

- LU Decomposition :-
 $AX=b$

① In this method the A matrix is Decomposed
in two matrices
 $A = LU$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

Lower triangular
matrix

upper triangular
matrix

② Solve $\bar{A}\bar{d} = \bar{b}$ by forward substitution,
Find $\bar{d}$, intermediate vector.

③ Find x , solve $\bar{U}\bar{x} = \bar{d}$ by Backward substitution

In this method assume the main diagonal for L matrix = 1.

Ex:

$$A = \begin{bmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ 5 & 4 & -1 \end{bmatrix} = L \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} U \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$U_{11} = 2$$

(2) $\leftarrow$ value of first element

الاول = العنصر الاول

$$3 = U_{12}$$

$$-2 = U_{13}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}$$

diagonal = 1

1 =

$$\text{Sol: } = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 2.5 & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -2 \\ 0 & 0.5 & 4 \\ 0 & 0 & 32 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ 5 & 4 & -1 \end{bmatrix}$$

$$-\frac{1}{2} R_1 + R_2 \rightarrow R_2$$

$$-\frac{5}{2} R_1 + R_3 \rightarrow R_3$$

$$-\frac{1}{2} \begin{pmatrix} 2 & 3 & -2 \end{pmatrix}$$

$$+ \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

$$0 \quad 0.5 \quad 4$$

$$-\frac{5}{2} \begin{pmatrix} 2 & 3 & -2 \end{pmatrix}$$

$$+ \begin{pmatrix} 5 & 4 & -1 \end{pmatrix}$$

$$0 \quad -3.5 \quad 4$$

$$\Rightarrow \begin{bmatrix} 2 & 3 & -2 \\ 0 & 0.5 & 4 \\ 0 & -3.5 & 4 \end{bmatrix}$$

$$\left(-\frac{3.5}{0.5}\right) R_2 + R_3 \rightarrow R_3$$

$$7 \begin{pmatrix} 0 & 0.5 & 4 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & -3.5 & 4 \end{pmatrix}$$

$$R_3 \leftarrow 0 \quad 0 \quad 32$$

$$\Rightarrow \begin{bmatrix} 2 & 3 & -2 \\ 0 & 0.5 & 4 \\ 0 & 0 & 32 \end{bmatrix}$$

$\Rightarrow$ upper
matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 2.5 & -7 & 1 \end{bmatrix}$$

$\Downarrow$ Lower
matrix

⇒

(Lower matrix)

Notes

دلیل این

U: A after applying Gauss Elimination و این را

L: Take the value of the elimination coefficient

Tuesday

Ex: Solve the following system using Gauss elimination

L.U. Decomposition

$$\begin{bmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ 5 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

①

U

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 2.5 & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -2 \\ 0 & 0.5 & 4 \\ 0 & 0 & 32 \end{bmatrix}$$

②

Ld=b

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 2.5 & -7 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

↓ to find d?

$$d_1 = 2$$

$$0.5d_1 + d_2 = 1$$

$$d_2 = 0$$

$$2.5d_1 - 7d_2 + d_3 = 2$$

$$d_3 = -3$$

smile for life

$$\textcircled{3} \quad U X = d$$

$$\begin{bmatrix} 2 & 3 & -2 \\ 0 & 0.5 & 4 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$$

$$32 x_3 = -3$$

$$x_3 = \frac{-3}{32} = -0.09375 = x_3$$

$$0.5 x_2 + 4 x_3 = 0 \Rightarrow x_2 = 0.75$$

$$2 x_1 + 3 x_2 - 2 x_3 = 2 \quad x_1 = -2.7187$$

Calculating Matrix Inverse Using LU Decomposition

$$A A^{-1} = I$$

$$\# A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$x_1 \quad x_2 \quad x_3$ $b_1 \quad b_2 \quad b_3$

$$A x_1 = b_1 \Rightarrow A^{-1} b_1$$

$$A x_2 = b_2$$

$$A x_3 = b_3$$

$A \otimes b$ معكوسة
 $b \leftarrow A^{-1} b$

Thursday

#Ex? Calculate the 3rd Column of the inverse matrix for

$$A = \begin{bmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ 5 & 4 & -1 \end{bmatrix} \quad ?$$

$$AX_1 = b_1 \quad b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$AX_2 = b_2 \quad b_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$AX_3 = b_3 \quad b_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \textcircled{1} \quad A\bar{X} = \bar{b} \quad \Rightarrow \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 2.5 & -7 & 1 \end{bmatrix} \quad \bar{b} = \begin{bmatrix} 2 & 3 & -2 \\ 0 & 0.5 & 4 \\ 0 & 0 & 32 \end{bmatrix}$$

$$\textcircled{2} \quad Ld = \bar{b}$$

$$\downarrow \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 2.5 & -7 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \begin{aligned} d_1 &= 0 \\ d_2 &= 0 \\ d_3 &= 1 \end{aligned}$$

$$\textcircled{3} \quad UX = d$$

$$\uparrow \begin{bmatrix} 2 & 3 & -2 \\ 0 & 0.5 & 4 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{aligned} x_3 &\cong \frac{1}{32} = 0.03125 \\ 0.5x_2 + 4x_3 &= 0 \quad \Rightarrow \quad x_2 = -0.125 \\ x_1 + 2x_2 - 2x_3 &= 0 \end{aligned}$$

$$x_1 = -2x_2 + 2x_3 = 0$$

$$x_1 = -0.0343$$

smile for life

$$X = \begin{bmatrix} -0.4671 \\ -0.25 \\ 0.03125 \end{bmatrix}$$

$\Rightarrow$ 3rd column in inverse matrix.

Jacobi and Gauss Seidel Iteration:

- These methods used to solve linear system:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$(1) \quad x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}} \Rightarrow x_1 = (g(x))$$

$$(2) \quad x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$

$$(3) \quad x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

Jacobi

$$(A) \quad x_{i+1} = \frac{b_1 - a_{12}x_{2,i} - a_{13}x_{3,i}}{a_{11}}$$

$$x_{2,i+1} = \frac{b_2 - a_{21}x_{1,i} - a_{23}x_{3,i}}{a_{22}}$$

$$x_{3,i+1} = \frac{b_3 - a_{31}x_{1,i} - a_{32}x_{2,i}}{a_{33}}$$

(B) Gauss Seidel's

$$x_1^{i+1} = \frac{b_1 - a_{12}x_{2,i} - a_{13}x_{3,i}}{a_{11}}$$

$$x_2^{i+1} = \frac{b_2 - a_{21}x_{1,i+1} - a_{23}x_{3,i}}{a_{22}}$$

$$x_3^{i+1} = \frac{b_3 - a_{31}x_{1,i+1} - a_{32}x_{2,i+1}}{a_{33}}$$

- convergence criteria

it is similar matrices manner as simple fixed point iteration

$$\left| \frac{dx_1}{dx_1} \right| + \left| \frac{dx_2}{dx_2} \right| + \left| \frac{dx_3}{dx_3} \right| < 1$$

$$\Rightarrow 0 + \left| -\frac{a_{12}}{a_{11}} \right| + \left| -\frac{a_{13}}{a_{11}} \right| < 1$$

$$\frac{a_{12}}{a_{11}} + \frac{a_{13}}{a_{11}} < 1 \Rightarrow \frac{a_{12} + a_{13}}{a_{11}} < 1$$

$$\boxed{|a_{12}| + |a_{13}| < |a_{11}|}$$

اگر حاصل (x_1) کلاً
ال x_1 متغیراً مروجی قانونه

No. _____

$$|a_{21}| + |a_{23}| < |a_{22}| \rightarrow (2)$$

$$|a_{31}| + |a_{32}| < |a_{33}| \rightarrow (3)$$

Exo $x_1 + 7x_2 + 3x_3 = 12 \Rightarrow x_2$ موصى العايرة

$$12x_1 + 4x_2 + x_3 = 11 \Rightarrow x_1 \text{ و } x_3$$

$$5x_1 + 3x_2 + 13x_3 = 5 \Rightarrow x_3 \text{ و } x_1$$

ترتيب اعداد حكيه بصرار (diagonal) الحكي (سي)

$$12x_1 + 4x_2 + x_3 = 11 \rightarrow x_1$$

$$x_1 + 7x_2 + 3x_3 = 12 \rightarrow x_2$$

$$5x_1 + 3x_2 + 13x_3 = 5 \rightarrow x_3$$

Sunday 3-11

- Curve fitting :-

$$S_r = \sum_{i=1}^n e_i^2 \text{ min}$$

$$e_i = y - \hat{y}$$

الحقيقية التقديرية

$$= \sum_{i=1}^n (y_i - \hat{y}_i)^2$$

n : عدد النقاط

$$S_r = \sum_{i=1}^n (y_i - a_0 - a_1 x_i)^2$$

$$\frac{dS_r}{da_0} = 0$$

$$\frac{dS_r}{da_1} = 0$$

من أجل إيجاد
المعادلة

$$0 = \sum 2(y_i - a_0 - a_1 x_i) \cdot (-1) \Rightarrow \frac{dS_r}{da_0} \quad (1)$$

$$0 = \sum 2(y_i - a_0 - a_1 x_i) \cdot (-x_i) \Rightarrow \frac{dS_r}{da_1} \quad (2)$$

$$\sum (a_0 + a_1 x_i - y_i) = 0$$

$$\sum (a_0 x_i + a_1 x_i^2 - y_i x_i) = 0$$

$$\Rightarrow \sum_{i=1}^n a_0 + \sum a_1 x_i - \sum y_i = 0$$

$$\Rightarrow \sum a_0 x_i + \sum a_1 x_i^2 - \sum y_i x_i = 0$$

$$a_0 n + \sum a_1 x_i = \sum y_i \Rightarrow a_0 n + a_1 \sum x_i = \sum y_i$$

$$\sum a_0 x_i + \sum a_1 x_i^2 = \sum x_i y_i \Rightarrow a_0 \sum x_i + a_1 \sum x_i^2 = \sum x_i y_i$$

No. _____

(a₀ & a₁) parameters of the regression line

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

Ex 2 Using linear Regression fit, the following data to a straight line

x	2	3.25	5.1
y	3.5	5.6	7.8

Sol:

$$n = 3$$

$$\sum x_i = 2 + 3.25 + 5.1 = 10.35$$

$$\sum x_i^2 = 2^2 + (3.25)^2 + (5.1)^2 = 40.57$$

$$\sum y_i = 3.5 + 5.6 + 7.8 = 16.9$$

$$\sum x_i y_i = 2 \times 3.5 + 3.25 \times 5.6 + 5.1 \times 7.8 = 64.98$$

$$= \begin{bmatrix} 3 & 10.35 \\ 10.35 & 40.57 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 16.9 \\ 64.98 \end{bmatrix}$$

$$a_0 = 0.899 \quad a_1 = 1.372$$

$$y = a_0 + a_1 x \Rightarrow y = 0.899 + 1.372x$$

هذا هو معادلة الخط المستقيم

Thursday 7-11

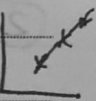
Correlation coefficient OR Factor R. ^{Prossed}
used to test the Goodness of the ^{Prossed} Fit

$$R^2 = \frac{st - sr}{st}$$

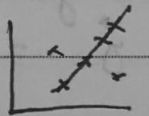
st: covariance

sr: $\sum e^2$: Residual Error.

note: ① if $R=1$, the Prossed Fit is said to be exact.
(the line will pass through all the given Data.)



② As R approaches 1, the Fit is referred to as Excellent fit.



③ As R approaches 0, the fit is referred to as poor. (see page 11)

④ If $R=0$, x and y are independent. (see page 11)

- For linear Regression R is given by:-

$$R = \frac{n \sum xy - \sum x \sum y}{\sqrt{n \sum x^2 - (\sum x)^2} \sqrt{n \sum y^2 - (\sum y)^2}}$$

Ex: For Previous Example 3:

$$\sum x_i = 10.35 \quad \sum x_i^2 = 40.57 \quad \sum y_i = 16.9 \quad \sum xy = 64.98$$

$$\sum y_i^2 = 3.5^2 + 5.6^2 + 7.8^2 = 104.45 \quad n=3 \Rightarrow$$

(see (2))

$$\# R = \frac{3 \times 64.98 - 10.35 \times 16.9}{\sqrt{3 \times (40.57) - (10.35)^2} \sqrt{3 \times (104.45) - (16.9)^2}} = 0.9952 \approx 1$$

smile for life

- Data Linearization :-

① Power :- $y = \alpha x^B$

$$y = \alpha x^B$$

$$\ln y = \ln \alpha + B \ln x$$

$$\ln y = \ln \alpha + B \ln x$$

$$y^* = a_0 + a_1 x^*$$

② Exponential

$$y = \alpha e^{Bx}$$

$$\ln e = 1$$

$$\ln y = \ln \alpha + Bx$$

$$y^* = a_0 + a_1 x^*$$

③ Growth Rate

$$y = \frac{\alpha x}{B + x}$$

$$\frac{1}{y} = \frac{B + x}{\alpha x} \Rightarrow \frac{1}{y} = \frac{B}{\alpha x} + \frac{x}{\alpha x}$$

$$\Rightarrow \frac{1}{y} = \frac{1}{\alpha} + \frac{B}{\alpha} \cdot \frac{1}{x}$$

$$y^* = a_0 + a_1 x^*$$

Ex: Using Linear Regression to find the values of α and B the following Relation
 $y = \alpha e^{Bx^2}$

X	2	3	5
y	49	8.7	13.5

?

* السؤال يعطي العلاقة

إذا كانت العلاقة كاول (الخطي)

أو أربعهم ماعرف العلاقة

Sol: $y = \alpha e^{Bx^2}$

$\ln y = \ln \alpha e^{Bx^2}$

$\ln y = \ln \alpha + Bx^2 (\ln e)$

II $\dot{y} = a_0 + B\dot{x}$

كل الميول (X)

(y)

2

$\dot{x} = x^2$	4	9	25
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كـ قيم a_0, a_1

$\dot{y} = \ln y$	1.5892	2.1633	2.6026
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$n=3 \quad \sum \dot{x} = 38 \quad \sum \dot{x}^2 = 722 \quad \sum \dot{y} = 6.3551$
 $\sum \dot{x} \dot{y} = 90.8915$

$$\begin{bmatrix} 3 & 38 \\ 38 & 722 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 6.3551 \\ 90.8915 \end{bmatrix}$$

$a_0 = 1.5713 \quad a_1 = 0.0431$

$\ln \alpha = a_0 \Rightarrow \alpha = e^{a_0} = e^{1.5713} = 4.8129$

$B = a_1 = 0.0431$

$y = 4.8129 e^{0.0431 x^2}$

Tuesday 12-11-2019

Polynomial Regression:-

- Given n Data Points to be Fitted to m^{th} order Polynomial of the form $y = a_0 + a_1x + a_2x^2 + \dots + a_mx^m$
- The Resulting system using Polynomial Regression will be:-

$$M \times 1 \begin{bmatrix} n & \sum X & \sum X^2 & \dots & \sum X^m \\ \sum X & \sum X^2 & \sum X^3 & \dots & \sum X^{m+1} \\ \sum X^2 & \dots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \dots & \sum X^{2m} \\ \sum X^m & \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum xy \\ \vdots \\ \sum x^m y \end{bmatrix}$$

- For Polynomial Regression $n \geq (m+1)$

نکات مهم در این روش

Ex: Using Regression analysis the 2nd order Polynomial coefficient using the following Data Points:-

X	2	3.25	4	5
y	5	9	7	4.25

$$y = a_0 + a_1x + a_2x^2$$

Sol:

$$m+1 = 3$$

$$M \times 1 = 3 \begin{bmatrix} n & \sum X & \sum X^{2+m} \\ \sum X & \sum X^2 & \sum X^3 \\ \sum X^2 & \sum X^3 & \sum X^{4=2m} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum xy \\ \sum x^2 y \end{bmatrix}$$

$$\begin{aligned}
 n &= 4 & \sum X &= 14.25 & \sum X^2 &= 55.56 & \sum X^3 &= 1008.5 \\
 \sum X^3 &= 231.32 & \sum X^4 &= 88.5 & \sum y &= 25.25 & \sum xy &= 88.5 \\
 & & & & \sum X^2 y &= 333.3 & &
 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 4 & 14.25 & 55.56 \\ 14.25 & 55.56 & 231.32 \\ 55.56 & 231.32 & 1008.56 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 25.25 \\ 88.5 \\ 333.3 \end{bmatrix}$$

$$a_0 = -10.75 \quad a_1 = 11.27 \quad a_2 = -1.66$$

$$y = -10.75 + 11.27X - 1.66X^2$$

$y(4.5) =$ تعويض بالقيمة

Thursday 14-11

Multiple Linear Regression :-

$$y = a_0 + a_1 X_1 + a_2 X_2 + \dots + a_n X_n$$

$X_1, X_2, X_3, \dots, X_n$, All are independent variables

For $y = a_0 + a_1 X_1 + a_2 X_2$, the Regression coefficient can be calculated using the following system.

$$\begin{bmatrix} n & \sum X_1 & \sum X_2 \\ \sum X_1 & \sum X_1^2 & \sum X_1 X_2 \\ \sum X_2 & \sum X_1 X_2 & \sum X_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum X_1 y \\ \sum X_2 y \end{bmatrix}$$

ex: Calculate $[a_0, a_1, a_2]$ for $y = a_0 + a_1x + a_2x^2$ using the following Data Points:

x_1	1	1	2
x_2	2	3	2
y	2	5	9

$$n = 3 \quad \sum x_1 = 4$$

$$\sum x_1 x_2 = 9 \quad \sum x_1^2 = 6$$

$$\sum y = 16 \quad \sum x_2 = 7$$

$$\sum x_1 y = 25 \quad \sum x_2^2 = 17$$

$$\sum x_2 y = 37$$

sol:

$$\begin{bmatrix} 3 & 4 & 7 \\ 4 & 6 & 9 \\ 7 & 9 & 17 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 16 \\ 25 \\ 37 \end{bmatrix}$$

$$a_0 = -11$$

$$a_1 = 7$$

$$a_2 = 3$$

$$y = -11 + 7x_1 + 3x_2$$

إذا كان في اختيار السردان يعطي
المستوية، فحينها مدد بتطبيقه

المحالة التي تم استخدامها

Sunday

Ch. 18

Interpolating Polynomials

Given a set of $(n+1)$ exact Data Points is used to create an n^{th} order interpolating which passes through all given Data Points, and can be used to Predict the behaviour of $f(x)$ within the interval $[x_0, x_n]$

① Newton Divided Difference Interpolating Polynomial
 - The General Formula for n^{th} order

- NDD Interpolating Polynomial is given by :-

$$f_n(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) + b_3(x-x_0)(x-x_1)(x-x_2) + \dots + b_n(x-x_0)(x-x_1)\dots(x-x_{n-1}).$$

012) 1st order: n

Calculating the Divided Difference is Done by :-

x_i	y_i	1 st $f(x_{i+1}, x_i)$	2 nd $f(x_{i+2}, x_{i+1}, x_i)$	3 rd $f(x_{i+3}, x_{i+2}, x_{i+1}, x_i)$
x_0	y_0	$\frac{y_1 - y_0}{x_1 - x_0}$	$\frac{f(x_2, x_1) - f(x_1, x_0)}{x_2 - x_0}$	$\frac{f(x_3, x_2, x_1) - f(x_2, x_1, x_0)}{x_3 - x_0}$
x_1	y_1	$\frac{y_2 - y_1}{x_2 - x_1}$	$\frac{f(x_3, x_2) - f(x_2, x_1)}{x_3 - x_1}$	
x_2	y_2	$\frac{y_3 - y_2}{x_3 - x_2}$		
x_3	y_3			

$$3^{\text{rd}} \Rightarrow \frac{f(x_3, x_2, x_1) - f(x_2, x_1, x_0)}{x_3 - x_0}$$

Ex: using NDD Interpolating polynomials estimate

$f_3(2.75)$ using the following Data Points

x	0	1	2.5	3	?
y	2	5	9	11	

	x_i	y_i	1 st	2 nd	3 rd	
x_0	0	2	$\frac{5-2}{1-0} = 3$	$\frac{2.666-3}{2.5-0} = -0.133$	$\frac{0.666-0.133}{3-0} = 0.2669$	اذا اطلبون قيمة ريم
	4			$= 0.133$	$= 0.2669$	موجود حواون ي
x_1	1	5	$\frac{4}{0.5} = 2.666$	$\frac{4-2.666}{3-1} = 0.666$		كلون من الجدول
x_2	2.5	9	4			
x_3	3	11				قيم $b \leftarrow$ اول صف

$$P_3(x) = 2 + 3(x-0) - 0.133(x-0)(x-1) + 0.2669(x-0)(x-1)(x-2.5)$$

$$P(2.75) = 2 + 3(2.75-0) - 0.133(2.75)(2.75-1) + 0.2669(2.75)(2.75-1)(2.75-2.5) = 9.927$$

Thursday 19-11

[2] Lagrange Interpolation Polynomial It defined by:-

$$F_n(x) = \sum_{i=0}^n L_i(x) f(x_i), \quad L_i(x) \text{ } i^{\text{th}} \text{ Lagrange}$$

Polynomial and it is defined as:-

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{(x-x_j)}{(x_i-x_j)}$$

Ex: using Data given in Previous Example estimate $P_3(2.75)$ using Lagrange Interpolation Polynomial?

No.

Tuesday 19-11

	x_0	x_1	x_2	x_3
x_i	0	1	2.5	3
y_i	2	5	9	11

$$L_0 = \frac{(x-1)(x-2.5)(x-3)}{(0-1)(0-2.5)(0-3)} = 0.0145$$

$$L_1 = \frac{(x-0)(x-2.5)(x-3)}{(1-0)(1-2.5)(1-3)} = -0.0572$$

$$L_2 = \frac{(x-0)(x-1)(x-3)}{(2.5-0)(2.5-1)(2.5-3)} = 0.6416$$

$$L_3 = \frac{(x-0)(x-1)(x-2.5)}{(3-0)(3-1)(3-2.5)} = 0.4010$$

$$F_n(2.75) = 0.0145 \times 2 + 0.0572 \times 5 + 0.6416 \times 9 + 0.4010 \times 11 = 9.92$$

طريقة NDD و طريقة لاجرانج

#Note:- For a given set of Data Points, the Resulting Interpolation polynomial will be unique (NDD and Lagrange Interpolation polynomials are equivalent if the same Data Points are used to create them)

$$P_n(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) \dots \quad \text{NDD}$$

$$P_n(x) = L_0 f(x_0) + L_1 f(x_1) + L_2 f(x_2) + \dots \quad \text{Lagrange}$$

كل صاري لكل المصطلح (كل terms) فيها، صيغة

L₁ و L₂ و L₃ و L₄ و L₅ و L₆ و L₇ و L₈ و L₉ و L₁₀ و L₁₁ و L₁₂ و L₁₃ و L₁₄ و L₁₅ و L₁₆ و L₁₇ و L₁₈ و L₁₉ و L₂₀ و L₂₁ و L₂₂ و L₂₃ و L₂₄ و L₂₅ و L₂₆ و L₂₇ و L₂₈ و L₂₉ و L₃₀ و L₃₁ و L₃₂ و L₃₃ و L₃₄ و L₃₅ و L₃₆ و L₃₇ و L₃₈ و L₃₉ و L₄₀ و L₄₁ و L₄₂ و L₄₃ و L₄₄ و L₄₅ و L₄₆ و L₄₇ و L₄₈ و L₄₉ و L₅₀ و L₅₁ و L₅₂ و L₅₃ و L₅₄ و L₅₅ و L₅₆ و L₅₇ و L₅₈ و L₅₉ و L₆₀ و L₆₁ و L₆₂ و L₆₃ و L₆₄ و L₆₅ و L₆₆ و L₆₇ و L₆₈ و L₆₉ و L₇₀ و L₇₁ و L₇₂ و L₇₃ و L₇₄ و L₇₅ و L₇₆ و L₇₇ و L₇₈ و L₇₉ و L₈₀ و L₈₁ و L₈₂ و L₈₃ و L₈₄ و L₈₅ و L₈₆ و L₈₇ و L₈₈ و L₈₉ و L₉₀ و L₉₁ و L₉₂ و L₉₃ و L₉₄ و L₉₅ و L₉₆ و L₉₇ و L₉₈ و L₉₉ و L₁₀₀ و L₁₀₁ و L₁₀₂ و L₁₀₃ و L₁₀₄ و L₁₀₅ و L₁₀₆ و L₁₀₇ و L₁₀₈ و L₁₀₉ و L₁₁₀ و L₁₁₁ و L₁₁₂ و L₁₁₃ و L₁₁₄ و L₁₁₅ و L₁₁₆ و L₁₁₇ و L₁₁₈ و L₁₁₉ و L₁₂₀ و L₁₂₁ و L₁₂₂ و L₁₂₃ و L₁₂₄ و L₁₂₅ و L₁₂₆ و L₁₂₇ و L₁₂₈ و L₁₂₉ و L₁₃₀ و L₁₃₁ و L₁₃₂ و L₁₃₃ و L₁₃₄ و L₁₃₅ و L₁₃₆ و L₁₃₇ و L₁₃₈ و L₁₃₉ و L₁₄₀ و L₁₄₁ و L₁₄₂ و L₁₄₃ و L₁₄₄ و L₁₄₅ و L₁₄₆ و L₁₄₇ و L₁₄₈ و L₁₄₉ و L₁₅₀ و L₁₅₁ و L₁₅₂ و L₁₅₃ و L₁₅₄ و L₁₅₅ و L₁₅₆ و L₁₅₇ و L₁₅₈ و 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L₈₇₄ و L₈₇₅ و L₈₇₆ و L₈₇₇ و L₈₇₈ و L₈₇₉ و L₈₈₀ و L₈₈₁ و L₈₈₂ و L₈₈₃ و L₈₈₄ و L₈₈₅ و L₈₈₆ و L₈₈₇ و L₈₈₈ و L₈₈₉ و L₈₉₀ و L₈₉₁ و L₈₉₂ و L₈₉₃ و L₈₉₄ و L₈₉₅ و L₈₉₆ و L₈₉₇ و L₈₉₈ و L₈₉₉ و L₉₀₀ و L₉₀₁ و L₉₀₂ و L₉₀₃ و L₉₀₄ و L₉₀₅ و L₉₀₆ و L₉₀₇ و L₉₀₈ و L₉₀₉ و L₉₁₀ و L₉₁₁ و L₉₁₂ و L₉₁₃ و L₉₁₄ و L₉₁₅ و L₉₁₆ و L₉₁₇ و L₉₁₈ و L₉₁₉ و L₉₂₀ و L₉₂₁ و L₉₂₂ و L₉₂₃ و L₉₂₄ و L₉₂₅ و L₉₂₆ و L₉₂₇ و L₉₂₈ و L₉₂₉ و L₉₃₀ و L₉₃₁ و L₉₃₂ و L₉₃₃ و L₉₃₄ و L₉₃₅ و L₉₃₆ و L₉₃₇ و L₉₃₈ و L₉₃₉ و L₉₄₀ و L₉₄₁ و L₉₄₂ و L₉₄₃ و L₉₄₄ و L₉₄₅ و L₉₄₆ و L₉₄₇ و L₉₄₈ و L₉₄₉ و L₉₅₀ و L₉₅₁ و L₉₅₂ و L₉₅₃ و L₉₅₄ و L₉₅₅ و L₉₅₆ و L₉₅₇ و L₉₅₈ و L₉₅₉ و L₉₆₀ و L₉₆₁ و L₉₆₂ و L₉₆₃ و L₉₆₄ و L₉₆₅ و L₉₆₆ و L₉₆₇ و L₉₆₈ و L₉₆₉ و L₉₇₀ و L₉₇₁ و L₉₇₂ و L₉₇₃ و L₉₇₄ و L₉₇₅ و L₉₇₆ و L₉₇₇ و L₉₇₈ و L₉₇₉ و L₉₈₀ و L₉₈₁ و L₉₈₂ و L₉₈₃ و L₉₈₄ و L₉₈₅ و L₉₈₆ و L₉₈₇ و L₉₈₈ و L₉₈₉ و L₉₉₀ و L₉₉₁ و L₉₉₂ و L₉₉₃ و L₉₉₄ و L₉₉₅ و L₉₉₆ و L₉₉₇ و L₉₉₈ و L₉₉₉ و L₁₀₀₀ و L₁₀₀₁ و L₁₀₀₂ و L₁₀₀₃ و L₁₀₀₄ و L₁₀₀₅ و L₁₀₀₆ و L₁₀₀₇ و L₁₀₀₈ و L₁₀₀₉ و L₁₀₁₀ و L₁₀₁₁ و L₁₀₁₂ و L₁₀₁₃ و L₁₀₁₄ و L₁₀₁₅ و L₁₀₁₆ و L₁₀₁₇ و L₁₀₁₈ و L₁₀₁₉ و L₁₀₂₀ و L₁₀₂₁ و L₁₀₂₂ و L₁₀₂₃ و L₁₀₂₄ و L₁₀₂₅ و L₁₀₂₆ و L₁₀₂₇ و L₁₀₂₈ و L₁₀₂₉ و L₁₀₃₀ و L₁₀₃₁ و L₁₀₃₂ و L₁₀₃₃ و L₁₀₃₄ و L₁₀₃₅ و L₁₀₃₆ و L₁