No. lec 7

Arc length

The Arc length Formula :-

If f' is continuous on $[a,b]$ then the length of the curve $y = f(x)$ is :

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

$\hookrightarrow \frac{dy}{dx}$

Example

Find the exact length of the curve

$f(x) = x^{3/2}, \quad 1 \leq x \leq 4$

$$f'(x) = \frac{3}{2} x^{\frac{1}{2}}$$

$$(f'(x))^2 = \frac{9}{4} x$$

$$L = \int_1^4 \sqrt{1 + \frac{9}{4}x}\, dx$$

$$L = \int_1^4 (1 + \frac{9}{4}x)^{\frac{1}{2}}\, dx$$

$$L = \left. \frac{(1 + 9/4\, x)^{3/2}}{\frac{3}{2} \cdot \frac{9}{4}} \right|_1^4$$

$$L = \frac{1}{27}(80\sqrt{10} - 13\sqrt{13})$$

* في حال كان ال curve مكتوب بدلالة y أي:
$x = g(y)$, فإن:

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\, dy \quad , \quad c \le y \le d$$

أو $\frac{dx}{dy}$

Example
set up the integral that represents the length of the curve: $x = y^2$ from $(0,0)$ to $(1,1)$?

$$\frac{dx}{dy} = 2y$$

$$\left(\frac{dx}{dy}\right)^2 = 4y^2$$

$$L = \int_0^1 \sqrt{1 + 4y^2}\, dy$$

Example
Find the arc length of the curve $y = \sqrt{4 - x^2}$

Domain: $4 - x^2 \ge 0 \longrightarrow x = \pm 2$

$$\begin{array}{ccccc} - & & + & & - \\ & -2 & & 2 & \end{array}$$

$\therefore$ Domain $[-2, 2]$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$ نجهز للقانون

$$\frac{dy}{dx} = -\cancel{2}x / \cancel{2}\sqrt{4 - x^2}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{4-x^2}$$

$$L = \int_{-2}^{2} \sqrt{1 + (x^2/4-x^2)}\, dx$$ نوحد المقامات

$$L = \int_{-2}^{2} \sqrt{\frac{4-\cancel{x^2}+\cancel{x^2}}{4-x^2}}\, dx$$

$$L = \int_{-2}^{2} \sqrt{\frac{4}{4-x^2}}\, dx$$

$$L = 2\int_{-2}^{2} \sqrt{\frac{1}{4-x^2}}\, dx \qquad \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$$

$$L = 2\int_{-2}^{2} \frac{1}{\sqrt{4-x^2}}\, dx$$

$$L = 2 \sin^{-1}\frac{x}{2}\Big|_{-2}^{2}$$

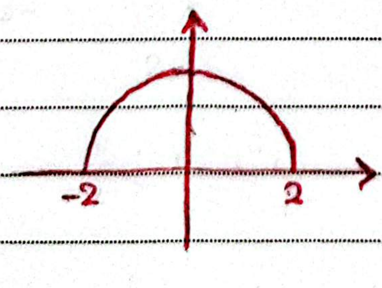
$$L = 2\left[\sin^{-1}\left(\frac{2}{2}\right) - \sin^{-1}\left(\frac{-2}{2}\right)\right]$$

الزاوية الي ساينها ١

$$L = 2\left(\sin^{-1}(1) - \sin^{-1}(-1)\right)$$

$$L = 2\left(\pi/2 \mp \pi/2\right) = 2\pi$$

note $\sqrt{a - x^2}$, $-\sqrt{a} \leq x \leq \sqrt{a}$



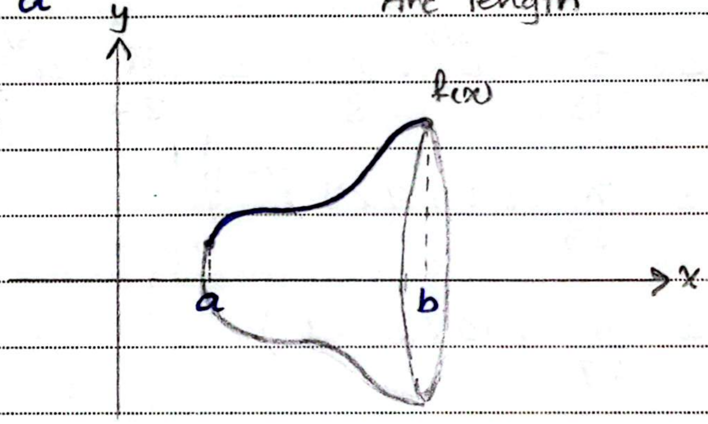
- semicurcle radias = 2
- $L = 2\pi r$
- $\frac{1}{2} L = \pi r = 2\pi$

Area of a Surface Revolution

f is positive and has a continuous derivative we define the surface area obtained by rotating the curve $y = f(x)$, $a \le x \le b$ about the x-axis.

دوران

$$S = \int_a^b 2\pi f(x) \underbrace{\sqrt{1 + (f'(x))^2}}_{\text{Arc length}} dx$$



Example

Find the exact area of the surface obtained by rotating the curve about the x-axis?

$f(x) = x^3$, $x \in [0, 1]$

$$S = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} \, dx$$ نجهز للقانون

$f'(x) = 3x^2$

$(f'(x))^2 = 9x^4$

$$S = 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4} \, dx$$ بالتعويض

No. lee.8

$u = 1 + 9x^4$

$du = 36x^3\, dx$

$dx = \frac{du}{36x^3}$

$S = 2\pi \int \cancel{x^3} \sqrt{u} \frac{du}{36\cancel{x^3}}$

$S = \frac{\cancel{2}\pi}{\cancel{36}\,18} \int u^{1/2}\, du$

$S = \frac{\pi}{18} u^{3/2} \cdot \frac{\cancel{2}}{3} = \frac{\pi}{27} u^{3/2}$

$S = \frac{\pi}{27} (1 + 9x^4)^{3/2} \Big|_0^1$

$S = \frac{\pi}{27} (10\sqrt{10} - 1)$

Example

$y = \sqrt{4 - x^2}$, $-2 \le x \le 2$

$\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}\sqrt{4-x^2}}$

$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{4-x^2}$

$S = \int_a^b 2\pi \sqrt{1 + (f'(x))^2}\, dx$

$S = \int_{-2}^{2} 2\pi \sqrt{4-x^2} \sqrt{1 + \frac{x^2}{4-x^2}}\, dx$

$$= 2\pi \int_{-2}^{2} \sqrt{4-x^2} \sqrt{\frac{4-\cancel{x^2}+\cancel{x^2}}{4-x^2}}\, dx$$

$$= 2\pi \int_{-2}^{2} \sqrt{4-x^2} \sqrt{\frac{4}{4-x^2}}\, dx$$

$$= 2\pi \int_{-2}^{2} \cancel{\sqrt{4-x^2}}\, \frac{2}{\cancel{\sqrt{4-x^2}}}\, dx$$

$$= 2\pi \int_{-2}^{2} 2\, dx = 4\pi \int_{-2}^{2} dx$$

$$= 4\pi(2--2) = 4\pi(4) = 16\pi$$

حل اخر :

$r = 2$

$S = 4r^2\pi$

$S = 4(2)^2\pi$

$= 16\pi$

Example

set up an Integral for the area of the surface obtained by rotating the curve about the x-axis?

$y = \tan x \quad , \quad 0 \leq x \leq \pi/3$

$$S = \int_a^b 2\pi f(x) \sqrt{1+(f'(x))^2}\, dx$$

$f(x) = \tan x$

$f'(x) = \sec^2 x$

$(f'(x))^2 = \sec^4 x$

$$S = \int_0^{\pi/3} 2\pi \tan x \sqrt{1 + \sec^4 x}\, dx$$

$$u = \sec^2 x$$

$$\frac{du}{dx} = 2 \sec x\, (\sec x \tan x)$$

$$\frac{du}{2u \tan x} = dx$$

$$\hookrightarrow S = \int_1^4 2\pi \tan x \sqrt{1 + (\sec^2 x)^2}\, dx$$

$$S = \int_1^4 2\pi \cancel{\tan x} \sqrt{1 + u^2}\, \frac{du}{2u \cancel{\tan x}}$$

$$S = \pi \int_1^4 \frac{\sqrt{1+u^2}}{u}\, du \quad \text{tri. sub.}$$

أو مباشرة $\longrightarrow$ $= \pi \left[\sqrt{1+u^2} - \ln \left| \frac{1+\sqrt{1+u^2}}{u} \right| \right]_1^4$

وعوض

$$u = \tan \theta$$

$$du = \sec^2 \theta\, d\theta$$

$$\sqrt{1+u^2} = \sqrt{1 + \tan^2 \theta} = \sqrt{\sec^2 \theta} = \sec \theta$$

$$\int \frac{\sqrt{1+u^2}}{u}\, du = \int \frac{\sec \theta}{\tan \theta} \sec^2 \theta\, d\theta = \int \frac{\sec^3 \theta}{\tan \theta}\, d\theta$$

$$= \int \frac{1}{\sin \theta \cos^2 \theta}\, d\theta = \text{partial Fraction...}$$

1. $S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\, dx$ $\quad y = f(x)$

2. $S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$ $\quad x = g(y)$

Example Find the exact area of the surface obtained by rotating the curve about the x-axis ?

$x = \frac{1}{3}(y^2 + 2)^{3/2}, \quad 1 \leq y \leq 2$ $\quad$ y [illegible]

(2) !.

$$\frac{dx}{dy} = \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{2}(y^2+2)^{1/2}(2y)$$

$$\frac{dx}{dy} = y(y^2+2)^{1/2}$$

$$\left(\frac{dx}{dy}\right)^2 = y^2(y^2+2) = y^4 + 2y^2$$

$$S = \int_1^2 2\pi y \sqrt{1 + y^4 + 2y^2}\, dy$$

$$S = \int_1^2 2\pi y \sqrt{(1+y^2)^{\cancel{2}}}\, dy$$

$$S = 2\pi \int_1^2 y(1+y^2)\, dy$$

$$S = 2\pi \int_1^2 y + y^3\, dy$$

$$S = 2\pi \left(\frac{y^2}{2} + \frac{y^4}{4}\right)\Big|_1^2 = \frac{21\pi}{4}$$

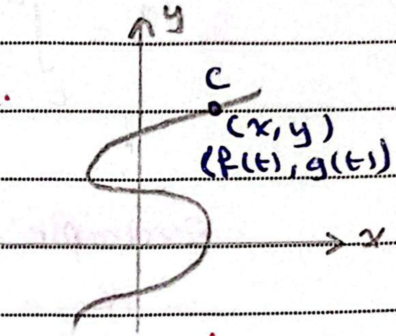
No. lec. 9

Curves defined by parametric equations
(الوسيط t)

new method for describing curves

t: parameter

$x = f(t)$ } parametric equ.
$y = g(t)$

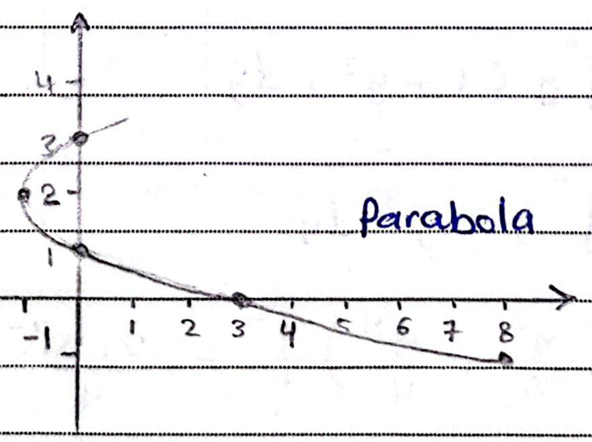


Example sketch the curve defined by the parametric equation :

$x = t^2 - 2t$

$y = t + 1$

t	x	y	(x, y)
-2	8	-1	(8, -1)
-1	3	0	(3, 0)
0	0	1	(0, 1)
1	-1	2	(-1, 2)
2	0	3	(0, 3)



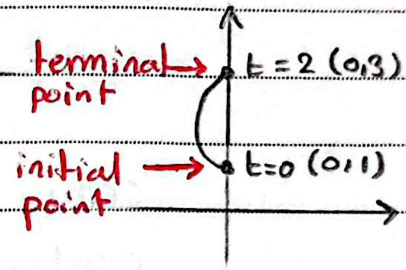
Parabola (قطع مكافئ)

Example

$x = t^2 - 2t \quad , \; 0 \leq t \leq 2$

$y = t + 1$

t	x	y	(x, y)
0	0	1	(0, 1)
1	-1	2	(-1, 2)
2	0	3	(0, 3)



Example What curve represented by the following parametric equations?

① $x = \cos t \quad , \; 0 \leq t \leq 2\pi$

$y = \sin t$

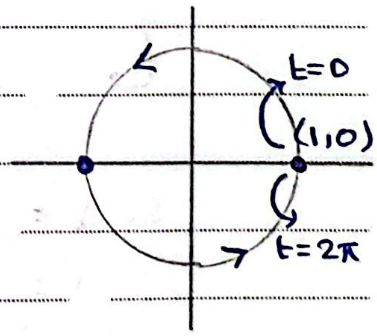
يمكن الحل بالجدول و الرسم

$\cos^2 t + \sin^2 t = 1$

$x^2 + y^2 = 1$ (معادلة الدائرة)

→ circle center (0, 0)

→ radius = 1



$t = 0 \longrightarrow x = \cos 0 = 1$

$y = \sin 0 = 0 \quad (1, 0)$

$t = \pi \longrightarrow x = \cos \pi = -1$

$y = \sin \pi = 0 \quad (-1, 0)$

$t = 2\pi \longrightarrow x = \cos 2\pi = 1$

$y = \sin 2\pi = 0 \quad (1, 0)$

No. lec 9

② $x = \cos t \quad , \quad 0 \leq t \leq \pi$

$y = \sin t$

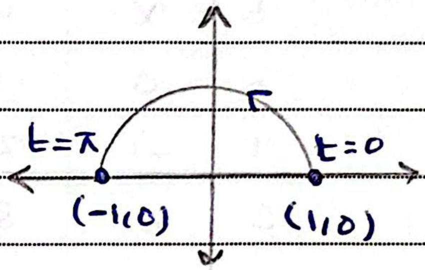
$\cos^2 t + \sin^2 t = 1$

$x^2 + y^2 = 1$

semi - circle

center $(0,0)$

radius $= 1$



تخلص   الوسيط t

Example Eliminate the parameter to find cartesian equations of the curve :

X Y

بدي الغي ال t

" " parameter ال

واكتبها كمعادلة بال cartesin

∴ بدور على علاقة بين x و y

1. $x = 3 - 4t$ المعادلات

$y = 2 - 3t$ كلاهما سهل

$x - 3 = -4t$

نعزل t $t = \dfrac{x-3}{-4} = \dfrac{3-x}{4}$

تعويض $\hookrightarrow \; y = 2 - 3\left(\dfrac{3-x}{4}\right)$ line

2. $x = 1 - t^2 \qquad -2 \leq t \leq 2$

$y = t - 2$ ← اسهل

نعزل t $y + 2 = t$

تعويض $x = 1 - (y+2)^2$ parabola

$t = -2 \longrightarrow y = t - 2 \longrightarrow -2-2 = -4$

$t = 2 \longrightarrow y = 2 - 2 = 0$

$\therefore -4 \leq y \leq 0$

3. $x = \sqrt{t} \longrightarrow t \geq 0$ لانها تحت الجذر

$y = 1 - t$ $\qquad \therefore x \geq 0$

$t = x^2$

$y = 1 - x^2 \ , \ x \geq 0$

4. $x = e^t \ , \ y = e^{-t} \ , \ t \in R$

$\ln x = t$

$y = e^{-t}$

$y = e^{-\ln x}$

$y = x^{-1} = \frac{1}{x}$

$\therefore y = \frac{1}{x} \ , \ x > 0$ $\qquad$ Range ال x $[0,\infty)$ لأن e ⊕

5. $x = 3\cos t \ , \ y = 4\sin t \ , \ 0 \leq t \leq 2\pi$

$\cos^2 t + \sin^2 t = 1$

$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$ Ellipse

6. $x = \cosh t \ , \ y = \sinh t$

Range ال $\cosh$ $[1,\infty)$

$\cosh^2 t - \sinh^2 t = 1$

$x^2 - y^2 = 1 \ \ x \geq 1$ Hyperbola

Example Find the parametric equations:

1. $x = y^2 - y$

$y = t$
$x = t^2 - t$

2. $y = 2x + 1 \quad , \quad 0 \le x \le 1$

$x = t$
$y = 2t + 1 \quad , \quad 0 \le t \le 1$

3. line segment from $(-2, 7)$ to $(3, -1)$

Point$_1$ (x_1, y_1) $\quad x = x_1 + (x_2 - x_1)t$
point$_2$ (x_2, y_2) $\quad y = y_1 + (y_2 - y_1)t$
$t \; [0, 1]$

$\hookrightarrow \overset{x_1}{(-2}, \overset{y_1}{7)} \longrightarrow \overset{x_2}{(3}, \overset{y_2}{-1)}$

$x = -2 + (3 - -2)t \longrightarrow x = -2 + 5t$
$y = 7 + (-1 - 7)t \longrightarrow y = 7 - 8t$
$0 \le t \le 1$

4. circle center $(5,2)$ radius 2

$$(x-5)^2 + (y-2)^2 = 4 \qquad \div 4$$

$$\frac{(x-5)^2}{4} + \frac{(y-2)^2}{4} = 1$$

$$\left(\frac{x-5}{2}\right)^2 + \left(\frac{y-2}{2}\right)^2 = 1$$

$$\cos^2 t + \sin^2 t = 1$$

$$\frac{x-5}{2} = \cos t \longrightarrow x = 2\cos t + 5$$

$$\frac{y-2}{2} = \sin t \longrightarrow y = 2\sin t + 2 \qquad 0 \leq t \leq 2\pi$$

solution 2

circle center (h, k), radius r

$$x = h + r\cos t$$

$$y = k + r\sin t \qquad 0 \leq t \leq 2\pi$$

Calculus with parametric Curves

$$\left.\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned}\right\} \text{ are differentable}$$

- $y' = \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$
- $y'' = \dfrac{dy'}{dx} = \dfrac{dy'/dt}{dx/dt}$

ما منشتق المشتقة الأولى لانه
متعامل مع para. equ

Example Find dy/dx , d^2y/dx^2 :

① $x = t^2 + 1$, $y = t^2 + t$

$\dfrac{dx}{dt} = 2t$ $\qquad \dfrac{dy}{dt} = 2t + 1$

$$\frac{dy}{dx} = \frac{2t+1}{2t}$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{\dfrac{(2t)(2) - (2t+1)(2)}{(2t)^2}}{2t}$$

$$= \frac{-1}{4t^3}$$

- Find $\left.\dfrac{dy}{dx}\right|_{t=1}$

$$= \frac{2t+1}{2t} = \frac{2+1}{2} = \frac{3}{2}$$

- Find $\left.\frac{d^2y}{dx^2}\right|_{t=1}$

$$\frac{d^2y}{dx^2} = -\frac{1}{4t^3} = -\frac{1}{4}$$

② $y = 2\sin t \quad , \quad x = 3\cos t$

$$\frac{dy}{dt} = 2\cos t \qquad \frac{dx}{dt} = -3\sin t$$

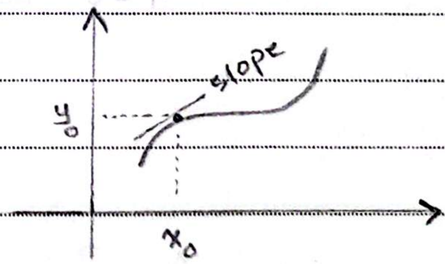
$$\frac{dy}{dx} = \frac{2\cos t}{-3\sin t} = -\frac{2}{3}\cot t$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{-\frac{2}{3}(-\csc^2 t)}{-3\sin t} = -\frac{2}{9}\csc^3 t$$

Equations of tangent line

- $y - y_0 = m(x - x_0)$

$$m = \text{slope} = \left.\frac{dy}{dx}\right|_{(x_0, y_0)}$$



- Curve has horizontal tangent

slope $= 0$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad , \quad \frac{dy}{dt} = 0 \quad , \quad \frac{dx}{dt} \neq 0$$

- Curve has vertical tangent

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad , \frac{dx}{dt} = 0 \quad , \frac{dy}{dt} \neq 0$$

•• $\frac{d^2y}{dx^2}$ → concarve up $\frac{d^2y}{dx^2} > 0$

→ concarve down $\frac{d^2y}{dx^2} < 0$

Example A curve defind by the paramitric equations :

$x = t^2 \ , \ y = t^3 - 3t$

① show that c has two tangent at the point (3,0) and find their equ:

1. متى يمر بـ (x,y)

= معادلة x , = معادلة y

$y = 0 \longrightarrow y = t^3 - 3t$

$0 = t^3 - 3t$

$0 = t(t^2 - 3)$

$t = 0 \ , \ t = \pm\sqrt{3}$ (بشيك عليهم)

2. أتأكد من القيم

$x = 3 \longrightarrow x = t^3$

$t = 0 \longrightarrow x = 0$ ✗ ($t = 0$ نهمل ∴)

$t = \sqrt{3} \longrightarrow x = 3$ ✓

$t = -\sqrt{3} \longrightarrow x = 3$ ✓

3. مشتقة

$$\frac{dx}{dt} = 2t \quad , \quad \frac{dy}{dt} = 3t^2 - 3$$

$$dy/dx = \frac{3t^2-3}{2t}$$

4. الحل

t	slope $= m = dy/dx = \frac{3t^2-3}{2t}$	$y-y_0=m(x-x_0)$ $(3,0)$
$\sqrt{3}$	$\frac{3(\sqrt{3})^2-3}{2\sqrt{3}} = \sqrt{3}$	$y-0=\sqrt{3}(x-3)$ $y=\sqrt{3}(x-3)$
$-\sqrt{3}$	$\frac{3(-\sqrt{3})^2-3}{-2\sqrt{3}} = -\sqrt{3}$	$y-0=-\sqrt{3}(x-3)$ $y=-\sqrt{3}(x-3)$

✡✡ ② ~~F~~ind the points on C, where the tangent is horizontal or vertal ?

$$\frac{dy}{dx} = \frac{3t^2-3}{2t}$$

point → tang. → horiz. $\frac{dy}{dx}=0$ $\boxed{\frac{dy}{dt}=0}$

→ vert. $\frac{dy}{dx}=\frac{1}{0}$ $\boxed{\frac{dx}{dt}=0}$

Horizontal $\rightarrow \frac{dy}{dt}=0 \rightarrow 3t^2-3=0$

$3t^2=3 \rightarrow t=\pm 1$

$t=1$ ✓ , $t=-1$ ✓

$2t=2\neq 0$ $\quad 2t=-2\neq 0$

$\therefore$ Horizontal at $t=1$, $t=-1$

$t=1 \rightarrow (1,-2)$ $\leftarrow$ $x=t^2$ نعوض في

$t=-1 \rightarrow (1,2)$ $\leftarrow$ $y=t^3-3$

$2t=0 \rightarrow t=0 \longrightarrow 3t^2-3 \xrightarrow{t=0} =-3\neq 0$ ✓

$\therefore$ Vertical at $t=0$ $\leftarrow$ x,y نعوض في

$t=0 \longrightarrow (0,0)$

No. lec-10

③ Determine where the curve is concave up or concave down?

من ٢٠١ الاولى

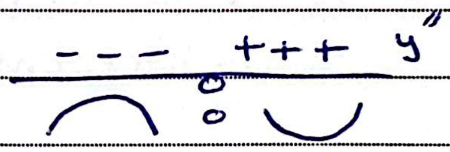
$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{6t^2+6}{4t^2}}{2t} = \frac{6t^2+6}{8t^3}$$

$$\frac{d^2y}{dx^2} = 0 \longrightarrow 6t^2+6 = 0 \longrightarrow 6t^2 = -6 \ \times$$

$$\qquad\qquad \hookrightarrow 8t^3 = 0 \longrightarrow t = 0 \ \checkmark$$

concave up $(0, \infty)$

" down $(-\infty, 0)$



Arc length

Theorem If a curve is described by parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$ where f' and g' are continuous on $[\alpha, \beta]$ and C traversed exactly one as t incr es from α to β then the length of C is :

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

Example Find the exact length of the curve $x = 1 + 3t^2$, $y = 4 + 2t^3$, $0 \leq t \leq 1$

$$\frac{dx}{dt} = 6t \quad , \quad \frac{dy}{dt} = 6t^2$$

$$L = \int_0^1 \sqrt{(6t)^2 + (6t^2)^2} \, dt$$

$$L = \int_0^1 \sqrt{36t^2 + 36t^4} \, dt$$

$$L = \int_0^1 \sqrt{36t^2(1+t^2)} \, dt$$

$$L = \int_0^1 6t\sqrt{1+t^2} \, dt \qquad t = \tan\theta$$

$$L = 4\sqrt{2} - 2$$

Surface Area

$x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$ is rotated about x-axis where f', g' are continuous and $g'(t) > 0$ then the area of the resulting surface is given by :

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example set up an Integral represents the area of the surface obtained by rotating the given curve about x-axis ?

$x = t\sin t$, $y = t\cos t$

$\frac{dx}{dt} = t\cos t + \sin t$, $\frac{dy}{dt} = -t\sin t + \cos t$

$$S = \int_{0}^{\pi/2} 2\pi t\cos t \sqrt{(t\cos t + \sin t)^2 + (-t\sin t + \cos t)^2}$$

فك فك

$$S = \int_{0} 2\pi t\cos t \sqrt{1 + t^2\cos^2 t + t^2\sin^2 t}$$

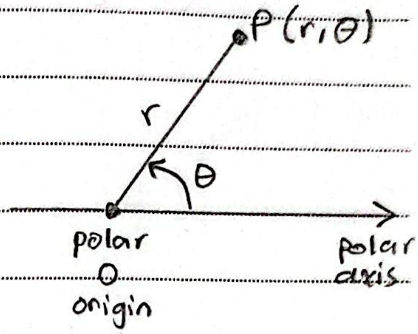
$$S = \int_{0} 2\pi t\cos t \sqrt{1 + t^2(\cos^2 t + \sin^2 t)}$$

$$S = \int_{0}^{\pi/2} 2\pi t\cos t \sqrt{1 + t^2}$$

No. lec 11

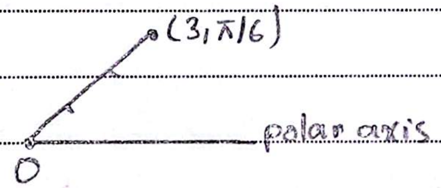
Polar Coordinates

(r, θ) polar coordinates
r: the distance from O to P.
θ: the angle between the polar axis and the line OP (radians).

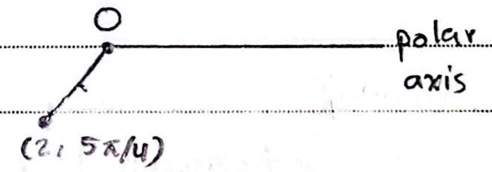


Example plot the points whose polar coordinates are given :

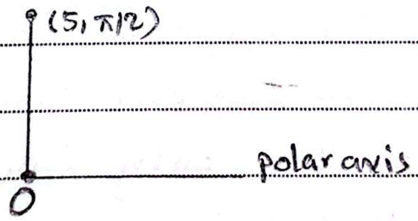
① $(\overset{r}{3}, \overset{\theta}{\frac{\pi}{6}})$



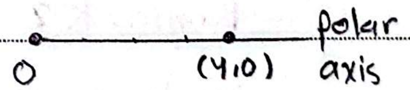
② $(\overset{r}{2}, \overset{\theta}{\frac{5\pi}{4}})$



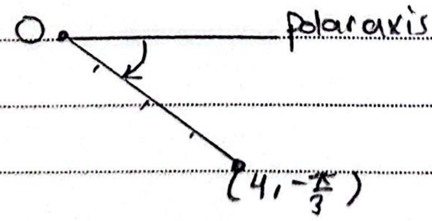
③ $(5, \pi/2)$



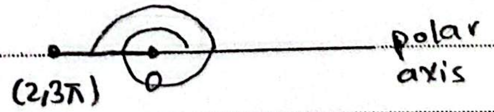
④ $(4, 0)$



⑤ $(4, -\frac{\pi}{3})$

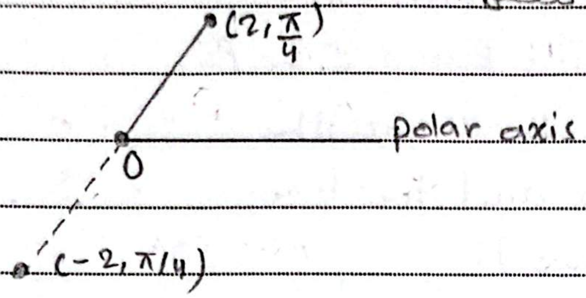


⑥ $(2, 3\pi)$

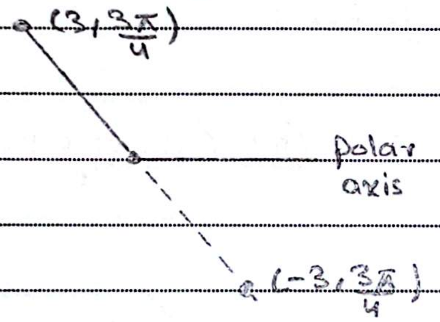


⑦ $(\ominus 2, \frac{\pi}{4})$ الاشارة السالبة فقط (عكس الاتجاه)

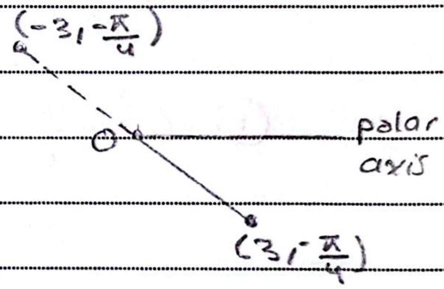
نحدد $(2, \frac{\pi}{4})$ اولاً ثم نعكس.



⑧ $(-3, \frac{3\pi}{4})$



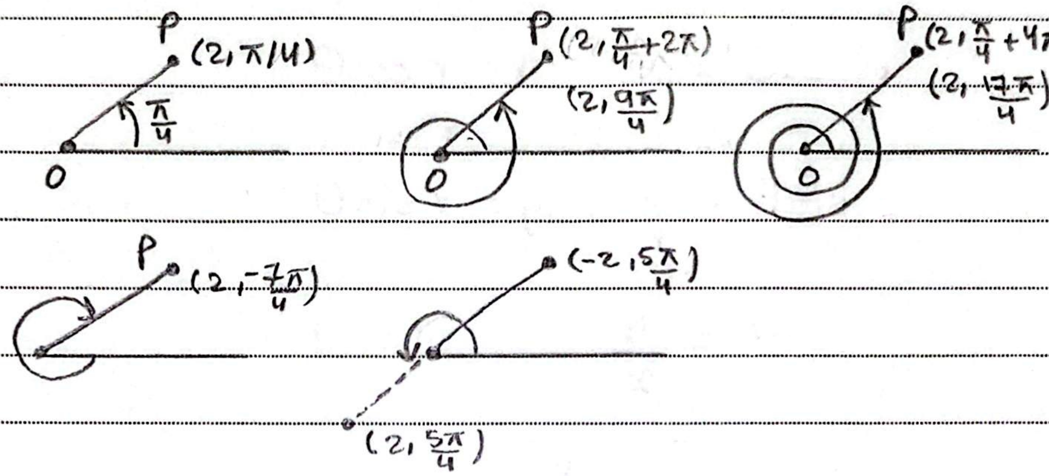
⑨ $(-3, -\pi/4)$



Remark 1 $(-r, \theta)$ represents the same point as $(r, \theta+\pi)$ or $(r, \theta-\pi)$

Remark 2 In the cartesian coordinate system every point has only one representation. But in the polar coordinate each point has many representations.

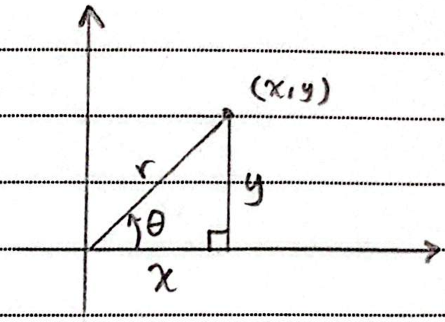
Example $p\ (2, \pi/4)$



- If (r, θ) is apoint in polar coordinate, then all polar coordinates of (r, θ) is :
  $(r, \theta + 2n\pi)$
  $(-r, \theta + \pi + 2n\pi)$
  $(-r, \theta + (2n+1)\pi)$

- Origin point $(0,0)$:
  O pole $(0, \theta)$, θ any value of θ

Relationship between Polar and Cartesian Coordinates



- $\cos\theta = \dfrac{x}{r}$, $x = r\cos\theta$
- $\sin\theta = \dfrac{y}{r}$, $y = r\sin\theta$
- $\tan\theta = \dfrac{y}{x}$
- $r^2 = y^2 + x^2$

Example convert the following points from polar to cartesian :

① $(\overset{r}{2}, \overset{\theta}{\pi/3}) \longrightarrow (x, y)$

$x = r\cos\theta \rightarrow 2\cos\frac{\pi}{3} = 1$

$y = r\sin\theta \rightarrow 2\sin\frac{\pi}{3} = \sqrt{3}$

$\therefore (x, y) = (1, \sqrt{3})$

② $(-6, 2\pi/3)$

$x = r\cos\theta = -6\cos 2\pi/3 = 3$

$y = r\sin\theta = -6\sin 2\pi/3 = -3\sqrt{3}$

$\therefore (3, -3\sqrt{3})$

No. lec 11

Example convert the following points from cartesian to polar coordinates:

-1	θ
+1	x2-1

① $\overset{x}{(1,}\overset{y}{-1)}^{*} \longrightarrow (r, \theta)$

$r^2 = x^2 + y^2$

$r^2 = 1 + 1 = 2$

$r = \sqrt{2}$

$\tan\theta = \frac{y}{x} = \frac{-1}{1}$

$\tan\theta = -1$

$\theta = 2\pi - \frac{\pi}{4}$ (4 ربع)*

$= \frac{7\pi}{4}$

$\therefore (r, \theta) = (\sqrt{2}, \frac{7\pi}{4})$

② $(-\sqrt{3}, 1)^{*}$

$r^2 = x^2 + y^2$

$r^2 = 3 + 1$

$r = 2$ (القيمة + دائماً)

$\tan\theta = \frac{y}{x} = \frac{1}{-\sqrt{3}}$

$\theta = \pi - \frac{\pi}{6}$ (2 ربع)*

$= \frac{5\pi}{6}$

$\therefore (2, \frac{5\pi}{6})$

Example Find the polar equation for the curve represented by given cartesian equ:

① $y = 2$ horizontal line

$y = r\sin\theta$

$2 = r\sin\theta \longrightarrow r = \frac{2}{\sin\theta} \Rightarrow r = 2\csc\theta$

② $xy = 4$ polar equ → (θ , r فيه)

$(r\cos\theta)(r\sin\theta) = 4$

$r^2 \cos\theta \sin\theta = 4$

$r^2 = \frac{4}{\sin\theta \cos\theta} = \frac{4}{\frac{1}{2}\sin 2\theta}$

$r^2 = \frac{8}{\sin 2\theta} = 8\csc 2\theta$

$r^2 = 8\csc 2\theta$

Example: Find the cartesian equation for the curve represented by given Polar equ:

① $r = 2$ $*2$

$r^2 = 4$

$x^2 + y^2 = 4$ circle with center (0,0) radius 2

② $r = \sec\theta$

$r = \frac{1}{\cos\theta}$

$r\cos\theta = 1$

$x = 1$ vertical line

③ $r = \sec\theta \tan\theta$

$r = \dfrac{1}{\cos\theta} \cdot \dfrac{y}{x} \cdot \cos\theta$ $\left(\tan\theta = \dfrac{y}{x}\right)$

$r\cos\theta = \dfrac{y}{x}$

$x = \dfrac{y}{x}$

$x^2 = y$ parabola

☆ ④ $r = 3\cos\theta$ $*r$

$r^2 = 3r\cos\theta$ $(r^2 = x^2 + y^2)$
$(x = r\cos\theta)$

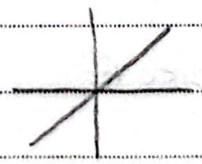
$x^2 + y^2 = 3x$
$x^2 - 3x + y^2 = 0$

⑤ $\theta = \dfrac{\pi}{4}$

$\tan\theta = \tan\dfrac{\pi}{4}$ $\left(\tan\theta = \dfrac{y}{x}\right)$

$\dfrac{y}{x} = 1$

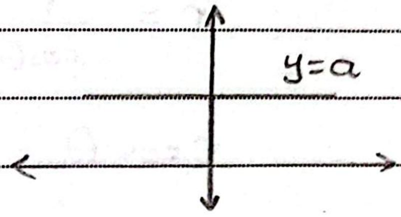
$y = x$ line



Polar Curves

- lines

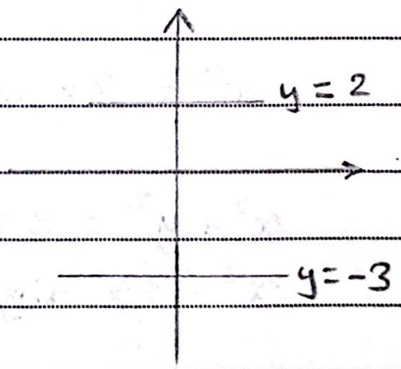
1. horizontal line



$$y = a$$
$$r\sin\theta = a$$
$$r = \frac{a}{\sin\theta} = a\csc\theta$$

$$r = a\csc\theta$$

Example



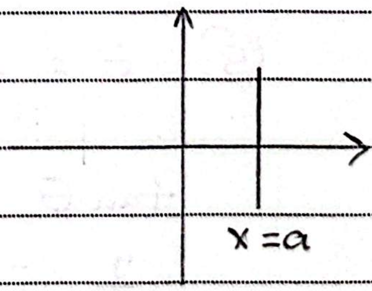
① $r = 2\csc\theta$

$\Rightarrow y = 2$

② $r = -3\csc\theta$

$\Rightarrow y = -3$

2. vertical line



$$x = a$$
$$r\cos\theta = a$$
$$r = \frac{a}{\cos\theta} = a\sec\theta$$

$$r = a\sec\theta$$

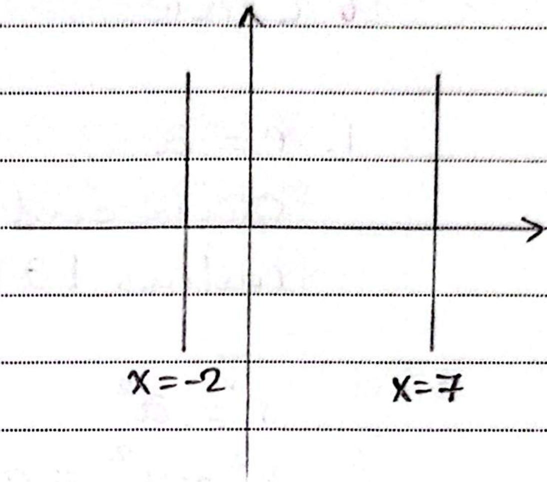
Example

① $r = 7 \sec \theta$

$\Rightarrow x = 7$

② $r = -2 \sec \theta$

$\Rightarrow x = -2$



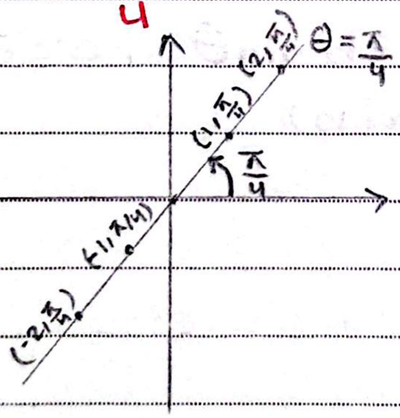
مستقيم

3. the straight line passes through the pole (0,0)

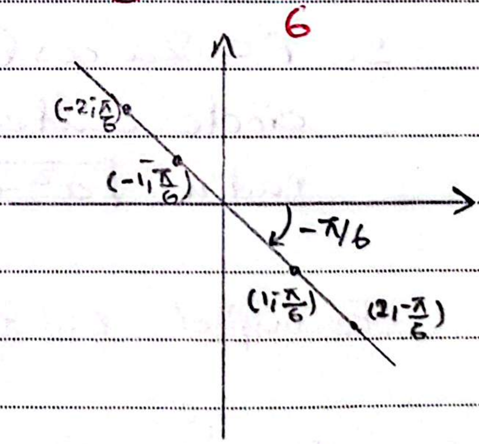
$\hookrightarrow \theta = \theta_0$

Example

$\theta = \frac{\pi}{4}$



$\theta = -\frac{\pi}{6}$



- Circle

1. $r = a$

circle center $(0,0)$

radius $|a|$

$r^2 = a^2$ ↲ الاثبات

$x^2 + y^2 = a^2$

Example

① $r = 5$

circle center $(0,0)$

radius $= 5$

② $r = -3$

circle center $(0,0)$

radius $= |-3| = 3$

2. $r = 2a\cos\theta + 2b\sin\theta$, $0 \le \theta \le \pi$

circle center (a,b)

radius $\sqrt{a^2+b^2}$

Example $r = 4\cos\theta + 6\sin\theta$

$r = 2(2\cos\theta + 3\sin\theta)$

circle center $(2,3)$

radius $\sqrt{4+9} = \sqrt{13}$

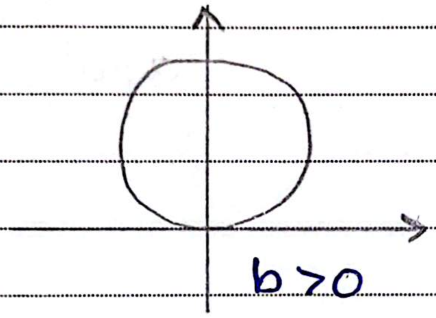
3. $r = 2a\cos\theta + 2b\sin\theta$, $0 \le \theta \le \pi$

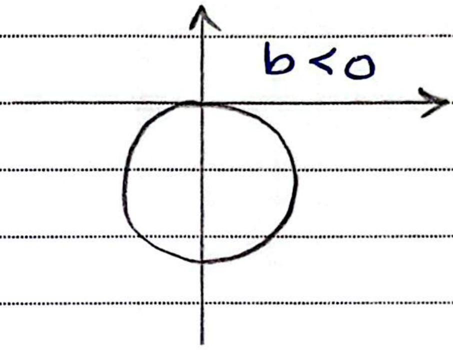
$a = 0$

$r = 2b\sin\theta$, $0 \le \theta \le \pi$

circle center $(0, b)$

radius $= |b|$





Example

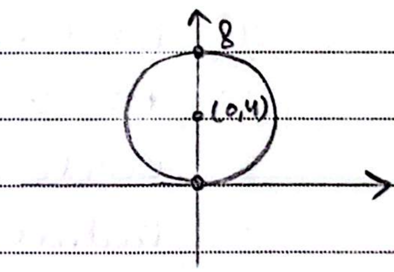
دائرة

① $r = 8\sin\theta$

$r = 2(4\sin\theta)$

circle center $(0, 4)$

radius $= 4$



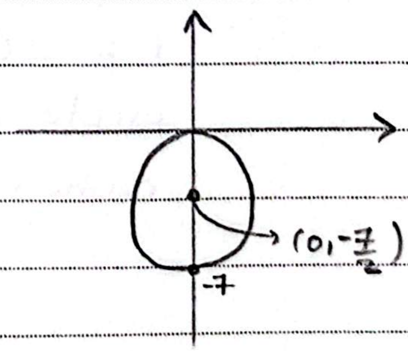
دائرة

② $r = -7\sin\theta$

$r = 2\left(-\frac{7}{2}\sin\theta\right)$

circle center $\left(0, -\frac{7}{2}\right)$

radius $= \left|-\frac{7}{2}\right| = \frac{7}{2}$



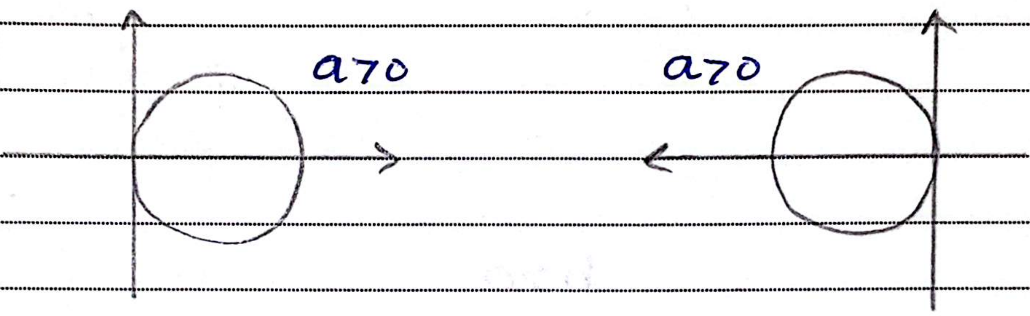
4. $r = 2a\cos\theta + 2b\sin\theta\ ,\ 0 \le \theta \le \pi$

$b = 0$

$r = 2a\cos\theta\ ,\ 0 \le \pi \le \pi$

circle center $(a, 0)$

radius $= |a|$



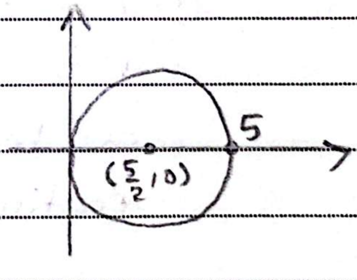
Example

① $r = 5\cos\theta$

$r = 2\left(\frac{5}{2}\cos\theta\right)$

circle center $\left(\frac{5}{2}, 0\right)$

radius $= \left|\frac{5}{2}\right| = \frac{5}{2}$

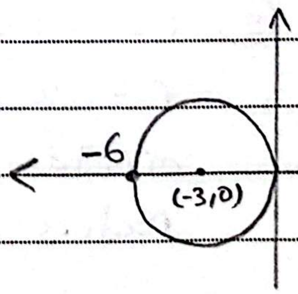


② $r = -6\cos\theta$

$r = 2(-3\cos\theta)$

circle center $(-3, 0)$

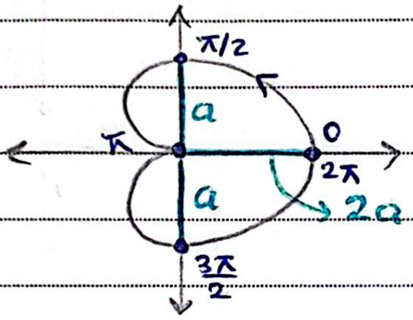
radius $= |-3| = 3$



- Cardioid

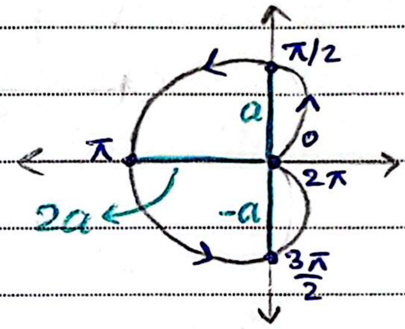
$r = a(1+\cos\theta)$

$r = -a(1-\cos\theta)$



$r = a(1-\cos\theta)$

$r = -a(1+\cos\theta)$

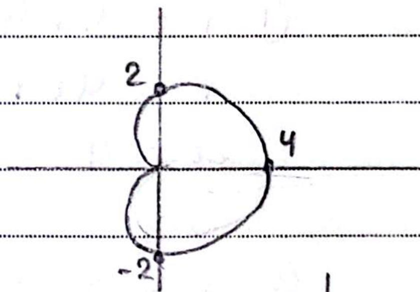


Example

① $r = 2+2\cos\theta$

$r = 2(1+\cos\theta)$

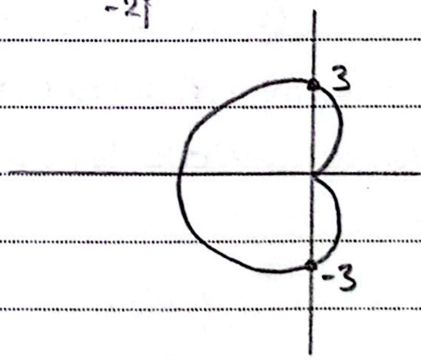
$a = 2$



② $r = 3-3\cos\theta$

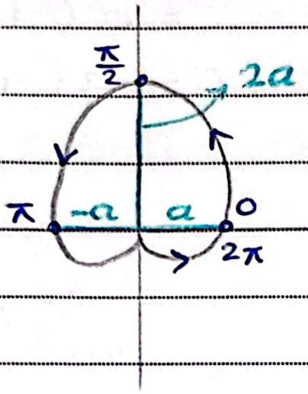
$r = 3(1-\cos\theta)$

$a = 3$



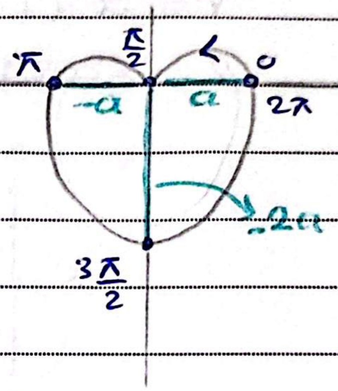
$r = a(1+\sin\theta)$

$r = -a(1-\sin\theta)$



$r = a(1-\sin\theta)$

$r = -a(1+\sin\theta)$

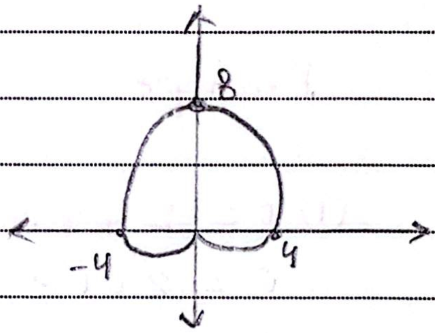


Example

① $r = 4 + 4\sin\theta$

$r = 4(1+\sin\theta)$

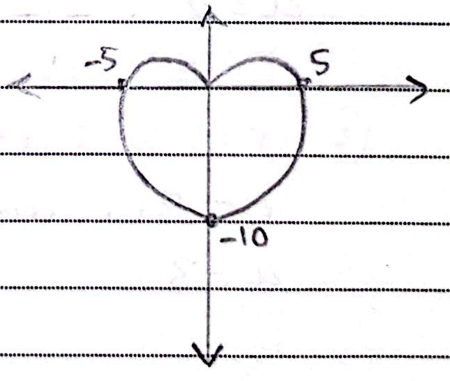
$a = 4$



② $r = 5 - 5\sin\theta$

$r = 5(1-\sin\theta)$

$a = 5$



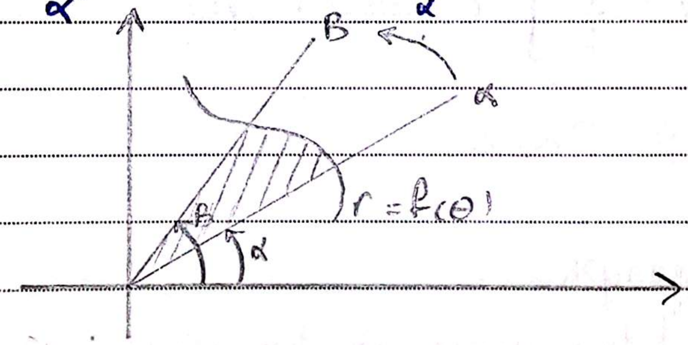
No. lec. 13

Area in Polar Coordinates

Suppose that α and β are angles that satisfy the condition $0 < \beta - \alpha \leq 2\pi$:

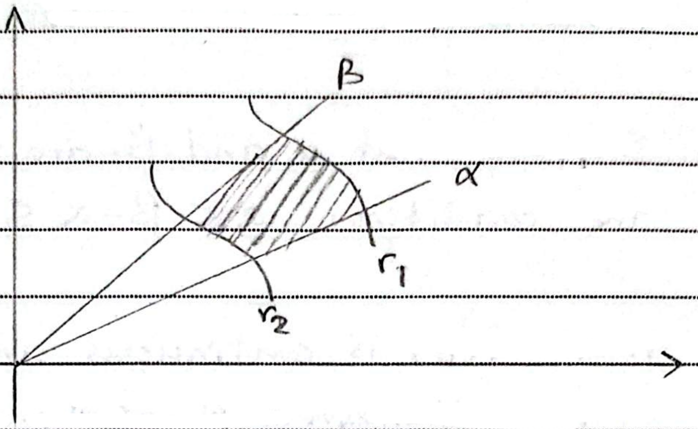
1curve 1. $r = f(\theta)$ is continuous and $r > 0$ for $\alpha \leq \theta \leq \beta$ then the area A of the region R enclosed by the polar curve $r = f(\theta)$ and lines $\theta = \alpha$ and $\theta = \beta$ is :

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2 \, d\theta$$



2curves 2. $r_1 = f(\theta)$, $r_2 = g(\theta)$ are continuous and $r_1 > 0$ and $r_2 > 0$ then the area A of the region R enclosed by the polar curves r_1 and r_2 and the lines $\theta = \alpha$ and $\theta = \beta$ is :

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \underbrace{(f(\theta))^2}_{\text{الاكبر}} - \underbrace{(g(\theta))^2}_{\text{الاصغر}} \, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r_1^2 - r_2^2 \, d\theta$$



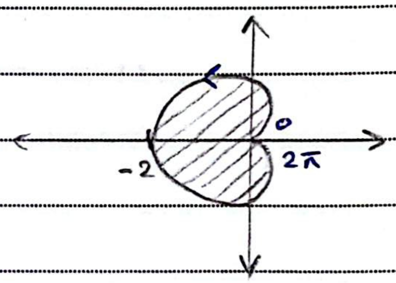
Example

Find the entire area within the Cardioid $r = 1 - \cos\theta$ المساحة الداخلية لـ

1. ارسم
2. قانون A
مسس $\frac{1}{2}$ >
curves

$$A = \frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2 \, d\theta$$

$$= 3\pi/2$$

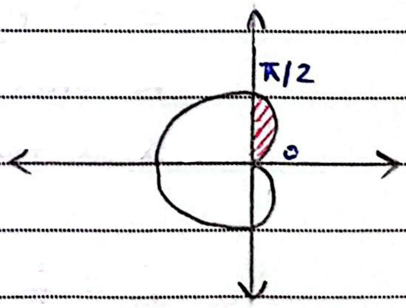


Example

Find the area of the region in the first quadrant that is within the cardioid $r = 1 - \cos\theta$

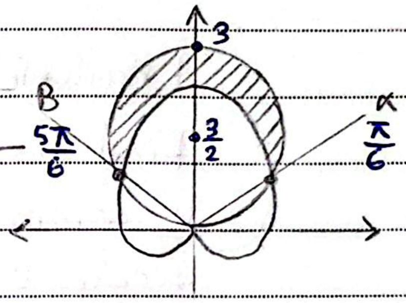
$$A = \frac{1}{2}\int_0^{\pi/2} (1-\cos\theta)^2 \, d\theta$$

$$= \frac{3}{8}\pi - 1$$



Example

Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1+\sin\theta$



لايجاد الزوايا نحسب نقاط التقاطع ← $\frac{5\pi}{6}$

بمساواة الـ 2curves

$3\sin\theta = 1+\sin\theta$

$2\sin\theta = 1$

$\sin\theta = 1/2$

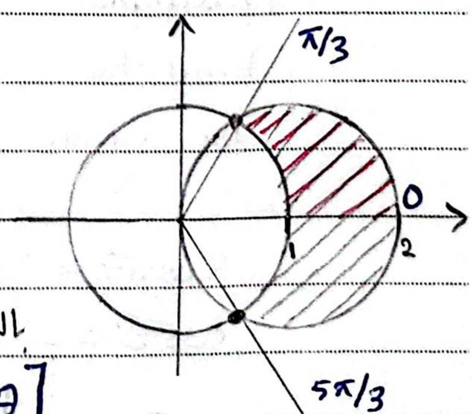
$\theta = \pi/6 \, , \, 5\pi/6$

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2 - (1+\sin\theta)^2 \, d\theta$$

$= \pi$

Example

Find the area of the region that lies outside $\underbrace{r=1}_{\text{circle}}$ and inside $\underbrace{r = 2\cos\theta}_{\text{circle}}$



$1 = 2\cos\theta$

$\cos\theta = 1/2$

$\theta = \pi/3 \, , \, 5\pi/3$

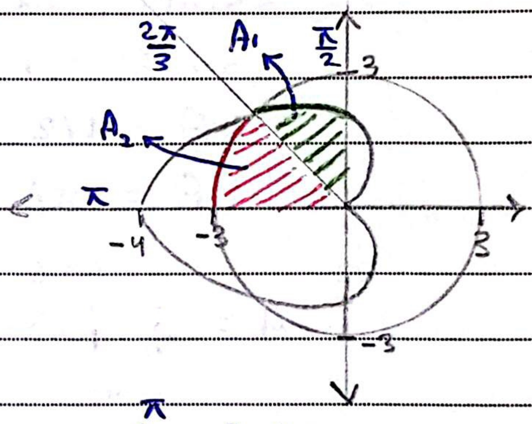
التماثل :

$$A = 2\left[\frac{1}{2}\int_{0}^{\pi/3} (2\cos\theta)^2 - (1)^2 \, d\theta\right]$$

or $A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2\cos\theta)^2 - (1)^2 \, d\theta$

Example

Find the area in the second quadrant that is common to the cardioid $r = 2-2\cos\theta$ and circle $r=3$



$3 = 2-2\cos\theta$

$\cos\theta = -1/2$

$\theta = 2\pi/3$

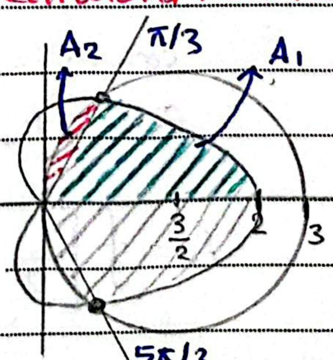
$A = A_1 + A_2$

$= \frac{1}{2}\int_{\pi/2}^{2\pi/3} (2-2\cos\theta)^2 d\theta + \frac{1}{2}\int_{3\pi/2}^{\pi} (3)^2 \, d\theta$

$=$

Example

Find the area of the region that is common to the circle $r = 3\cos\theta$ and the cardioid $r = 1+\cos\theta$



$3\cos\theta = 1+\cos\theta$

$\cos\theta = \frac{1}{2} \longrightarrow \theta = \pi/3 , 5\pi/3$

No. lec.14

Sequences

Sequence : list of numbers written in a definite order.

$a_1 , a_2 , a_3 , \dots , a_n , \dots$

first term second term general term (nth term)

Example

① $2 , 4 , 6 , 8 , \dots , 2n$

$a_1 \quad a_2 \quad a_3 \quad a_4$

$a_1 \Rightarrow n=1 \Rightarrow a_1 = 2 \times 1 = 2$

$a_2 \Rightarrow n=2 \Rightarrow a_2 = 2 \times 2 = 4 \quad \dots$

② $1 , 1/2 , 1/3 , 1/4 , 1/5 , \dots , \frac{1}{n}$

$a_1 \quad a_2 \quad a_3 \quad a_4$

- Def: An infinit seq is a function whose domain is the positive integers:
  $f(n) = a_n \quad , n = 1, 2, 3, 4, \dots$
- notation : $\{a_1 , a_2 , a_3 , \dots\} , f(n) = a_n$
  $\{a_n\}$ or $\{a_n\}_{n=1}^{+\infty}$

Example Find the first five terms in the following :

① $a_n = \frac{n}{n+1}$, $n = 1, 2, \dots$

$n = 1, 2, 3, 4, 5$

$a_1 = \frac{1}{1+1} = \frac{1}{2}$

$a_2 = \frac{2}{2+1} = \frac{2}{3}$

$a_3 = \frac{3}{3+1} = \frac{3}{4}$

$a_4 = \frac{4}{4+1} = \frac{4}{5}$

$a_5 = \frac{5}{5+1} = \frac{5}{6}$

② $a_n = \sqrt{n-3}$, $n \geq 3$ $n = 3, 4, 5, 6, 7$

$a_3 = \sqrt{3-3} = 0$

$a_4 = \sqrt{4-3} = 1$

$a_5 = \sqrt{5-3} = \sqrt{2}$

$a_6 = \sqrt{6-3} = \sqrt{3}$

$a_7 = \sqrt{7-3} = 2$

No. lec. 14

Example Find the general term of the seq:

① $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

$a_1 \quad a_2 \quad a_3 \quad a_4$

$2^1 \quad 2^2 \quad 2^3 \quad 2^4$

$\therefore a_n = \frac{1}{2^n}$

② $\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \ldots$

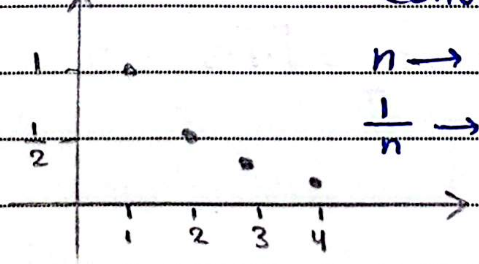
$a_1 \quad a_2 \quad a_3 \quad a_4$

odd even $(-1)^4$

$\therefore a_n = \frac{(-1)^{n+1} n}{n+1}$

Graph of sequences

- $a_n = \frac{1}{n}$, $n = 1, 2, 3$



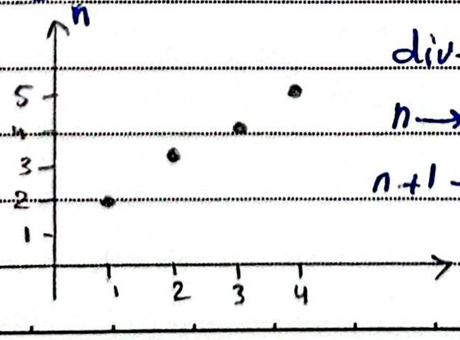
Conv. to 0

$n \to +\infty$

$\frac{1}{n} \to 0$

$a_1 = 1$	$(1, 1)$
$a_2 = \frac{1}{2}$	$(2, 1/2)$
$a_3 = \frac{1}{3}$	$(3, 1/3)$
$a_4 = \frac{1}{4}$	$(4, 1/4)$

- $a_n = n + 1$



divergent

$n \to +\infty$

$n+1 \to +\infty$

$a_1 = 2$	$(1, 2)$
$a_2 = 3$	$(2, 3)$
$a_3 = 4$	$(3, 4)$
$a_4 = 5$	$(4, 5)$

No. lec. 14

- $a_n = (-1)^{n+1}$ d.n.e $n \to +\infty$

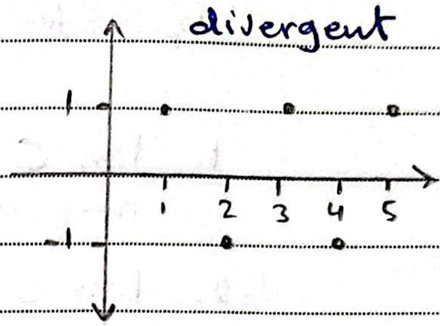
divergent

$a_1 = (-1)^2 = 1$ (1,1)

$a_2 = (-1)^3 = -1$ (2,-1)

$a_3 = (-1)^4 = 1$ (3,1)

$a_4 = (-1)^5 = -1$ (4,-1)



Def: A seq $\{a_n\}$ is said to be convergent to L if $\lim_{n \to +\infty} a_n = L$. ($a_n \to L$ as $n \to +\infty$)

otherwise, we say the seq divergent.

Theorem: If $\{a_n\}$, $\{b_n\}$, $a_n \to L$ and $b_n \to M$ as $n \to +\infty$, then:

1. $\lim_{n\to+\infty} C = C$, $C \in \mathbb{R}$
2. $\lim_{n\to+\infty} C a_n = cL$
3. $\lim_{n\to+\infty} (a_n \pm b_n) = L \pm M$
4. $\lim_{n\to+\infty} (a_n b_n) = LM$
5. $\lim_{n\to+\infty} (a_n / b_n) = \frac{L}{M}$, $M \neq 0$

Theorem: If $\lim_{n\to+\infty} a_n = L$ and the function f is continuous at L, then:

$$\lim_{n\to+\infty} f(a_n) = f(L) = f(\lim_{n\to+\infty} a_n)$$

L'Hopital's Rule

No. lec: 15

Example Determine whether the seq conv or div ??

: لازم ناخذ الـ $\lim_{n\to\infty}$

① $\{n^2+2n+1\}$

اكبر قوة ومعاملها

$$\lim_{n\to+\infty} n^2+2n+1 = \lim_{n\to+\infty} n^2 = +\infty \quad \text{div}$$

② $a_n = \dfrac{n^2}{n+1}$

اكبر قوة بالبسط على
" " بالمقام

$$\lim_{n\to+\infty} \frac{n^2}{n+1} = \lim_{n\to+\infty} \frac{n^2}{n} = \lim_{n\to+\infty} n = +\infty \quad \text{div}$$

③ $\{ne^{-n}\}$

$$\lim_{n\to+\infty} ne^{-n} = (+\infty)(0) \quad L'H$$

$$\lim_{n\to+\infty} \frac{n}{e^n} \xrightarrow{L'H} \lim_{n\to+\infty} \frac{1}{e^n} = 0 \quad \text{Conv to } 0$$

④ $\{\ln(2n+1) - \ln n\}$

$$\lim_{n\to+\infty} (2n+1) - \ln n = \infty - \infty \quad L'H$$

$$\lim_{n\to+\infty} \ln\left(\frac{2n+1}{n}\right) = \ln\left(\lim_{n\to+\infty} \frac{2n+1}{n}\right)$$

اكبر قوة من
البسط مع معاملها
واكبر قوة ...

$$= \ln\left(\lim_{n\to+\infty} \frac{2n}{n}\right) = \ln 2 \quad \text{conv to } \ln 2$$

⑤ $a_n = \sin n$

فتذبذب

$\lim\limits_{n \to +\infty} \underbrace{\sin n}$ = d.n.e div

⑥ $a_n = \left(1 + \frac{2}{n}\right)^{4n}$

$\lim\limits_{n \to +\infty} \left(1 + \frac{2}{n}\right)^{4n} = e^{8}$ conv to e^{8}

** $\lim\limits_{x \to +\infty} \left(1 + \frac{a}{x}\right)^{bx} = e^{ab}$

⑦ $a_n = \frac{(2n-1)!}{(2n+1)!}$

$\lim\limits_{n \to +\infty} \frac{(2n-1)!}{(2n+1)!} = \lim\limits_{n \to +\infty} \frac{\cancel{(2n-1)!}}{(2n+1)(2n)\cancel{(2n-1)!}}$

$= \lim\limits_{n \to +\infty} \frac{1}{(2n+1)(2n)} = \frac{1}{\infty} = 0$ Conv to 0

⑧ $a_n = (3^n + 5^n)^{\frac{1}{n}}$

$\lim\limits_{n \to +\infty} (3^n + 5^n)^{\frac{1}{n}} = \infty^{0}$ L'H

or $\lim\limits_{n \to +\infty} \left(5^n\left(\left(\frac{3}{5}\right)^n + 1\right)\right)^{1/n} = \lim\limits_{n \to +\infty} 5\left(\cancel{\left(\frac{3}{5}\right)^n}^{\,0} + 1\right)^{\frac{1}{n}}$

$= (5)(1)^{0} = 5$ conv to 5

Theorem: The sequeezing Thm for seq:

let $\{a_n\}, \{b_n\}, \{c_n\}$ (3 seq) such that $a_n \le b_n \le c_n$. If $\lim\limits_{n\to+\infty} a_n = \lim\limits_{n\to+\infty} c_n$, then $\lim\limits_{n\to+\infty} b_n = L$

Example Determine whether the seq conv or div ??

$a_n = \frac{\sin n}{n}$

$\lim\limits_{n\to+\infty} \frac{\sin n}{n}$

$= 0$

$\therefore$ conv. to 0

$-1 \le \sin n \le 1 \quad \div n$

$\frac{-1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$

$\lim\limits_{n\to+\infty}$ ↓ 0, ↓, $\lim\limits_{n\to+\infty}$ ↓ 0

$\therefore 0$

Alternating Sequences

+ , − , + , − ...

$\{(-1)^n a_n\} = \{-a_1, a_2, -a_3, a_4 \ldots\}$

or $\hookrightarrow (-1)^{n\pm1}$

Example

$a_n = (-1)^n = \{-1, 1, -1, 1, \ldots\}$ div

$a_n = (\cos n\pi)2^n = (\cos\pi)2^1, (\cos\pi)2^2, \ldots$

$= -2, 2^2, \ldots$ div

Thm 1: If $\lim_{n\to+\infty} |a_n| = 0$ then $\lim_{n\to+\infty} a_n = 0$

Example Determine whether the seq div or conv??

$\left\{\frac{(-1)^n}{n}\right\} = \left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots\right\}$

$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore \lim_{n\to\infty} \frac{(-1)^n}{n} = 0 \quad \therefore$ conv to 0

Thm 2: A seq to L iff the seq of even numbered terms and odd-numbered terms converge to L

$\Rightarrow a_1, a_2, a_3, a_4, a_5, a_6 \dots$

$a_1, a_3, a_5, \dots \longrightarrow \lim = L$

$a_2, a_4, a_6, \dots \longrightarrow \lim = L$

$\therefore \lim \text{seq} = L$

Example Determine conv or div?

$$a_n = (-1)^n \frac{n^2+1}{2n^2+3n+5}$$

$$\lim_{n\to+\infty} \left| (-1)^n \frac{n^2+1}{2n^2+3n+5} \right| = \lim_{n\to+\infty} \frac{n^2+1}{2n^2+3n+5}$$

$= \frac{1}{2} \neq 0$ fail (Thm 1)

(Thm 2)

$$\text{even} = a_{2n} = \frac{n^2+1}{2n^2+3n+5} \longrightarrow \lim_{n\to\infty} \frac{n^2+1}{2n^2+3n+5} = \frac{1}{2}$$

$$\text{odd} = a_{2n+1} = (-1)\frac{n^2+1}{2n^2+3n+5} \longrightarrow \lim_{n\to\infty} (-1)\frac{n^2+1}{2n^2+3n+5} = \frac{-1}{2}$$

$\neq$

$\Rightarrow a_n = (-1)^n \dfrac{n^2+1}{2n^2+3n+5}$ $\therefore$ div

No. lec. 17

The Sequence r^n

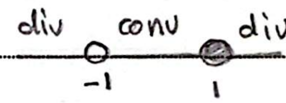
r^n, $r \in \mathbb{R}$

$\{2^1, 2^2, 2^3, 2^4, \ldots$ div

$(1/5)^n = \{1/5, (1/5)^2, (1/5)^3, \ldots$ conv to 0

For what values of r is the seq r^n conv

** $r^n = \begin{cases} \text{conv}, & -1 < r \leq 1 \\ \text{div}, & \text{otherwise} \end{cases}$



Example

- $(\frac{1}{2})^n \longrightarrow$ conv $\quad -1 < \frac{1}{2} \leq 1$
- $4^n \longrightarrow$ div $\quad 4 > 1$
- $-5^n \longrightarrow$ div $\quad -5 \leq -1$
- $(\frac{e}{\pi})^n \longrightarrow$ conv $\quad -1 < \frac{e}{\pi} \leq -1$ (e: 2.71, π: 3.14)

** conv $\begin{cases} -1 < r \leq -1 & \text{conv to } 0 \\ r = 1 & \text{conv to } 1 \end{cases}$

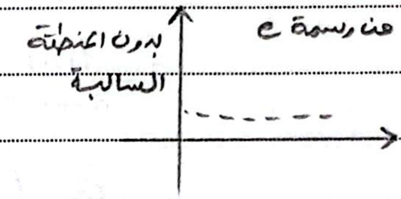
FIVE APPLE

$-1 < r \leq 1$ conv ?

- $r = 1 \Rightarrow r^n = \{1, 1, 1, \dots$ conv to 1
- $r = 0 \Rightarrow r^n = \{0, 0, 0, \dots$ conv to 0

- $0 < r < 1$ $\qquad n \to +\infty$

$\lim_{n \to +\infty} r^n = 0 \qquad r^n \to 0$



- $-1 < r < 0 \to 0 < |r| < 1$

$\lim_{n \to +\infty} |r^n| = \lim_{n \to +\infty} |r|^n = 0$

وحسب Thm 1 :

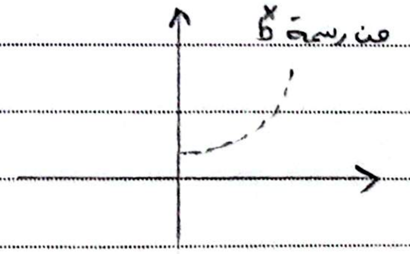
$\lim_{n \to +\infty} |a_n| = 0 \longrightarrow \lim_{n \to +\infty} a_n = 0$

$\lim_{n \to +\infty} r^n = 0 \quad \therefore$ conv

div ?

- $r > 1$

$\lim_{n \to +\infty} r^n = +\infty$ div



- $r \leq -1$

$r = -1 \Longrightarrow (-1)^n = \{-1, 1, -1, 1 \dots$ div

$r < -1 \Longrightarrow (-2)^n = \{-2, 4, -8 \dots$ div

↳ Alternating

Example: Find the value of a , if the following seq is conv :

$\left\{\frac{a^{n+1}}{2^n}\right\}$ conv $\therefore \lim_{n\to+\infty} \frac{a^{n+1}}{2^n}$ exists

$= \lim_{n\to+\infty} \frac{a\,a^n}{2^n}$ exists

$= a \lim_{n\to+\infty} \left(\frac{a}{2}\right)^n$ $\rightarrow r^n \rightarrow r^n$ conv ?

$\therefore -1 < \frac{a}{2} \leq 1$ $\qquad -1 < r \leq 1$

$-2 < a \leq 2$

$a \in (-2, 2]$ seq conv.

No. lec 18

Recursive Sequences

Example $a_1 = 3,\ a_{n+1} = a_n + 2$

$a_1 = 3$

$a_2 = a_{1+1} = a_1 + 2 = 3 + 2 = 5$

$a_3 = a_{2+1} = a_2 + 2 = 5 + 2 = 7$

$a_4 = a_{3+1} = a_3 + 2 = 7 + 2 = 9$

Example $a_1 = 1,\ a_{n+1} = \frac{1}{2}(a_n + 3)$

① Find the first four terms?

$a_1 = 1$

$a_2 = a_{1+1} = \frac{1}{2}(a_1 + 3) = \frac{1}{2}(1+3) = 2$

$a_3 = a_{2+1} = \frac{1}{2}(a_2 + 3) = \frac{1}{2}(2+3) = 5/2$

$a_4 = \frac{1}{2}(a_3 + 3) = \frac{1}{2}(5/2 + 3) = 11/4$

$\therefore$ 1, 2, 5/2, 11/4

② Assuming that the seq converges, find the limit?

conv $\therefore$ lim exists (limit = L)

$a_n \longrightarrow L$ لما $n \longrightarrow +\infty$

$\lim_{n \to +\infty} a_{n+1} = L$

$$\rightarrow \lim_{n \to +\infty} a_{n+1} = \lim_{n \to +\infty} \frac{1}{2}(a_n + 3)$$

$$L = \frac{1}{2} \lim_{n \to +\infty} (a_n + 3)$$

$$L = \frac{1}{2}(L + 3)$$

$$2L = L + 3$$

$$L = 3$$

$$\rightarrow \text{limit} = L = 3$$

Example $a_1 = 1$, $a_2 = 1$

$a_1 = 1$

$a_2 = 1$

$a_3 = a_{1+2} = a_1 + a_2 = 2$

$a_4 = a_3 + a_2 = 1 + 2 = 3$

$a_5 = a_4 + a_3 = 2 + 3 = 5$

$a_6 = a_5 + a_4 = 5 + 3 = 8$

$\rightarrow 1, 1, 2, 3, 5, 8 \ldots$

Example Determine the seq. defined as follows is conv or div?

1. $a_1 = 1$, $a_{n+1} = 4 - a_n$

$a_1 = 1$ الحد الي قبله

$a_2 = 4 - a_1 = 3$ متذبذبة

$a_3 = 4 - a_2 = 4 - 3 = 1$ div ∴

$a_4 = 4 - a_3 = 4 - 1 = 3$ ∴ $1, 3, 1, 3, \ldots$

No. lec 18

② $a_1 = 2, \quad a_{n+1} = 4 - a_n$

$a_1 = 2$

$a_2 = 4 - a_1 = 4 - 2 = 2$

$a_3 = 4 - a_2 = 4 - 2 = 2$ $\qquad$ ∴ conv to 2

$a_4 = 4 - a_3 = 4 - 2 = 2$ $\qquad$ 2, 2, 2, 2 ...

Example Consider the seq $a_1 = \sqrt{6}$,

$a_2 = \sqrt{6 + \sqrt{6}}$, $a_3 = \sqrt{6 + \sqrt{6 + \sqrt{6}}}$

① Find a recursion formula for a_{n+1}?

$a_1 = \sqrt{6}$

$a_2 = \sqrt{6 + \sqrt{6}} = \sqrt{6 + a_1}$

$\therefore a_{n+1} = \sqrt{6 + a_n}$ اي قبلها

② Assuming that the seq conv, find lim?

conv $\xrightarrow{\therefore}$ lim seq exists

$$\left.\begin{array}{l} \lim\limits_{n \to +\infty} a_n = L \\ \\ \lim\limits_{n \to +\infty} a_{n+1} = L \end{array}\right\}$$

لانه ما بفرق الحد المليون عن المليون و واحد

$a_{n+1} = \sqrt{6 + a_n}$

$\lim\limits_{n \to +\infty} a_{n+1} = \lim\limits_{n \to +\infty} \sqrt{6 + a_n}$

$L = \sqrt{6 + L}$

$L^2 = 6 + L$

$L^2 - L - 6 = 0$

$(L+2)(L-3) = 0$

$L = -2 \qquad L = 3$ seq منرجع الى صدر

$\therefore L = 3$ $\sqrt{6}\ /\ \sqrt{6+\sqrt{6}}$ + كله

No. lec. 19

Monotonic Sequences

1. Increasing seq :

$a_1 < a_2 < a_3 < a_4 \dots a_n < a_{n+1}$

Ex: 2, 4, 6, 8, 10, ...

2. Decreasing seq :

$a_1 > a_2 > a_3 > a_4 \dots > a_n > a_{n+1}$

Ex: 1, 1/2, 1/3, 1/4, 1/5 ...

3. Not inc, Not Dec (Not monotonic seq)

Ex: $(-1)^n = \{-1, 1, -1, 1 \dots\}$ متذبذب

Seq	Difference	Ratio (terms +)	Derivative
Inc	$a_{n+1} - a_n > 0$	$\frac{a_{n+1}}{a_n} > 1$	$f'(x) > 0$
Dec	$a_{n+1} - a_n < 0$	$\frac{a_{n+1}}{a_n} < 1$	$f'(x) < 0$

Example Determine whether the seq is inc, dec or not monotonic ?

① $a_n = \frac{3}{n+5} = a_{n+1} - a_n$

$a_{n+1} = \frac{3}{n+1+5}$

$\frac{3}{n+6} - \frac{3}{n+5}$ نوحد

$= \frac{3}{n+6}$

No. lec. 19

$$= \frac{3(n+5) - 3(n+6)}{(n+6)(n+5)}$$

$$= \frac{3n + 15 - 3n - 18}{(n+6)(n+5)}$$

$$= \frac{-3 \leftarrow -}{(n+6)(n+5) \leftarrow +} < 0 \quad \therefore \text{Dec}$$

في القوة
دالـ!
نستخدم
القسمة

(2) $a_n = \dfrac{n^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \div \frac{n^n}{n!}$$

$$= \frac{(n+1)^n \cancel{(n+1)}}{\cancel{(n+1)}\, \cancel{n!}} \times \frac{\cancel{n!}}{n}$$

$$= \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n$$

$$= \left(1 + \frac{1}{n}\right)^n \leftarrow + > 1 \therefore \text{Inc}$$

(3) $a_n = \tan^{-1} n$

$f(x) = \tan^{-1} x$

$f'(x) = \dfrac{1}{1+x^2} > 0 \quad \therefore \text{Inc}$

(4) $a_n = (-2)^{n+1}$

$4, -8, 16, -32, \dots \quad \therefore$ not monotonic

No. lec 19

5) $a_n = \frac{n!}{6^n}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{6^{n+1}} \div \frac{n!}{6^n}$$

$$= \frac{(n+1)\cancel{n!}}{\cancel{6^n}\,6} \cdot \frac{\cancel{6^n}}{\cancel{n!}} = \frac{n+1}{6}$$

$$\frac{n+1}{6} > 1 \rightarrow n+1 > 6 \rightarrow n > 5 \rightarrow n \geq 6$$

$\therefore$ inc $n \geq 6$ $[6, \infty)$

6) $a_n = \frac{\ln n}{n}$

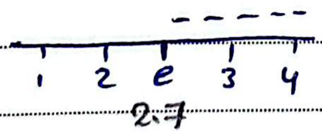
$\rightarrow$ Domain $(0, \infty)$

$$f(x) = \frac{\ln x}{x} \rightarrow f'(x) = \frac{1 - \ln x}{x^2}$$

$1 - \ln x = 0$ $\qquad$ $x^2 = 0$

$1 = \ln x$ $\qquad$ $x = 0 \notin$ Dom

$x = e \in$ Dom



$\therefore$ Dec $n \geq 3$

ماحنا دايماً ال n عدد
صحيح