أن النسبة لا تؤول لـ L

Fail

$$= \frac{n^2 + 1}{2n^2 + 3n + 5}$$

↓

$$\lim \dots$$

↓

$$\lim \dots$$

$$\lim_{n \rightarrow +\infty} \frac{n^2 + 1}{2n^2 + 3n + 5}$$

$n \rightarrow +\infty$

$$= \lim_{n \rightarrow +\infty} \frac{n^2}{2n^2} = \left[\frac{1}{2} \right]$$

$$\lim_{n \rightarrow +\infty} \frac{(-1)^n n^2 + 1}{2n^2 + 3n + 5}$$

$n \rightarrow +\infty$

$$\lim_{n \rightarrow +\infty} \frac{(-1)^n n^2}{2n^2} = \left[-\frac{1}{2} \right]$$

$\neq$

$$a_n = \frac{(-1)^n n^2 + 1}{2n^2 + 3n + 5} \text{ div}$$

The sequence r^n

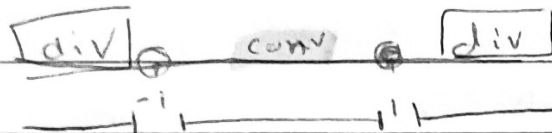
$$r^n, r \in \mathbb{R} \rightarrow r \text{ const \& } n \text{ var}$$

$$\rightarrow \{2, 2^2, 2^3, 2^4, 2^5, \dots\} \rightarrow \text{div} \rightarrow \infty$$

$$\rightarrow \left(\frac{1}{2}\right)^n = \left\{\frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \dots\right\} \rightarrow \text{conv to zero}$$

* For what values of r is the seq r^n convergent?

$$r^n = \begin{cases} -1 < r < 1 \rightarrow \text{conv to } 0 \\ r = 1 \rightarrow \text{conv to } 1 \\ \text{otherwise} \rightarrow \text{div} \end{cases}$$



$$\rightarrow \left(\frac{1}{2}\right)^n \rightarrow -1 < \frac{1}{2} < 1 \rightarrow \text{conv to } 0$$

$$\rightarrow 4^n \rightarrow 4 > 1 \rightarrow \text{div}$$

$$\rightarrow (-5)^n \rightarrow -5 < -1 \rightarrow \text{div}$$

$$\rightarrow \left(\frac{e}{\pi}\right)^n \rightarrow \begin{matrix} e \approx 2.71 \\ \pi \approx 3.14 \end{matrix} \quad \frac{e}{\pi} < 1 \quad \text{conv to zero}$$

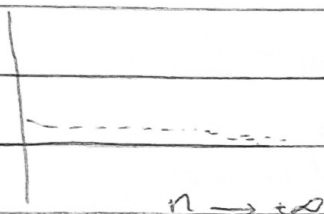
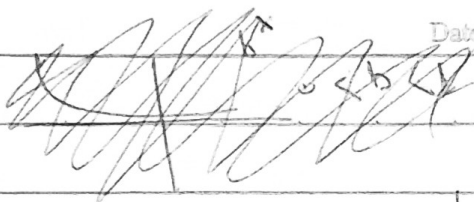
$$* \text{ ~~conv~~ } \rightarrow -1 < r < 1 \text{ conv?}$$

$$\bullet r = 1 \rightarrow r^n = \{1, 1, 1, \dots\} \rightarrow \text{conv to } 1$$

$$\bullet r = 0 \rightarrow \{0, 0, 0, \dots\} \rightarrow \text{conv to } 0$$

~~not a~~

$$\rightarrow 0 < r < 1$$



$$n \rightarrow +\infty$$

$$r^n \rightarrow 0$$

$$\lim_{n \rightarrow +\infty} r^n = 0$$

(conv)

$$b^x$$

$$0 < b < 1$$

$$\lim_{x \rightarrow +\infty} b^x = 0$$

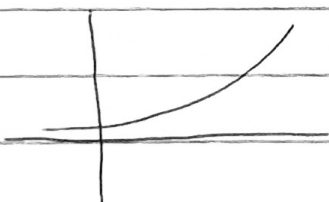
$$\rightarrow -1 < r < 0 \rightarrow 0 < |r| < 1$$

$$\lim_{n \rightarrow +\infty} |r^n| = \lim_{n \rightarrow +\infty} |r|^n = 0$$

$$\rightarrow \lim_{n \rightarrow +\infty} r^n = 0$$

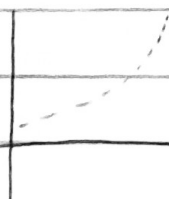
conv

div



$$b > 1$$

$$\lim_{x \rightarrow +\infty} b^x = +\infty$$



$$r > 1$$

$$\lim_{n \rightarrow +\infty} r^n = +\infty$$

div

$$r < -1$$

$$r = -1 \rightarrow r = (-1)^n = \{-1, 1, -1, 1, \dots\}$$

$$n \rightarrow +\infty$$

div

$$r = -2, -3, \dots < -1$$

$$(-2)^n = \{-2, (-2)^2, (-2)^3, \dots\}$$

$$\left\{ \begin{array}{cccc} -2 & 4 & -8 & 16 & \dots \\ a_1 & a_2 & a_3 & a_4 & \dots \end{array} \right.$$

نجد

$$-2, -8, \dots \rightarrow a_{2n+1} \rightarrow -\infty \text{ as } n \rightarrow +\infty$$

$$4, 16, \dots \rightarrow a_{2n} \rightarrow +\infty \text{ as } n \rightarrow +\infty$$

div

Ex: Find the value of a , If the following seq is conv?

$$a \left(\frac{a^{n+1}}{2^n} \right) \text{ conv } a = ?$$

$$\text{Con} \leftarrow \frac{a^{n+1}}{2^n} \text{ قابل التقارب } \left(\frac{a^{n+1}}{2^n} \right)$$

$$\lim_{n \rightarrow +\infty} \left(\frac{a^{n+1}}{2^n} \right) \text{ exists } \text{ان الـ } \frac{a^{n+1}}{2^n} \text{ convergible}$$

$$\lim_{n \rightarrow +\infty} \frac{a \cdot a^n}{2^n} \rightarrow [a] \lim_{n \rightarrow +\infty} \left(\frac{a}{2} \right)^n \quad \left(\frac{a}{2} \right)^n = (r)^n$$

$$-1 < r < 1$$

$$\text{لك } 1 \text{ conv } \text{as } r^n \text{ as}$$

$$-1 < \frac{a}{2} < 1$$

$$a \in (-2, 2)$$

$$-2 < a < 2$$

$$\rightarrow a \in (-2, 2] \text{ seq conv}$$

$$\text{div} \leftarrow \frac{a}{2} \text{ conv}$$

$$a \in (-\infty, -2] \cup (2, \infty)$$

Seq
div

div $\leftarrow$ seq قابل التقارب

Recursive sequences

الحد الأول

$$a_1 = 3$$

القاعدة

$$a_{n+1} = a_n + 2$$

$$a_2 = a_{1+1} = a_1 + 2 \rightarrow 3 + 2 = 5$$

$$a_3 = a_{2+1} = a_2 + 2 \rightarrow 5 + 2 = 7$$

$$a_4 = a_{3+1} = a_3 + 2 \rightarrow 7 + 2 = 9$$

Ex: $a_1 = 1$ $a_{n+1} = \frac{1}{2}(a_n + 3)$

① Find the first 4 terms

② Assuming that the seq converges find the Limit

$$1- a_2 = a_{1+1} = \frac{1}{2}(a_1 + 3) \rightarrow \frac{1}{2}(1 + 3) = 2$$

$$a_3 = a_{2+1} = \frac{1}{2}(a_2 + 3) \rightarrow \frac{1}{2}(2 + 3) = \frac{5}{2}$$

$$a_4 = a_{3+1} = \frac{1}{2}(a_3 + 3) \rightarrow \frac{1}{2}\left(\frac{5}{2} + 3\right) = \frac{11}{4}$$

$$a_1, a_2, a_3, a_4$$

$$1, 2, \frac{5}{2}, \frac{11}{4}$$

✓

2. convergent $\rightarrow$ Limit exist

$$\text{Limit} = L$$

$$\lim_{n \rightarrow +\infty} a_n = L$$

$$n \rightarrow +\infty$$

$$a_n \rightarrow L$$

$$\lim_{n \rightarrow +\infty} a_{n+1} = L$$

$$a_{n+1} = \frac{1}{2}(a_n + 3)$$

$$n \rightarrow +\infty$$

$$\lim_{n \rightarrow +\infty} a_{n+1} = \lim_{n \rightarrow +\infty} \frac{1}{2}(a_n + 3)$$

$$n \rightarrow +\infty$$

$$n \rightarrow +\infty$$

$$L = \frac{1}{2} \lim_{n \rightarrow +\infty} (a_n + 3)$$

$$L = \frac{1}{2}(L + 3)$$

$$\boxed{L = 3}$$

$$\text{Limit} = L = 3$$

$$\text{Ex: } a_1 = 1, a_2 = 1 \quad \Leftrightarrow \quad a_{n+2} = a_n + a_{n+1}$$

$$a_1 = 1$$

$$a_4 = a_2 + a_3 = \boxed{3}$$

$$a_2 = 1$$

$$a_5 = a_3 + a_4 = \boxed{5}$$

$$a_3 = a_1 + a_2 = \boxed{2}$$

$$a_6 = a_4 + a_5 = \boxed{8}$$

* $\{1, 1, 2, 3, 5, 8, \dots\} \rightarrow$ Fibonacci sequence

Ex: Determine the seq defined as follows is convergent or divergent

① $a_{n+1} \rightarrow a_{n+1} = 4 - a_n$

$$a_2 = 4 - a_1 = 4 - 1 = 3$$

$$a_3 = 4 - a_2 = 4 - 3 = 1$$

$$a_4 = 4 - a_3 = 4 - 1 = 3$$

1, 3, 1, 3, ...

div

② $a_1 = 2, a_{n+1} = 4 - a_n$

$$a_2 = 4 - a_1 = 4 - 2 = 2$$

$$a_3 = 4 - a_2 = 4 - 2 = 2$$

$$a_4 = 4 - a_3 = 4 - 2 = 2$$

2, 2, 2, ...

Conv to (2)

Ex: consider the seq

$$a_1 = \sqrt{6}, a_2 = \sqrt{6 + \sqrt{6}}, a_3 = \sqrt{6 + \sqrt{6 + \sqrt{6}}}$$

seq

① Find a recursion formula for a_{n+1}

② Assuming that the seq converges, find the Limit

$$L = a_1 = \sqrt{6} \rightarrow a_{n+1} = ?$$

$$a_2 = \sqrt{6 + \sqrt{6}}$$

$$= \sqrt{6 + a_1}$$

$$a_{n+1} = \sqrt{6 + a_n}$$

Ex 2- conv $\rightarrow$ Limit seq exists

$$\lim_{n \rightarrow +\infty} a_n = L$$

$$n \rightarrow +\infty$$

$$\lim_{n \rightarrow +\infty} a_{n+1} = L$$

$$n \rightarrow +\infty$$

$$a_{n+1} = \sqrt{6 + a_n} \rightarrow \lim_{n \rightarrow +\infty} a_{n+1} = \lim_{n \rightarrow +\infty} \sqrt{6 + a_n}$$

$$L = \sqrt{6 + L} \rightarrow L^2 = 6 + L \rightarrow L^2 - L - 6 = 0$$

$$L^2 - L - 6 = 0 \rightarrow L = -2 \text{ or } L = 3$$

ما يلي هو السلسلة التي يرجع إليها Seq (شروط ضرورية)

$$a_1 = \sqrt{6} \text{ or } a_2 = \sqrt{6 + \sqrt{6}} \text{ or } a_3 = \sqrt{6 + \sqrt{6 + \sqrt{6}}} \text{ or } \dots$$

القيم المتكررة (2000)

$$L = -2 \text{ or } L = 3$$

$$\text{Limit} = 3$$

Monotonic Sequences

① Increasing seq:

$$a_1 < a_2 < a_3 < a_4 \dots a_n < a_{n+1}$$

inc $\leftarrow$ $\leftarrow$ $\leftarrow$ $\leftarrow$

Ex $2, 4, 6, 8, 10, \dots$

$$2 < 4 < 6 < 8 < 10$$

$$\textcircled{2} \frac{1}{2} < \frac{1}{3} < \frac{1}{4} < \dots$$

② decreasing seq:

$$a_1 > a_2 > a_3 > a_4 \dots a_n > a_{n+1}$$

dec $\leftarrow$ $\leftarrow$ $\leftarrow$ $\leftarrow$

Ex $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

$$1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} \dots$$

$$\textcircled{2} \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \dots$$

$$\frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \dots$$

③ Not inc, Not dec (Not monotonic seq)

Ex: $(-1)^n = \{-1, 1, -1, 1, \dots\}$



وکیف بدی اسی حالت

Sequence

Difference

Ratio (terms +)

Derivative

increasing

$$a_{n+1} - a_n > 0$$

$$\frac{a_{n+1}}{a_n} > 1$$

$$f'(x) > 0$$

decreasing

$$a_{n+1} - a_n < 0$$

$$\frac{a_{n+1}}{a_n} < 1$$

$$f'(x) < 0$$

طرح اسی حد میں متالیاں

میب الی

(H.P, D.P)

Ex: Determine whether the seq is inc or dec or not monotonic seq

$$① a_n = \frac{3}{n+5}$$

Q

$$\rightarrow a_{n+1} = \frac{3}{(n+1)+5}$$

$$② a_{n+1} - a_n = \frac{3}{n+6} - \frac{3}{n+5}$$

$$a_{n+1} = \frac{3}{n+6}$$

$$= \frac{3(n+5) - 3(n+6)}{(n+6)(n+5)} = \frac{3n+15 - 3n-18}{(n+6)(n+5)}$$

$$= \frac{-3}{(n+6)(n+5)} < 0$$

$$a_n = \frac{3}{n+5} \text{ decreasing}$$

$$② a_n = \frac{n^n}{n!}$$

→

a_{n+1} > a_n

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \div \frac{n^n}{n!}$$

$$a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$$

$$= \frac{(n+1)^n (n+1)}{(n+1) \cdot n!} \times \frac{n!}{n^n}$$

(n+1)(n+1)!

$$= \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n > 1$$

$$a_n = \frac{n^n}{n!}$$

Increasing

③ $a_n = \tan^{-1} n \rightarrow$ ~~increasing~~ ~~seq~~ ~~not~~ ~~monotonic~~

$\rightarrow f(x) = \tan^{-1} x$

$f'(x) = \frac{1}{1+x^2} \oplus > 0$

$a_n = \tan^{-1} n$ increasing

④ $a_n = (-2)^{n+1}$

$a_1 = -4, a_2 = -8, a_3 = 16$

$4, -8, 16, \dots$

not monotonic
seq

⑤ $a_n = \frac{n!}{6^n}$

$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{6^{n+1}} \div \frac{n!}{6^n}$

$\rightarrow a_{n+1} = \frac{(n+1)!}{6^{n+1}}$

$= \frac{(n+1)n!}{6 \cdot 6^n} \times \frac{6^n}{n!}$

$= \frac{n+1}{6} \quad ? > 1$
or
 < 1

$\frac{n+1}{6} > 1$

$n > 5 \rightarrow n \geq 6 \rightarrow [6, \infty)$

(n) ~~increasing~~ ~~seq~~ ~~not~~ ~~monotonic~~

تكون متناقص (5)

تكون متناقص (1)

~~$\frac{n+1}{6} > 1$~~

$a_1, a_2, a_3, a_4, a_5, a_6, a_7, \dots \infty$

dec, inc ~~seq~~ ~~not~~ ~~monotonic~~ Inc

$$Q) a_n = \frac{\ln n}{n} \quad D_f = (0, \infty)$$

$$f(x) = \frac{\ln x}{x} \rightarrow f'(x) = (x)^{-\frac{1}{2}} - \frac{\ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

critical number نقطة التوقف

$$1 - \ln x = 0 \rightarrow 1 = \ln x \Rightarrow e = x \in D_f$$

$$x^2 = 0 \rightarrow x = 0 \notin D_f$$

$$\begin{array}{c} \text{dec} \\ \Rightarrow \\ f' \end{array}$$

$$f'(e^2) = \frac{1 - \ln e^2}{(e^2)^2}$$

$$a_n = \frac{\ln n}{n} \text{ decreasing}$$

$$= \frac{1-2}{e^4}$$

$$n \geq 3 \rightarrow \text{27 bits} = \frac{-1}{e^4}$$

(n-1, 27) bits

- Infinite Series:

$-a_1, a_2, a_3, a_4, \dots$ seq

$\Downarrow$

$-a_1 + a_2 + a_3 + a_4 + \dots$ series $\rightarrow \sum_{n=1}^{+\infty} a_n = \sum a_n$

Ex ① $1 + 2 + 3 + 4 + \dots \rightarrow \sum_{n=1}^{+\infty} n$

② $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \rightarrow \sum_{n=1}^{+\infty} \frac{1}{n}$

Def: Let $\sum_{n=1}^{+\infty} a_n$, the seq of partial sum

$\{s_n\}_{n=1}^{+\infty}$
 $- \{s_1, s_2, s_3, \dots\}$

$$s_1 = a_1 \rightarrow s_1 = a_1$$

$$s_2 = a_1 + a_2 \rightarrow s_2 = s_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3 \rightarrow s_3 = s_2 + a_3$$

$$s_4 = a_1 + a_2 + a_3 + a_4 \rightarrow s_4 = s_3 + a_4$$

$\vdots$

$$s_n = a_1 + a_2 + \dots + a_{n-1} + a_n \rightarrow s_n = s_{n-1} + a_n$$

$$\{s_n\}_{n=1}^{+\infty} \text{ conv } (\lim_{n \rightarrow +\infty} s_n = L)$$

$$\Rightarrow \sum_{n=1}^{+\infty} a_n \text{ conv } \Rightarrow \sum_{n=1}^{+\infty} a_n = L$$

conv $\rightarrow$ limit

div $\rightarrow$ not limit

$$\sum_{n=1}^{+\infty} \{a_n\} \text{ div} \Rightarrow \sum_{n=1}^{+\infty} a_n \text{ div (has no sum)}$$

Ex: ① If the series $\sum_{n=1}^{+\infty} a_n$ has the n th partial sum

$$S_n = 2 + \frac{3n}{n+1}, \text{ find the sum of } \sum_{n=1}^{+\infty} a_n$$

(Lim) $\rightarrow$ $\sum_{n=1}^{+\infty} a_n$ div or conv $\leftarrow (S_n)$ اولها به چه اعدادی می رسد

$$\lim_{n \rightarrow +\infty} S_n = \lim_{n \rightarrow +\infty} 2 + \frac{3n}{n+1} = 2 + 3 = 5 \text{ conv}$$

المجموع له

$\leftarrow (5)$ لایحه

$$\sum_{n=1}^{+\infty} a_n \text{ conv}, \sum_{n=1}^{+\infty} a_n = 5$$

Ex: conv or div?

$$\textcircled{1} \sum_{n=1}^{+\infty} (-1)^n = -1 + 1 - 1 + 1 - \dots$$

partial sum

$$S_1 = a_1 = -1$$

$$S_2 = 0$$

$$S_3 = -1$$

$$S_4 = 0$$

$$\left\{ -1, 0, -1, 0, \dots \right\}$$

$$\lim_{n \rightarrow +\infty} S_n \text{ does not exist}$$

$$\therefore \sum_{n=1}^{+\infty} (-1)^n \text{ div has no sum}$$

$$\textcircled{2} \sum_{n=1}^{+\infty} 1 = 1 + 1 + 1 + \dots$$

$$S_1 = 1, S_2 = 2, S_3 = 3, S_4 = 4, \dots, S_n = n$$

$$\lim_{n \rightarrow +\infty} S_n = \lim_{n \rightarrow +\infty} n = +\infty \quad \text{div}$$

$$\sum_{n=1}^{+\infty} 1 = \text{div no sum}$$

Ex: Determine whether the series is convergent or divergent if it is convergent, find the sum?

(telescoping) ~~جدا + يبقى~~ sum

$$\textcircled{a} \sum_{n=1}^{+\infty} \frac{1}{n} - \frac{1}{n+1}$$

$$S_1 = \frac{1}{1} - \frac{1}{2}, S_2 = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{1}{1} - \frac{1}{3}$$

$$S_3 = 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}, S_4 = 1 - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} = 1 - \frac{1}{5}$$

$$S_n = 1 - \frac{1}{n+1}$$

$$\lim_{n \rightarrow +\infty} 1 - \frac{1}{n+1} = \boxed{1} \quad \text{conv}$$

$$\sum_{n=1}^{+\infty} \frac{1}{n} - \frac{1}{n+1} = \boxed{1}$$

$$② \sum_{n=1}^{+\infty} \ln\left(\frac{n+1}{n}\right)$$

$$\sum_{n=1}^{+\infty} \ln(n+1) - \ln n, \rightarrow S_1 = \ln 2 - \cancel{\ln 1} = \ln 2$$

$$S_2 = \ln 2 + \ln 3 - \cancel{\ln 2} = \ln 3$$

$$S_3 = \ln 2 + \ln 3 + \ln 4 - \cancel{\ln 3} = \ln 4$$

$$S_n = \ln(n+1) \rightarrow \lim_{n \rightarrow +\infty} \ln(n+1) = +\infty \text{ div}$$

$$\rightarrow \sum_{n=1}^{+\infty} \ln\left(\frac{n+1}{n}\right) \text{ div} \rightarrow \text{has no sum}$$

لل

$$③ \sum_{n=1}^{+\infty} \frac{1}{n^2 + 3n - 2} \rightarrow \text{يجب ان اهل كسور جزئية}$$

$$\sum \frac{1}{n^2 + 3n - 2} = \frac{1}{(3n-1)(3n+2)} = \frac{A}{3n-1} + \frac{B}{3n+2}$$

$$\frac{A(3n+2) + B(3n-1)}{(3n-1)(3n+2)} = \frac{1}{n^2 + 3n - 2}$$

$$A(3n+2) + B(3n-1) = 1 \rightarrow A = \frac{1}{3} \text{ \& } B = -\frac{1}{3}$$

$$\sum_{n=1}^{+\infty} \frac{1}{3(3n-1)} - \frac{1}{3(3n+2)}$$

✓

$$S_1 = \frac{1}{6} - \frac{1}{15}, \quad S_2 = \frac{1}{6} - \frac{1}{15} + \frac{1}{15} - \frac{1}{24} = \frac{1}{6} - \frac{1}{24}$$

$$S_3 = \frac{1}{6} - \frac{1}{24} + \frac{1}{24} - \frac{1}{33} = \frac{1}{6} - \frac{1}{33}$$

$$S_n = \frac{1}{6} - \frac{1}{3(3n+1)}$$

← الحد الثاني

$$\lim_{n \rightarrow \infty} \frac{1}{6} - \frac{1}{3(3n+1)} = \frac{1}{6} \text{ conv}$$

$$\rightarrow \sum_{n=1}^{+\infty} \frac{1}{6n^2 + 3n - 2} = \left(\frac{1}{6} \right)$$

- Geometric Series

$$\text{Ex: } 5 + 5^2 + 5^3 + 5^4 + 5^5 + \dots$$

$$\frac{5^2}{5} = 5, \quad \frac{5^3}{5^2} = 5, \quad \frac{5^4}{5^3} = 5 \rightarrow \text{ratio}$$

إذا قسمنا كل حد في الحد الذي

قبله حصلنا نفس الجواب

نسبة الجواب (ratio)

$$\sum_{n=m}^{+\infty} r^n \quad r: \text{ratio}$$

Ex: which of the following is G.S?

① $\sum_{n=5}^{\infty} 4^n = 4^5 + 4^6 + 4^7 + 4^8 + \dots$ G.S

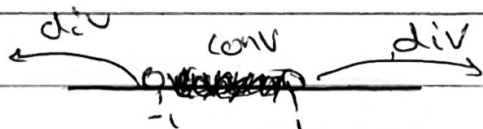
② $\sum_{k=1}^{+\infty} \frac{1}{3^{k+1}} = \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \dots$ G.S

③ $\sum_{k=1}^{\infty} k^3 = 1^3 + 2^3 + 3^3 + 5^3 + \dots$ Not G.S
 Not G.S
 Not G.S
 Not G.S

④ $\sum_{k=1}^{+\infty} (2k)^k = 2^1 + 4^2 + 6^3 + 8^4 + \dots$

Not G.S

Then $\sum_{n=m}^{\infty} r^n = \begin{cases} \text{conv if } |r| < 1 \quad (-1 < r < 1) \\ \text{div if } |r| > 1 \quad (r > 1 \text{ or } r < -1) \end{cases}$



If $\sum_{n=m}^{+\infty} r^n$ conv then

$$\sum_{n=m}^{+\infty} r^n = \frac{r^m}{1-r}$$

first term

ratio

Ex: determine whether the series conv or div
If conv find its sum?

$$\textcircled{1} \sum_{n=0}^{+\infty} \left(-\frac{1}{5}\right)^n \rightarrow \text{conv} \quad -1 < -\frac{1}{5} < 1$$

$$\rightarrow \sum_{n=0}^{+\infty} \left(-\frac{1}{5}\right)^n = \frac{\left(-\frac{1}{5}\right)^0}{1 - \left(-\frac{1}{5}\right)} = \frac{1}{1 + \frac{1}{5}} = \frac{5}{6}$$

$$\textcircled{2} \sum_{k=1}^{+\infty} \left(\frac{\pi}{e}\right)^k \rightarrow \frac{\pi}{e} \approx \frac{3.14}{2.7} > 1 \rightarrow \text{div}$$

$$\textcircled{3} \sum_{n=1}^{+\infty} 5^n 7^{-n+2} \rightarrow \sum_{n=1}^{+\infty} 5^n \cdot 7^{-n} \cdot 7^2 = \sum_{n=1}^{\infty} 7^2 \left(\frac{5}{7}\right)^n$$

$$7^2 \sum_{n=1}^{\infty} \left(\frac{5}{7}\right)^n = 49 < 1 \rightarrow \text{conv}$$

$$7^2 \left(\frac{\left(\frac{5}{7}\right)^1}{1 - \frac{5}{7}} \right) = \left[\frac{245}{2} \right]$$

Ex: If $\sum_{n=2}^{\infty} 2n^n = \textcircled{1}$ find it?

conv

←

Thm 1: $\sum a_n$ conv Then $\sum c a_n = c \sum a_n$

$$2 \sum_{n=1}^{+\infty} r^n = 1 \rightarrow \sum_{n=1}^{+\infty} r^n = \frac{1}{2} \rightarrow G.S$$

$$\frac{r^2}{1-r} = \frac{1}{2} \rightarrow 2r^2 + r - 1 = 0$$

$$(2r-1)(r+1) = 0 \rightarrow r = \frac{1}{2}, r = -1$$

اذا $|r| < 1$ $\sum r^n$ conv $\forall r$

$$\boxed{r = \frac{1}{2}} \text{ is conv}$$

The Divergence test: $\rightarrow D.T$

⊗ $\sum a_n$, If $\lim_{n \rightarrow +\infty} a_n \neq 0$ then $\sum a_n$ diverges

⊗ if $\lim_{n \rightarrow +\infty} a_n = 0 \rightarrow$ (Test fail), $\sum a_n$ may either con or div

Ex: con or div?

$$\textcircled{1} \sum_{n=1}^{+\infty} \frac{2n-1}{3n+4} / \lim_{n \rightarrow +\infty} \frac{2n-1}{3n+4} = \frac{2}{3} \neq 0$$

by D.T $\sum \frac{2n-1}{3n+4}$ div

$$\textcircled{2} \sum_{n=1}^{+\infty} \left(1 - \frac{3}{n}\right)^n, \lim_{n \rightarrow +\infty} \left(1 - \frac{3}{n}\right)^n = \underline{e^{-3}} \neq 0$$

by D.T $\sum \left(1 - \frac{3}{n}\right)^n$ div

$$\textcircled{3} \sum_{n=1}^{+\infty} \cos n\pi \rightarrow \lim_{n \rightarrow +\infty} \cos n\pi = \text{d.n.e} \neq 0$$

by D.T $\sum \cos n\pi$ div

$$\left\{ \cos \pi, \cos 2\pi, \cos 3\pi, \dots \right\}$$

$$-1, 1, -1, \dots$$

Lim = d.n.e

$$\textcircled{4} \sum_{n=1}^{\infty} \ln n \rightarrow \lim_{n \rightarrow +\infty} \ln n = \infty \neq 0 \text{ div}$$

by D.T

$$\textcircled{5} \sum_{n=1}^{\infty} \frac{1}{e^n} \rightarrow \lim_{n \rightarrow +\infty} \frac{1}{e^n} = \underline{0} \text{ (Test fail)}$$

بهره گیری از تست

$$\rightarrow \sum_{n=1}^{+\infty} \left(\frac{1}{e}\right)^n \rightarrow \text{G.S}$$

$$1 < \frac{1}{e} < 1 \rightarrow \text{conv}$$

$$\textcircled{6} \sum_{n=1}^{\infty} \frac{1}{n} \rightarrow \lim_{n \rightarrow +\infty} \frac{1}{n} = 0 \text{ (Test fail)}$$

$$\rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \text{ div (p-series)}$$

The Integral Test ^{conv or div} ^{تحويل أو تقارب}

$$\rightarrow \sum_{n=m}^{+\infty} a_n, \quad a_n = f(n) \quad [m, \infty)$$

3 شروط I.T ^{شروط}

① $f \rightarrow$ positive

② f continuous ^{cts}

③ f decreasing

$D [m, \infty)$

$$\text{if } \int_m^{+\infty} f(x) dx \text{ conv} \Rightarrow \sum_m^{+\infty} a_n \text{ conv}$$

$$\int_m^{+\infty} f(x) dx \text{ div} \Rightarrow \sum_m^{+\infty} a_n \text{ div}$$

Ex: Conv or div?

$$\text{II) } \sum_{n=3}^{+\infty} \frac{\ln n}{n} \rightarrow$$

$D [3, \infty)$

$$f(x) = \frac{\ln x}{x} \rightarrow$$

لنطبق I.T ^{نطبق}

div or conv ^{تقارب أو تباعد}

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n} = \frac{\infty}{\infty} \rightarrow \text{L'H}$$

نتحقق من 3 شروط ^{نتحقق}

$$\text{① } \frac{\ln x}{x} \rightarrow + \quad x \rightarrow +\infty + \quad \rightarrow 0 \rightarrow \text{Positive} \quad \checkmark$$

$$\text{② } f(x) = \frac{\ln x}{x}, \quad x=0 \notin [3, \infty) \quad f(x) \text{ cts } [3, \infty) \quad \checkmark$$

$$\text{③ } f(x) = \frac{\ln x}{x}, \quad f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} \rightarrow f'(x) = \frac{1 - \ln x}{x^2}$$

$$1 - \ln x = 0 \rightarrow x = e \quad \leftarrow \frac{1}{e} \rightarrow \text{dec} \quad \text{تقارب}$$

تجسس ال (3) دى ان راسم (Integral Test)

$$\int_3^{+\infty} \frac{\ln x}{x} dx = \lim_{t \rightarrow +\infty} \int_3^t \frac{\ln x}{x} dx \rightarrow \int \frac{\ln x}{x} dx$$

$$\rightarrow \lim_{t \rightarrow +\infty} \left[\frac{(\ln x)^2}{2} \right]_3^t$$

$$u = \ln x$$

$$du = \frac{1}{x} dx$$

$$\int u du = \frac{u^2}{2} = \frac{(\ln x)^2}{2}$$

$$\lim_{t \rightarrow +\infty} \frac{(\ln t)^2}{2} - \frac{(\ln 3)^2}{2}$$

$$= +\infty$$

$$\text{So } \int_3^{+\infty} \frac{\ln x}{x} dx \text{ div} \rightarrow \sum_{n=3}^{\infty} \frac{\ln n}{n} \text{ div by I.T}$$

$$\textcircled{2} \sum_1^{\infty} n e^{-n^2} \rightarrow f(x) = x e^{-x^2}$$

$$\textcircled{1} \underbrace{x}_+ \underbrace{e^{-x^2}}_+ > 0 \text{ positive } [1, \infty)$$

$$\textcircled{2} f(x) = \frac{x}{e^{x^2}}, e^{x^2} \neq 0 \text{ cts } [1, \infty)$$

$$\textcircled{3} f(x) = x e^{-x^2} \rightarrow f'(x) = x(-2x e^{-x^2}) + e^{-x^2}$$

$$\rightarrow f'(x) = e^{-x^2}(-2x^2 + 1) = 0$$

$$e^{-x^2} \neq 0 \quad -2x^2 + 1 = 0 \rightarrow x = \pm \frac{1}{\sqrt{2}}$$

$$f' \leftarrow \frac{1}{\sqrt{2}} \rightarrow \text{dec}$$

$$\int_1^{\infty} x e^{-x^2} dx \rightarrow \lim_{t \rightarrow +\infty} \int_1^t x e^{-x^2} dx$$

$$\lim_{t \rightarrow +\infty} \left[-\frac{1}{2} e^{-x^2} \right]_1^t$$

$$\lim_{t \rightarrow +\infty} \left(-\frac{1}{2} e^{-t^2} + \frac{1}{2} e^{-1} \right) = \frac{1}{2} e^{-1}$$

$$\sum_{n=1}^{+\infty} n e^{-n^2} \text{ conv by I.T conv}$$

$$\int x e^{-x^2} dx$$

$$u = x^2$$

$$du = 2x dx$$

$$\frac{du}{2x} = dx$$

$$\int x e^{-u} \frac{du}{2x}$$

$$\frac{1}{2} \int e^{-u} du = -\frac{1}{2} e^{-u} + C = -\frac{1}{2} e^{-x^2} + C$$

(P-Series)

$$\textcircled{1} \sum_{n=1}^{+\infty} \frac{1}{n} \text{ conv, div}$$

$$\textcircled{1} f(x) = \frac{1}{x} > 0 \text{ Positive } [1, \infty) \checkmark$$

$$\textcircled{2} f(x) = \frac{1}{x} \rightarrow x=0 \notin [1, \infty) \text{ TS } \checkmark$$

$$\textcircled{3} f(x) = \frac{1}{x} \rightarrow f'(x) = -\frac{1}{x^2} < 0 \text{ dec}$$

$$\int_1^{\infty} \frac{1}{x} dx \rightarrow \lim_{t \rightarrow +\infty} \int_1^t \frac{1}{x} dx = \lim_{t \rightarrow +\infty} \ln|x| \Big|_1^t$$

$$\rightarrow \lim_{t \rightarrow +\infty} \ln t = \ln \infty \rightarrow +\infty \text{ div}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ div} \rightarrow (\text{harmonic series}) \text{ تنص}$$

$$\rightarrow p\text{-Series} \rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$$

قاعدة اختبار المقارنة div or conv من خلال p

$$\left\{ \text{conv} \rightarrow p > 1, \text{div}, p \leq 1 \right\}$$

$$\hookrightarrow \sum_{n=1}^{\infty} \frac{1}{n^i} \rightarrow \text{div by } p\text{-Series}$$

Ex: conv or div?

$$\textcircled{1} \sum_{n=1}^{\infty} \frac{1}{n^2} \rightarrow 2 > 1 \text{ conv by } p\text{-Series}$$

$$\textcircled{2} \sum_{k=1}^{\infty} \frac{1}{k^{\pi}} \rightarrow \pi > 1 \text{ conv " " "}$$

$$\textcircled{3} \sum_{n=3}^{\infty} \frac{1}{n^{\frac{2}{5}}} \rightarrow \frac{2}{5} < 1 \text{ div " " "}$$

$$\textcircled{4} \sum_{n=2}^{\infty} \frac{1}{\sqrt[n+3]{n}} = \sum_{n=5}^{\infty} \frac{1}{n^{\frac{1}{2}}} \quad \frac{1}{2} < 1 \text{ div ...}$$

The Limit Comparison Test

$\sum a_n$, $\sum b_n$ are Series with positive terms

سلسلة ذات حدود موجبة (+)

$$c = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$$

$\rightarrow c > 0$ & $c \neq 0$. Then both series are conv or " " " div

Ex: conv or div?

$$e \sum_{n=1}^{+\infty} \frac{3n^3 - 2n^2 + 4}{n^7 - n^3 + 2}$$

سلسلة الحدود الموجبة
القطر

$$\rightarrow \sum b_n = \sum \frac{3n^3}{n^7}$$

لذلك نتحقق $\sum b_n$
وصالحا القوة بالبو

$$\rightarrow \sum \frac{3}{n^4} \rightarrow p=4 > 1 \rightarrow \text{conv}$$

مع القوة المقام

by p-series

الحل:

$$c = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} \rightarrow \lim_{n \rightarrow \infty} \frac{3n^3 - 2n^2 + 4}{n^7 - n^3 + 2} \cdot \frac{n^4}{3}$$

$$\lim_{n \rightarrow \infty} \frac{3n^7 - 2n^6 + 4n^4}{3n^7 - 3n^3 + 6} \rightarrow \lim_{n \rightarrow \infty} \frac{3n^7}{3n^7} = \boxed{1} > 0 \neq \infty$$

$\sum a_n$, $\sum b_n$ are conv by L.C.T

$$\sum \frac{3n^3 - 2n^2 + 4}{n^7 - n^3 + 2} \text{ conv.}$$

$$② \sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5+n^5}} \quad , \quad \sum b_n = \sum \frac{2n^2}{n^{\frac{5}{2}}} = \sum \frac{2}{n^{\frac{1}{2}}}$$

$\frac{1}{2} < 1$ div by p -series

$$C = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{2n^2 + 3n}{\sqrt{5+n^5}} = \frac{\sqrt{n}}{2}$$

$$\frac{1}{2} \lim_{n \rightarrow \infty} \frac{\sqrt{n} \cdot n^2 (2 + \frac{3}{n})}{\sqrt{n^5 (\frac{5}{n^5} + 1)}} \rightarrow \lim_{n \rightarrow \infty} \frac{\sqrt{n} (2 + \frac{3}{n})}{\sqrt{\frac{5}{n^5} + 1}}$$

$$\rightarrow \frac{1}{2} \lim_{n \rightarrow \infty} \frac{(2 + \frac{3}{n})}{\sqrt{\frac{5}{n^5} + 1}} \rightarrow \frac{1}{2} (2) \cdot \sqrt{1} \neq 0$$

$$\rightarrow \sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5+n^5}} \text{ div by L.C.T}$$

$$③ \sum_{n=1}^{\infty} \frac{3 + 9^n}{5 + 10^n} \rightarrow \sum b_n = \sum \frac{9^n}{10^n} \rightarrow \sum \left(\frac{9}{10}\right)^n$$

$$-1 < \frac{9}{10} < 1$$

conv by (G.S)

$$C = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{3 + 9^n}{5 + 10^n} = \frac{9^n}{10^n} \rightarrow \lim_{n \rightarrow \infty} \frac{9^n (\frac{3}{9^n} + 1)}{10^n (\frac{5}{10^n} + 1)} \cdot 9^n$$

←

$$\lim_{n \rightarrow \infty} \frac{\cancel{\frac{3}{n^3}} + 1}{\cancel{\frac{1}{n^3}} + 1} \rightarrow \boxed{\infty} \neq 0$$

$$\sum_{n=1}^{\infty} \frac{3n^4}{5n^5} \text{ conv by L.C.T}$$

→ C.T

The comparison Test → انجزيه، بالاجزاء

يا انا يا انا يا انا

— Suppose that $\sum a_n$ & $\sum b_n$ are series with positive terms, such that

$$\sum a_n \leq \sum b_n$$

ان انا انا انا انا

— If $\sum b_n$ is convergent then $\sum a_n$ is also conv

— " $\sum a_n$ is div " $\sum b_n$ is div

Q conv or div?

$$\text{Ex ① } \sum_{n=1}^{\infty} \frac{5}{2n^3 + n + 3}$$

$$\frac{5}{2n^3 + n + 3}$$

$$< \frac{5}{2n^2}$$

$$\rightarrow \text{ان انا انا انا انا}$$

$$\frac{5}{2n^2 + n + 3}$$

$$\rightarrow \sum_{n=1}^{\infty} \frac{5}{2n^2} \rightarrow \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \rightarrow 2 > 1 \text{ by P-series (conv)}$$

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + n + 3}$$

$$< \sum_{n=1}^{\infty} \frac{5}{2n^2}$$

$$\text{So } \sum_{n=1}^{\infty} \frac{5}{2n^2 + n + 3}$$

conv

conv

by the C.T

$$\text{Ex: } \sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$$

$$-1 \leq \sin n \leq 1 \quad \}^2$$

$$0 \leq \sin^2 n \leq 1$$

$\propto$ major term

$$\sin^2 n \leq 1 \quad \} \div n^2$$

$$\frac{\sin^2 n}{n^2} \leq \frac{1}{n^2}$$

$$\sum \frac{1}{n^2} \quad \text{P-Series} \rightarrow \text{conv} \quad 2 > 1$$

$$\sum \frac{\sin^2 n}{n^2} \leq \underbrace{\sum \frac{1}{n^2}}_{\text{conv}}$$

$$\text{So } \sum \frac{\sin^2 n}{n^2} \text{ conv by the C.T}$$

$$\text{Ex: } \sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^3} \rightarrow \underbrace{-\frac{\pi}{2}}_{\propto} < \tan^{-1} n \leq \underbrace{\frac{\pi}{2}}_{\checkmark} \quad \} \div n^3$$

$$\frac{\tan^{-1} n}{n^3} < \frac{\frac{\pi}{2}}{n^3} \rightarrow \sum \frac{\tan^{-1} n}{n^3} < \sum \frac{\frac{\pi}{2}}{n^3}$$

$$\sum \frac{\pi}{2} \sum \frac{1}{n^3} \rightarrow 3 > 1 \quad \text{conv by P-series}$$

$$\text{So } \sum \frac{\tan^{-1} n}{n^3} \quad \text{conv by The C.T}$$

$$\text{Ex } \sum_{n=2}^{\infty} \frac{1}{\ln n}$$

$$\rightarrow \ln n \leftarrow n$$

(نقطه)

$$\frac{1}{\ln n} > \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \rightarrow p\text{-series div}$$

$$\text{So } \sum \frac{1}{\ln n} \text{ div by (C.T)}$$

$$\text{Ex: } \sum_{n=1}^{\infty} \frac{5^n + 1}{2^n - 1}$$

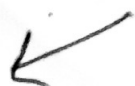
$$\rightarrow 2^n - 1 < 2^n \rightarrow \frac{1}{2^n - 1} > \frac{1}{2^n} \quad * (5^n + 1)$$

$$\rightarrow \frac{5^n + 1}{2^n - 1} > \frac{5^n + 1}{2^n} \quad \frac{5^n + 1}{2^n} \boxed{>} \frac{5^n}{2^n}$$

$$\frac{5^n + 1}{2^n - 1} > \frac{5^n}{2^n} > \frac{5^n}{2^n}$$

$$\rightarrow \frac{5^n + 1}{2^n - 1} > \frac{5^n}{2^n}$$

$$\sum \left(\frac{5}{2}\right)^n \rightarrow \frac{5}{2} > 1 \text{ div by G.S}$$



So $\sum \frac{5^n + 1}{2^n - 1}$ div by C.T

- Alternating Series Test (A.S.T)

$$\sum_{n=1}^{+\infty} (-1)^n a_n, a_n > 0 \rightarrow \text{Ex: } 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

for (2) $\rightarrow$ ∞

① $\lim_{n \rightarrow \infty} a_n = 0$

② a_n decreasing

$\rightarrow \sum (-1)^n a_n$ is convergent

Ex: conv or div?

① $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \rightarrow \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n}\right)$

1- $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ ✓

2- $f(x) = \frac{1}{x} \Rightarrow f'(x) = -\frac{1}{x^2}$ dec ✓

So $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ conv by A.S.T

(alternating harmonic series) $\rightarrow$ ∞

② $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)} = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{\ln(n+4)}$

① $\lim_{n \rightarrow \infty} \frac{1}{\ln(n+4)} = 0$ ✓

2- $f(x) = \frac{1}{\ln(x+4)} \Rightarrow f'(x) = \frac{-1}{(\ln(x+4))^2}$ dec ✓

$\hookrightarrow \frac{1}{x} \rightarrow$ dec

$$So \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{Ln(n+1)} \text{ conv by A.S.T}$$

$$(3) \sum_{n=1}^{\infty} \frac{(-1)^n}{e^n} \rightarrow \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{e^n}$$

$$1- \lim_{n \rightarrow \infty} \frac{1}{e^n} = \frac{1}{e^{\infty}} = \boxed{0}$$

$$2- f(x) = \frac{1}{e^x} = e^{-x}$$

$$f'(x) = \boxed{-e^{-x}}$$

$$So \sum_{n=1}^{\infty} \frac{(-1)^n}{e^n} \text{ conv by A.S.T}$$

... + = -

dec ✓

$$(4) \sum_{n=1}^{\infty} (-1)^n \frac{n+1}{2n+1}$$

$$1- \lim_{n \rightarrow \infty} \frac{n+1}{2n+1} = \lim_{n \rightarrow \infty} \frac{n}{2n} = \frac{1}{2} \neq 0$$

(Test fail)

$$\text{fail (A.S.T)} \Rightarrow \text{D.T}$$

$$\lim_{n \rightarrow \infty} (-1)^n \cdot \frac{n+1}{2n+1} \left\{ \begin{array}{l} \text{even } \lim_{n \rightarrow \infty} \frac{n+1}{2n+1} = \boxed{\frac{1}{2}} \\ \text{odd } \lim_{n \rightarrow \infty} -\frac{n+1}{2n+1} = \boxed{-\frac{1}{2}} \end{array} \right\} \text{ d.n.p}$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{2n+1} \text{ div by D.T}$$

- Ratio test :-

قانون ليم (Lim) لحاصل، ومفيدة للقيمة المطلقة

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = C$$

1- $C < 1 \rightarrow \sum a_n \text{ conv. (Absolutely conv)}$

2- $C > 1 \rightarrow \sum a_n \text{ div}$

or
 $C = \infty$

3- $C = 1 \rightarrow \text{fail (no conclusion)}$

Ex: Conv or div?

① $\sum_{n=1}^{\infty} \frac{3^n}{n!}$

→ Ratio test

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \rightarrow \lim_{n \rightarrow \infty} \left| \frac{3^{n+1}}{(n+1)!} \div \frac{3^n}{n!} \right|$$

$$\rightarrow \lim_{n \rightarrow \infty} \frac{3^{n+1}}{(n+1)!} \times \frac{n!}{3^n} \rightarrow \lim_{n \rightarrow \infty} \frac{3 \cdot n!}{(n+1)n!} = \frac{3}{n+1}$$

$$\lim_{n \rightarrow \infty} \frac{3}{n+1} = 0 \rightarrow C < 1 \text{ conv}$$

$$\rightarrow \sum_{n=1}^{\infty} \frac{3^n}{n!} \text{ conv by ratio test.}$$

$$\textcircled{2} \sum_{n=1}^{\infty} \frac{n!}{n^3} \rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \rightarrow \lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{(n+1)^3} \cdot \frac{n^3}{n!} \right|$$

$$\rightarrow \lim_{n \rightarrow \infty} \frac{(n+1)n^3}{(n+1)^3} = \lim_{n \rightarrow \infty} \frac{n^3}{(n+1)^2}$$

$$\rightarrow \lim_{n \rightarrow \infty} \frac{n^3}{(n+1)^2} \rightarrow \lim_{n \rightarrow \infty} \frac{n^3}{n^2 + 2n + 1} \rightarrow \lim_{n \rightarrow \infty} \frac{n}{1} = \infty$$

$$\sum_{n=1}^{\infty} \frac{n!}{n^3} \text{ div by Ratio test } (C = \infty)$$

$$\textcircled{3} \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{n^n}{n!} \rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \rightarrow \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2} (n+1)^{n+1}}{(-1)^{n+1} n^n} \right|$$

|| $\lim_{n \rightarrow \infty} \frac{n+1}{n} = 1$

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2} (n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^{n+1} n^n} \right| \rightarrow \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$$

|| $\lim_{n \rightarrow \infty} \frac{n+1}{n} = 1$

$$\rightarrow \lim_{n \rightarrow \infty} \frac{(n+1)(n+1)^n}{(n+1)!} \cdot \frac{1}{n^n} \rightarrow \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{(n+1)!} = e > 1$$

< 1 div

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^n}{n!} \text{ div by Ratio Test}$$

Root test:

$$\lim_{n \rightarrow \infty} (|a_n|)^{\frac{1}{n}} \rightarrow \sqrt[n]{|a_n|}$$

1 - $c < 1 \rightarrow \sum a_n$ conv (Abs. conv)

2 - $c > 1$ or $c = \infty \rightarrow \sum a_n$ div

3 - $c = 1$ Test Fail (no conclusion)

Ex: conv or div?

① $\sum_{n=2}^{\infty} \left(\frac{2n+1}{4n-5} \right)^n \rightarrow \lim_{n \rightarrow \infty} \left(\left| \frac{2n+1}{4n-5} \right|^n \right)^{\frac{1}{n}}$

$$\rightarrow \lim_{n \rightarrow \infty} \left(\frac{2n+1}{4n-5} \right)^{\frac{1}{n}} \rightarrow \lim_{n \rightarrow \infty} \frac{2n}{4n} = \frac{1}{2} < 1$$

conv by root test

② $\sum_{n=1}^{\infty} \left(\frac{n}{100} \right)^n \rightarrow \lim_{n \rightarrow \infty} \left(\left| \frac{n}{100} \right|^n \right)^{\frac{1}{n}} \rightarrow \lim_{n \rightarrow \infty} \left(\frac{n}{100} \right)^{\frac{1}{n}}$

$$= \infty \rightarrow \boxed{c = \infty}$$

div by root test

③ $\sum_{n=2}^{\infty} \left(\frac{-1}{\ln n} \right)^n \rightarrow \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} \cdot \left(\frac{1}{\ln n} \right)^n$

↙

1/2021

$$\lim_{n \rightarrow +\infty} \left(\left| \left(\frac{1}{L_n n} \right)^n \right| \right)^{\frac{1}{n}} \rightarrow \lim_{n \rightarrow +\infty} \left(\left(\frac{1}{L_n n} \right)^n \right)^{\frac{1}{n}}$$

$$\lim_{n \rightarrow +\infty} \frac{1}{L_n n} = 0 < 1 \quad \text{conv by root test}$$

Algebraic Properties of infinite series
if $\sum a_n$ & $\sum b_n$ are conv, then series

$$\sum a_n + \sum b_n \text{ \& } \sum a_n - \sum b_n \text{ are conv}$$

$$\bullet \sum a_n + b_n = \sum a_n + \sum b_n$$

$$\bullet \sum a_n - b_n = \sum a_n - \sum b_n$$

* If c is a non zero constant, then

$$\bullet \text{ if } \sum a_n \text{ conv} \rightarrow \sum c a_n \text{ conv}$$

$$\sum c a_n = c \sum a_n$$

$$\bullet \text{ if } \sum a_n \text{ div} \rightarrow \sum c a_n \text{ div}$$

$$\text{Ex: } \sum_{n=1}^{\infty} \frac{3}{4^n} - \frac{2}{5^{n-1}} \rightarrow \sum_{n=1}^{\infty} 3\left(\frac{1}{4}\right)^n - 2\left(\frac{1}{5}\right)^{n-1}$$

↓
G.S

↓
G.S

$$\rightarrow 3 \sum \left(\frac{1}{4}\right)^n - 2 \sum \left(\frac{1}{5}\right)^{n-1}$$

↓

↓

$$\text{conv} \rightarrow -1 < \frac{1}{4} < 1$$

$$\text{conv} \rightarrow -1 < \frac{1}{5} < 1$$

$$\rightarrow \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \left(\frac{1}{3}\right)$$

$$\rightarrow \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^{n-1} \cdot \frac{\left(\frac{1}{5}\right)^0}{1 - \frac{1}{5}} = \boxed{\frac{5}{4}}$$

$$\rightarrow 3\left(\frac{1}{3}\right) - 2\left(\frac{5}{4}\right) = \frac{-3}{2}$$

Q $\sum_{n=1}^{\infty} \left[\frac{1}{e^n} + \frac{1}{n(n+1)} \right]$ conv

$\downarrow$ $\downarrow$

Q.S $\rightarrow \left(\frac{1}{e}\right)^n$ telescoping sum

$\text{Conv } V \rightarrow \text{rank } K \leq 1$
 ~~$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} L_n$~~
 ~~$= \lim_{n \rightarrow \infty} S_n$~~

$$\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n = \frac{\frac{1}{e}}{1 - \frac{1}{e}} = \boxed{\frac{1}{e-1}}$$

Conviv

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum \frac{A}{n} + \frac{B}{n+1}$$

$$A(n+1) + nB = 1$$

$$B = -1 \quad A = 1$$

$$\hookrightarrow \sum \frac{1}{n(n+1)} = \sum \frac{1}{n} - \frac{1}{n+1}$$

$$s_1 = \frac{1}{1} - \frac{1}{2}, \quad s_2 = 1 - \frac{1}{2} - \frac{1}{3}$$

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} 1 - \frac{1}{n+1} = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{e^n} + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{e-1} + 1 = \boxed{\frac{e}{e-1}}$$

(2) $\sum a_n$ conv $\sum b_n$ div, then

$$\sum a_n \mp b_n \rightarrow \text{div}$$

Ex: $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \frac{1}{n} \rightarrow \text{conv or div?}$

G.S conv

$$-1 < \frac{1}{2} < 1$$

div (Harmonic series)

$$\rightarrow \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \frac{1}{n} \rightarrow \text{div}$$

(3) If $\sum a_n$ & $\sum b_n$ are div, then

$\sum a_n \mp b_n$ may either conv or div

Ex: (a) $\sum_{n=1}^{\infty} -1 = \text{div}$ $\rightarrow -1 + -1 + -1 \dots \rightarrow (-\infty)$

$\sum_{n=1}^{\infty} 1 = \text{div}$ $\rightarrow 1 + 1 + 1 \dots \rightarrow (\infty)$

$\sum_{n=1}^{\infty} -1 + 1 = \sum_{n=1}^{\infty} 0 = 0 + 0 + 0 \dots$

$\rightarrow \boxed{\text{conv}}$

② $\sum_{n=1}^{\infty} \frac{1}{n}$ div (P-series)

$\sum_{n=1}^{\infty} \frac{5}{n}$ div (Harmonic series)

$\sum_{n=1}^{\infty} \frac{1}{n} + \frac{5}{n} = \sum_{n=1}^{\infty} \frac{6}{n} \rightarrow \text{div (Harmonic Series)}$

Strategy for test series

① Geometric series $\sum_{n=m}^{\infty} r^n \begin{cases} \text{conv} & \text{if } |r| < 1 \\ \text{div} & \text{if } |r| > 1 \end{cases}$

② P-Series $\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{conv} & , p > 1 \\ \text{div} & , p \leq 1 \end{cases}$

③ Divergence test $\rightarrow$ If $\lim_{n \rightarrow \infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ div

④ Integral test $\rightarrow \sum_{n=m}^{\infty} a_n$, $f(n) = a_n$

If $\begin{cases} \text{① } f(n) \text{ Positive} \\ \text{② } f(n) \text{ continuous} \\ \text{③ } f(n) \text{ decreasing} \end{cases} \rightarrow \begin{cases} \int_m^{\infty} f(x) \text{ div} \\ \sum_{n=m}^{\infty} a_n \text{ div} \\ \int_m^{\infty} f(x) \text{ conv} \leftrightarrow \sum_{n=m}^{\infty} a_n \text{ conv} \end{cases}$

⑤ Limit comparison Test

$$c = \lim_{n \rightarrow +\infty} \frac{a_n}{b_n}, \text{ if } c > 0$$

& $c \neq \infty$, then both series are conv or div

⑥ Alternating series Test $\rightarrow$ A.S.T

$$\sum_{n=1}^{\infty} (-1)^n a_n, a_n > 0$$

if ① $\lim_{n \rightarrow +\infty} a_n = 0$ ② a_n decreasing ✓

Then $\sum (-1)^n a_n$ convergent

⑦ Ratio test $\rightarrow c = \lim_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right|$ if

$$① c < 1 \rightarrow \sum a_n \text{ conv}$$

$$② c > 1 \rightarrow \sum a_n \text{ div}$$

or $c = \infty \rightarrow$

$$\rightarrow \frac{n!}{n^n}$$

$$③ c = 1 \rightarrow \text{Test fail}$$

Ⓐ Root Test

$$c = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \quad \text{if } 0 < c < 1 \Rightarrow \sum a_n \text{ conc}$$

$n \rightarrow \infty$

$$\textcircled{2} c > 1 \text{ or } c = \infty \rightarrow \sum a_n \text{ div}$$

$$\textcircled{3} c = 1 \rightarrow \text{Test fail!}$$

Ⓐ telescoping sum if $\lim_{n \rightarrow \infty} S_n = L$

then $\sum_{n=1}^{\infty} a_n = \textcircled{L}$

Ex: Test the series for conc or div?

$$\sum_{n=1}^{\infty} \frac{1}{n \sqrt[n]{n}} \quad (\text{I.T}) \quad \sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{2n}} \quad (\text{Root test})$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \quad (\text{p-series}) \quad \sum_{n=1}^{\infty} \frac{1}{3^{n-1}} \quad (\text{G.S.})$$

$$\sum_{n=1}^{\infty} \frac{2^n n!}{(n+2)!} \quad (\text{Ratio T}) \quad \sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3 + 2n^2 + 5} \quad \xrightarrow{\text{limit comparison}} \text{L.C.T.}$$

test

$$\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 + 1} \quad (\text{L.C.T}) \quad \sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right) \quad (\text{Divergent Test})$$

$$\sum_{n=1}^{\infty} \left(\sqrt[n]{2-1}\right)^n \quad (\text{Root test}) \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{\cosh n} \quad \text{A.S.T}$$

$$\sum_{n=1}^{\infty} e^{\frac{1}{n}} - e^{\frac{1}{n+1}} \quad \text{(Telescoping Sum)} \quad \sum_{n=1}^{\infty} \left(\frac{1}{n+1} \right)^n \quad \text{(Root test)}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+5} \quad \sum_{n=1}^{\infty} \frac{(-5)^n}{n!} \rightarrow \frac{(-1)^n (5)^n}{n!} \quad \text{(Ratio test)}$$

(A.S.T) (Ratio test)

$$\sum_{n=1}^{\infty} \frac{n+1}{2n+1} \quad \text{(div. T)} \quad \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{n+1}}$$

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} \quad \sum_{n=1}^{\infty} \frac{n^2}{(-5)^n}$$

$$\sum_{n=1}^{\infty} \frac{(-3)^n (-3)^n}{4^n} = \sum_{n=1}^{\infty} (-3)^{2n} (4)^n$$

(C.S.) or Root test

or

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{n+1}} \quad \sum_{n=1}^{\infty} \left(\frac{6}{n} \right)^n$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n (5)^n}{4^n}$$

Root test

$$\sum_{n=1}^{\infty} (-1)^n n^2 \left(\frac{2}{3} \right)^n$$

Root test & ratio test & A.S.T



- Absolute convergence, conditional convergence.

$$\sum a_n = a_1 + a_2 + a_3 + \dots$$

$$\sum |a_n| = |a_1| + |a_2| + |a_3| + \dots$$

} conv &
div

$$\rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots \text{ conv by A.S.T}$$

$$\rightarrow \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \text{ div}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \rightarrow \text{div by P-S}$$

[1] If $\sum |a_n|$ is conv $\Rightarrow \sum a_n$ Absolutely convergent

[2] " " " div $\Rightarrow \sum a_n$ conv $\Rightarrow \sum a_n$ conditionally conv

[3] $\Leftarrow \sum |a_n|$ is div & $\sum a_n$ div $\Rightarrow \sum a_n$ divergent

Thm: $\sum a_n$ Abs. conv $\rightarrow \sum a_n$ conv

Ex: Determine whether the series is absolutely convergent, conditionally convergent, or divergent?

$\leftarrow$

$$\textcircled{1} \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{2}\right)^n$$

$$\rightarrow \sum_{n=1}^{\infty} |(-1)^n \left(\frac{1}{2}\right)^n| = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \rightarrow \text{conv G.S. } -1 < \frac{1}{2} < 1$$

$$\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{2}\right)^n \text{ Abs. conv} \Rightarrow \boxed{\text{conv}}$$

$$\textcircled{2} \sum_{n=1}^{\infty} \frac{(-1)^n n^3}{3^n}$$

$$\rightarrow \sum_{n=1}^{\infty} \left| \frac{(-1)^n n^3}{3^n} \right| = \sum_{n=1}^{\infty} \frac{n^3}{3^n} \quad ? \begin{matrix} \text{conv} \\ \text{or} \\ \text{div} \end{matrix} \rightarrow \text{Ratio test}$$

$$\rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \rightarrow \lim_{n \rightarrow \infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$

$$\rightarrow \lim_{n \rightarrow \infty} \frac{(n+1)^3}{3 \cdot n^3} \xrightarrow{\text{zero}} \lim_{n \rightarrow \infty} \frac{1}{3} \left(1 + \frac{1}{n}\right)^3 = \boxed{\frac{1}{3}} < 1$$

conv

$$\rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n n^3}{3^n} \text{ Abs. conv} \rightarrow \boxed{\text{conv}}$$

$$\textcircled{3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$\rightarrow \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| \rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \rightarrow \text{div P-series}$$

veral p > 1

$$\rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \rightarrow \text{A.S.T.}$$

. (series) > 1



① $\lim \frac{1}{n} = 0$ ② $\frac{1}{n}$ dec

$\sum \frac{(-1)^n}{n}$ conv by A.S.T

ال (series) المتكبة (conv) ولما اخذت القيمة المتكبة فالت

$\sum \frac{(-1)^n}{n} \rightarrow$ conditionally convergent.

$\leftarrow$ (conv)

④ $\sum \frac{(-1)^n (2n-1)!}{3^n} \rightarrow \sum \left| \frac{(-1)^n (2n-1)!}{3^n} \right|$

$\rightarrow \sum \frac{(2n-1)!}{3^n} \leftarrow \begin{matrix} \text{conv} \\ \text{or} \\ \text{div} \end{matrix} ? \text{ Ratio test}$

$\rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \rightarrow \lim_{n \rightarrow \infty} \frac{(2(n+1)-1)!}{3^{n+1}} \cdot \frac{3^n}{(2n-1)!}$

$\rightarrow \lim_{n \rightarrow \infty} \frac{(2n+1)! \cdot 3^n}{3^{n+1} \cdot (2n-1)!} \rightarrow \lim_{n \rightarrow \infty} \frac{(2n+1)(2n)}{3(2n-1)}$

$\rightarrow \lim_{n \rightarrow \infty} \frac{1}{3} (2n+1) 2n = \infty \rightarrow \text{div by Ratio Test}$

$\rightarrow \sum \frac{(-1)^n (2n-1)!}{3^n}$ div by Ratio Test

لما استخدم طريقة (Ratio test) كان، رجعنا لـ (series) لـ
div or conv

$$(5) a_1 = 2, a_{n+1} = \frac{5n+1}{4n+3} a_n$$

$$\rightarrow (\text{Ratio test}) \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{5n+1}{4n+3} \cdot \frac{a_n}{a_n} \right|$$

$$\rightarrow \lim_{n \rightarrow \infty} \frac{5n+1}{4n+3} = \lim_{n \rightarrow \infty} \frac{5n}{4n} = \boxed{\frac{5}{4}} > 1$$

by Ratio Test divergent.

Convergent or ~~div~~ (div) or ~~series~~ (series)
 Ratio test

$$\sum a_n$$

$$\sum |a_n|$$

conv

conv

$\sum a_n$ Abs. conv

conv

div

$\sum a_n$ conditionally convergent

div

div

$\sum a_n$ div

div

conv

X



Power Series

$$\sum_{n=0}^{\infty} C_n X^n = C_0 + C_1 X + C_2 X^2 + C_3 X^3 + \dots$$

X : variable C_n : Constant (coefficients)

$$\rightarrow \sum_{n=0}^{\infty} C_n (X-a)^n \text{ power series centered at } a. \quad x-a=0 \rightarrow x=a$$

$$C_0 + C_1 (X-a)^1 + C_2 (X-a)^2 + C_3 (X-a)^3 + \dots$$

Ex: ① $\sum_{n=0}^{\infty} \frac{X^n}{5^n} \rightarrow \text{centered at } (0) \quad (X=0)$

$$\begin{array}{ccccccc} 1 & + & \frac{X}{5} & + & \frac{X^2}{5^2} & + & \frac{X^3}{5^3} & + & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \textcircled{1} = 1 & & C_1 = \frac{1}{5} & & C_2 = \frac{1}{25} & & C_3 = \frac{1}{125} & & \end{array}$$

② $\sum_{n=1}^{\infty} n X^n \rightarrow \text{centered at } 0$

$$= X + 2X^2 + 3X^3 + 4X^4 + \dots$$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$
 $C_1 \quad C_2 \quad C_3 \quad C_4$

③ $\sum_{n=1}^{\infty} \frac{(X-3)^n}{n+1} \rightarrow \text{centered at } 3$

←

$$\frac{x-3}{2} + \frac{(x-3)^2}{3} + \frac{(x-3)^3}{4} + \dots$$

⑧ $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{\sqrt{n}} \rightarrow \text{centered at } \frac{1}{2}$

$$\frac{2x-1}{1} + \frac{(2x-1)^2}{\sqrt{2}} + \frac{(2x-1)^3}{\sqrt{3}} + \dots$$

$f(x) = \sum_{n=1}^{\infty} C_n x^n$, Domain: The Set of all x for which the series converges.

الحوال هو جمع قيم x التي يتقارب فيها السلسلة (series) $\leftarrow \text{conv}$

Ex: $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$

$x=1 \rightarrow \sum \frac{1}{2^n} \rightarrow \text{conv G.S. } -1 < \frac{1}{2} < 1$

$x=6 \rightarrow \sum \frac{6^n}{2^n} \rightarrow \sum 3^n \text{ div G.S.}$

$\rightarrow \sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n$

conv or div } x - قيم

$-1 < \frac{x}{2} < 1 \rightarrow -2 < x < 2 \rightarrow \text{Domain } (-2, 2)$

conv $\leftarrow$ Series يتقارب فيها

div $\leftarrow$ Series لا يتقارب فيها

Thm: $\sum_{n=0}^{\infty} C_n (x-a)^n$

عبداللہ 3

① → The series converges only when $x=a$

② → " " " " for all x

③ → There is a positive number (R) such that
series converges if $|x-a| < R$
& div $|x-a| > R$

Radius & Interval of conv

← لیٹے والے

سوال	Radius of conv	Interval of conv
①	$R=0$	$\{a\}$
②	$R=+\infty$	$(-\infty, \infty)$
③	R	$[a-R, a+R)$ $(a-R, a+R)$ $(a-R, a+R]$ $[a-R, a+R]$

① Ratio Test ← نوے ایک سے زیادہ

or

② Root Test

Ex: Find the radius of conv & interval of conv

$$\textcircled{1} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n (n+1)}$$

$$\rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \rightarrow \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{3^{n+1} (n+2)} \cdot \frac{3^n (n+1)}{(-1)^n x^n} \right|$$

$$\rightarrow \lim_{n \rightarrow \infty} \left| \frac{x^{n+1} \cdot 3^n (n+1)}{3^{n+1} (n+2) x^n} \right| \rightarrow \lim_{n \rightarrow \infty} \left| \frac{x (n+1)}{3 (n+2)} \right|$$

$$\rightarrow \lim_{n \rightarrow \infty} \frac{1}{3} \cdot \frac{n+1}{n+2} |x| \rightarrow \frac{1}{3} |x| \rightarrow \text{Ratio test } c < 1$$

conv

$$\frac{1}{3} |x| < 1 \rightarrow |x| < 3$$

Radius of conv $R = 3$

$$\rightarrow \text{Interval of conv } |x| < 3 \rightarrow -3 < x < 3$$

نقطة انحدار في اطراف الفترة (endpoint) $(-3, 3)$

$$\boxed{x = -3} \rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n (-3)^n}{3^n (n+1)} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n}$$

$\rightarrow$ p-series $\rightarrow$ div

$$\boxed{x = 3} \rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{(n+1) 3^n} \rightarrow \sum_{n=0}^{\infty} \frac{-1^n}{n+1} \text{ conv by A.S.T}$$

$$\rightarrow -3 < x < 3 \rightarrow (-3, 3]$$

Interval of conv

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$$\textcircled{1} \sum_{n=1}^{\infty} \frac{(x-2)^n}{(n^2+1)}$$

Ratio Test

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \rightarrow \lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{(x-2)^n} \right|$$

cancel

$$\rightarrow \lim_{n \rightarrow \infty} \frac{n^3+1}{n^3+2n+2} |x-2| = |x-2| \quad \text{R.T.} \rightarrow \begin{cases} < 1 \\ \text{conv} \end{cases}$$

$$|x-2| < 1 \rightarrow \textcircled{R=1} \rightarrow \text{Radius of conv}$$

$$|x-2| < 1 \rightarrow -1 < x-2 < 1 \rightarrow \boxed{1 < x < 3}$$

$$\rightarrow \sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$$

$$\boxed{x=1} \rightarrow \sum_{n=1}^{\infty} \frac{-1^n}{n^2+1} \rightarrow \text{div of conv} \rightarrow \text{A.S.T} \rightarrow \textcircled{1} \lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$$

② $\frac{1}{n^2}$ dec

$\rightarrow \text{conv by A.S.T}$

$$\boxed{x=3} \rightarrow \sum_{n=1}^{\infty} \frac{1^n}{n^2+1} \rightarrow \sum b_n = \sum \frac{1}{n^2}, \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$$

$$\rightarrow \lim_{n \rightarrow \infty} \frac{1}{n^2+1} \cdot n^2 \rightarrow \lim_{n \rightarrow \infty} \frac{n^2}{n^2+1} = \textcircled{1} > 0$$

conv by L.C.T

$$1 \leq x \leq 3 \rightarrow \boxed{[1, 3]}$$

Interval of conv

$$\textcircled{3} \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$$\rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$

$$\rightarrow \lim_{n \rightarrow \infty} \left| \frac{\cancel{x^n} \cdot x \cdot \cancel{n!}}{(n+1) \cancel{n!} \cdot \cancel{x^n}} \right| \rightarrow \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right|$$

$$\rightarrow \lim_{n \rightarrow \infty} \frac{0}{n+1} |x| = 0 |x| = \boxed{0} < 1$$

وبما ان (0) عين الـ (series) الـ متقاربة دائما (conv)

— converges for all values of x .

— Interval of con $(-\infty, \infty)$

— Radius of con $R = \infty$

$$\textcircled{4} \sum_{n=1}^{\infty} n^n (x-3)^n \rightarrow$$

نسبة
Test

Root test $\lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$

$$\rightarrow \lim_{n \rightarrow \infty} \left| n^n (x-3)^n \right|^{\frac{1}{n}} \rightarrow \lim_{n \rightarrow \infty} |n(x-3)|$$

$$\lim_{n \rightarrow \infty} \frac{n}{n} |x-3| = \boxed{+\infty} \rightarrow \boxed{\text{diverge}}$$

لواجبة $\rightarrow \lim_{n \rightarrow \infty} n |x-3| \rightarrow x=3 \rightarrow \lim_{n \rightarrow \infty} n(0) \rightarrow \boxed{0}$

conv

→ series conv → $\boxed{x=3}$ → radius of con

($R=\infty$)

Interval of con $\{3\}$

Taylor & MacLaurin series

$$\rightarrow \text{Taylor Series: } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$f(x) = \underbrace{f(a)}_{c_0} + \underbrace{\frac{f'(a)}{1!}}_{c_1} (x-a)^1 + \underbrace{\frac{f''(a)}{2!}}_{c_2} (x-a)^2 + \dots$$

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$\rightarrow$ If $a=0$, we call it MacLaurin series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$f(x) = \underbrace{f(0)}_{1!} + \underbrace{\frac{f'(0)}{1!}}_{1!} x^1 + \underbrace{\frac{f''(0)}{2!}}_{2!} x^2 + \underbrace{\frac{f'''(0)}{3!}}_{3!} x^3 + \dots$$

Ex: find the Maclaurin series for the function

$$f(x) = e^x$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$f(x) = e^x \rightarrow f(0) = 1$$

$$f'(x) = e^x \rightarrow f'(0) = 1$$

$$f''(x) = e^x \rightarrow f''(0) = 1$$

$$f^{(n)}(x) = e^x \rightarrow f^{(n)}(0) = 1$$

$$\rightarrow f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \dots + \frac{f^{(n)}(0)}{n!} x^n$$

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \left[\frac{x^n}{n!} \right]$$

(Maclaurin series) $\Rightarrow$ Taylor series $\leftarrow$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Ex: find maclaurin series for the function

$$f(x) = \sin x$$

$$f(x) = \sin x \rightarrow f(0) = 0$$

$$f'(x) = \cos x \rightarrow f'(0) = 1$$

$$f''(x) = -\sin x \rightarrow f''(0) = 0$$

$$f'''(x) = -\cos x \rightarrow f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \rightarrow f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \cos x \rightarrow f^{(5)}(0) = 1$$

$$f^{(6)}(x) = -\sin x \rightarrow f^{(6)}(0) = 0$$

$$f^{(7)}(x) = -\cos x \rightarrow f^{(7)}(0) = -1$$

0, 1, 0, -1

0, 1, 0, -1

$$f^{(2n)}(x) = \sin x (-1)^n \rightarrow f^{(2n)}(0) = (-1)^n \sin(0) = 0$$

$$f^{(2n+1)}(x) = \cos x (-1)^n \rightarrow f^{(2n+1)}(0) = (-1)^n \cos(0) = (-1)^n$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \frac{f^{(4)}(0)}{4!} x^4 + \dots$$

$$+ \frac{f^{(2n)}(0)}{2n!} x^{2n} + \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} + \dots$$

$$0 + x + 0 + \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + \dots + 0 + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

$$\rightarrow \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

maclaurin series (sin x)

Maclaurin Series

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$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

$R=1$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$R=\infty$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$R=\infty$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$R=\infty$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$R=1$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$R=1$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots$$

$R=1$

(maclaurin series) سلسلہ

پہلیں 5

Ex: Find the Maclurin series

$$\textcircled{1} f(x) = \sin 3x \rightarrow \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin 3x = \sum_{n=0}^{\infty} \frac{(-1)^n 3x^{2n+1}}{(2n+1)!} \quad 3x \leftarrow \text{في الجواب}$$

$$\textcircled{2} \frac{1}{1+x^2} = f(x) \rightarrow \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\downarrow$$

$$f(x) = \frac{1}{1-(x^2)} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{1} \rightarrow \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\textcircled{3} f(x) = \frac{1}{x+2} \rightarrow \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$f(x) = \frac{1}{2(\frac{x}{2}+1)} \rightarrow \frac{1}{2(1-(\frac{-x}{2}))} = \frac{1}{2} \left(\frac{1}{1-(\frac{-x}{2})} \right)$$

$$\rightarrow \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{-x}{2} \right)^n \rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1}}$$

$$\textcircled{4} f(x) = x e^{-x}$$

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ e^{-x} &= \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \\ x e^{-x} &= x \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \\ x e^{-x} &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n!} \end{aligned}$$

$$⑤ f(x) = \sin^2 x \rightarrow \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

~~$$\sin^2 x = \frac{1}{2} \left(1 - \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} \right)$$~~

$$\sin^2 x = \frac{1}{2} \left(1 - \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} \right)$$

$$\sin^2 x = \frac{1}{2} \left(1 - 1 - \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} \right)$$

لأن $n=0$ هو 1

$$\sin^2 x = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} \rightarrow \frac{2^{2n} x^{2n}}{(2n)!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1) (-1)^n 2^{2n} x^{2n}}{(2n)!} \rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!}$$

$$⑥ f(x) = \cosh x \rightarrow \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\rightarrow e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

$$\begin{array}{l} e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots \\ e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots \end{array}$$



$$\left] \div \frac{1}{2} \right.$$

$$e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \frac{2x^6}{6!} + \dots$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots + \frac{x^{2n}}{(2n)!}$$

PRENKO

ارجع للفيديو (الخريستو) طرفية ١
Lecture #34 Part 2

$$\frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

- Find the sum of the series.

$$\textcircled{1} \sum_{n=0}^{\infty} \frac{2^n}{n!} \rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \rightarrow \boxed{e^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!}}$$

$$\textcircled{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!} \rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\downarrow$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x^4)^n}{n!}$$

$$\downarrow$$

$$\sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} \rightarrow \boxed{e^{-x^4}}$$

$$\textcircled{3} \sum_{n=0}^{\infty} (-1)^{n-1} \frac{3^n}{n 5^n} \rightarrow \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{3}{5}\right)^n}{n} \rightarrow \ln\left(1 + \frac{3}{5}\right) \rightarrow \ln \frac{8}{5}$$

$$\textcircled{4} \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+2}}{4^{2n+2} (2n+1)!} \rightarrow \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\downarrow$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{4}\right)^{2n+2}}{(2n+1)!}$$

$$\frac{\pi}{4} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!} = \frac{\pi}{4} \sin \frac{\pi}{4}$$

$$\downarrow$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{4}\right) \left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!}$$

$$= \frac{\pi}{4} \cdot \frac{1}{\sqrt{2}} = \frac{\pi}{4\sqrt{2}}$$

اعداد فردية

②
$$-\frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!}$$

ببساطة هذا لأنه كما أعرف
 $-\frac{\pi^3}{3!}$ يطرح (1) ..

نفس $\sin$ لكن ناقصة الحد الأول. $\downarrow$
 $n=0$
 $\sin$ تبدأ من (0)

$$\sin \pi = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!}$$

لأنه $\rightarrow n=0 \rightarrow \boxed{\pi}$
 $\rightarrow n=1 \rightarrow -\frac{\pi^3}{3!}$
 الحاصل

نقلنا الحد الأول

$$\sin \pi \rightarrow \pi + \sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} = \sin \pi$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} = \sin \pi - \pi$$

$$= 0 - \pi$$

$$= \boxed{-\pi}$$