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دفتر

تفاضل وتكامل 2 (الفيرست)

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No. Lec. 1

Integration by Parts:

$$\frac{d}{dx}\left(f(x)\,g(x)\right) = f(x)\,g'(x) + g(x)\,f'(x)$$

$$\cancel{\int \frac{d}{dx}}\left(f(x)\,g(x)\right) = \int f(x)\,g'(x)\,dx + \int g(x)\,f'(x)\,dx$$

موضوع القانون

$$f(x)\,g(x) = \int \underline{f(x)\,g'(x)\,dx} + \int g(x)\,f'(x)\,dx$$

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int g(x)\,f'(x)\,dx$$

$u = f(x)$, $v = g(x)$

$du = f'(x)\,dx$, $dv = g'(x)\,dx$

$$\therefore \int u\,dv = u \cdot v - \int v\,du$$

Example :

1. $\int x \sin x\,dx$

اشتقاق $u = x \xrightarrow{\ '\ } du = 1\,dx$

تكامل $dv = \sin x\,dx \xrightarrow{\int} v = -\cos x$

$-x\cos x - \int \underline{-\cos x\,dx}$ لازم يكون اسهل من التكامل الاصلي

$-x\cos x + \sin x + C$

② $\int \ln x \, dx$

$u = \ln x \xrightarrow{'} du = 1/x \, dx$

$dv = 1 \, dx \xrightarrow{\int} v = x$

$= x \ln x - \int x \cdot \frac{1}{\cancel{x}} dx$

$= x \ln x - \int 1 \, dx$

$= x \ln x - x + C$

③ $\int x^2 e^x \, dx$

$u = x^2 \xrightarrow{'} du = 2x \, dx$

$dv = e^x dx \xrightarrow{\int} v = e^x$

$= x^2 e^x - \int \overset{\text{parts}}{2 e^x x \, dx}$

$= x^2 e^x - 2 \int x e^x \, dx$

$u = x \xrightarrow{'} du = dx$

$dv = e^x dx \xrightarrow{\int} v = e^x$

$x e^x - \int e^x \, dx$

$x e^x - e^x$

No. Lec.1

$= x^2e^x - 2(xe^x - e^x) + c$

$= x^2e^x - 2xe^x + 2e^x + c$

Tabular method

for repeated Integration by parts.

Example

① $\int x^2 e^x \, dx$

$\downarrow '$

u		dv
x^2		e^x
$2x$	$\searrow +$	e^x
2	$\searrow -$	e^x
0	$\searrow +$	e^x

$\uparrow \int$

$= xe^x - 2xe^x + 2e^x + c$

* When do we use tabular method ?

$\int \text{Polynomial} \cdot e^{ax+b}$

$\downarrow$ u

or $\cos(ax+b)$

or $\sin(ax+b)$

or $\sqrt{ax+b}$

$\downarrow$ dv

② $\int \underset{}{\overset{\text{poly}}{\underline{x^4}}}\ \overset{\text{sin}}{\underline{\sin}}\, 2x \; dx$ ∴ tabular method.

$$= -x^4 \frac{\cos 2x}{2} + \cancel{4}x^3 \frac{\sin 2x}{\cancel{4}}$$

$$+ \frac{\overset{3}{\cancel{12}} x^2 \cos 2x}{\underset{2}{\cancel{8}}} - \frac{\overset{3}{\cancel{24}} x \sin 2x}{\underset{2}{\cancel{16}}}$$

$$- \frac{\overset{3}{\cancel{24}} \cos 2x}{\underset{4}{\cancel{32}}} + c$$

u	dv
x^4	$\sin 2x$
$4x^3$	$-\cos 2x/4$
$12x^2$	$-\sin 2x/4$
$24x$	$\cos 2x/8$
24	$\sin 2x/16$
0	$-\cos 2x/32$

$$= \left(x^3 - \frac{3}{2}x\right)\sin 2x - \left(\frac{x^4}{2} - \frac{3x^2}{2} + \frac{3}{4}\right)$$

$$\cos 2x \quad + c$$

③ $\int e^x \sin x \; dx$

$$u = e^x \xrightarrow{\;'\;} du = e^x\,dx$$

$$dv = \sin x \xrightarrow{\int} v = -\cos x$$

$$\int e^x \sin x = -e^x \cos x - \int (-\cos x)\, e^x\, dx$$

$$\int e^x \sin x = -e^x \cos x + \underbrace{\int e^x \cos x \; dx}_{\text{by parts}} \rightarrow$$

No. Lec.1

$$\int e^x \sin x = -e^x \cos x + \underbrace{\int e^x \cos x \, dx}_{\text{by parts}}$$

$u = e^x \longrightarrow du = e^x \, dx$

$dv = \cos x \longrightarrow v = \sin x$

$e^x \sin x - \int e^x \sin x \, dx$

$$\int e^x \sin x = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

$$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x$$

$$\int e^x \sin x \, dx = \frac{1}{2} (-e^x \cos x + e^x \sin x) + c$$

FIVE APPLE

No. Lec. 1

Exercises

① $\int_0^{\pi} e^{\cos x} \sin 2x \, dx$ بجرب بالتعويض

$y = \cos x \longrightarrow dy = -\sin x \, dx$

$dx = \dfrac{dy}{-\sin x}$

$\therefore \int_1^{-1} e^y \sin 2x \dfrac{dy}{-\sin x}$ $x = 0 \longrightarrow y = 1$

$x = \pi \longrightarrow y = -1$

$\int_1^{-1} e^y \, 2\cancel{\sin x} \cos x \cdot \dfrac{dy}{-\cancel{\sin x}}$

$-2 \int_1^{-1} e^y \cos x \, dy$

$-2 \int_1^{-1} e^y \, y \, dy \longrightarrow$ by parts

$u = y \xrightarrow{'} du = dy$

$dv = e^y \xrightarrow{\int} v = e^y$

$y e^y - \int e^y dy$

$y e^y - e^y$

$-2 (y e^y - e^y) \Big|_1^{-1}$

$-2 \left((-e^{-1} - e^{-1}) - (\cancel{e^1 - e^1}) \right) = \dfrac{4}{e}$

دطلي ← اجزاء

No. Lec. 1

② $\int \frac{xe^{2x}}{(1+2x)^2} dx$

$u = xe^{2x} \xrightarrow{'} du = x(2e^{2x}) + e^{2x}$

$dv = (1+2x)^{-2} \longrightarrow v = \frac{(1+2x)^{-1}}{(-1)(2)}$

$(xe^{2x})\frac{(1+2x)^{-1}}{-2} \ominus \int \frac{(1+2x)^{-1}}{\ominus 2} \cdot 2xe^{2x} + e^{2x} dx$

$\frac{-xe^{2x}(1+2x)^{-1}}{2} + \frac{1}{2}\int \cancel{(1+2x)^{-1}}\, e^{2x}\cancel{(2x+1)}$

$\frac{-xe^{2x}(1+2x)^{-1}}{2} + \frac{1}{2}\int e^{2x} dx$

$\frac{-xe^{2x}(1+2x)^{-1}}{2} + \frac{1}{2}\,\frac{e^{2x}}{2} + c$

③ $\int x^4 (\ln x)^2 dx$

$y = \ln x$

$dy = \frac{1}{x} dx$

$x\,dy = dx$

$\int x^4\, y^2\, x\,dy$

$\int x^5\, y^2\, dy$ الى الفرض

$e^y = e^{\ln x}$

$e^y = x$

$\int (e^y)^5\, y^2\, dy$

$\int e^{5y} \underbrace{y^2}_{\text{Poly}} dy$ ∴ repeated

No. Lec1

u	dv
y^2	e^{5y}
$2y$	$+\ e^{5y}/5$
2	$-\ e^{5y}/25$
0	$+\ e^{5y}/125$

$$= y^2 \frac{e^{5y}}{5} - 2y \frac{e^{5y}}{25} + \frac{2e^{5y}}{125} + C$$

$$= e^{5y}\left(\frac{y^2}{5} - \frac{2y}{25} + \frac{2}{125} \right) + C$$

$$= x^5\left(\frac{(\ln x)^2}{5} - \frac{2(\ln x)}{25} + \frac{2}{125} \right) + C$$

No. Lec. 2

Integration power of sine and cosin

① • $\int \sin x \, dx = -\cos x + C$ (x حفظية)

• $\int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x) \, dx$

$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C$

★ $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

• $\int \sin^{\overset{\text{odd}}{③}} x \, dx = \int \sin x \cdot \sin^2 x \, dx$

$= \int \sin x \, (1 - \cos^2 x) \, dx$

بالتعويض:

$u = \cos x$

$dx = \frac{du}{-\sin x}$

$= \int \cancel{\sin x} \, (1 - u^2) \cdot \frac{du}{\cancel{-\sin x}}$

$= \int u^2 - 1 \, du$

$= \frac{u^3}{3} - u + C$

$= \frac{\cos^3 x}{3} - \cos x + C$

★ $\sin^2 x + \cos^2 x = 1$

No. Lec. 2

- $\int \sin^{\overset{\text{even}}{④}} x \, dx = \int (\sin^2 x)^2 \, dx$

$= \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 dx$

$= \frac{1}{4} \int 1 - 2\cos 2x + \cos^2 2x \, dx$

$= \frac{1}{4} \int 1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)$

$= \frac{1}{4} \int \frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x \, dx$

$= \frac{1}{4}\left(\frac{3}{2}x - \cancel{2}\frac{\sin 2x}{\cancel{2}} + \frac{1}{2}\,\frac{\sin 4x}{4}\right) + c$

$= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{\sin 4x}{8}\right) + c$

★ $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

② • $\int \cos x \, dx = \sin x + c$ (خطية x)

- $\int \cos^2 x \, dx = \int \frac{1}{2}(1 + \cos 2x) \, dx$

$= \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + c$

No. Lec.2

odd

$$\bullet \int \cos^{\textcircled{3}} x \, dx = \int \cos x \cdot \cos^2 x \, dx$$

$$= \int \cos x (1 - \sin^2 x) \, dx$$

$u = \sin x$ بالتعويض :

$du = \cos x \, dx$

$dx = \frac{du}{\cos x}$

$$= \int \cancel{\cos x} (1 - u^2) \frac{du}{\cancel{\cos x}}$$

$$= u - \frac{u^3}{3} + c$$

$$= \sin x - \frac{\sin^3 x}{3} + c$$

even

$$\bullet \int \cos^{\textcircled{4}} x \, dx = \int (\cos^2 x)^2 \, dx$$

$$= \int \left(\frac{1}{2} (1 + \cos 2x) \right)^2 dx$$

$$= \frac{1}{4} \int 1 + 2\cos 2x + \cos^2 2x \, dx$$

$$= \frac{1}{4} \int 1 + 2\cos 2x + \frac{1}{2} (1 + \cos 4x) \, dx$$

$$= \frac{1}{4} \int \frac{3}{2} + 2\cos 2x + \frac{1}{2} \cos 4x \, dx$$

$$= \frac{1}{4} \left(\frac{3}{2} x + \sin 2x + \frac{1}{8} \sin 4x \right) + c$$

- $\int \sin^5 x \, dx$

$= \int \sin x \cdot \sin^4 x \, dx$

$= \int \sin x \, (\sin^2 x)^2 \, dx$

$= \int \sin x \, (1 - \cos^2 x)^2 \, dx \ \ldots$

- $\int \cos^6 x \, dx$

$= \int (\cos^2 x)^3 \, dx$

$= \int \left(\frac{1}{2} (1 + \cos 2x) \right)^3 dx \ \ldots$

Reduction Formulas

- $\int \sin^n x \, dx =$ (زاوية x فردية)

$$= -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

- $\int \cos^n x \, dx$ (زاوية x فردية)

$$= \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

Example

$\int \sin^5 x \, dx \qquad n=5$

$$= -\frac{1}{5} \sin^4 x \cos x + \frac{4}{5} \overset{R.F \quad n=3}{\underbrace{\int \sin^3 x \, dx}}$$

$$= -\frac{1}{5} \sin^4 x \cos x + \frac{4}{5}\left(-\frac{1}{3} \sin^2 x \cos x + \frac{2}{3} \int \sin x \, dx \right)$$

$$= -\frac{1}{5} \sin^4 x \cos x + \frac{4}{5}\left(-\frac{1}{3} \sin^2 x \cos x - \frac{2}{3} \cos x \right)$$

$$= -\frac{1}{5} \sin^4 x \cos x + \frac{-4}{15} \sin^2 x \cos x - \frac{8}{15} \cos x + C$$

Example

$\int \cos^6 \underline{\underline{2x}}\, dx$ $\qquad u = 2x$

$\qquad du = 2dx$

$\int \cos^6 u \, \frac{du}{2}$

$\frac{1}{2} \int \cos^6 u \, du \qquad n = 6$

R.F $\quad n = 4$

$= \frac{1}{2}\left(\frac{1}{6} \cos^5 u \sin + \frac{5}{6} \underline{\int \cos^4 u \, du} \right)$

$= \frac{1}{2}\left(\frac{1}{6} \cos^5 u \sin u + \frac{5}{6}\left(\frac{1}{4} \cos^3 u \sin u + \right.\right.$

$\left.\left. \frac{3}{4} \int \cos^2 u \, du \right)\right)$

$= \frac{1}{2}\left(\frac{1}{6} \cos^5 u \sin u + \frac{5}{6}\left(\frac{1}{4} \cos^3 u \sin u + \right.\right.$

$\frac{3}{4} \int \underline{\frac{1}{2}(1 + \cos 2u) du} \longrightarrow \frac{1}{2}\left(u + \frac{\sin 2u}{2} \right)$

$= \frac{1}{2}\left(\frac{1}{6} \cos^5 u \sin u + \frac{5}{24} \cos^3 u \sin u + \right.$

$\left. \frac{15}{24} \left(\frac{1}{2}\left(u + \frac{\sin 2u}{2} \right)\right)\right) + c$

$= \frac{1}{2}\left(\frac{1}{6} \cos^5 2x \sin 2x + \frac{5}{24} \cos^3 2x \sin 2x + \right.$

$\left. \frac{15}{48} (2x) + \frac{15}{48} \sin 4x \right) + c$

No. Lec.2

note :

$$\int \sin^{\overset{\text{even}}{n}} x \xrightarrow{R.F} \int \sin^2 x$$

$$\int \sin^{\overset{\text{odd}}{n}} x \xrightarrow{R.F} \int \sin x$$

$$\int \cos^{\overset{\text{even}}{n}} x \xrightarrow{R.F} \int \cos^2 x$$

$$\int \cos^{\overset{\text{odd}}{n}} x \xrightarrow{R.F} \int \cos x$$

- $\int \tan^n x \, dx$

$$= \frac{\tan^{\boxed{n-1}} x}{\boxed{n-1}} - \int \tan^{\boxed{n-1}} x \, dx$$

- $\int \sec^n x \, dx$

$$= \frac{\sec^{\boxed{n-2}} x \tan x}{\boxed{n-1}} + \frac{\boxed{n-2}}{\boxed{n-1}} \int \sec^{\boxed{n-2}} x \, dx$$

No. Lec. 2

Integration product of sines and cosines

if m and n positive integers, then the integral :

$$\int \sin^m \underline{x} \cdot \cos^n \underline{x} \, dx$$

Example

$$\int \sin^4 x \, \underline{\cos^5} x \, dx$$ ١. البحث عن القوة الفردية

$$\int \sin^4 x \cos x \cos^4 x \, dx$$ 2. نفكها ثم متطابقة

$$\int \sin^4 x \, \cos x \, (1-\sin^2 x)^2 dx$$

$$u = \sin x \longrightarrow \frac{du}{\cos x} = dx$$

$$\int u^4 \cos x \, (1-u^2)^2 \frac{du}{\cos x}$$

$$\int u^4 (1-u^2)^2 du$$

$$\int u^4 (1-2u^2+u^4) \, du$$

$$\int u^4 - 2u^6 + u^8 \, du$$

$$= \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + C$$

$$= \frac{(\sin x)^5}{5} - \frac{2(\sin x)^7}{7} + \frac{(\sin x)^9}{9} + C$$

Example

$$\int \sin^3 x \cos^6 x \, dx$$ ١. ابحث عن القوة الفردية

$$\int \sin x \sin^2 x \cos^6 x \, dx$$ 2. نفكها ثم متطابقة

$$\int \sin x (1-\cos^2 x) \cos^6 x \, dx$$

$$u = \cos x \longrightarrow \frac{du}{-\sin x} = dx$$

$$\int \cancel{\sin x}(1-u^2)\, u^6 \frac{du}{-\cancel{\sin x}}$$

$$\int -(1-u^2)u^6 \, du$$

$$-\int u^6 - u^8 \, du$$

$$= \int u^8 - u^6$$

$$= \frac{u^9}{9} - \frac{u^7}{7} + C$$

$$= \frac{\cos^9 x}{9} - \frac{\cos^7 x}{7} + C$$

No. Lec.2

Example

$\int \sin^3 x \cos^5 x \, dx$ كلاهما فردي ∴ نفك القوة الاقل

$\int \sin x \sin^2 x \cos^5 x \, dx$

$\int \sin x \, (1 - \cos^2 x) \cos^5 x \, dx$

$u = \cos x \longrightarrow \frac{du}{-\sin x} = dx$

$\int \cancel{\sin x} (1 - u^2) \, u^5 \cdot \frac{du}{-\cancel{\sin x}}$

$\int (u^2 - 1) \, u^5 \, du$

$\int u^7 - u^5 \, du$

$= \frac{u^8}{8} - \frac{u^6}{6} + c$

$= \frac{\cos^8 x}{8} - \frac{\cos^6 x}{6} + c$

Example

$\int \sin^4 x \cos^4 x \, dx$ كلاهما زوجي ∴ نستخدم المتطابقات

$\int \left(\frac{1}{2}(1-\cos 2x)\right)^2 \left(\frac{1}{2}(1+\cos 2x)\right)^2 dx$

$\frac{1}{16} \int \left((1-\cos 2x)(1+\cos 2x)\right)^2 dx$

$\frac{1}{16} \int \left(1-\cos^2 2x\right)^2 dx$ نفك أو متطابقة

$\frac{1}{16} \int \left(\sin^2 2x\right)^2 dx$ (الحالات)

$\frac{1}{16} \int \sin^4 2x \, dx$ R. F

$u = 2x \longrightarrow dx = \frac{du}{2}$

$\frac{1}{16} \int \sin^4 u \, \frac{du}{2}$

$\frac{1}{32} \int \sin^4 u \, du$

$\frac{1}{32} \left(\frac{-1}{4} \sin^3 u \cos u + \frac{3}{4} \int \sin^2 u \, du \right.$

$\frac{1}{32} \left(-\frac{1}{4} \sin^3 u \cos u + \frac{3}{4} \left(\frac{1}{2} \left(u - \frac{\sin 2u}{2} \right) \right) \right)$

$-\frac{1}{128} \sin^3 u \cos u + \frac{3}{256} u - \frac{3 \sin 2u}{512} + c$

$$= \frac{-1}{128} \sin^3 2x \cos 2x + \frac{3}{256}(2x) - \frac{3\sin 4x}{512}$$

* ?

$$= \frac{3}{128} x - \frac{1}{128} \sin 4x + \frac{1}{1024} \sin 8x + C$$

sol. 2

$$\int \sin^4 x \cos^4 x \, dx$$

$$\int (\sin x \cos x)^4 \, dx$$

$$\int \left(\frac{1}{2} \sin 2x\right)^4 dx$$

$$\frac{1}{16} \int (\sin^4 2x) \, dx \quad (R.F)$$

No. Lec. 2

$$= \frac{-1}{128} \sin^3 2x \cos 2x + \frac{3}{256}(2x) - \frac{3\sin 4x}{512}$$

$$= \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C$$

sol. 2

$$\int \sin^4 x \cos^4 x \, dx$$

$$\int (\sin x \cos x)^4 \, dx$$

$$\int (\tfrac{1}{2} \sin 2x)^4 \, dx$$

$$\frac{1}{16}\int (\sin^4 2x) \, dx \quad (R.F)$$

- $\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$
- $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$
- $\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$

Example

① $\int \sin 4x \cos 5x \, dx$

$= \int \frac{1}{2}\left[\sin(4x-5x) + \sin(4x+5x)\right] dx$

$= \frac{1}{2} \int \sin(-x) + \sin 9x \quad dx$

$= \frac{1}{2}\left(\frac{\cancel{-}\cos(-x)}{\cancel{-}1} + \frac{-\cos 9x}{9} \right) + c$

② $\int x \sin 3x \sin 5x \, dx$

$u = x \longrightarrow du = dx$

$dv = \sin 3x \sin 5x \rightarrow v = \int \sin 3x \sin 5x$

$v = \int \frac{1}{2}\left(\cos(\cancel{-}2x) - \cos(8x)\right) dx$

$v = \frac{1}{2}\left(\frac{\sin 2x}{2} - \frac{\sin 8x}{8} \right)$

No. lec. 2

$$U = \frac{\sin 2x}{4} - \frac{\sin 8x}{16}$$

$$= x\left(\frac{\sin 2x}{4} - \frac{\sin 8x}{16}\right) - \int \frac{\sin 2x}{4} - \frac{\sin 8x}{16}$$

$$= x\left(\frac{\sin 2x}{4} - \frac{\sin 8x}{16}\right) - \left(\frac{-\cos 2x}{8} + \frac{\cos 8x}{128}\right)$$

$$= \frac{x\sin 2x}{4} - \frac{x\sin 8x}{16} + \frac{\cos \;\; x}{8} - \frac{\cos 8x}{128} + C$$

Integration products of tangents and secants

* $\sec^2 x = 1 + \tan^2 x$
* $\tan^2 x = \sec^2 x - 1$

① $\int \tan x \, dx$

$= \int \frac{\sin x}{\cos x} \, dx$ $\qquad u = \cos x$

$\qquad dx = \frac{du}{-\sin x}$

$= \int \frac{\cancel{\sin x}}{u} \frac{du}{-\cancel{\sin x}}$

$= \int -\frac{1}{u} \, du = -\ln|u| + c$

$= -\ln|\cos x| + c = \ln|\cos x|^{-1} + c$

$= \ln|\sec x| + c$

② $\int \tan^2 x \, dx$

$= \int \sec^2 x - 1 \, dx$

$= \tan x - x + c$

No. lec. 2

③ $\int \sec x \, dx$

$$= \int \sec x \left(\overset{1}{\frac{\sec x + \tan x}{\sec x + \tan x}} \right) dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

تعويض

المقام مشتقة البسط

$$= \ln | \sec x \tan x | + C$$

• $\int \tan^m x \sec^n x \, dx$

if the power of secant is even :

Example

$$\int \tan^4 x \sec^{\textcircled{6}} x \, dx$$

$$\int \tan^4 x \sec^2 x \underline{\sec^4 x} \, dx$$ متطابقة

$$\int \tan^4 x \sec^2 x (1+\tan^2 x)^2 \, dx$$

$u = \tan x$ بالتعويض

$dx = \frac{du}{\sec^2 x}$

$$\int u^4 \cancel{\sec^2 x} (1+u^2)^2 \frac{du}{\cancel{\sec^2 x}}$$

$= \int u^4 (1+u^2)^2 \, du$

$= \int u^4 (1 + 2u^2 + u^4) \, du$

$= \int u^4 + 2u^6 + u^8 \, du$

$= \frac{u^5}{5} + 2\frac{u^7}{7} + \frac{u^9}{9} + C$

$= \frac{\tan^5 x}{5} + \frac{2\tan^7 x}{7} + \frac{\tan^9 x}{9} + C$

- $\int \tan^m x \sec^n x \, dx$

if the power of tangent is odd :

Example

$\int \tan^5 \sec^7 x \, dx$ نفك وحدة من ال tan

و وحدة من ال sec

$\int \tan x \sec x \underbrace{\tan^4 x} \sec^6 x \, dx$ (متطابقة)

$\int \tan x \sec x \, (\sec^2 x - 1)^2 \sec^6 x \, dx$

$u = \sec x$

$dx = \frac{du}{\sec x \tan x}$

No. Lec. 2

$$= \int \cancel{\tan x \sec x} \, (u^2-1)^2 \, u^6 \; \frac{du}{\cancel{\sec x \tan x}}$$

$$= \int u^{10} - 2u^8 + u^6 \; du$$

$$= \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} + c$$

$$= \frac{\sec^{11} x}{11} - \frac{2}{9} \sec^9 x + \frac{\sec^7 x}{7} + c$$

No. lec. 2

Integration powers of tangent and secant

Example

$$\int \tan^3 x \, dx$$

$$\int \tan x \cdot \tan^2 x \, dx$$ متطابقة

$$\int \tan x \, (\sec^2 x - 1) \, dx$$

$$\int \tan x \sec^2 x - \tan x \, dx$$

$$\int \underbrace{\tan x \sec^2 x \, dx}_{\downarrow} - \int \tan x \, dx$$

$u = \tan x$ $\qquad - \ln|\sec x|$

$$dx = \frac{du}{\sec^2 x}$$

$$\int u \cdot \cancel{\sec^2 x} \cdot \frac{du}{\cancel{\sec^2 x}}$$

$$= \frac{u^2}{2} - \ln|\sec x| + c$$

$$= \frac{\tan^2 x}{2} - \ln|\sec x| + c$$

No. Lec. 2

Example

$$\int \tan^4 x \, dx$$

$$\int \tan^2 x \cdot \tan^2 x \, dx$$

$$\int \tan^2 x \, (\sec^2 x - 1) \, dx$$

$$\int \tan^2 x \sec^2 x - \tan^2 x \, dx$$

$$\int \underbrace{\tan^2 x \sec^2 x \, dx}_{\downarrow} - \int \underbrace{\tan^2 x \, dx}_{\downarrow}$$

$u = \tan x$ $\quad\quad$ $- \int \sec^2 x - 1 \, dx$

$dx = \dfrac{du}{\sec^2 x}$ $\quad\quad$ $-(\tan x - x)$

$$\int u^2 \cancel{\sec^2 x} \, \frac{du}{\cancel{\sec^2 x}}$$

$$= \frac{u^3}{3}$$

$$= \frac{\tan^3 x}{3} - \tan x + x + c$$

Example

$\int \sec^3 x \, dx$

$\int \sec x . \sec^2 x \, dx$ (by parts)

$u = \sec x \longrightarrow du = \sec x \tan x$

$dv = \sec^2 x \, dx \longrightarrow v = \tan x$

$= \sec x \tan x - \int \sec x \tan^2 x \, dx$

$\int \sec^3 x = \sec x \tan x - \int \sec x \, (\sec^2 x - 1)$

$\int \sec^3 x = \sec x \tan x - \int \sec^3 x - \sec x$

$\int \sec^3 x = \sec x \tan x - \int \sec^3 x + \int \sec x$

$2 \int \sec^3 x \, dx = \sec x \tan x + \ln |\sec x + \tan|$

$\int \sec^3 x = \frac{1}{2} (\sec x \tan x + \ln|\sec x + \tan x|$

$+ c$

Example

$$\int \sec^4 x \, dx$$

$$\int \sec^2 x \sec^2 x \, dx$$

$$\int \sec^2 x \left(\tan^2 x + 1 \right) dx$$

$$\int \underbrace{\sec^2 x \tan^2 x}_{\downarrow} + \int \underbrace{\sec^2 x}_{\downarrow} \, dx$$

$u = \tan x$ $\qquad$ $+ \tan x$

$$dx = \frac{du}{\sec^2 x}$$

$$\int \cancel{\sec^2 x} \, u^2 \, \frac{du}{\cancel{\sec^2 x}}$$

$$= \frac{u^3}{3}$$

$$= \frac{\tan^3 x}{3} + \tan x + C$$

No. Lec. 3

Trigonometric Substitution

$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$0 \leqslant \theta \leqslant \frac{\pi}{2}$ or $\pi \leqslant \theta \leqslant \frac{3\pi}{2}$

Example

$$\int \frac{\sqrt{9-x^2}}{x^2}\, dx$$

$\sqrt{9-x^2} = \sqrt{a^2-x^2}$

$a = 3$

$x = a\sin\theta$

$x = 3\sin\theta$

$dx = 3\cos\theta$

$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)}$

$= 3\sqrt{\cos^2\theta} = 3|\cos\theta| = 3\cos\theta$

$\hookrightarrow \int \frac{\cancel{3}\cos\theta}{\cancel{9}\sin^2\theta} * \cancel{3}\cos\theta\, d\theta$

$= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta = \int \cot^2\theta\, d\theta$

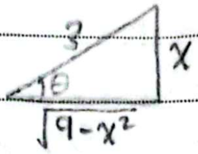
$= \int (\csc^2\theta - 1)\, d\theta$

$= -\cot\theta - \theta + C$ (لسه ما رجعنا لـ x)

No. Lec.3

$x = 3\sin\theta$

$\frac{x}{3} = \sin\theta$



$\hookrightarrow -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + c$

Example

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$$

العوض

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

نبسط الجذر

$\sqrt{x^2+4} = \sqrt{4\tan^2\theta+4} = \sqrt{4(\tan^2\theta+1)}$

$= 2\sqrt{\sec^2\theta} = 2|\sec\theta| = 2\sec\theta$

نرجع

$\hookrightarrow \int \frac{\cancel{2}\sec^{\cancel{2}}\theta\,d\theta}{4\tan^2\theta \times \cancel{2\sec\theta}} = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$

$= \frac{1}{4}\int \frac{1}{\cancel{\cos\theta}} \cdot \frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$

$= \frac{1}{4}\int \frac{\cos\theta}{u^2} \cdot \frac{du}{\cos\theta}$

$u = \sin\theta$

$\frac{du}{\cos\theta} = d\theta$

$= \frac{1}{4}\int u^{-2}\,du$

$= \frac{1}{4}\,\frac{u^{-1}}{-1} + c$

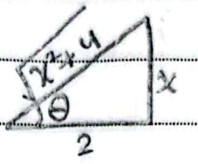
No. Lec.3

$$= -\frac{1}{4u} + c$$

$$= -\frac{1}{4\sin\theta} + c = -\frac{1}{4}\csc\theta + c$$

$$x = 2\tan\theta$$

$$\frac{x}{2} = \tan\theta$$



(نعوض بدلالة x)
مرجع للمرجنا

$$\hookrightarrow -\frac{1}{4}\,\frac{\sqrt{x^2+4}}{x} + c$$

Example

$$\int \frac{\sqrt{x^2-9}}{x}\,dx$$

$$\sqrt{x^2-9} = \sqrt{x^2-a^2}$$

$$x = a\sec\theta$$

$$x = 3\sec\theta$$

$$dx = 3\sec\theta\tan\theta\,d\theta$$

$$\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9} = \sqrt{9(\sec^2\theta - 1)}$$

$$= 3\sqrt{\tan^2\theta} = 3|\tan\theta| = 3\tan\theta$$

$$\hookrightarrow \int \frac{3\tan\theta}{\cancel{3\sec\theta}} * \cancel{3\sec\theta}\tan\theta\,d\theta$$

$$= 3\int \tan^2\theta\,d\theta = 3\int \sec^2\theta - 1\,d\theta$$

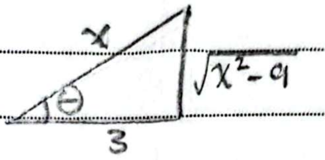
$$= 3(\tan\theta - \theta) + c$$

$$= 3\tan\theta - 3\theta + c$$

No. Lec.3

$x = 3\sec\theta$

$\frac{x}{3} = \sec\theta$



↳ $\frac{\cancel{3}\sqrt{x^2-9}}{\cancel{3}} - 3\sec^{-1}\frac{x}{3} + c$

$\sqrt{x^2-9} - 3\sec^{-1}\frac{x}{3} + c$

No. Lec. 3

Trigonometric Substitution

Example

$\int e^x \sqrt{1-e^{2x}}\ dx$

$u = e^x$

$\frac{du}{e^x} = dx$

$\int u \sqrt{1-u^2} \cdot \frac{du}{u}$

$\int \sqrt{1-u^2}\ du$

$x = a\sin\theta$

$u = \sin\theta$

$du = \cos\theta\ d\theta$

$\hookrightarrow \int \sqrt{1-\sin^2\theta}\ \cos\theta\ d\theta$

$\int \sqrt{\cos^2\theta} \cdot \cos\theta\ d\theta$

$\int \cos\theta \cdot \cos\theta\ d\theta$

$\int \cos^2\theta\ d\theta$

$\int \frac{1}{2}(1+\cos 2\theta)\ d\theta$

$\frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + c$

Trigonometric Substitution

Example

$\int e^x \sqrt{1-e^{2x}}\ dx$

$u = e^x$

$\frac{du}{e^x} = dx$

$\int u\sqrt{1-u^2} \cdot \frac{du}{u}$

$\int \sqrt{1-u^2}\ du$

$x = a\sin\theta$

$u = \sin\theta$

$du = \cos\theta\ d\theta$

$\hookrightarrow \int \sqrt{1-\sin^2\theta}\ \cos\theta\ d\theta$

$\int \sqrt{\cos^2\theta} \cdot \cos\theta\ d\theta$

$\int \cos\theta \cdot \cos\theta\ d\theta$

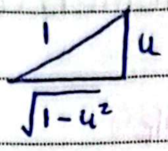
$\int \cos^2\theta\ d\theta$

$\int \frac{1}{2}(1+\cos 2\theta)\ d\theta$

$\frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + c$

No. Lec.3

$$\frac{u}{1} = \sin\theta$$



$$\hookrightarrow \frac{1}{2}\theta + \frac{1}{2}\,\frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}} + C$$

$$\hookrightarrow \frac{1}{2}\sin^{-1}u + \frac{1}{2}\cdot u\cdot\sqrt{1-u^2} + C$$

$$\hookrightarrow \frac{1}{2}\sin^{-1}(e^x) + \frac{e^x}{2}\sqrt{1-e^{2x}} + C$$

Example

$$\int \frac{x}{\sqrt{3-2x-x^2}}\,dx$$

الاقتران في المقام لا ينضبط
على اي مرض لذلك نلجأ إلى
اكمال المربع .

$3-2x-x^2$

$-(x^2+2x-3)$ نضيف ونطرح $\left(\frac{b}{2}\right)^2$

$-(\underline{x^2+2x+1}-1-3)$

$-((x+1)^2-4)$

$4-(x+1)^2$

$$\hookrightarrow \int \frac{x}{\sqrt{4-(x+1)^2}}\,dx$$

$u = x+1$

$du = dx$

$x = u-1$

$$\hookrightarrow \int \frac{u-1}{\underbrace{\sqrt{4-u^2}}}\,du$$

sin العزم

No. Lec. 3

$u = 2\sin\theta$

$du = 2\cos\theta \, d\theta$

$\sqrt{4-u^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4(1-\sin^2\theta)}$

$= 2\sqrt{1-\sin^2\theta} = 2 \cdot \cos\theta$

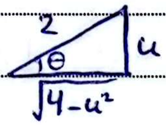
↳ $\int \frac{2\sin\theta - 1}{2\cos\theta} \cdot 2\cos\theta \, d\theta$

↳ $\int 2\sin\theta - 1 \, d\theta$

$= -2\cos\theta - \theta + c$

$u = 2\sin\theta$

$\frac{u}{2} = \sin\theta$



$= -\cancel{2}\frac{\sqrt{4-u^2}}{\cancel{2}} - \sin^{-1}(\frac{u}{2}) + c$

$= -\sqrt{4-(x+1)^2} - \sin^{-1}(\frac{x+1}{2}) + c$

No. Lec 3

Example

$$\int \frac{dx}{x^4+2x^2+1}$$

* يمكن استخدام طريقة Tri. Sub في الاسئلة التي لا تحتوي جذراً .

$$\int \frac{dx}{(x^2+1)^2}$$

$x = a\tan\theta$

$x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

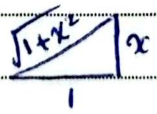
$$\hookrightarrow \int \frac{\sec^2\theta \, d\theta}{(\tan^2\theta+1)^2} = \int \frac{\sec^2\theta \, d\theta}{(\sec^2\theta)^2}$$

$$= \int \frac{1}{\sec^2\theta} d\theta = \int \cos^2\theta \, d\theta$$

$$= \frac{1}{2} \int 1+\cos 2\theta \, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$\hookrightarrow \frac{x}{1} = \tan\theta$



$$\hookrightarrow \frac{1}{2}\left(\tan^{-1}x + \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}}\right) + C$$

$$= \frac{1}{2}\left(\tan^{-1}x + \frac{x}{1+x^2}\right) + C$$

No. Lec. 3

Example

$$\int \frac{dx}{(1-x^2)^{3/2}}$$

$x = \sin\theta$

$dx = \cos\theta \; d\theta$

$\hookrightarrow \int \frac{\cos\theta \; d\theta}{(1-\sin^2(\theta))^{3/2}} = \int \frac{\cos\theta}{(\cos^{\cancel{2}}\theta)^{3/\cancel{2}}} \, d\theta$

$\hookrightarrow -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

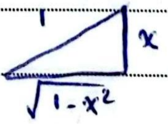
$= \int \frac{\cos\theta}{\cos^3\theta} \, d\theta = \int \frac{1}{\cos^2(\theta)} \, d\theta$

$= \int \sec^2\theta \; d\theta$

$= \tan\theta + c$

$\hookrightarrow x = \sin\theta$

$\frac{x}{1} = \sin\theta$



$\hookrightarrow \frac{x}{\sqrt{1-x^2}} + c$

No. Lec. 4

Integration of Rational (النسبية) functions by partial fractions الكسور الجزئية

Example

$$\int \frac{3x+3}{x^2+x-2}\,dx \qquad \begin{aligned} &x^2+x-2\\ &=(x-1)(x+2)\end{aligned}$$

$$= \int \frac{2}{x-1} + \frac{1}{x+2}\,dx$$

$$= 2\ln|x-1| + \ln|x+2| + c$$

method of partial fractions

when $f(x) = \frac{P(x)}{Q(x)}$:

1. $\deg(P) \geq \deg(Q)$

$$Q(x)\overline{)P(x)} = \overset{\text{الناتج}}{S(x)} + \frac{R(x)}{Q(x)} \quad \text{الباقي}$$

then $\deg(R(x)) < \deg(Q(x))$
then partial fractions

2. $\deg(P) < \deg(Q)$
then partical fraction.

No. Lec. 4

Example of (1)

$$\int \frac{x^3+x}{x-1}\,dx$$

$$= \int x^2+x+2+\frac{2}{x-1}\,dx$$

$$= \frac{x^3}{3}+\frac{x^2}{2}+2x+2\ln|x-1|$$

$$\begin{array}{r|l} & x^2+x+2 \\ \hline x-1 & \cancel{x^3}+x \\ & -\cancel{x^3}+x^2 \\ \hline & \cancel{x^2}+x \\ & -\cancel{x^2}-x \\ \hline & 2\cancel{x} \\ & -2\cancel{x}+2 \\ \hline & 2 \end{array}$$

* partical friction

factor the denominator $Q(x)$ حلل المقام

$(ax+b)$ or (ax^2+bx+c)

↳ liner ↳ irr quadratic $(\Delta<0)$

4 cases according to the denominator factors :

Case 1

liner factors

$$\frac{1}{x(x-1)(x+5)} = \frac{\overset{\text{const.}}{A}}{x} + \frac{B}{x-1} + \frac{\overset{\text{const.}}{C}}{x+5}$$

case 2

repeated liner factor :

- $$\frac{1}{(x-3)^2(2x+1)^3} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{c}{2x+1} + \frac{D}{(2x+1)^2} + \frac{E}{(2x+1)^3}$$

- $$\frac{1}{x^2(x+5)^4} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+5} + \frac{D}{(x+5)^2} + \frac{E}{(x+5)^3} + \frac{H}{(x+5)^4}$$

case3

irreducible quadratic factors :

$$\frac{x}{(x^2+1)(x^2+x+1)} = \frac{Ax+B}{x^2+1} + \frac{cx+D}{x^2+x+1}$$

بنواتج التحليل لما له المقام تربيعي :. البسط خطي.

case 4

repeated irr quadratic factor :

$$\frac{1}{(x^2+4)^2(x^2+1)^3} = \frac{A_1x+B_1}{x^2+4} + \frac{A_2x+B_2}{(x^2+4)^2} + \frac{A_3x+B_3}{(x^2+4)} + \frac{A_4x+B_4}{(x^2+4)^2} + \frac{A_5x+B_5}{(x^2+4)^3}$$

- وبعد تحديد الحالة نجد الثوابت - A ,B

No. Lec. 4

Example

write out the form of the partial fraction decomposition of the function :

1. $\dfrac{1-3x^{\textcircled{4}}}{(x-2)^{\textcircled{1}}(x^{\textcircled{2}}+1)^{\textcircled{2}}}$ درجة البسط 4 < درجة المقام 5

$$= \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+F}{(x^2+1)^2}$$

2. $\dfrac{x^4-2x^3+x^2+2x-1}{x^2-2x+1}$ درجة البسط > درجة المقام

$$\begin{array}{r|l} & x^2 \\ \hline x^2-2x+1 & \cancel{x^4}-\cancel{2x^3}+\cancel{x^2}+2x-1 \\ & -\cancel{x^4}+\cancel{2x^3}+\cancel{x^2} \\ \cline{2-2} & 2x-1 \end{array}$$

$$= x^2 + \frac{2x-1}{x^2-2x+1}$$

$$= x^2 + \frac{2x-1}{(x-1)(x-1)}$$

$$= x^2 + \frac{2x-1}{(x-1)^2}$$

$$= x^2 + \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

No. Lec. 4

Example درجة البسط < درجة المقام

$$\int \frac{dx}{x^2+x-2}$$

$x^2 + x - 2$

$(x+2)(x-1)$

$$\frac{1}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1}$$

$1 = A(x-1) + B(x+2)$ $\quad x-1=0$

$\boxed{x=1} \longrightarrow 1 = 0 + 3B$ $\quad x=1$

$B = 1/3$ $\quad x+2=0$

$\boxed{x=-2} \longrightarrow 1 = -3A + 0$ $\quad x=-2$

$A = -1/3$

$$\frac{1}{x^2+x-2} = \frac{-1/3}{x+2} + \frac{1/3}{x-1}$$

$$\int \frac{-1/3}{x+2} + \frac{1/3}{x-1} \, dx$$

$$= -1/3 \ln|x+2| + 1/3 \ln|x-1| + c$$

$$= \frac{1}{3} \ln\left|\frac{x-1}{x+2}\right| + c$$

No. Lec. 4

Example درجة البسط < درجة المقام

$$\int \frac{1}{x^3+x}\,dx$$

x^3+x

$x(x^2+1)$

$$\frac{1}{x^3+x} = \frac{A}{x} + \frac{Bx+c}{x^2+1}$$

$$1 = A(x^2+1) + x(Bx+c)$$

$\boxed{x=0} \longrightarrow 1 = A + 0 \quad \boxed{A=1}$

عوض A $\quad 1 = \underline{x^2}+1 + B\underline{x^2}+c$

في المعادلة $\quad \cancel{1} = (1+B)x^2 + cx + \cancel{1}$

جمع المتشابه $\quad 0 = (1+B)x^2 + cx$

نقارن $\quad 0x^2 + 0x = (1+B)x^2 + cx$

$0 = 1 + B \qquad \boxed{c=0}$

$\boxed{B=-1}$

$$\int \frac{1}{x^3+x}\,dx = \int \frac{1}{x} - \frac{x}{x^2+1}\,dx$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| + C$$

No. Lec. 4

Example درجة البسط > درجة المقام

$$\int \frac{x^{\textcircled{4}} - 2x^2 + 4x + 1}{x^{\textcircled{3}} - x^2 - x + 1}\, dx$$

$$\begin{array}{r|l} & x+1 \\ \hline x^3-x^2-x+1 & \cancel{x^4}-2x^2+4x+1 \\ & -\cancel{x^4}+x^3+x^2+x \\ \hline & \cancel{x^3}-\cancel{x^2}+3x+\cancel{1} \\ & -\cancel{x^3}+\cancel{x^2}+x-\cancel{1} \\ \hline & 4x \end{array}$$

$$= \int x + 1 + \frac{4x}{x^3 - x^2 - x + 1}\, dx$$

$$= \int x + 1\, dx + \int \frac{4x}{x^3 - x^2 - x + 1}\, dx$$

$$= \frac{x^2}{2} + x \quad + \quad \text{كسور جزئية}$$

$$\int \frac{4x}{x^3 - x^2 - x + 1}\, dx$$

$$x^3 - x^2 - x + 1$$

$$(x-1)(\text{قسمة طويلة})$$

$$\begin{array}{r|l} & x^2-1 \\ \hline x-1 & x^3-x^2-x+1 \\ & -x^3+x^2 \\ \hline & -x+1 \\ & +x-1 \\ \hline & 0 \end{array}$$

$$\int \frac{4x}{(x-1)^2(x+1)}$$

$$= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$\therefore (x-1)(x^2-1)$$

$$(x-1)(x+1)(x-1)$$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

No. Lec. 4

$$4x = A(x-1)(x+1) + B(x+1) + c(x-1)^2$$

بمساواة الأقواس

$$\left\{ \begin{array}{l} \boxed{x=1} \rightarrow 4 = 2B \rightarrow B = 2 \\ \boxed{x=-1} \rightarrow -4 = 4c \rightarrow c = -1 \end{array} \right.$$

نعوض ←

$$4x = A(x-1)(x+1) + 2(x+1) - (x-1)^2$$

C/B

$$4x = A(x^2-1) + 2x + 2 - (x^2 - 2x + 1)$$

بالمعادلة

$$4x = Ax^2 - A + 2x + 2 - x^2 + 2x - 1$$

نجمع الحدود المتشابهة ←

$$\cancel{4x} = (A-1)x^2 + \cancel{4x} - A + 1$$

$$0 = (A-1)x^2 - A + 1$$

$$0x^2 + 0 = (A-1)x^2 - A + 1$$

$$0 = A - 1 \qquad -$$

$$A = 1$$

$$= \int \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{-1}{x+1} \, dx$$

طرفي -

تتحول ÷

$$= \ln|x-1| + \frac{2(x-1)^{-1}}{-1} - \ln|x+1| + c$$

$$= \ln\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1} + c$$

$$= \frac{x^2}{2} - x + \ln\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1} + c$$

No. Lec. 4

note poly / poly

بعض الاسئلة لا نحتاج الى الكسور الجزئية لحلها
لذلك دائماً نتأكد من درجة البسط والمقام

Example

- $\int \frac{3x^2+2}{x^3+2x-8}\,dx \rightarrow \int \frac{g'}{g}$

$= \ln|x^3+2x-8| + c$

- $\int \frac{2x-1}{x^2+1}\,dx$ (نوزع المقام)

$= \int \frac{2x}{x^2+1} - \frac{1}{x^2+1}\,dx$

$= \ln|x^2+1| - \tan^{-1}x + c$

No. Lec. 5

Strategy of integration

- Basic Rules of integration.
- The substitution rules.
- Integration by parts.
- Trigonometric substitution.
- partial fractions.

Example

نبسّط

$$\int \frac{\tan^3 x}{\cos^3 x}\,dx$$

$$= \int \tan^3 x \cdot \frac{1}{\cos^3 x}\,dx$$

$$= \int \tan^{\overset{\text{odd}}{\textcircled{3}}} x \cdot \sec^3 x\,dx \;\ldots$$

$$\underline{\text{or}} = \int \frac{\sin^3 x}{\cos^3 x} \cdot \frac{1}{\cos^3 x}\,dx$$

$$= \int \frac{\sin^3 x}{\cos^6 x}\,dx \qquad u = \cos x \qquad \frac{du}{-\sin x} = dx$$

$$= \int \frac{\sin^3 x}{u^6}\,\frac{du}{-\sin x} = \int \frac{u^2-1}{u^6}\,du$$

$$= \int \frac{u^2}{u^6}\,du - \int u^{-6}\,du$$

$$= \frac{u^{-3}}{-3} - \frac{u^{-5}}{-5} + c = \frac{-1}{3\cos^3 x} + \frac{1}{5\cos^5 x} + C$$

No. Lec. 5

Example

$$\int \frac{1}{1-\cos x}\,dx$$ نضرب بالمرافقة

$$\int \frac{1}{1-\cos x} * \frac{1+\cos x}{1+\cos x}\,dx$$

$$\int \frac{1+\cos x}{1-\cos^2 x}\,dx$$

$$\int \frac{1+\cos x}{\sin^2 x} = \int \frac{1}{\sin^2 x} + \frac{\cos x}{\sin^2 x}\,dx$$

$$\int \csc^2 x + \frac{\cos x}{\sin^2 x}\,dx \qquad u = \sin x$$

$$\frac{du}{\cos x} = dx$$

$$-\cot x + \int \frac{\cancel{\cos x}}{u^2}\,\frac{du}{\cancel{\cos x}}$$

$$-\cot x + \int \frac{1}{u^2}\,du$$

$$-\cot x + \int u^{-2}\,du$$

$$-\cot x + \frac{u^{-1}}{-1} + c$$

$$-\cot x - \csc x + c$$

Example

$$\int \frac{x^3}{(4x^2+9)^{3/2}}\,dx$$

عبارة تربيعية وجذر ∴
Tri.sub أو تعويض

$$2x = 3\tan\theta$$

or $u = 4x^2 + 9$

$$\frac{du}{8x} = dx$$

$$\int \frac{x^{\cancel{3}\,2}}{u^{\frac{3}{2}}} \frac{du}{8\cancel{x}} = \frac{1}{8}\int \frac{x^2}{u^{3/2}}\,du$$

$$= \frac{1}{8}\int \frac{\frac{u-9}{4}}{u^{3/2}}\,du = \frac{1}{32}\int \frac{u-9}{u^{3/2}}\,du$$

$$= \frac{1}{32}\int \frac{u}{u^{3/2}} - \frac{9}{u^{3/2}}\,du$$

$$= \frac{1}{32}\int u^{-1/2} - 9\,u^{-3/2}\,du$$

$$= \frac{1}{32}\left(\frac{u^{1/2}}{1/2} - 9\,\frac{u^{-1/2}}{-1/2}\right) + C$$

$$= \frac{1}{32}\left(2\sqrt{u} + \frac{18}{\sqrt{u}}\right) + C$$

$$= \frac{1}{32}\left(2\sqrt{4x^2+9} + \frac{18}{\sqrt{4x^2+9}}\right) + C$$

Example

$\int \sqrt{1-\sin x}\ dx$ الضرب بالمرافقة

$$\int \sqrt{1-\sin x} * \frac{\sqrt{1+\sin x}}{\sqrt{1+\sin x}}\ dx$$

$$\int \frac{\sqrt{1-\sin^2 x}}{\sqrt{1+\sin x}}\ dx = \int \frac{\sqrt{\cos^2 x}}{\sqrt{1+\sin x}}\ dx$$

$$= \int \frac{\cos x}{\sqrt{1+\sin x}}\ dx$$

$u = 1+\sin x$

$du = \cos x\ dx$

$dx = \frac{du}{\cos x}$

$$= \int \frac{\cancel{\cos x}}{\sqrt{u}} \cdot \frac{du}{\cancel{\cos x}}$$

$$= \int u^{-1/2}\ du$$

$$= 2\ u^{1/2} + c$$

$$= 2\sqrt{u} + c$$

$$= 2\sqrt{1+\sin x} + c$$

No. Lec. 5

Example

$\int e^{\sqrt{x}}\, dx$ $\qquad\qquad y = \sqrt{x}$

$2y\, dy = dx$

$\int e^{y} \cdot 2y\, dy$

$2\int y e^{y}\, dy$ py parts

$u = y \longrightarrow du = 1\, dy$

$dv = e^{y} dy \longrightarrow v = e^{y}$

$2\left(ye^{y} - \int e^{y} \right)$

$2ye^{y} - 2e^{y} + c$

$2\sqrt{x}\, e^{\sqrt{x}} - 2e^{\sqrt{x}} + c$

Example

$\int \sqrt{\frac{1-x}{1+x}}\, dx$

sol.1 $\quad u = \sqrt{\frac{1-x}{1+x}}$ ...

sol.2 ضرب بالمرافق

No. Lec. 5

$$\int \frac{\sqrt{1-x}}{\sqrt{1+x}} * \frac{\sqrt{1-x}}{\sqrt{1-x}}\, dx$$

$$\int \frac{1-x}{\sqrt{1-x^2}}\, dx$$

$$\int \frac{1}{\sqrt{1-x^2}} + \frac{-x}{\sqrt{1-x^2}}\, dx$$

by Tri. sub.

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

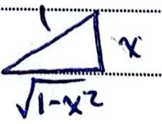
$$= \sin^{-1} x +$$

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$$

$$\int \frac{-\sin\theta}{\cos\theta} \cos\theta\, d\theta$$

$$= \cos\theta + c$$

$\frac{x}{1} = \sin\theta$



$$\sqrt{1-x^2} + c$$

$$= \sin^{-1} x + \sqrt{1-x^2} + c$$

Example

$$\int \frac{1}{\sqrt{x} - \sqrt[3]{x}}\, dx$$

$x = u^6$

$6u^5\, dy = dx$

$$\int \frac{1}{\sqrt{u^6} - \sqrt[3]{u^6}} \cdot 6u^5\, du$$

$$\int \frac{6u^{\textcircled{5}}\, du}{u^{\textcircled{3}} - u^2}$$

$$\begin{array}{r|l} & 6u^2+6u+6 \\ \hline u^3-u^2 & \cancel{6u^5} \\ & \underline{-\cancel{6u^5}+6u^4} \\ & \cancel{6u^4} \\ & \underline{-\cancel{6u^4}+6u^3} \\ & \cancel{6u^3} \\ & \underline{-\cancel{6u^3}+6u^2} \\ & 6u^2 \end{array}$$

$$\int 6u^2+6u+6 + \frac{6u^2}{u^3-u^2}\, du$$

$$\int 6u^2+6u+6 + \frac{6\cancel{u^2}}{\cancel{u^2}(u-1)}\, du$$

$$= \frac{\overset{2}{\cancel{6}}u^3}{\cancel{3}} + \frac{\overset{3}{\cancel{6}}u^2}{\cancel{2}} + 6u + 6 \ln|u-1| + c$$

$$= 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6 \ln|\sqrt[6]{x} - 1| + c$$

$x = u^6$

$\sqrt[6]{x} = u$

$\sqrt{x} = u^3$

$\sqrt[3]{x} = u^2$

No. Lec 5

Example

$$\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx$$

$u = \sqrt{1+\sqrt{x}}$

$u^2 = 1+\sqrt{x}$

$2u\,du = \frac{1}{2\sqrt{x}}\,dx$

$4u\sqrt{x}\,du = dx$

$4u(u^2-1)\,du = dx$

$(u^2-1)^2 = x$

$$\int \frac{u}{x} \cdot 4u(u^2-1)\,du$$

$$\int \frac{4u^2\cancel{(u^2-1)}}{(u^2-1)^2}\,du$$

$$\int \frac{4u^2}{u^2-1}\,du$$

$$4\int \frac{u^2}{u^2-1}\,du$$

$$\begin{array}{r|l} & 1 \\ \hline u^2-1 & u^2 \\ & -u^2+1 \\ \hline & 1 \end{array}$$

$$4\int 1 + \frac{1}{u^2-1}\,du$$

$4($u + by partial fractions$) + c$

$$4\left(u + \int\frac{A}{u-1} + \int\frac{B}{u+1}\right) + C$$

$1 = A(u+1) + B(u-1)$

$u=1 \rightarrow 1 = 2A \quad A = \frac{1}{2}$

$u=-1 \rightarrow 1 = -2B \quad B = -\frac{1}{2}$

$$4\left(u + \int\frac{1/2}{u-1} - \int\frac{1/2}{u+1}\right) + c$$

[illegible] $4\left(u + \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1|\right) + c$

Example

$$\int \frac{xe^{x}}{\sqrt{1+e^{x}}}\,dx$$

$y = \sqrt{1+e^{x}}$

$dy = \frac{e^{x}}{2\sqrt{1+e^{x}}}\,dx$

$dx = \frac{2\sqrt{1+e^{x}}}{e^{x}}\,dy$

$dx = \frac{2y}{e^{x}}\,dy$

$$\int \frac{x\cancel{e^{x}}}{\cancel{y}} \cdot \frac{2\cancel{y}}{\cancel{e^{x}}}\,dy$$

$2\int x . dy$ (ادي التعويض)

$y^{2} = 1+e^{x}$

$y^{2} - 1 = e^{x}$

$\ln(y^{2}-1) = x$

$2\int \ln(y^{2}-1)\,dy$ (by parts)

$u = \ln(y^{2}-1) \longrightarrow du = \frac{2y}{y^{2}-1}$

$dv = dy \longrightarrow v = y$

قسمة طويلة

$$2\left[y\ln(y^{2}-1) - \int \frac{2y^{2}}{y^{2}-1}\,dy \right]$$

كسور جزئية

$$2\left[y\ln(y^{2}-1) - \int 2 + \frac{2}{y^{2}-1} \right]$$

$$\begin{array}{r|l} & 2 \\ \hline y^{2}-1 & 2y^{2} \\ & -2y^{2}+2 \\ \hline & 2 \end{array}$$

$$2\left[y\ln(y^{2}-1) - 2y + \ln\left|\frac{y-1}{y+1}\right| \right] + c$$

مربع y

No. Lec. 5

Example

$$\int \frac{e^{2x}}{e^{2x}+3e^{x}+2}\,dx$$

$u = e^{x}$

$du = e^{x}dx$

$\frac{du}{e^{x}} = dx$

$dx = \frac{du}{u}$

$$\int \frac{u^{\cancel{2}}}{u^{2}+3u+2}\,\frac{du}{\cancel{u}}$$

$$\int \frac{u}{u^{2}+3u+2}\,du$$

$u^{2}+3u+2$

$(u+2)(u+1)$

partial.frac.

$$\int \frac{u}{(u+2)(u+1)} = \frac{A}{u+2} + \frac{B}{u+1}$$

$u = A(u+1) + B(u+2)$

$u=-2 \longrightarrow +2 = +A \longrightarrow A = 2$

$u=-1 \longrightarrow -1 = B$

$$= \int \frac{-1}{u+1} + \int \frac{2}{u+2}$$

$$= -1 \ln|u+1| + 2\ln|u+2| + C$$

$$= -1 \ln|e^{x}+1| + 2\ln|e^{x}+2| + C$$

No. Lec.6

Improper Integrals

التكامل المحدود

type1 infinite intervals

نستبدل ∓∞ بمتغير، ونحسب lim

$$1.\ \int_a^{+\infty \text{ المشكلة}} f(x)\,dx = \lim_{t\to\infty} \int_a^{t \text{ متغير}} f(x)\,dx$$

$$2.\ \int_{-\infty \text{ المشكلة}}^{a} f(x)\,dx = \lim_{t\to-\infty} \int_{t}^{a} f(x)\,dx$$

↳ called convergent if limit exist and divergent if the limit d.n.e

$$3.\ \int_{-\infty\ 〃}^{+\infty\ 〃} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{+\infty} f(x)\,dx$$

a → اي عدد

Example

improper inte.

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{t\to+\infty} \int_1^{t} \frac{1}{x}\,dx$$

$$= \lim_{t\to+\infty} \ln|x| \Big|_1^{t}$$

$$= \lim_{t\to+\infty} \ln|t| - \overset{\text{Zero}}{\cancel{\ln 1}}$$

$$= +\infty$$

$$\int_1^{+\infty} \frac{1}{x}\,dx = \text{divergent}$$

ماله جواب

No. Lec. 6

Example

$$\int_{1}^{+\infty} \frac{1}{x^2}\,dx = \lim_{t\to+\infty} \int_{1}^{t} \frac{1}{x^2}\,dx$$

$$= \lim_{t\to+\infty} \int_{1}^{t} x^{-2}\,dx$$

$$= \lim_{t\to+\infty} -x^{-1}\Big|_{1}^{t}$$

$$= \lim_{t\to+\infty} -\overset{\frac{1}{\infty}=0}{\frac{1}{t}} - -1$$

$$= 1$$

$$\int_{1}^{+\infty} 1/x^2\,dx = 1 \quad \text{convergent}$$ (موجودة lim)

Thrm

$$\int_{1}^{+\infty} \frac{1}{x^p}\,dx = \begin{cases} p>1, & \text{convergent to } \frac{1}{p-1} \\ p\le 1, & \text{divergent} \end{cases}$$

Example

- $\int_{1}^{+\infty} \frac{1}{x^3}\,dx = p>1 \therefore$ convergent to $\frac{1}{3-1} = \frac{1}{2}$
- $\int_{1}^{+\infty} \frac{1}{x^{10}}\,dx =$ convergent to $\frac{1}{9}$

No. Lec. 6

- $\int_{1}^{+\infty} \frac{1}{\sqrt{x}}\, dx$ = divergent

- $\int_{1}^{+\infty} x^{-2/3}\, dx = \int_{1}^{+d} \frac{1}{x^{2/3}}$ = divergent

Example

$$\int_{-\infty}^{0} x e^{x}\, dx = \lim_{t \to -\infty} \int_{t}^{0} x e^{x}\, dx$$

$$\int x e^{x} dx$$

$u = x \to du = dx$

$dv = e^{x} \to v = e^{x}$

$xe^{x} - \int e^{x} dx$

$xe^{x} - e^{x}$

$$= \lim_{t \to -\infty} xe^{x} - e^{x} \Big|_{t}^{0}$$

$$= \lim_{t \to -\infty} (-1-(\underset{0}{\cancel{te^{t}}} - \underset{0}{\cancel{e^{t}}}))$$

$= -1$ ∴ convergent.

- $\lim_{t \to -\infty} e^{t} = 0$

- $\lim_{t \to -\infty} te^{t} = \underbrace{-\infty \cdot 0}_{\text{L'H rules}}$

$\lim_{t \to -\infty} \frac{t}{e^{-t}}$ بجعلوا كسر بشتقة

$\lim_{t \to -\infty} \frac{1}{-e^{-t}}$

$\lim_{t \to -\infty} -e^{t} = 0$

No. Lec. 6

Example اذا واحد منهم طلع con بكمل اذا di بوقف ⟵

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \overset{\text{كال}}{\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx} + \overset{\text{كال}}{\int_{0}^{+\infty} \frac{1}{1+x^2}\,dx}$$

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty} \tan^{-1}x \Big|_{t}^{0}$$

$$= \lim_{t\to-\infty} (0 - \tan^{-1}t)$$

$$= - -\frac{\pi}{2} = \frac{\pi}{2} \text{ con.}$$

$$\int_{0}^{+\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to+\infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to+\infty} \tan^{-1}x \Big|_{0}^{t}$$

$$= \lim_{t\to+\infty} \tan^{-1}t - \overset{0}{\cancel{\tan^{-1}0}}$$

$$= \frac{\pi}{2} \text{ con.}$$

$\hookrightarrow \frac{\pi}{2} + \frac{\pi}{2} = \pi$ convergent.

No. Lec. 6

Type 2 discontinuous integrands

1. If f is continuous on $[a, b)$ and is discontinuous at b, then

المشكلة ←

$$\int_a^{\boxed{b}} f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

2. If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_{\boxed{a}}^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

↳ called convergent if limit exists and divergent if the limit d.n.e.

3. If f(x) is discontinuous at c, such that $a < c < b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example

1. $\int_{\boxed{2}}^5 \frac{1}{\sqrt{x-2}}\,dx$ $\quad \frac{1}{\sqrt{x-2}}$ disconts at 2

انه
صفر مقام

$\therefore$ improper inte.

$$\lim_{t\to 2^+} \int_t^5 (x-2)^{-1/2}\,dx$$

$$\lim_{t\to 2^+} 2(x-2)^{1/2} \Big|_t^5$$

FIVE APPLE

No. Lec. 6

$$\lim_{t\to 2^+} 2(3)^{1/2} - 2(t-2)^{1/2}$$

$$\lim_{t\to 2^+} 2(3)^{\frac{1}{2}} - 2\overset{0}{\cancel{\sqrt{t-2}}}$$

$= 2\sqrt{3}$ ∴ convergent

2. $\int_0^{\pi/2} \sec x \, dx$ $\qquad \sec\frac{\pi}{2} = \frac{1}{\cos\frac{\pi}{2}} = \frac{1}{0}$

$= \lim_{t\to\frac{\pi}{2}^-} \int_0^t \sec x \, dx$ $\qquad$ ∴ improper inte.

بنطلعوا ال lim

$$= \lim_{t\to\frac{\pi}{2}^-} \ln|\sec x + \tan x| \Big|_0^t$$

$$= \lim_{t\to\frac{\pi}{2}^-} \underset{\ln\infty}{\ln|\overset{\infty}{\sec t} + \overset{\infty}{\tan t}|} - \ln|\overset{1}{\sec 0} + \overset{0}{\sec 0}|$$

$= +\infty$ divergent

3. $\int_0^3 \frac{dx}{x-1}$ $\qquad \frac{1}{x-1}$ discont. at $x = 1$

improper inte.

$$\int_0^1 \frac{1}{x-1} dx + \int_1^3 \frac{1}{x-1} dx$$

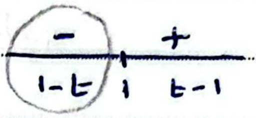
$$\int_0^1 \frac{1}{x-1} dx = \lim_{t\to 1^-} \int_0^t \frac{1}{x-1} dx$$

$$= \lim_{t\to 1^-} \ln|x-1| \Big|_0^t$$

No. Lec. 6

$$= \lim_{t \to 1^-} \ln|t-1| - \overset{0}{\cancel{\ln|\overset{-1}{0-1}|}}$$

$$= \lim_{t \to 1^-} \ln|t-1|$$



$$= \lim_{t \to 1^-} \ln|1-t|$$

$= -\infty$ divergent ∴ لاداعي اني اكمل

$\therefore \int_0^3 \frac{1}{x-1}\,dx$ divergent

Exercises

Find the values of p for which the integral converges :- (limit exist)

1) $\int_{e}^{+\infty} \frac{1}{x(\ln x)^{2p}} dx$

$u = \ln x$

$x\,du = dx$

$x = e \rightarrow u = 1$

$x = \infty \rightarrow u = \infty$

$$\int_{1}^{+\infty} \frac{1}{\cancel{x}\, u^{2p}} \cancel{x}\, du$$

$$\int_{1}^{+\infty} \frac{1}{u^{2p}} du \quad \xrightarrow{\text{remember}} \quad \int_{1}^{+\infty} \frac{1}{x^p} = \begin{cases} p \leq 1 & \text{div} \\ p > 1 & \text{conv} \end{cases}$$

$\therefore \quad 2p > 1$

$p > 1/2 \qquad (\frac{1}{2}, \infty)$

2) $\int_{a}^{b} \frac{1}{(x-a)^{p-1}} dx$ for $a < b$

$$\lim_{t \to a^+} \int_{t}^{b} \frac{1}{(x-a)^{p-1}} dx$$

$$\lim_{t \to a^+} \int_{t}^{b} (x-a)^{1-p} dx$$

$$\lim_{t \to a^+} \frac{(x-a)^{1-p+1}}{1-p+1} \Big|_{t}^{b}$$

$$\lim_{t \to a^+} \frac{(x-a)^{2-p}}{2-p} \Big|_{t}^{b}$$

No. ……………

$\overset{?}{\text{exist}}$ $= \lim_{t\to a^+} \frac{1}{2-p}\left[(b-a)^{2-p} - (t-a)^{2-p}\right]$

$2-p=0$

$2 \neq 0$ مو مقام

$(+)^{2-p}$

+number

عند تعويض t

0^{2-p}

$+$ | $-$

$2-p>0$ | $2-p<0$

$0^{+}\ 0^{2}\ 0^{3}$ | $0^{-}\ 0^{-2}\ 0^{-3}$

$=0$ | $=\frac{1}{0}$

convergent ✓ | divergent

$\therefore\ 2-p>0$

$-p>-2$

$p<2 \qquad (-\infty,2)$