CIVIL ENGINEERING DEPARTMENT

Introduction to Earthquake Engineering

Done by :
Yousef Al Mashaqbah .

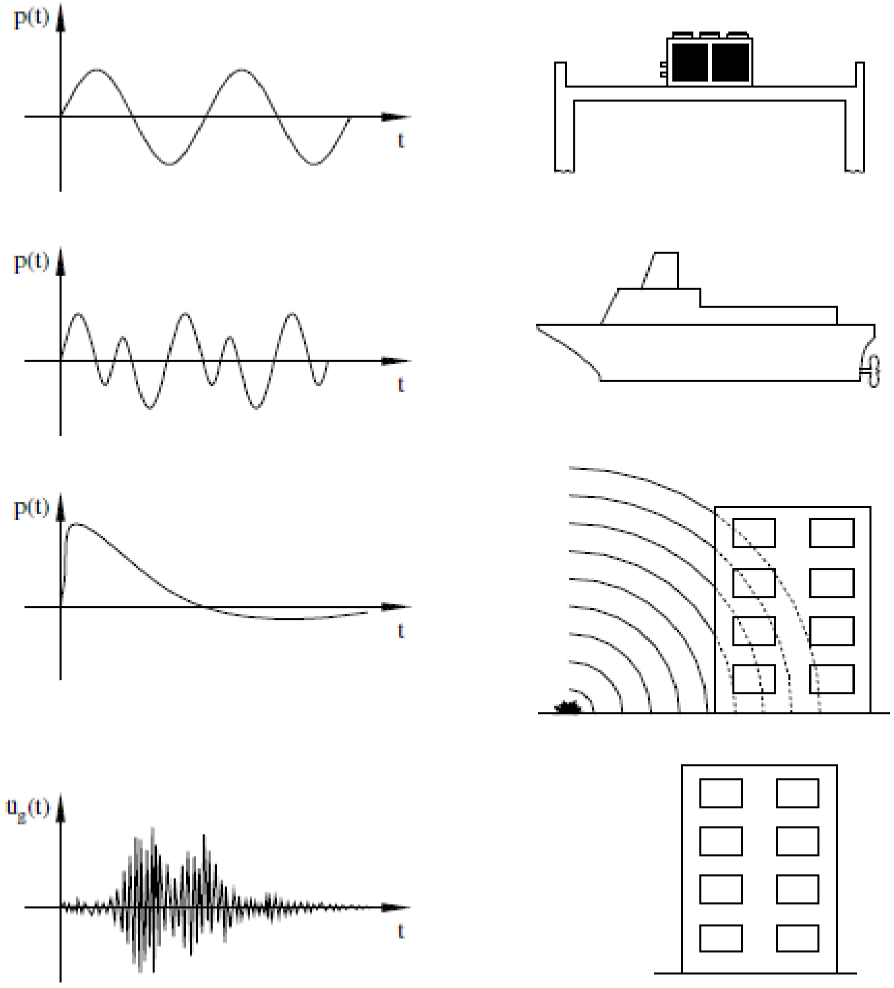
Outlines

- Introduction to earthquake engineering
- Introduction to earthquake seismology
- Dynamics of structures – linear analysis
- Dynamics of structures – nonlinear analysis
- Ductility and nonlinear behavior
- Design response spectra (UBC & IBC)
- Method of analysis – Force based
  (equivalent lateral force)
- beam, column and joint design issues

Dynamics of structures

Dynamics of structures

- Dynamics of structures ⇒ determination of response of structures under the effect of dynamic loading
- Dynamic load is one whose magnitude, direction, sense and point of application changes in time



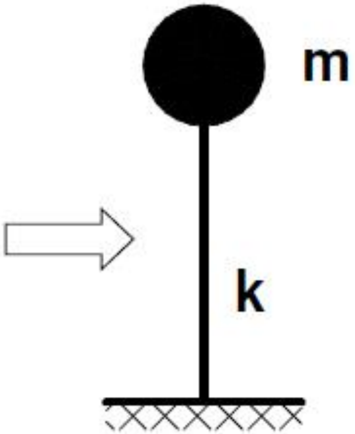
Dynamics of structures

Single degree of freedom systems

- Simple structures:
  - mass *m*
  - stiffness *k*

} Single degree of freedom systems (SDOF)

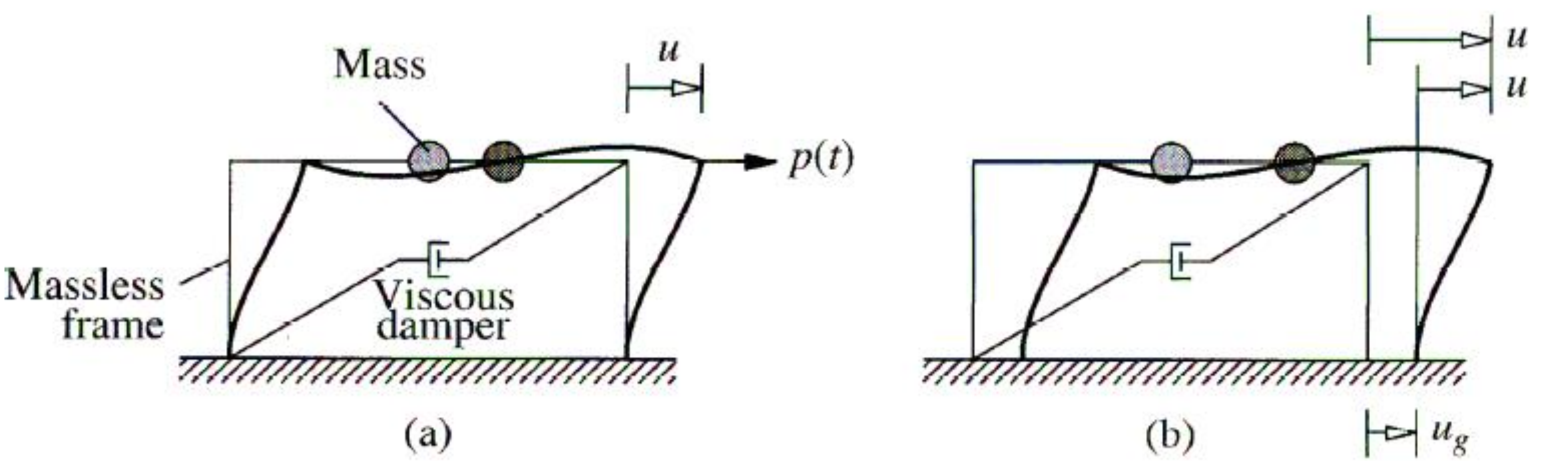
- Objective: find out response of SDOF system under the effect of:
  - a dynamic load acting on the mass
  - a seismic motion of the base of the structure
- The number of degree of freedom (DOF) necessary for dynamic analysis of a structure is the number of independent displacements necessary to define the displaced position of masses with respect to their initial position



Dynamics of structures

Single degree of freedom systems

- One-storey frame =
  - mass component
  - stiffness component
  - damping component
- Number of dynamic degrees of freedom = 1
- Number of static degrees of freedom = ?



•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

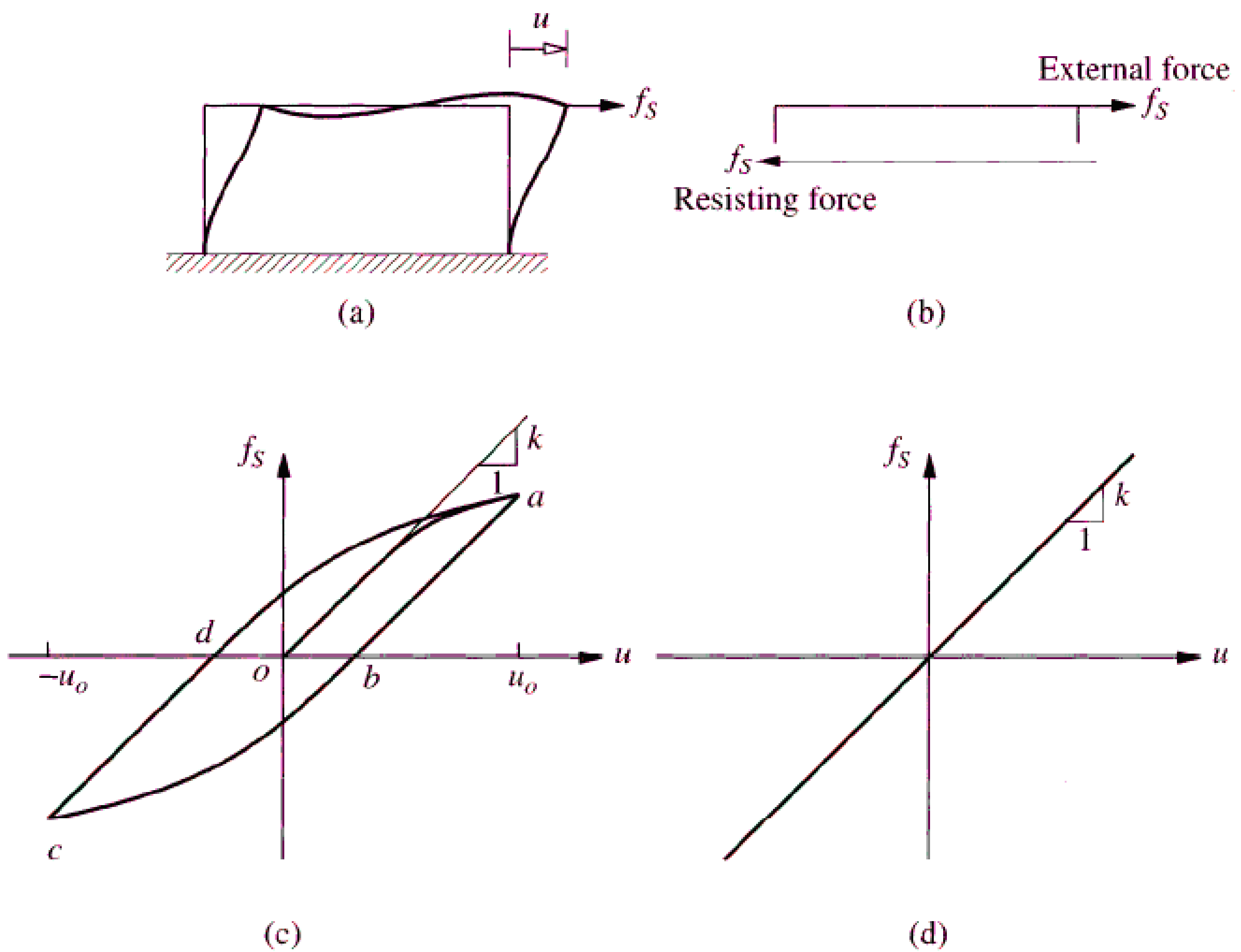
•Design response spectra (UBC & IBC)

•Method of analysis – Force based (equivalent lateral force)

•beam, column and joint design issues

Dynamics of structures

Force-displacement relationship



Dynamics of structures

Force-displacement relationship

- Linear elastic system:
  - elastic material
  - first order analysis

$$f_S = k \cdot u$$

- Inelastic system:
  - plastic material
  - First-order or second-order analysis

$$f_S = f_S(u, \dot{u})$$

•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

•Design response spectra (UBC & IBC)

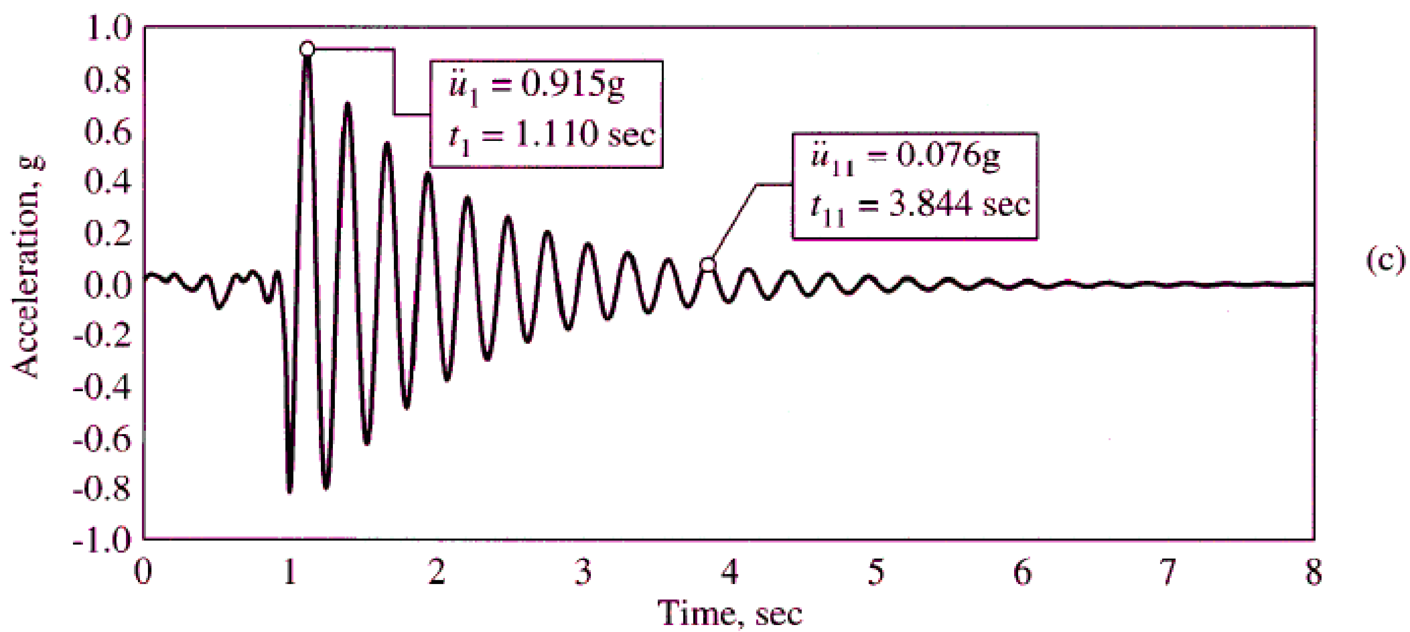
•Method of analysis – Force based (equivalent lateral force)

•beam, column and joint design issues

Dynamics of structures

Damping force

- Damping: decreasing with time of amplitude of vibrations of a system let to oscillate freely
- Cause: thermal effect of elastic cyclic deformations of the material and internal friction



Dynamics of structures

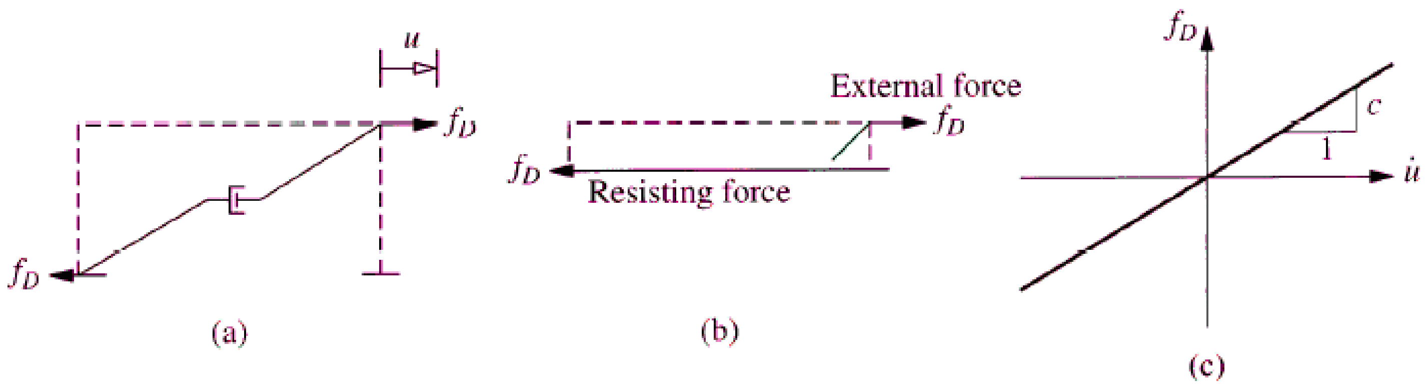
Damping

- Damping in real structures:
  - friction in steel connections
  - opening and closing of microcracks in r.c. elements
  - friction between structural and non-structural elements
- Mathematical description of these components impossible
- Modelling of damping in real structures $\Rightarrow$ equivalent viscous damping

Dynamics of structures

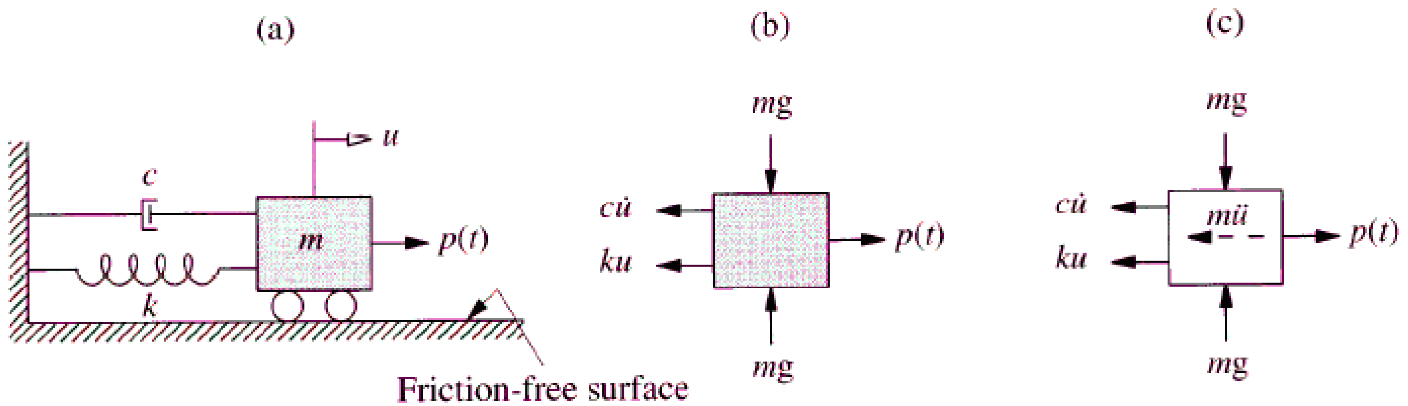
Damping

- Relationship between damping force and velocity: $f_D = c \cdot \dot{u}$
- *c* - viscous damping coefficient
  units: (Force x Time / Length)
- Determination of viscous damping:
  - free vibration tests
  - forced vibration tests
- Equivalent viscous damping ⇒ modelling of the energy dissipated by the structure in the elastic range



Dynamics of structures

SDOF systems: classical representation



•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

•Design response spectra (UBC & IBC)

•Method of analysis – Force based (equivalent lateral force)

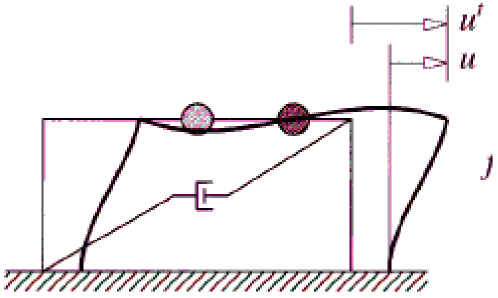
•beam, column and joint design issues

Dynamics of structures

Equation of motion: seismic excitation

- Dynamics of structures in the case of seismic motion ⇒ determination of structural response under the effect of seismic motion applied at the base of the structure
- Ground displacement u_g
- Total (or absolute) displacement of the mass u^t
- Relative displacement between mass and ground u

$$u^t(t) = u(t) + u_g(t)$$



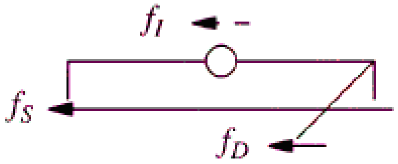
(a)

- Introduction to earthquake engineering
- Introduction to earthquake seismology
- linear analysis
- nonlinear analysis
- Ductility and nonlinear behavior
- Design response spectra (UBC & IBC)
- Method of analysis – Force based (equivalent lateral force)
- beam, column and joint design issues

Dynamics of structures

Equation of motion: seismic excitation

- D'Alambert principle of dynamic equilibrium
- Elastic forces ⇒ relative displacement ***u*** $f_S = k \cdot u$
- Damping forces ⇒ relative displacement ***u*** $f_D = c \cdot \dot{u}$
- Inertia force⇒ total displacement ***u^t^*** $f_I = m\ddot{u}^t$



$$f_I + f_S + f_D = 0$$

$$m\ddot{u}^t + c\dot{u} + ku = 0$$

$$u^t(t) = u(t) + u_g(t)$$

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g$$

•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

•Design response spectra (UBC & IBC)

•Method of analysis – Force based (equivalent lateral force)

•beam, column and joint design issues

Dynamics of structures

Equation of motion: seismic excitation

- Equation of motion in the case of an external force

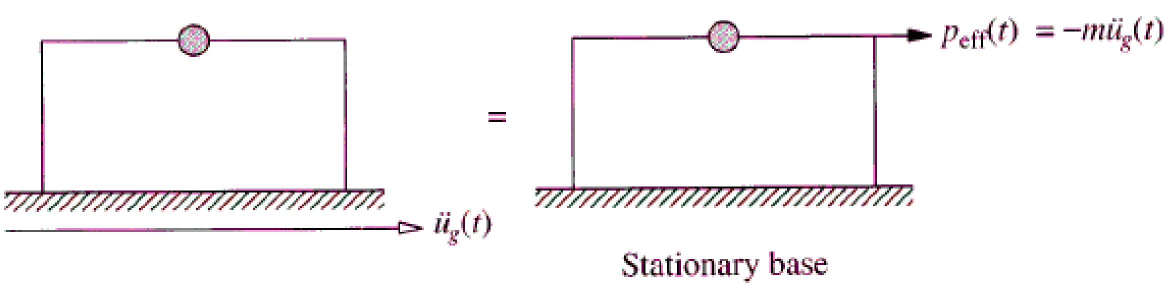
$$m\ddot{u} + c\dot{u} + ku = p(t)$$

- Equation of motion in the case of seismic excitation

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g$$

- Equation of motion for a system subjected to seismic motion described by ground acceleration $\ddot{u}_g$ is identical to that of a system subjected to an external force $-m\ddot{u}_g$
- Effective seismic force

$$p_{eff}(t) = -m\ddot{u}_g(t)$$



Dynamics of structures

Problem formulation

- Fundamental problem in dynamics of structures: determination of the response of a (SDOF) system under a dynamic excitation
  - a external force
  - ground acceleration applied to the base of the structure
- "Response" $\Rightarrow$ any quantity that characterizes behaviour of the structure
  - displacement
  - velocity
  - mass acceleration
  - forces and stresses in structural members

Dynamics of structures

Determination of element forces

- Solution of the equation of motion of the SDOF system ⇒ displacement time history $u(t)$
- Displacements ⇒ forces in structural elements
  - Imposed displacements ⇒ forces in structural elements
  - Equivalent static force: an external static force f_S that produces displacements u determined from dynamic analysis

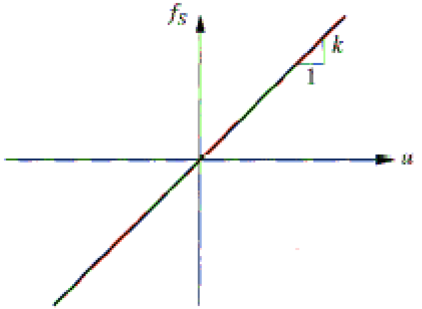
$$f_s(t) = ku(t)$$

Forces in structural elements ⇒ by static analysis of the structure subjected to equivalent seismic forces f_S

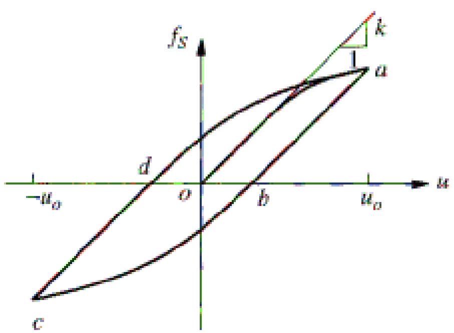
Dynamics of structures

Combination of static and dynamic response

- **Linear elastic systems: superposition of effects possible ⇒ total response can be determined through the superposition of the results obtained from:**
  - static analysis of the structure under permanent and live loads, temperature effects, etc.
  - dynamic response of the structure



- **Inelastic systems: superposition of effects NOT possible ⇒ dynamic response must take account of deformations and forces existing in the structure before application of dynamic excitation**



•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

•Design response spectra (UBC & IBC)

•Method of analysis – Force based (equivalent lateral force)

•beam, column and joint design issues

Dynamics of structures

Solution of the equation of motion

- Equation of motion of a SDOF system

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = p(t)$$

differential linear non-homogeneous equation of second order

- In order to completely define the problem:
  - initial displacement $u(0)$
  - initial velocity $\dot{u}(0)$

- Solution methods:
  - Classical solution
  - Duhamel integral
  - Numerical techniques

Dynamics of structures

Classical solution

- Complete solution $u(t)$ of a linear non-homogeneous differential equation of second order is composed of
  - complementary solution $u_c(t)$ and
  - particular solution $u_p(t)$
    $u(t) = u_c(t) + u_p(t)$
- Second order equation $\Rightarrow$ 2 integration constants $\Leftarrow$ initial conditions
- Classical solution useful in the case of
  - free vibrations
  - forces vibrations, when dynamic excitation is defined analytically

Dynamics of structures

Classical solution: example

- Equation of motion of an undamped (c=0) SDOF system excited by a step force $p(t)=p_0$, $t\geq 0$:

$$m\ddot{u} + ku = p_0$$

- Particular solution: $u_p(t) = \dfrac{p_0}{k}$
- Complementary solution: $u_c(t) = A\cos\omega_n t + B\sin\omega_n t$
- where A and B are integration constants and $\omega_n = \sqrt{k/m}$
- The complete solution $u(t) = A\cos\omega_n t + B\sin\omega_n t + \dfrac{p_0}{k}$
- Initial conditions: for t=0 we have $u(0)=0$ and $\dot{u}(0)=0 \Rightarrow$

$A = -\dfrac{p_0}{k}$ $\quad B = 0$ the eq. of motion $u(t) = \dfrac{p_0}{k}(1-\cos\omega_n t)$

Dynamics of structures

Undamped free vibrations

- General form of the equation of motion: $m\ddot{u} + c\dot{u} + ku = p(t)$
- Equation of motion in the case of undamped free vibrations:
  $m\ddot{u} + ku = 0$
- Vibrations $\Rightarrow$ the system disturbed from the static equilibrium position by
  - initial displacement $u(0)$
  - initial velocity $\dot{u}(0)$
- Classical solution $\Rightarrow$

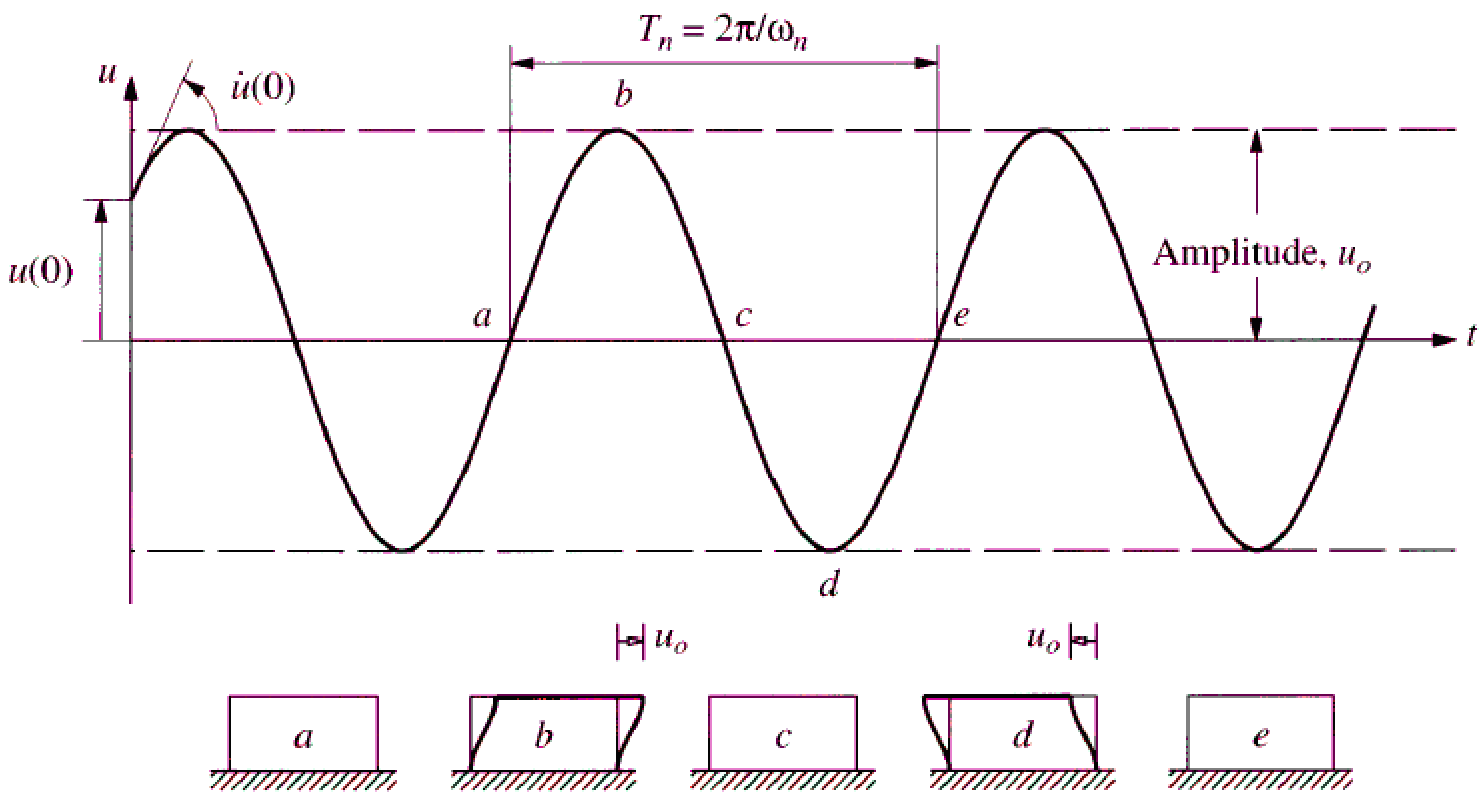
$$u(t) = u(0)\cos\omega_n t + \frac{\dot{u}(0)}{\omega_n}\sin\omega_n t$$

where $\omega_n = \sqrt{k/m}$

Dynamics of structures

Undamped free vibrations

- Simple harmonic motion $u(t) = u(0)\cos\omega_n t + \frac{\dot{u}(0)}{\omega_n}\sin\omega_n t$



Dynamics of structures

Undamped free vibrations

- Natural period of vibration - T_n - time needed for an undamped SDOF system to perform a complete cycle of free vibrations
- Natural circular frequency $$T_n = \frac{2\pi}{\omega_n}$$
- Natural frequency of vibration f_n represents the number of complete cycles performed by the system in one second $$f_n = \frac{1}{T_n} \qquad f_n = \frac{\omega_n}{2\pi}$$

$$\omega_n = \sqrt{k/m}$$

mass stiffness

- "Natural" - depends only on the properties of the SDOF system

•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

•Design response spectra (UBC & IBC)

•Method of analysis – Force based (equivalent lateral force)

•beam, column and joint design issues

Dynamics of structures

Undamped free vibrations

- Alternative expressions for ω_n, f_n, T_n:

$$\omega_n = \sqrt{\frac{g}{\delta_{st}}} \qquad f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}} \qquad T_n = 2\pi\sqrt{\frac{\delta_{st}}{g}}$$

$\delta_{st} = mg/k$ elastic deformation of a SDOF system under a static force equal to *mg*

- Amplitude: magnitude of oscillations

$$u_0 = \sqrt{\left[u(0)\right]^2 + \left[\frac{\dot{u}(0)}{\omega_n}\right]^2}$$

Dynamics of structures

Damped free vibrations

- General form of the eq. of motion: $m\ddot{u}+c\dot{u}+ku=p(t)$
- Equation of motion in the case of damped free vibrations:
  $m\ddot{u}+c\dot{u}+ku=0$
- Dividing eq. by *m* we obtain $\ddot{u}+2\xi\omega_n\dot{u}+\omega_n^2u=0$
- with the notations: $\omega_n=\sqrt{k/m}$ $\quad \xi=\dfrac{c}{2m\omega_n}=\dfrac{c}{c_{cr}}$
- Critical damping coefficient $c_{cr}=2m\omega_n=2\sqrt{km}=\dfrac{2k}{\omega_n}$
- Damping coefficient *c* - a measure of the energy dissipated in a complete cycle
- ξ - critical damping ratio: a non-dimensional measure of damping, which depends on the stiffness and mass as well

•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

•Design response spectra (UBC & IBC)

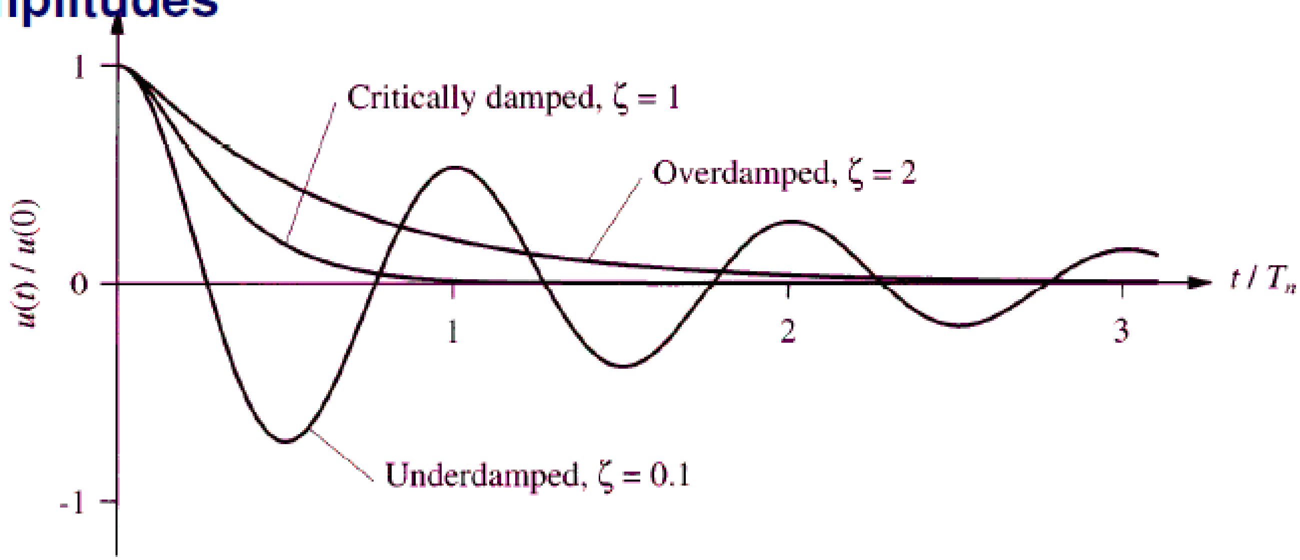
•Method of analysis – Force based (equivalent lateral force)

•beam, column and joint design issues

Dynamics of structures

Types of motion

- $c = c_{cr}$ or $\xi = 1 \Rightarrow$ the system returns to the position of equilibrium without oscillation
- $c > c_{cr}$ or $\xi > 1 \Rightarrow$ the system returns to the position of equilibrium without oscillation, but slower
- $c < c_{cr}$ or $\xi < 1 \Rightarrow$ the system oscillates with respect to the equilibrium position with progressively decreasing amplitudes



- •Introduction to earthquake engineering
- •Introduction to earthquake seismology
- •linear analysis
- •nonlinear analysis
- •Ductility and nonlinear behavior
- •Design response spectra (UBC & IBC)
- •Method of analysis – Force based (equivalent lateral force)
- •beam, column and joint design issues

Dynamics of structures

Types of motion

- c_{cr} - the smallest value of the damping coefficient that completely prevents oscillations
- Most engineering structures - underdamped ($c<c_{cr}$)
- Few reasons to study:
  - critically damped systems ($c=c_{cr}$)
  - overdamped systems ($c>c_{cr}$)

•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

•Design response spectra (UBC & IBC)

•Method of analysis – Force based (equivalent lateral force)

•beam, column and joint design issues

Dynamics of structures

Damping for engineering structures

stress level	structural type	ξ (%)
stress level below 0.5 times the yield strength	welded steel structures, prestressed concrete structures, strongly reinforced concrete structures (limited cracks)	2-3
	reinforced concrete structures with significant cracking	3-5
	steel structures with bolted or riveted connections, wood structures connected with screws or nails	5-7
stresses close to the yield strength	welded steel structures, prestressed concrete structures (without total loss of prestress)	5-7
	prestressed concrete structures with total loss of prestress	7-10
	reinforced concrete structures	7-10
	steel structures with bolted or riveted connections, wood structures connected with screws	10-15
	wood structures connected with nails	15-20

•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

•Design response spectra (UBC & IBC)

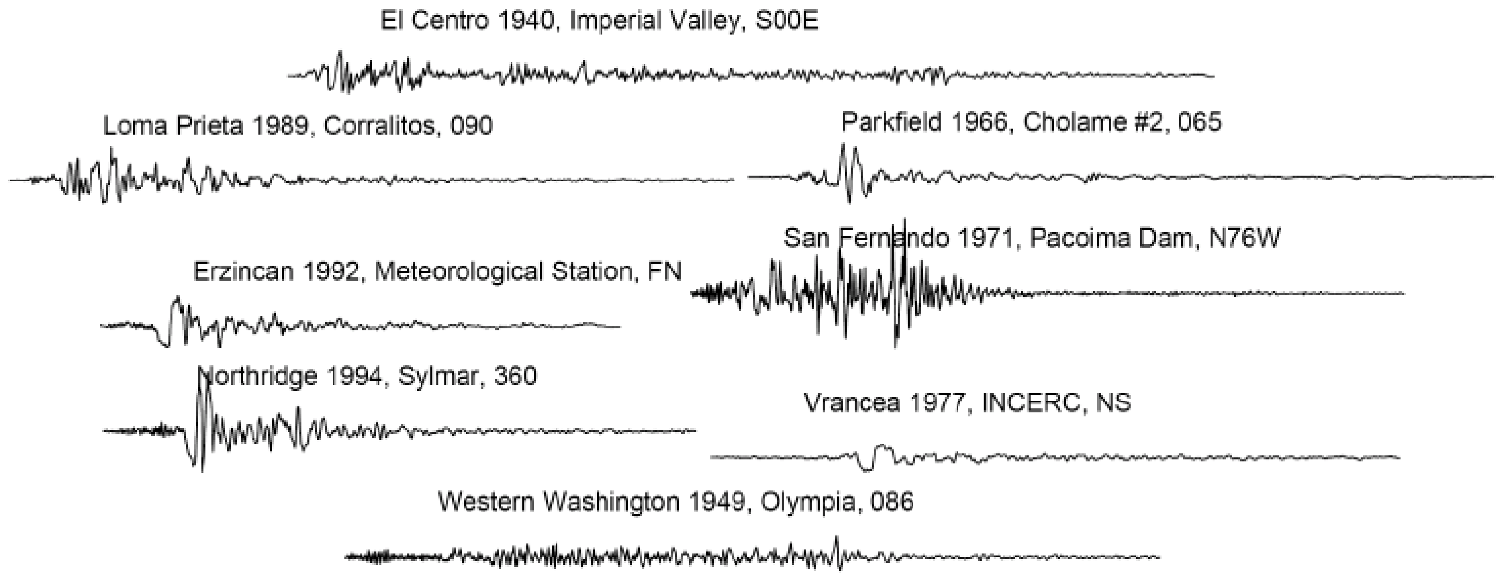
•Method of analysis – Force based (equivalent lateral force)

•beam, column and joint design issues

Dynamics of structures

Seismic action

- Ground acceleration: accelerogram



- Properties of a SDOF system (m, c, k) + $\ddot{u}_g(t) \Rightarrow$

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g$$

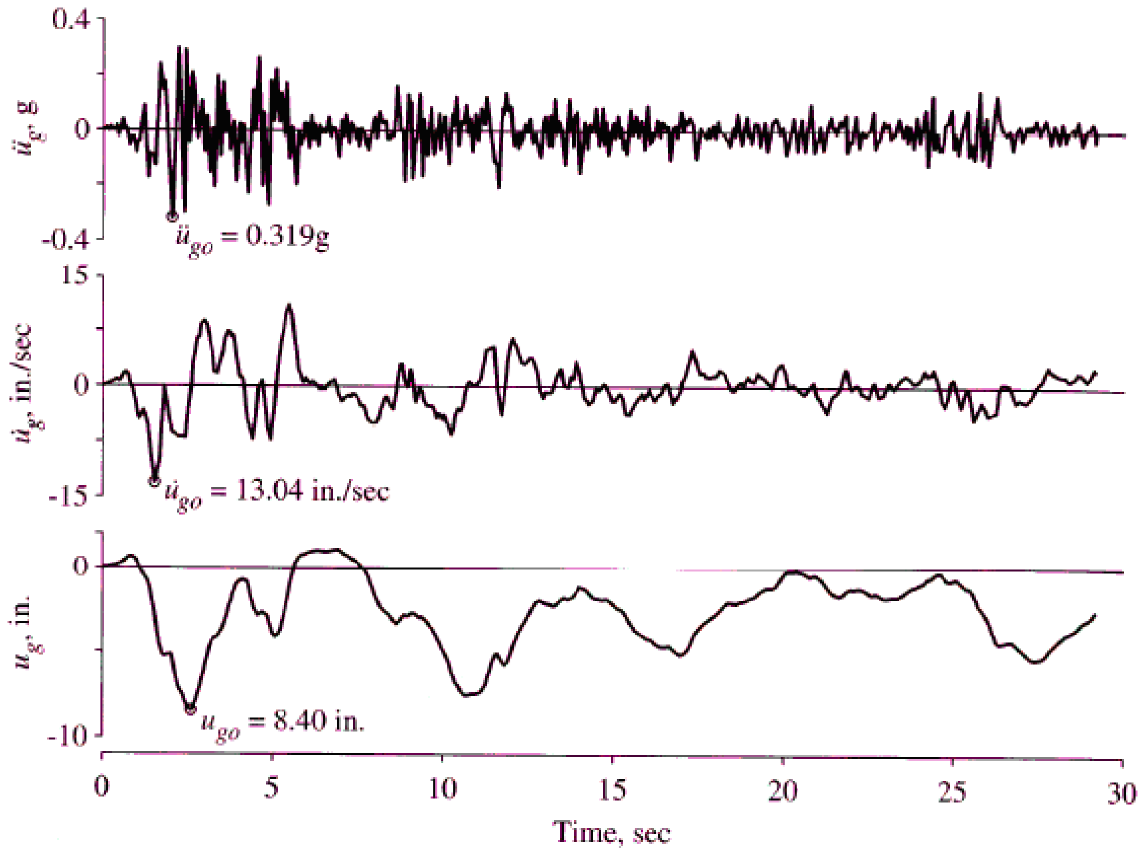
- Relative displacement, velocity and acceleration of a SDOF system

- Introduction to earthquake engineering
- Introduction to earthquake seismology
- linear analysis
- nonlinear analysis
- Ductility and nonlinear behavior
- Design response spectra (UBC & IBC)
- Method of analysis – Force based (equivalent lateral force)
- beam, column and joint design issues

Dynamics of structures

Seismic action

- North-south component of the El Centro, California record during Imperial Valley earthquake from 18.05.1940



Dynamics of structures

Determination of seismic response

- **Equation of motion:** $m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g$
- **/m:** $\ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2 u = -\ddot{u}_g$
- **Numerical methods**
  - central difference method
  - Newmark method
  - ...

$$\Rightarrow \quad u \equiv u(t, T_n, \xi)$$

- **Response depends on:**
  - natural circular frequency ω_n (or natural period T_n)
  - critical damping ratio ξ
  - ground motion $\ddot{u}_g$

•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

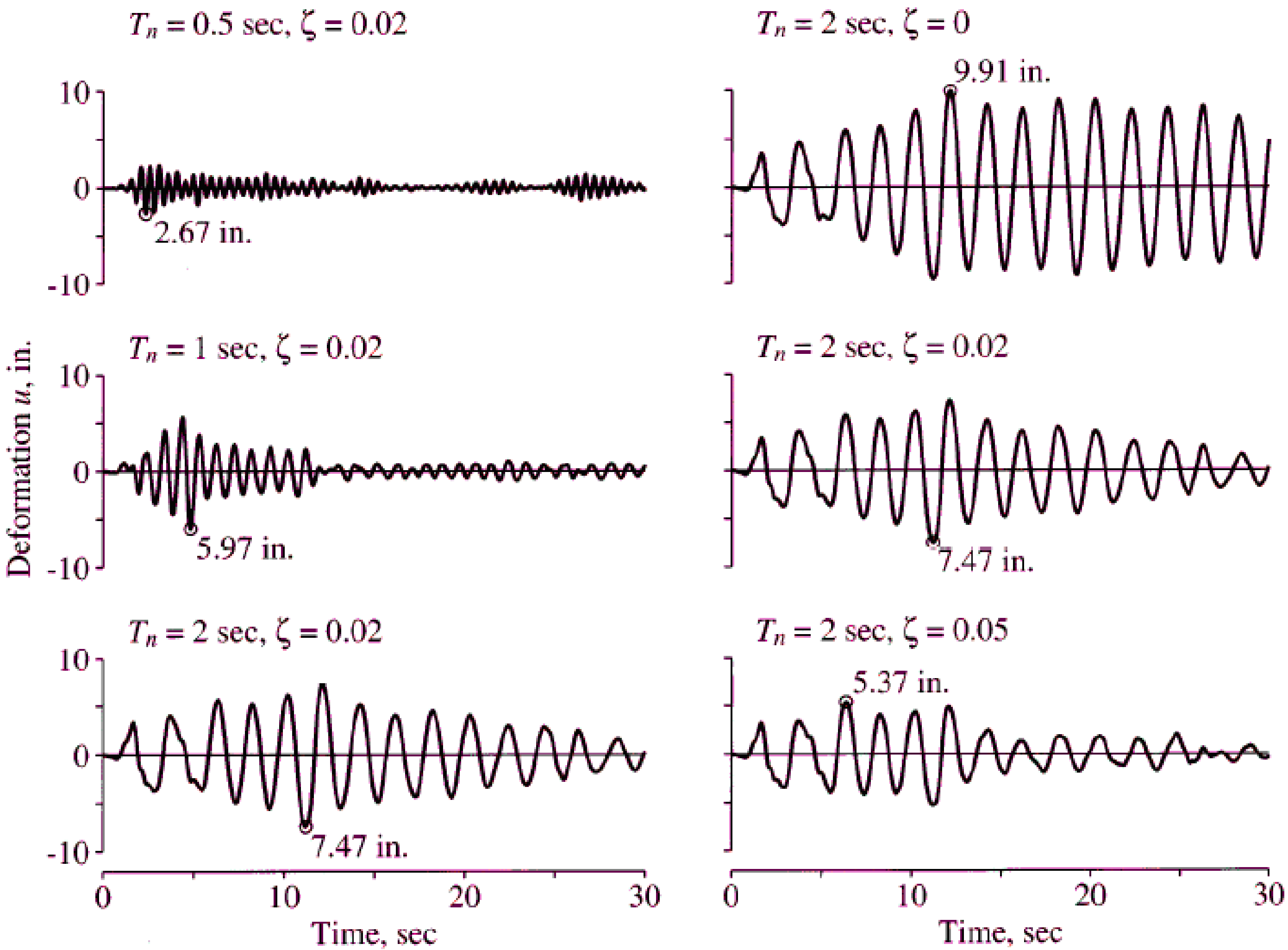
•Design response spectra (UBC & IBC)

•Method of analysis – Force based (equivalent lateral force)

•beam, column and joint design issues

Dynamics of structures

Seismic response



- Introduction to earthquake engineering
- Introduction to earthquake seismology
- linear analysis
- nonlinear analysis
- Ductility and nonlinear behavior
- Design response spectra (UBC & IBC)
- Method of analysis – Force based (equivalent lateral force)
- beam, column and joint design issues

Dynamics of structures

Elastic response spectra

- Response spectrum: representation of peak values of seismic response (displacement, velocity, acceleration) of a SDOF system versus natural period of vibration, for a given critical damping ratio

$$u_0(T_n,\xi) = \max_t \left|u(t,T_n,\xi)\right|$$

$$\dot{u}_0(T_n,\xi) = \max_t \left|\dot{u}(t,T_n,\xi)\right|$$

$$\ddot{u}_0^t(T_n,\xi) = \max_t \left|\ddot{u}^t(t,T_n,\xi)\right|$$

•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

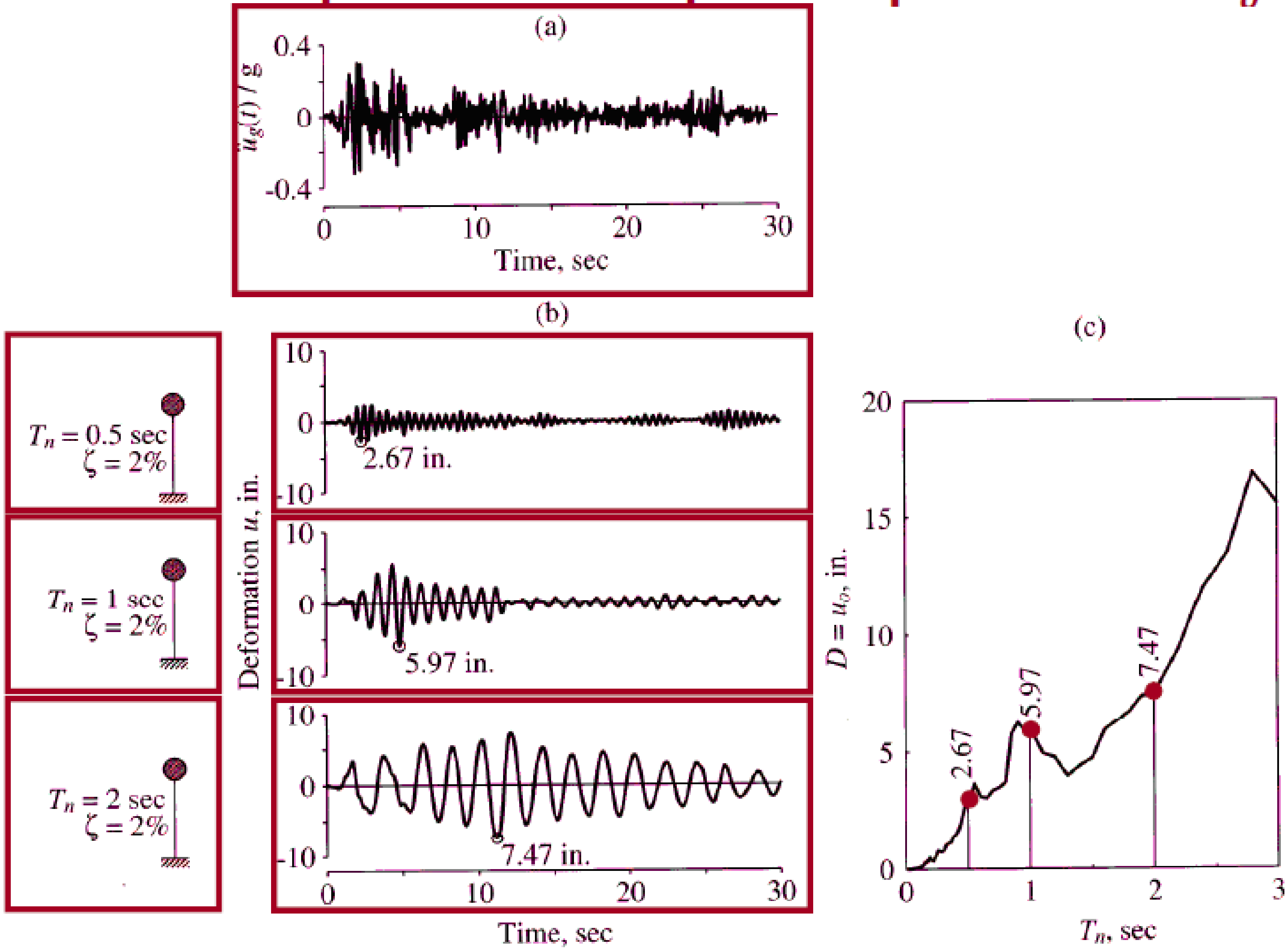
•Design response spectra (UBC & IBC)

•Method of analysis – Force based (equivalent lateral force)

•beam, column and joint design issues

Dynamics of structures

Elastic displacement response spectrum: $D \equiv u_0$



Dynamics of structures

Pseudo-velocity and pseudo-acceleration

- **Spectral pseudo-velocity:**
  - units of velocity
  - different from peak velocity

$$V = \omega_n D = \frac{2\pi}{T_n} D$$

- **Strain energy**

$$E_{S0} = \frac{k u_0^2}{2} = \frac{k D^2}{2} = \frac{k \left(V/\omega_n\right)^2}{2} = \frac{m V^2}{2}$$

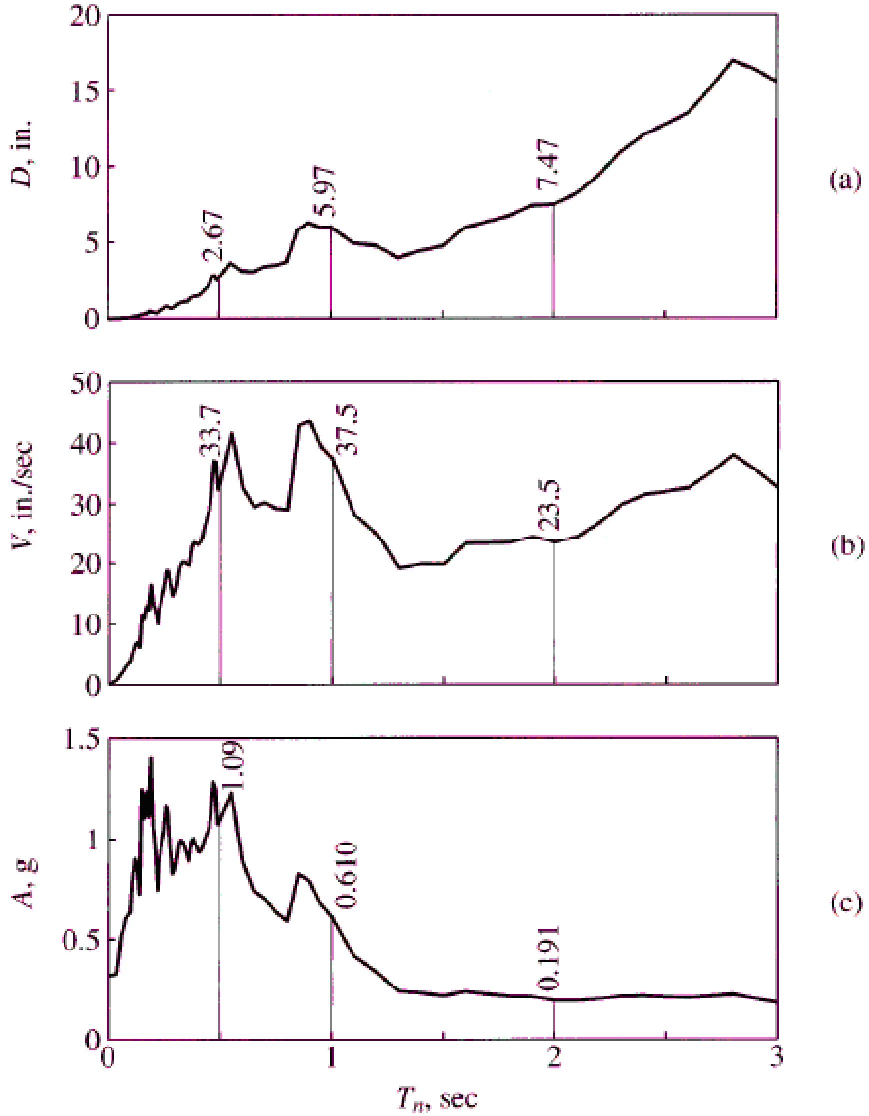
- **Spectral pseudo-acceleration:**

$$f_{S0} = k u_0 = m\omega_n^2 u_0 = mA \Rightarrow A = \omega_n^2 u_0 = \omega_n^2 D = \left(\frac{2\pi}{T_n}\right)^2 D$$

  - units of acceleration
  - different from peak acceleration

- Introduction to earthquake engineering
- Introduction to earthquake seismology
- **linear analysis**
- nonlinear analysis
- Ductility and nonlinear behavior
- Design response spectra (UBC & IBC)
- Method of analysis – Force based (equivalent lateral force)
- beam, column and joint design issues

Dynamics of structures



$$D$$

$$V = \omega_n D = \frac{2\pi}{T_n} D$$

$$A = \omega_n^2 D = \left(\frac{2\pi}{T_n}\right)^2 D$$

Dynamics of structures

Combined D-V-A spectrum

- Displacement, pseudo-velocity and pseudo-acceleration spectra:
  - same information
  - different physical meaning

$$\frac{A}{\omega_n} = V = \omega_n D \quad or \quad \frac{T_n}{2\pi} A = V = \frac{2\pi}{T_n} D$$

$$(T_n / 2\pi) A = V \Longrightarrow \lg T_n + \lg A - \lg 2\pi = \lg V$$

- A line inclined at +45º for lg*A* - lg2π = const. $\Rightarrow$ spectral pseudo-acceleration: an axis inclined to -45º

- Similarly, spectral displacement: an axis inclined to +45º

•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

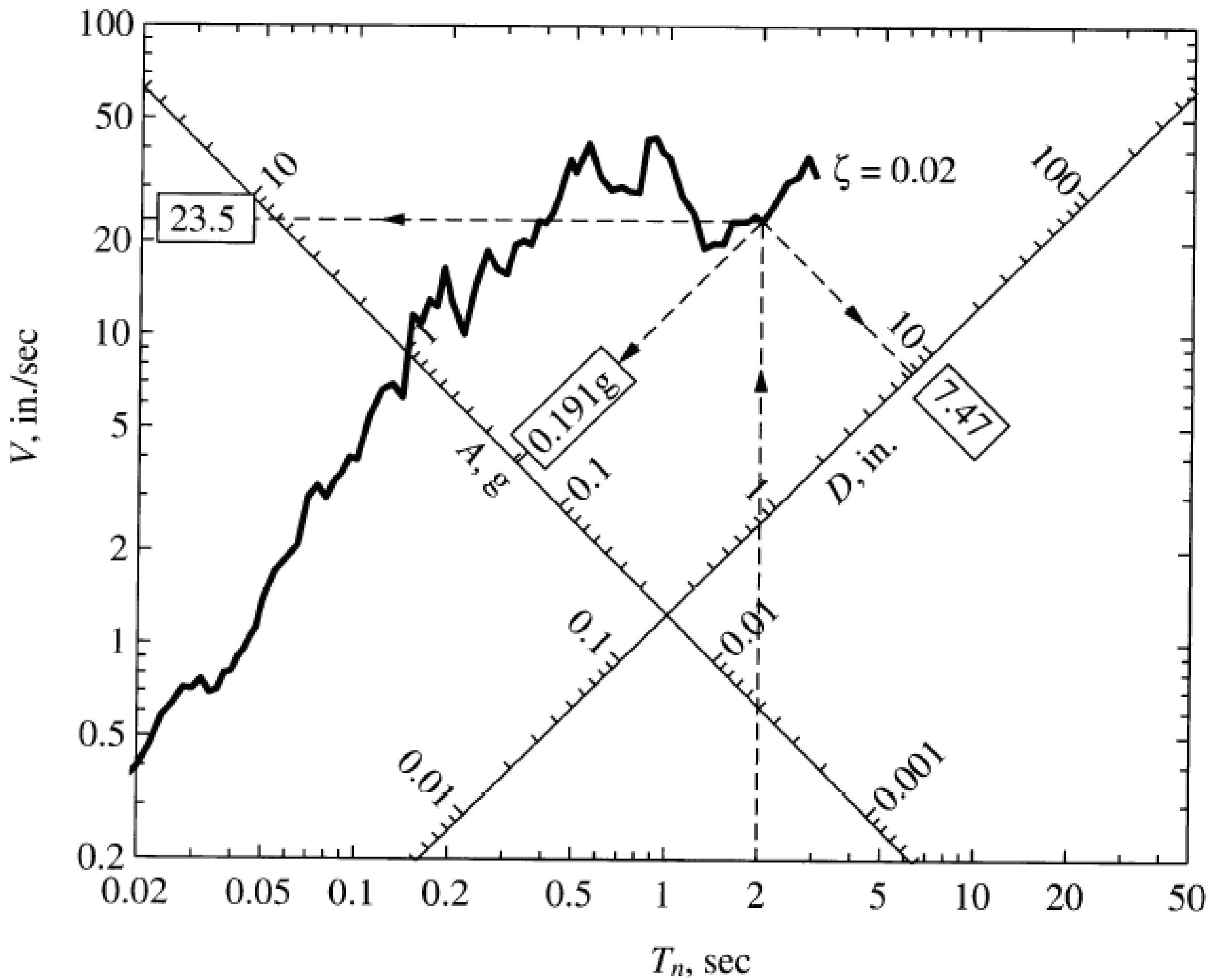
•Design response spectra (UBC & IBC)

•Method of analysis – Force based
(equivalent lateral force)

•beam, column and joint design issues

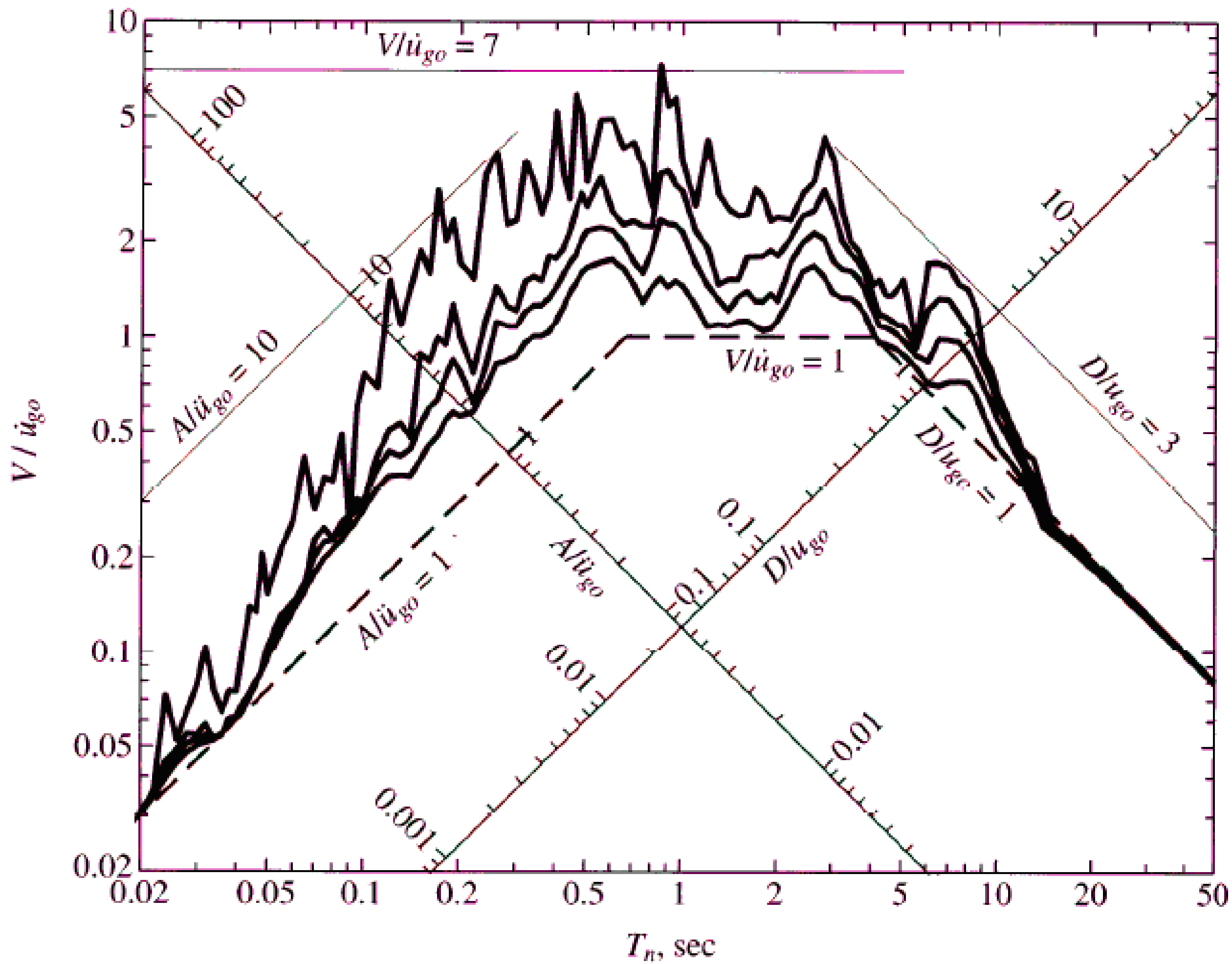
Dynamics of structures

Combined D-V-A spectrum



Dynamics of structures

Characteristics of elastic response spectra



•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

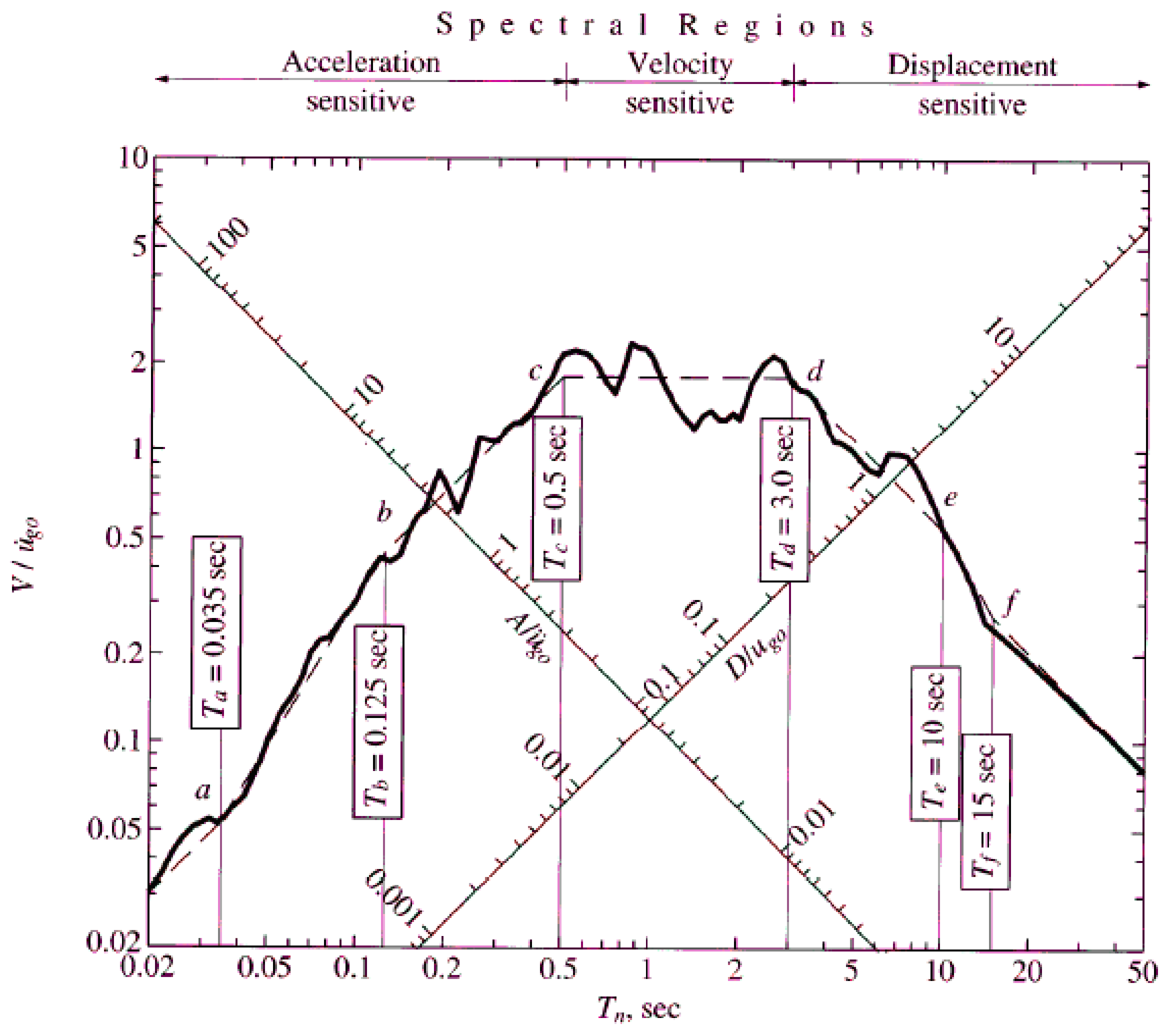
•Design response spectra (UBC & IBC)

•Method of analysis – Force based (equivalent lateral force)

•beam, column and joint design issues

Dynamics of structures

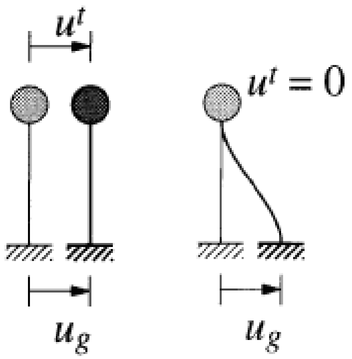
Characteristics of elastic response spectra



Dynamics of structures

Characteristics of elastic response spectra

- For $T_n < T_a$
  - pseudo-acceleration A is close to $\ddot{u}_{g0}$
  - spectral displacement D is small
- For $T_n > T_f$
  - spectral displacement D is close to u_{g0}
  - spectral pseudo-acceleration A is small
- Between T_a and $T_c \Rightarrow A > \ddot{u}_{g0}$
- Between T_b and $T_c \Rightarrow A$ can be considered constant
- Between T_d and $T_f \Rightarrow D > u_{g0}$
- Between T_d and $T_e \Rightarrow D$ can be considered constant
- Between T_c and $T_d \Rightarrow V > \dot{u}_{g0}$
- Between T_c and $T_d \Rightarrow V$ can be considered constant



Dynamics of structures

Characteristics of elastic response spectra

- $T_n > T_d \Rightarrow$ response region sensible to displacements
- $T_n < T_c \Rightarrow$ response region sensible to accelerations
- $T_c < T_n < T_d \Rightarrow$ response region sensible to velocity

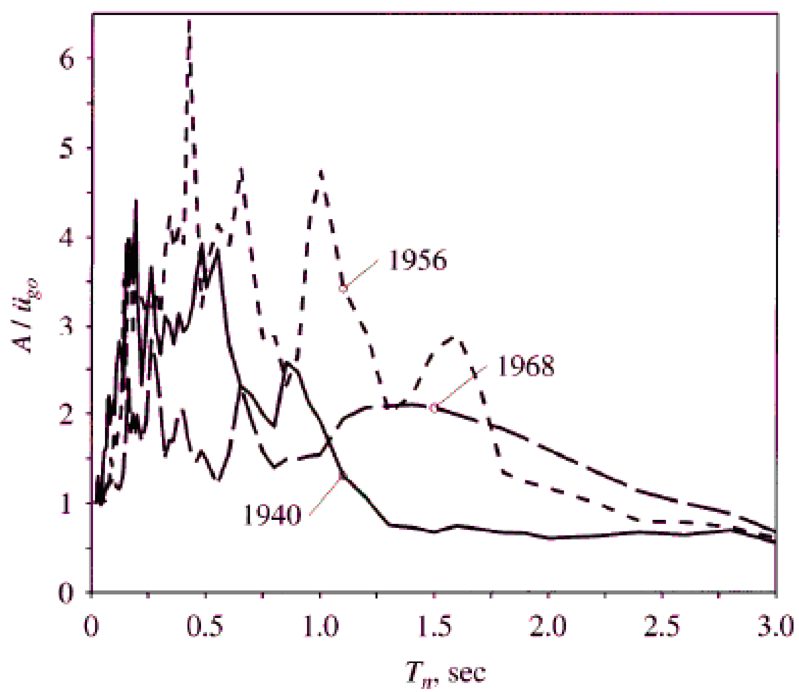
- Larger damping:
  - smaller values of displacements, pseudo-velocity and pseudo-acceleration
  - more "smooth" spectra

- Effect of damping:
  - insignificant for $T_n \rightarrow 0$ and $T_n \rightarrow \infty$,
  - important for $T_b < T_n < T_d$

Dynamics of structures

Elastic design spectra

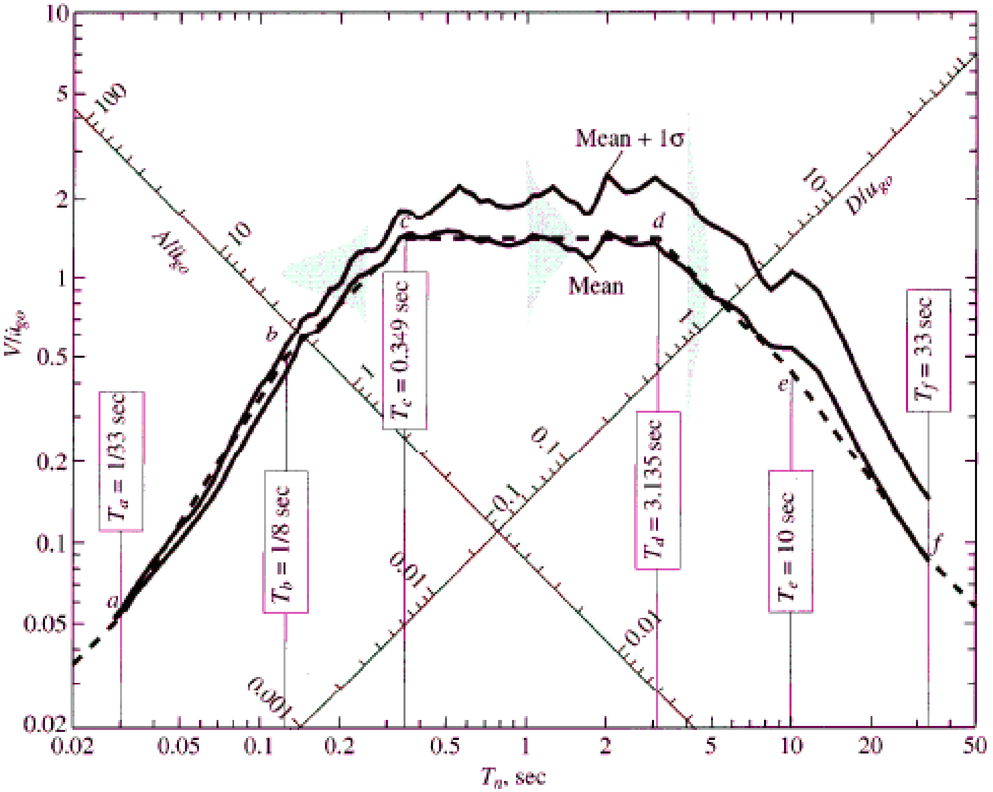
- Spectra of past ground motions:
  - jagged shape
  - variation of response for different earthquakes
  - areas where previous data is not available



Dynamics of structures

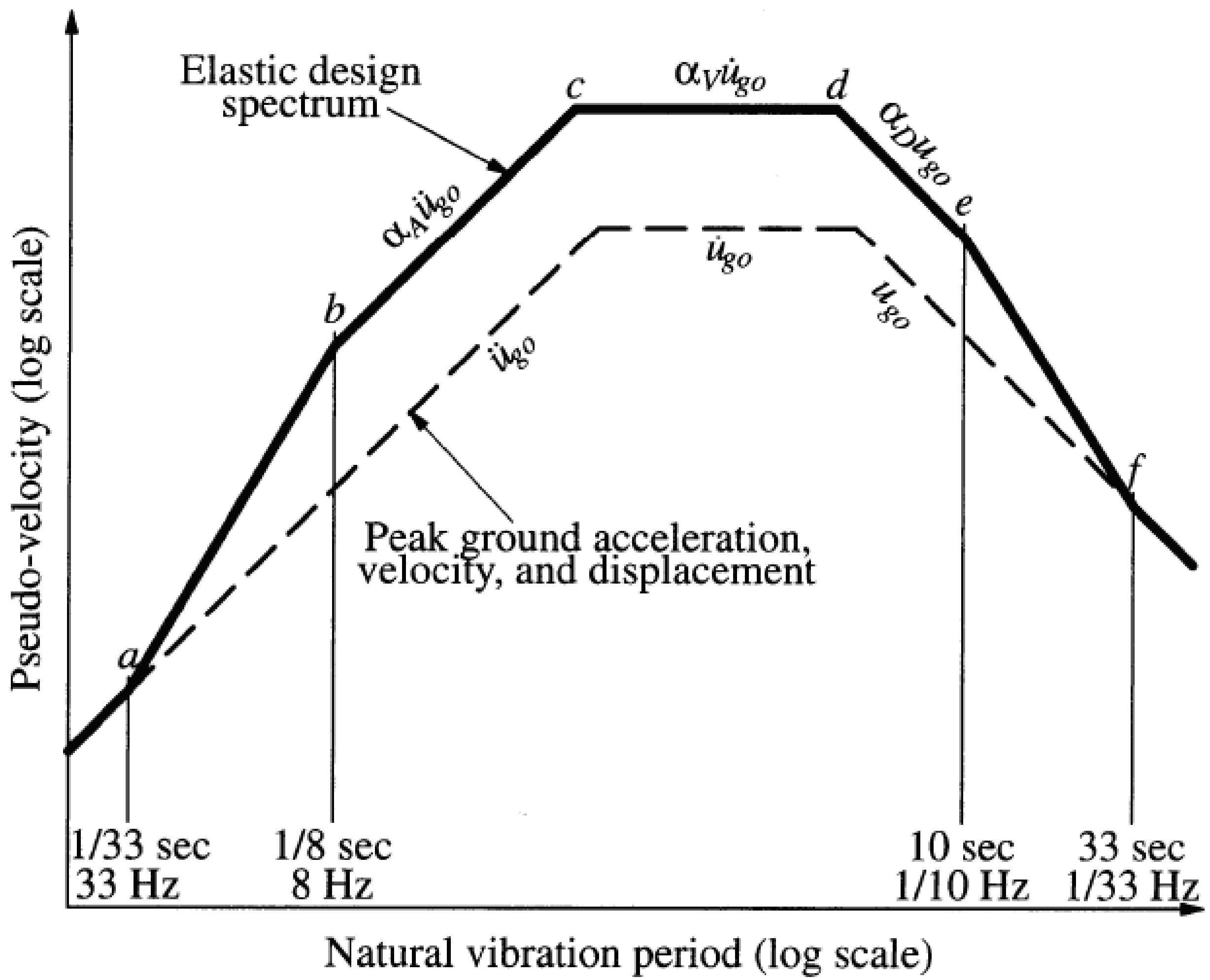
Elastic design spectra

- idealized "smooth" spectra
- based on statistical interpretation (median; median plus standard deviation) of several records characteristic for a given site



Dynamics of structures

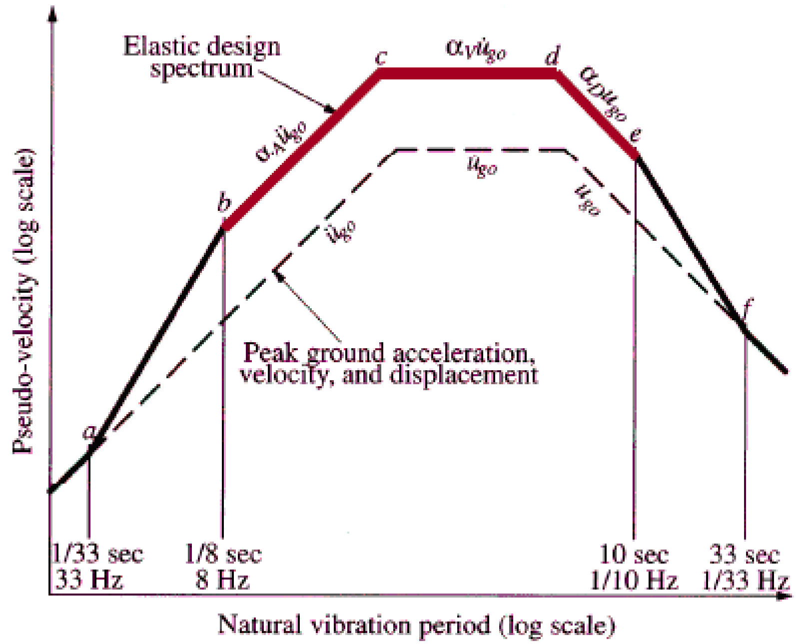
Elastic design spectra

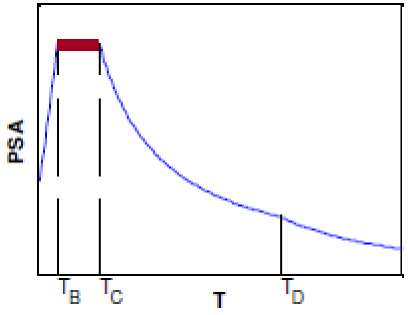


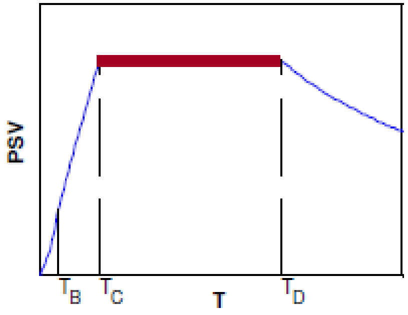
- •Introduction to earthquake engineering
- •Introduction to earthquake seismology
- •linear analysis
- •nonlinear analysis
- •Ductility and nonlinear behavior
- •Design response spectra (UBC & IBC)
- •Method of analysis – Force based (equivalent lateral force)
- •beam, column and joint design issues

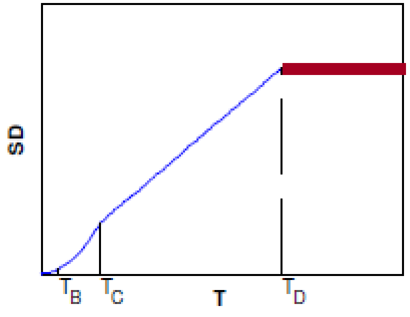
Dynamics of structures

Elastic design spectra





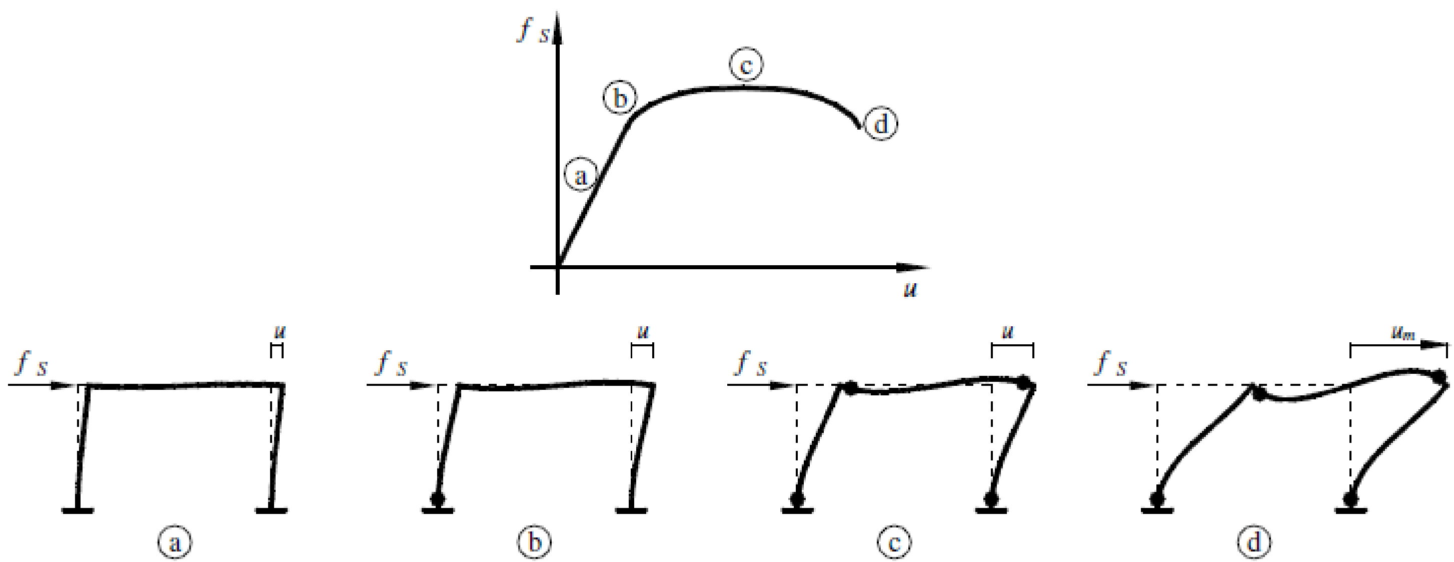




Dynamics of structures

Inelastic response of SDOF systems

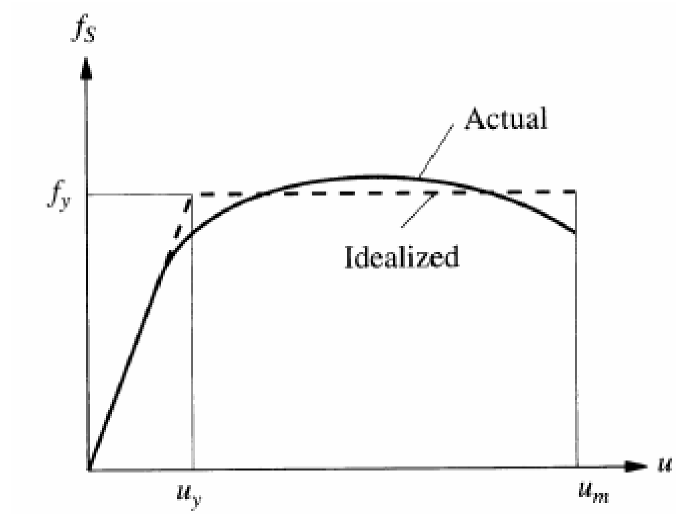
- Most structures designed for seismic forces lower than the ones assuring an elastic response during the design earthquake
  - design of structures in the elastic range for rare seismic events considered uneconomical
  - in the past, structures designed for a fraction of the forces necessary for an elastic response, survived major earthquakes



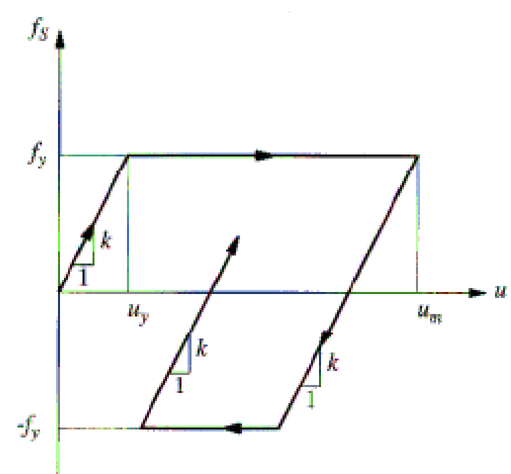
Dynamics of structures

Inelastic response of SDOF systems

- Elasto-plastic system:
  - stiffness k
  - yield force f_y
  - yield displacement u_y
- Elasto-plastic idealization: equal area under the actual and idealised curves up to the maximum displacement u_m



- Cyclic response of the elasto-plastic system



Dynamics of structures

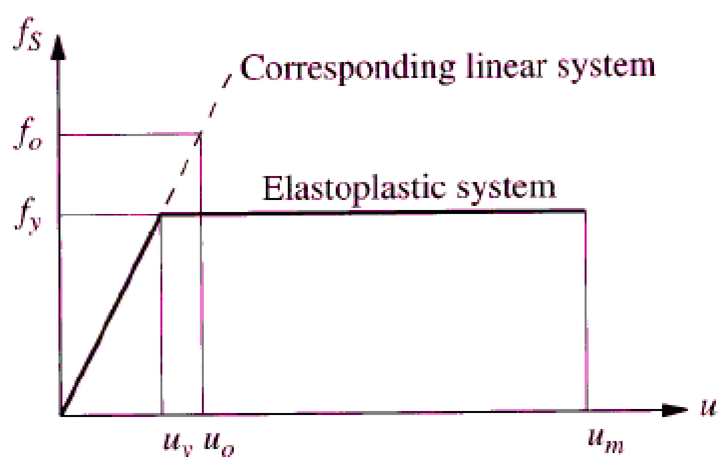
Corresponding elastic system

- Corresponding elastic system:
  - same stiffness
  - same mass
  - same damping

  } the same period of vibration (at small def.)

- Inelastic response:
  - yield force reduction factor R_y

$$R_y = \frac{f_0}{f_y} = \frac{u_0}{u_y}$$



  - ductility factor

$$\mu = \frac{u_m}{u_y}$$

Dynamics of structures

Equation of motion

- **Equation of motion:** $m\ddot{u} + c\dot{u} + f_S(u,\dot{u}) = -m\ddot{u}_g$

- /m $\Rightarrow$ $\ddot{u} + 2\xi\omega_n\dot{u} + \omega_n^2 u_y \tilde{f}_S(u,\dot{u}) = -\ddot{u}_g$

  $$\tilde{f}_S(u,\dot{u}) = f_S(u,\dot{u}) / f_y$$

- **Seismic response of an inelastic SDOF system depends on:**
  - natural circular frequency of vibration ω_n
  - critical damping ratio ξ
  - yield displacement u_y
  - force-displacement shape $\tilde{f}_S(u,\dot{u})$

•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

•Design response spectra (UBC & IBC)

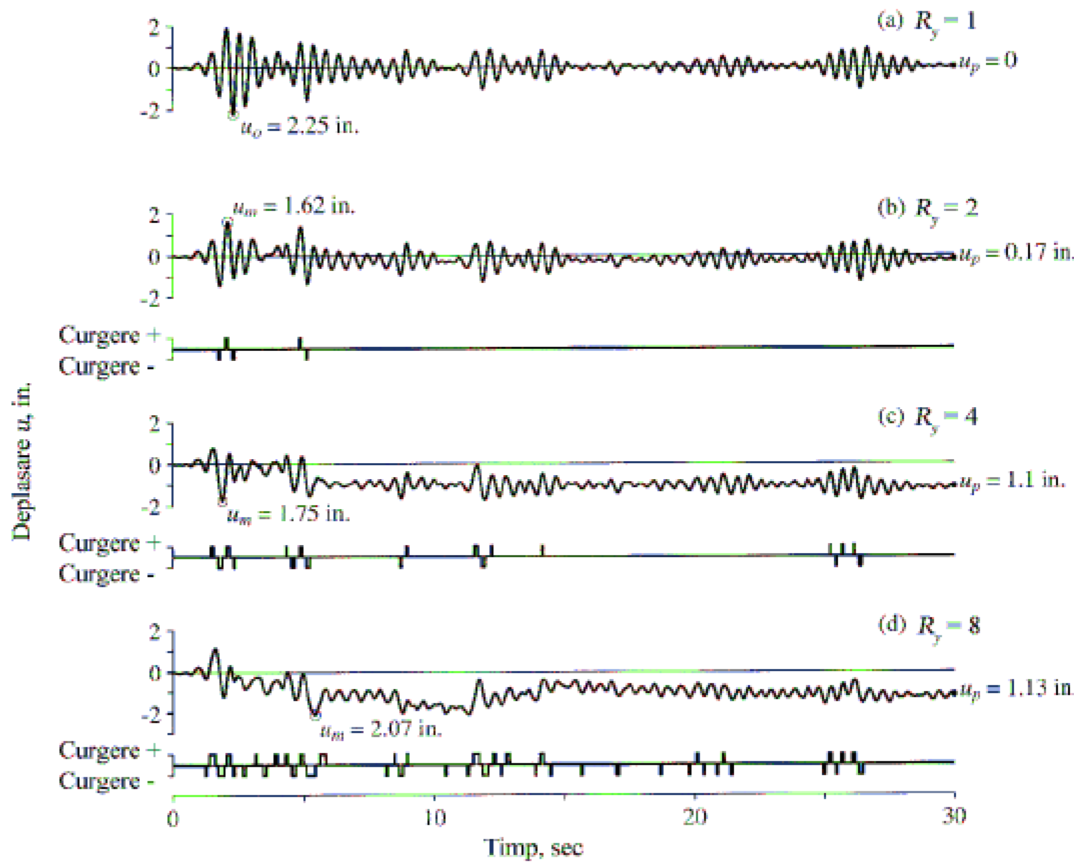
•Method of analysis – Force based (equivalent lateral force)

•beam, column and joint design issues

Dynamics of structures

Effects of inelastic force-displacement relationship

- 4 SDOF systems (El Centro):
  - T_n = 0.5 sec
  - ξ = 5%
  - R_y = 1, 2, 4, 8
- Elastic system:
  - vibr. about the initial position of equilibrium
  - u_p=0
- Inelastic syst.:
  - vibr. about a new position of equilibrium
  - $u_p \neq 0$

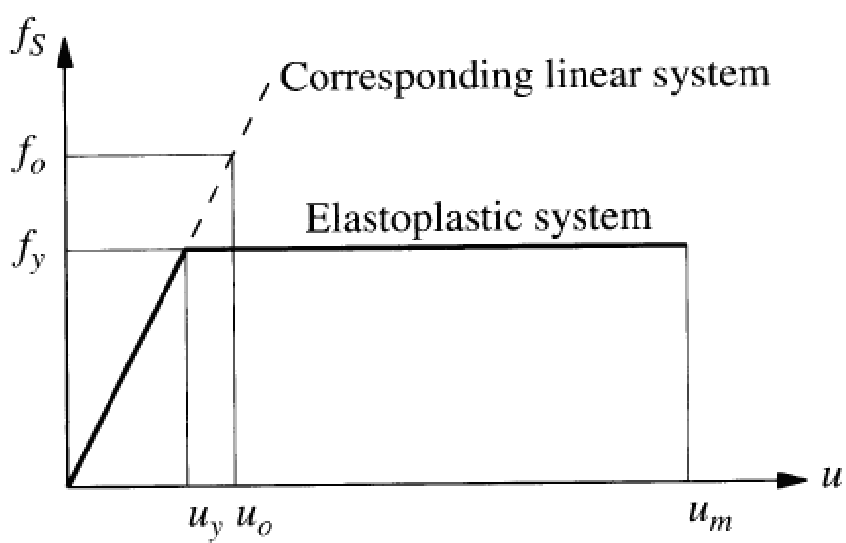


Dynamics of structures

Elastic ⇔ inelastic

- Design of a structure responding in the elastic range: $f_0 \leq f_{Rd}$
- Design of a structure responding in the inelastic range: $u_m \leq u_{Rd}$ $\mu \leq \mu_{Rd}$

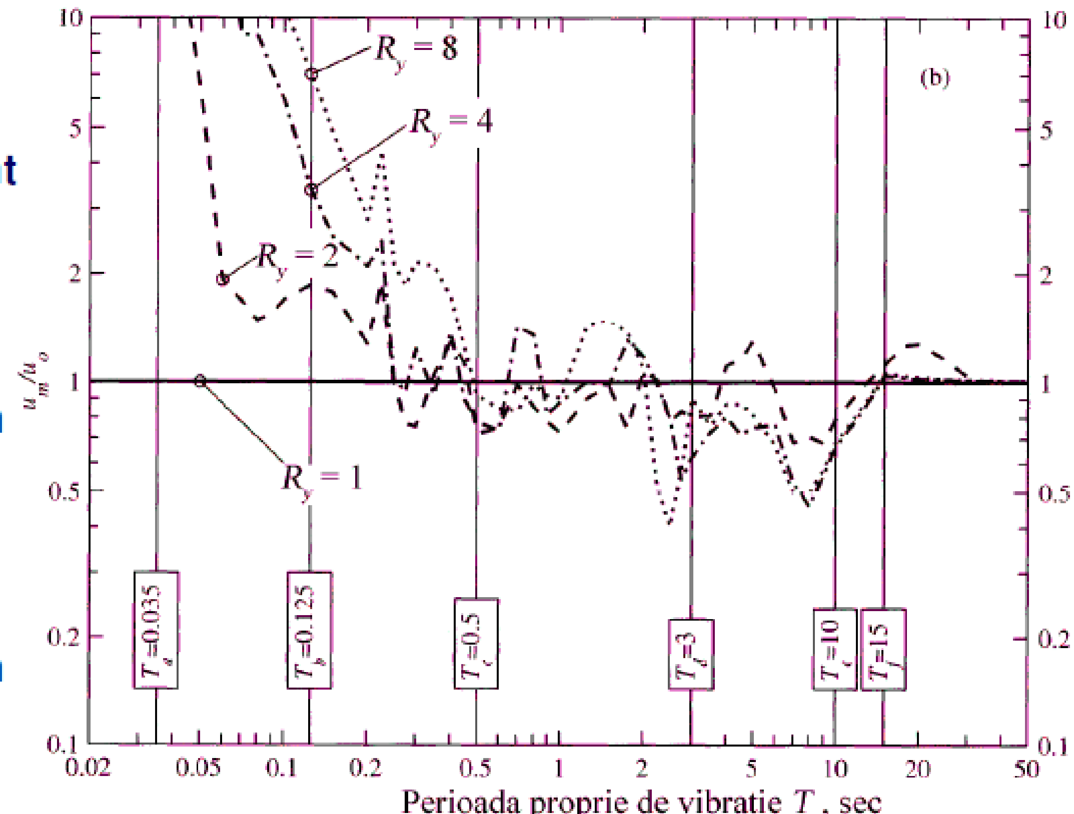
ductility demand ductility capacity



Dynamics of structures

u_m/u_0 ratio

- El Centro ground motion
  - $\xi = 5\%$
  - $R_y = 1, 2, 4, 8$
- $T_n > T_f$
  - u_m independent of R_y
  - $u_m \cong u_0$
- $T_n > T_c$
  - u_m depends on R_y
  - $u_m \cong u_0$
- $T_n < T_c$
  - u_m depends on R_y
  - $u_m > u_0$



•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

•Design response spectra (UBC & IBC)

•Method of analysis – Force based
(equivalent lateral force)

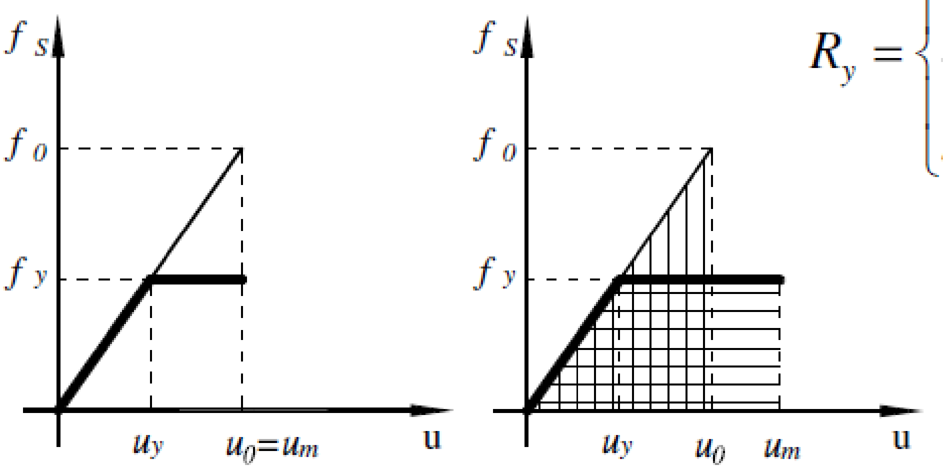
•beam, column and joint design issues

Dynamics of structures

R_y - μ relationship: idealisation

- T_n in the displacement- and velocity-sensitive region:
  - "equal displacement" rule $u_m/u_0=1 \Rightarrow R_y=\mu$
- T_n in the acceleration-sensitive region:
  - "equal energy" rule $u_m/u_0>1 \Rightarrow$ $R_y = \sqrt{2\mu - 1}$
- $T_n < T_a$:
  - small deformations, elastic response $\Rightarrow R_y=1$

$$R_y = \begin{cases} 1 & T_n < T_a \\ \sqrt{2\mu - 1} & T_b < T_n < T_{c'} \\ \mu & T_n > T_c \end{cases}$$



•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

•Design response spectra (UBC & IBC)

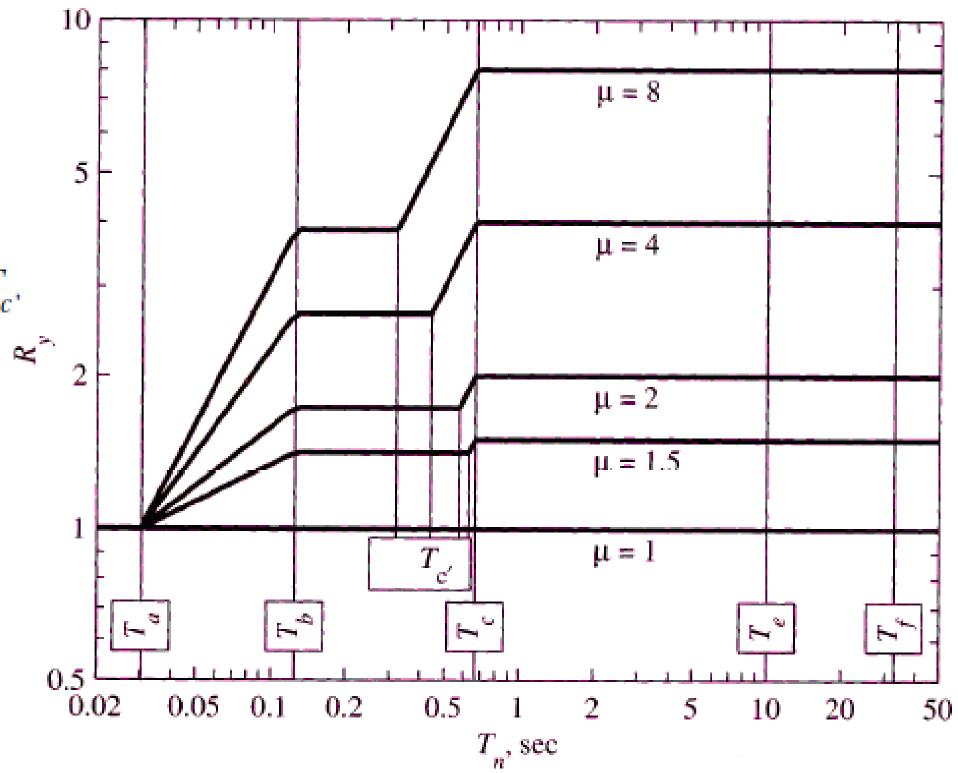
•Method of analysis – Force based (equivalent lateral force)

•beam, column and joint design issues

Dynamics of structures

R_y - μ relationship: idealisation

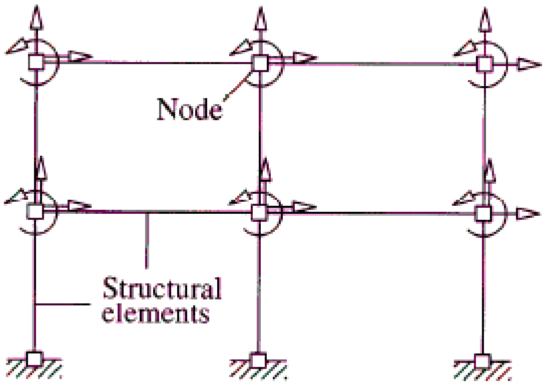
$$R_y = \begin{cases} 1 & T_n < T_a \\ \sqrt{2\mu - 1} & T_b < T_n < T_{c'} \\ \mu & T_n > T_c \end{cases}$$

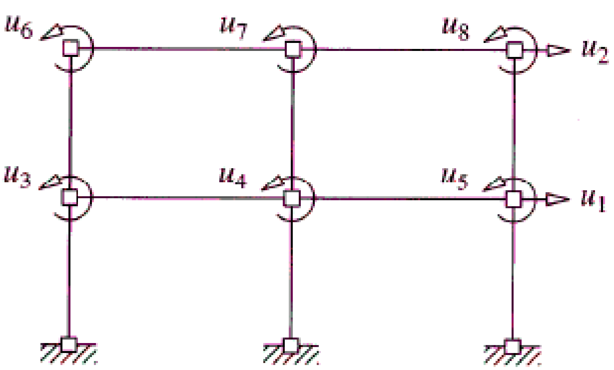


Dynamics of structures

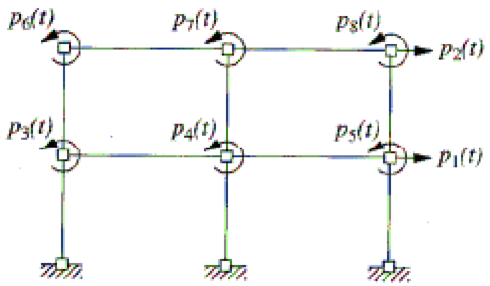
Multi degree of freedom systems

- Structure idealisation: elements interconnected at nodes
- Degrees of freedom: node displacements/rotations
  - 3 DOF for two-dimensional frames
  - 6 DOF for three-dimensional frames





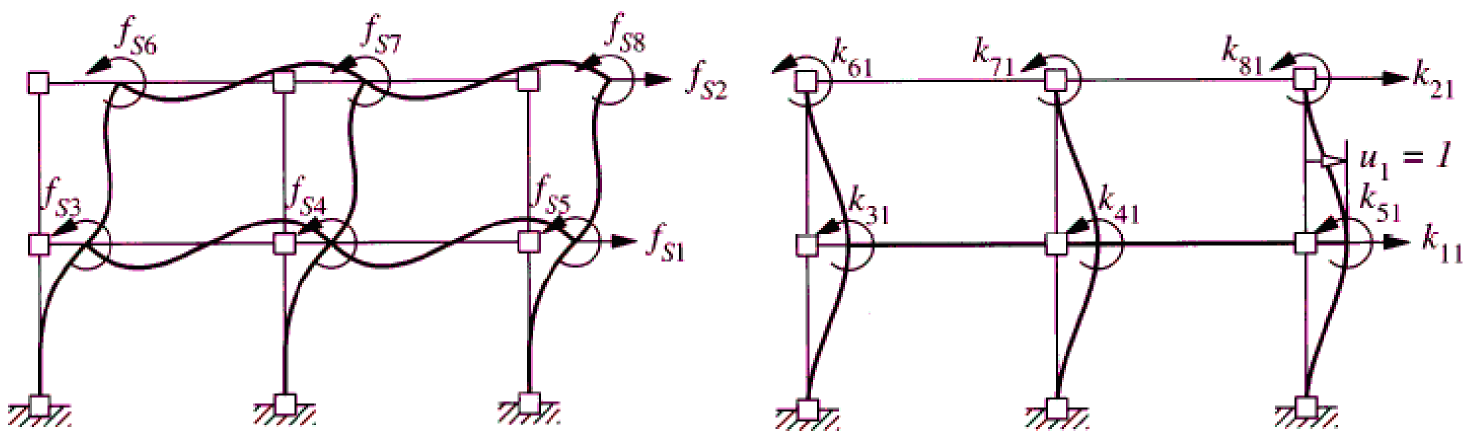
- Forces applied at nodes



Dynamics of structures

Elastic forces

- Node displacements $u_j \Leftrightarrow$ nodal forces f_{Sj}
- Linear systems: nodal forces determined based on
  - superposition principle
  - stiffness coefficients
- Stiffness coefficient k_{ij} is equal to the force along DOF i due to a unit displacement along DOF j



- Introduction to earthquake engineering
- Introduction to earthquake seismology
- linear analysis
- nonlinear analysis
- Ductility and nonlinear behavior
- Design response spectra (UBC & IBC)
- Method of analysis – Force based (equivalent lateral force)
- beam, column and joint design issues

Dynamics of structures

Elastic forces

- Knowing stiffness coefficients k_{ij}, nodal forces f_{Si} along DOF i, associated with displacements u_j, j=1, 2, …, N are obtained by superposition

$$f_{Si} = k_{i1}u_1 + k_{i2}u_2 + \ldots + k_{ij}u_j + \ldots + k_{iN}u_N$$

- Equations corresponding to i=1, 2, …, N can be written in matrix form

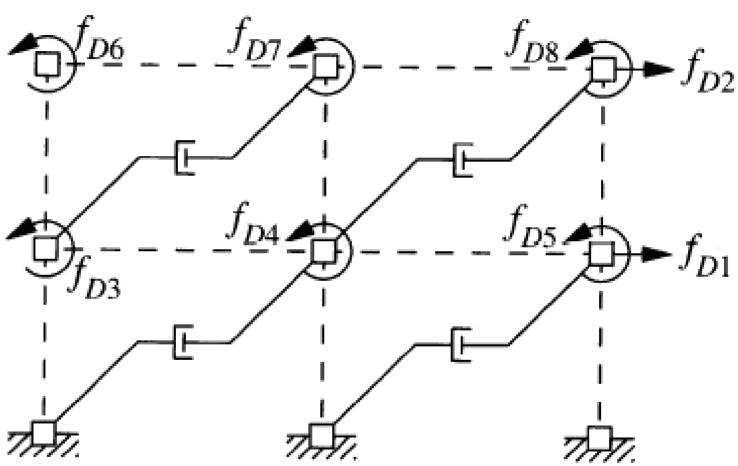
$$\begin{bmatrix} f_{S1} \\ f_{S2} \\ \vdots \\ f_{SN} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1j} & \cdots & k_{1N} \\ k_{21} & k_{22} & \cdots & k_{2j} & \cdots & k_{2N} \\ \vdots & \vdots & & \vdots & & \vdots \\ k_{N1} & k_{N2} & \cdots & k_{Nj} & \cdots & k_{NN} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{Bmatrix}$$

$$[f_S] = [k]\{u\}$$

Dynamics of structures

Damping forces

- A unit velocity along DOF *j*, generates forces along considered DOFs
- Damping coefficient c_{ij} is the force along DOF *i* due to a unit velocity along DOF *j*



Dynamics of structures

Damping forces

- Knowing damping coefficients c_{ij}, nodal forces f_{Di} along DOF i, associated to velocity $\dot{u}_j$ j=1, 2, …, N are obtained by superposition

$$f_{Di} = c_{i1}\dot{u}_1 + c_{i2}\dot{u}_2 + \ldots + c_{ij}\dot{u}_j + \ldots + c_{iN}\dot{u}_N$$

- Equations corresponding to i=1, 2, …, N can be written in matrix form

$$\begin{bmatrix} f_{D1} \\ f_{D2} \\ \vdots \\ f_{DN} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1N} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2N} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{N1} & c_{N2} & \cdots & c_{Nj} & \cdots & c_{NN} \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \vdots \\ \dot{u}_N \end{Bmatrix}$$

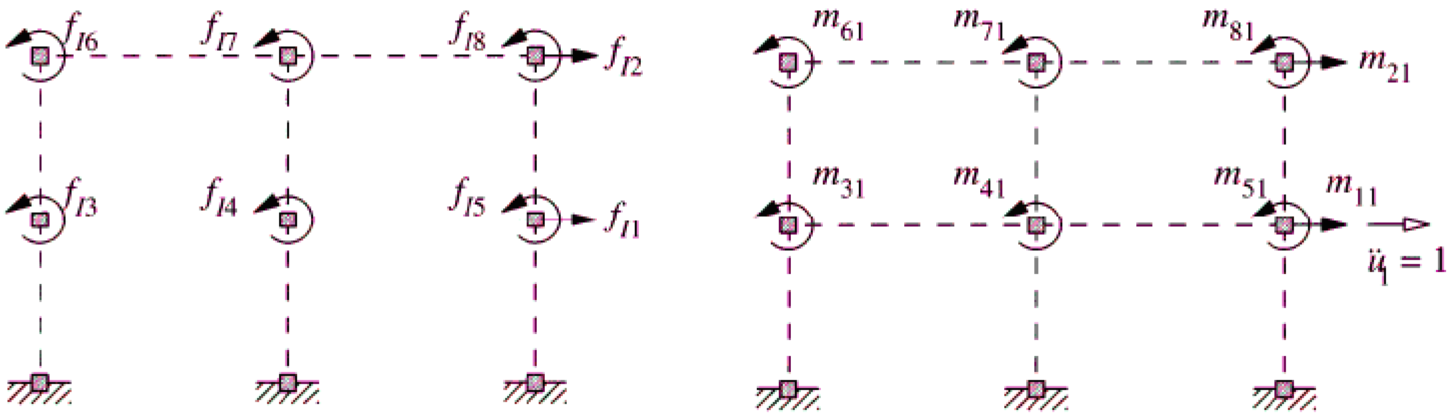
$$[f_D] = [c]\{\dot{u}\}$$

Dynamics of structures

Inertia forces

- A unit acceleration along DOF *j*, according to D'Alambert principle will generate fictitious inertia forces along the considered DOFs

- The mass coefficient m_{ij} is the force along DOF *i* due to a unit acceleration along DOF *j*



Dynamics of structures

Inertia forces

- Knowing the mass coefficients m_{ij}, nodal forces f_{Ii} along DOF i, associated to acceleration $\ddot{u}_j$ j=1, 2, …, N are obtained by superposition

$$f_{Ii} = m_{i1}\ddot{u}_1 + m_{i2}\ddot{u}_2 + \ldots + m_{ij}\ddot{u}_j + \ldots + m_{iN}\ddot{u}_N$$

- Equations corresponding to i=1, 2, …, N can be written in matrix form

$$\begin{bmatrix} f_{I1} \\ f_{I2} \\ \vdots \\ f_{IN} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1j} & \cdots & m_{1N} \\ m_{21} & m_{22} & \cdots & m_{2j} & \cdots & m_{2N} \\ \vdots & \vdots & & \vdots & & \vdots \\ m_{N1} & m_{N2} & \cdots & m_{Nj} & \cdots & m_{NN} \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \vdots \\ \ddot{u}_N \end{Bmatrix}$$

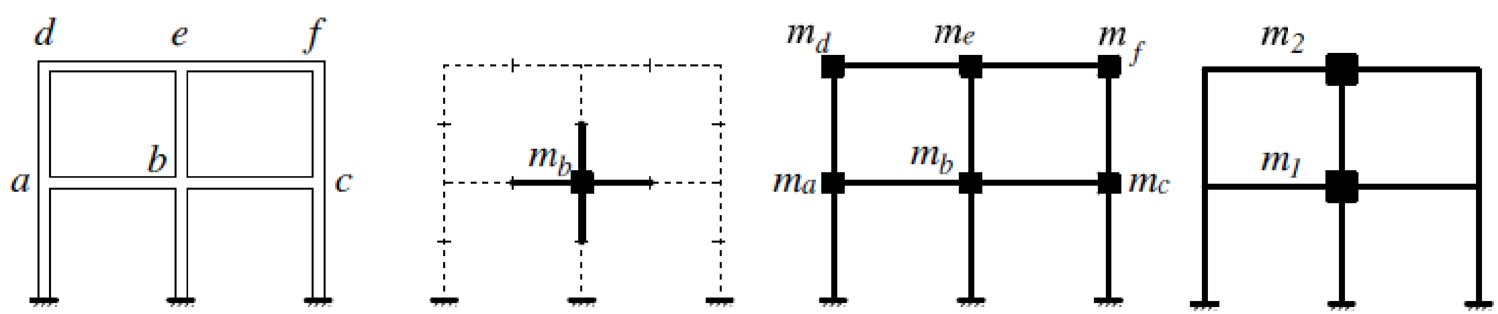
$$[f_I] = [m]\{\ddot{u}\}$$

Dynamics of structures

Mass idealisation: 2D structures

- Generally mass is distributed through the structure
- In a simplified manner $\Rightarrow$ concentrated in nodes
- Rotational component: generally neglected
- Neglecting axial deformations of members $\Rightarrow$ masses can be considered concentrated at the floor levels
- In general, for masses lumped in nodes, the mass matrix is diagonal

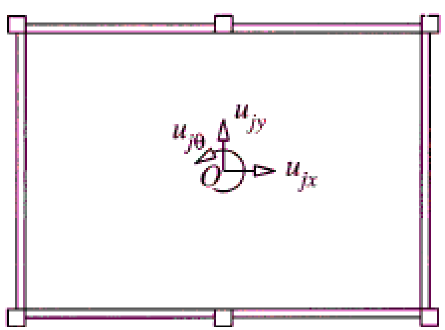
$$m_{ij} = 0 \ for \ i \neq j \quad and \quad m_{jj} = m_j \ or \ 0$$

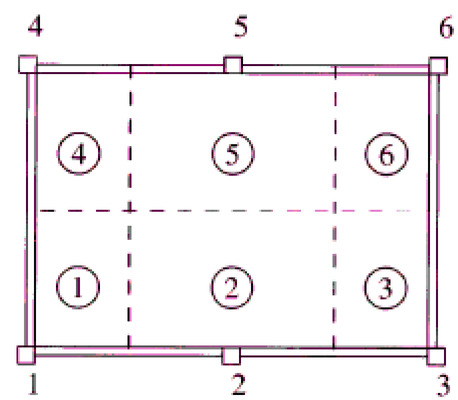


Dynamics of structures

Mass idealisation: 3D structures

- Multistorey three-dimensional structures: number of elements in the mass matrix can be reduced due to floor diaphragm effect
  - infinite in-plane stiffness
  - flexible out of plane
- Rigid floor diaphragms ⇒ 3 DOFs defined at the center of mass: u_x, u_y, u_θ
- Flexible floor diaphragms ⇒ masses should be assigned to each node , proportionally to their tributary area





•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

•Design response spectra (UBC & IBC)

•Method of analysis – Force based (equivalent lateral force)

•beam, column and joint design issues

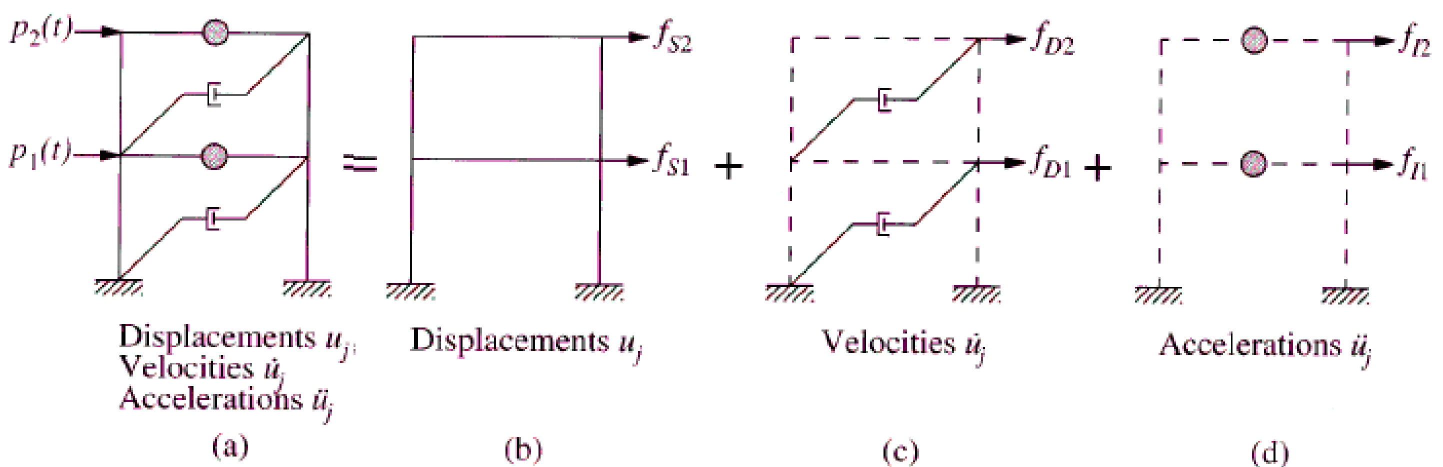
Dynamics of structures

Equation of motion: dynamic force

- Dynamic forces $\{p(t)\}$ can be considered distributed to:
  - stiffness component $\{f_S(t)\}$
  - damping component $\{f_D(t)\}$
  - mass component $\{f_I(t)\}$

$$\{f_I(t)\}+\{f_D(t)\}+\{f_S(t)\}=\{p(t)\}$$

- Equation of motion: $[m]\{\ddot{u}\}+[c]\{\dot{u}\}+[k]\{u\}=\{p(t)\}$

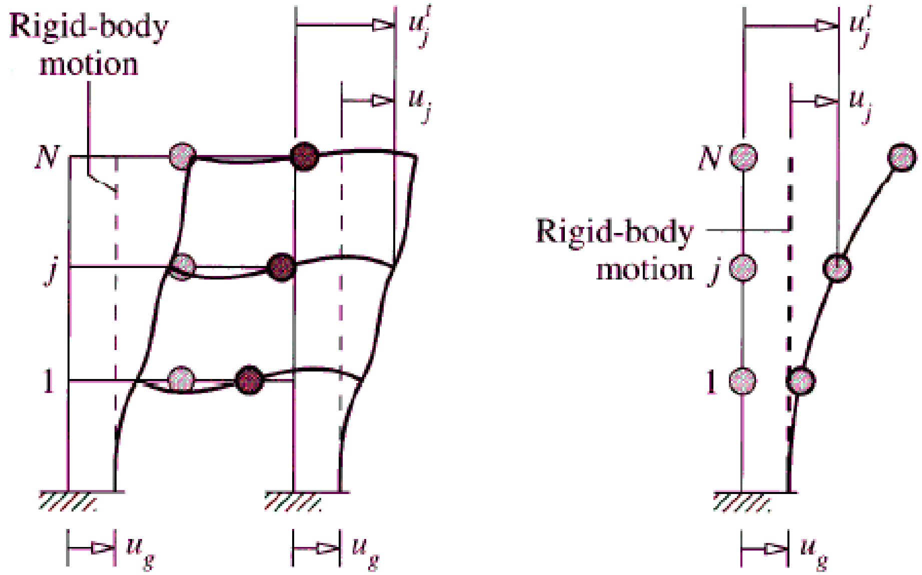


Dynamics of structures

Equation of motion: ground motion

- MDOF systems with al DOFs displacements in the same direction with ground motion
  - ground displacement: u_g
  - total displacement of mass m_j: u_j^t
  - relative displacement between mass and ground: u_j

$$\{u_j^t(t)\} = \{u_j(t)\} + u_g(t)\{1\}$$



Dynamics of structures

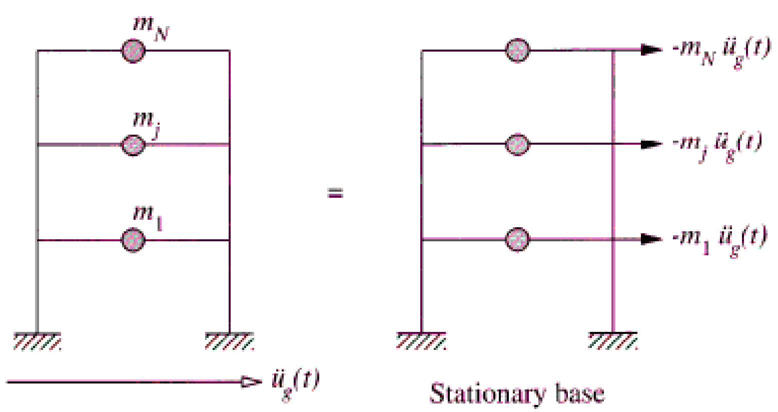
Equation of motion: ground motion

- In case of ground motion dynamic forces $\{p(t)\}=0$

$$\{f_I(t)\}+\{f_D(t)\}+\{f_S(t)\}=\{0\}$$

$$[m]\{\ddot{u}^t\}+[c]\{\dot{u}\}+[k]\{u\}=\{0\} \qquad \{u_j^t(t)\}=\{u_j(t)\}+u_g(t)\{1\}$$

$$[m]\{\ddot{u}\}+[c]\{\dot{u}\}+[k]\{u\}=-[m]\{1\}\ddot{u}_g(t)$$



$$\{p_{eff}(t)\}=-[m]\{1\}\ddot{u}_g(t)$$

•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

•Design response spectra (UBC & IBC)

•Method of analysis – Force based (equivalent lateral force)

•beam, column and joint design issues

Dynamics of structures

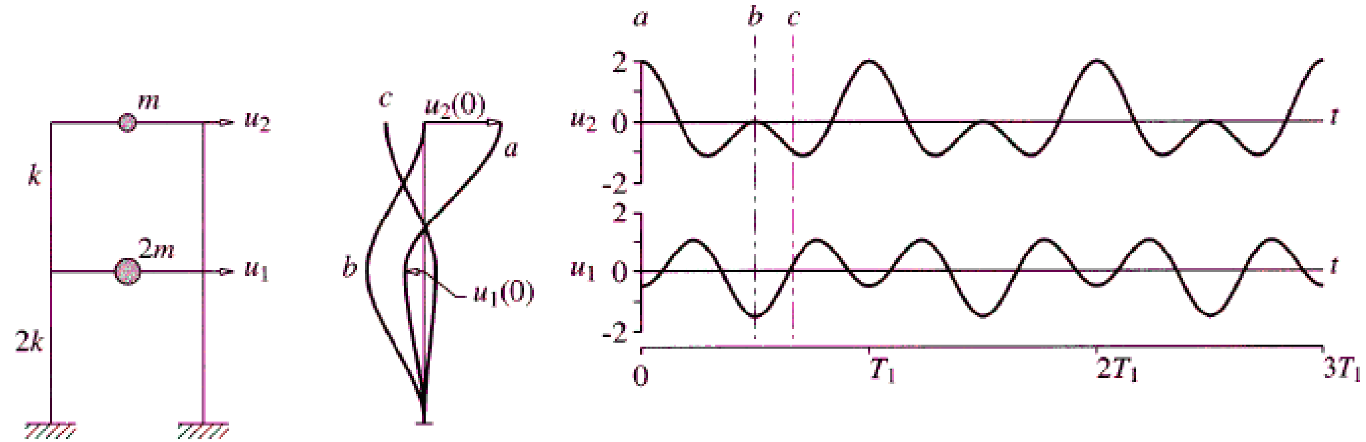
Free vibrations of MDOF systems

- Equation of motion : $[m]\{\ddot{u}\}+[k]\{u\}=\{0\}$

  $\Downarrow$

  a system of N homogeneous differential equations

- Initial conditions: $\{u\}=\{u(0)\}$ $\{\dot{u}\}=\{\dot{u}(0)\}$



- The motion is NOT harmonic
- Deformed shape of the structure changes in time

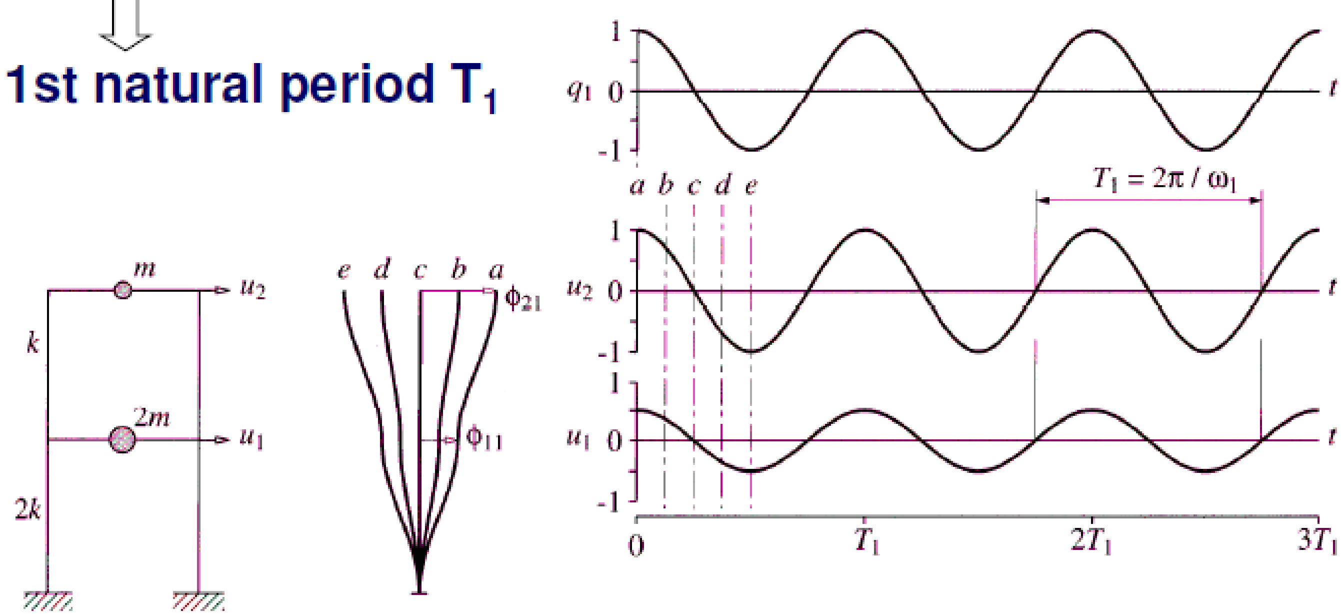
Dynamics of structures

Free vibrations of MDOF systems

- For an appropriate distribution of initial deformations: $\{u\} = \{u(0)\}$
- Vibrations ARE harmonic
- Deformed shape does NOT change in time
- 1st Distribution ⇨ 1st natural mode $\{\phi\}_1$

⇩

1st natural period T_1



•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

•Design response spectra (UBC & IBC)

•Method of analysis – Force based (equivalent lateral force)

•beam, column and joint design issues

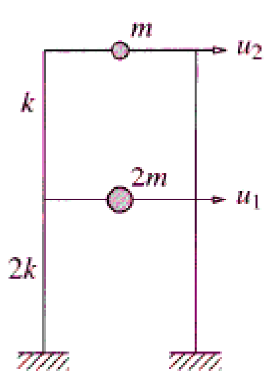
Dynamics of structures

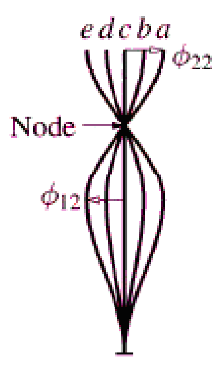
Free vibrations of MDOF systems

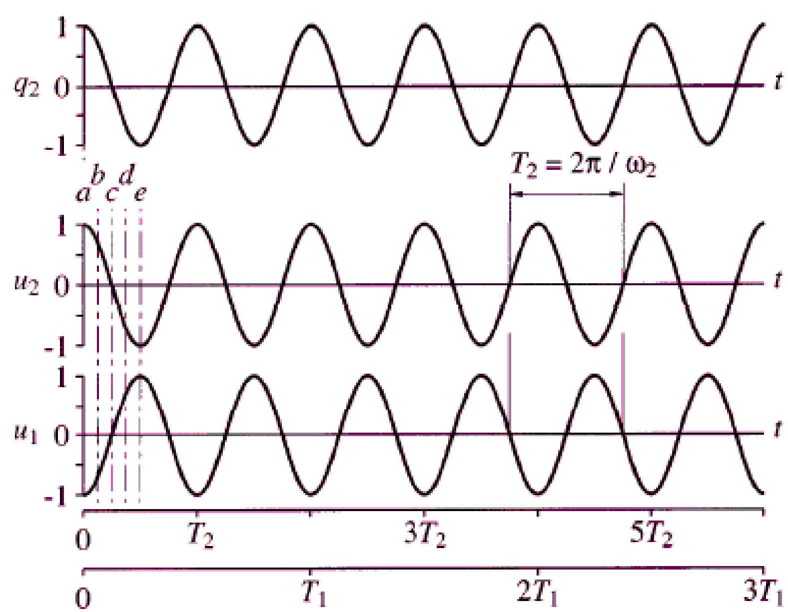
- For an appropriate distribution of initial deformations: $\{u\} = \{u(0)\}$
- Vibrations ARE harmonic
- Deformed shape does NOT change in time
- 2nd Distribution ⇨ 2nd natural mode $\{\phi\}_2$

⇩

2nd natural period T_2







- Introduction to earthquake engineering
- Introduction to earthquake seismology
- linear analysis
- nonlinear analysis
- Ductility and nonlinear behavior
- Design response spectra (UBC & IBC)
- Method of analysis – Force based (equivalent lateral force)
- beam, column and joint design issues

Dynamics of structures

Natural modes of undamped MDOF systems

- Vibrations in n-th natural mode: $\{u(t)\}_n = q_n(t)\{\phi\}_n$
  - deformed shape: $\{\phi\}_n$
  - time response: $q_n(t) = A_n \cos\omega_n t + B_n \sin\omega_n t$

$$\underbrace{\{u(t)\}_n = \{\phi\}_n\left(A_n \cos\omega_n t + B_n \sin\omega_n t\right) \qquad [m]\{\ddot{u}\} + [k]\{u\} = \{0\}}$$

$$\left[-\omega_n^2 [m]\{\phi\}_n + [k]\{\phi\}_n\right] q_n(t) = \{0\}$$

$q_n(t)=0$
trivial solution

$[k]\{\phi\}_n = \omega_n^2 [m]\{\phi\}_n$ or $\left([k] - \omega_n^2 [m]\right)\{\phi\}_n = \{0\}$
non-trivial solution
eigenvalue problem

determination of scalars ω_n and vectors $\{\phi\}_n$

$\det\left([k] - \omega_n^2 [m]\right) = 0$

Dynamics of structures

Natural modes of undamped MDOF systems

- Expanding the determinant $\det\left([k]-\omega_n^2[m]\right)=0$
- Characteristic equation: polynomial of order *N* in ω_n^2
  - *N* eigenvalues ω_n^2
  - *N* eigenmodes $\{\phi\}_n$ (relative values - just shape)

- Matrix notation:
  - eigenmodes: $$[\Phi]=\{\{\phi\}_1 \ \cdots \ \{\phi\}_n\}=\begin{bmatrix}\phi_{11} & \cdots & \phi_{1N}\\ \vdots & \ddots & \vdots \\ \phi_{N1} & \cdots & \phi_{NN}\end{bmatrix}$$
  - eigenvalues: $$\left[\Omega^2\right]=\begin{bmatrix}\omega_1^2 & & \\ & \ddots & \\ & & \omega_N^2\end{bmatrix}$$
  - eigenproblem: $$[k][\Phi]=[m][\Phi]\left[\Omega^2\right]$$

Dynamics of structures

Orthogonality of natural modes

- Eigenproblem: $[k]\{\phi\}_n = \omega_n^2 [m]\{\phi\}_n$
- Multiplying to the left by $\{\phi\}_r^T$ ($r \neq n$):

$$\{\phi\}_r^T [k]\{\phi\}_n = \omega_n^2 \{\phi\}_r^T [m]\{\phi\}_n \xrightarrow{\text{transpose}} \{\phi\}_n^T [k]\{\phi\}_r = \omega_n^2 \{\phi\}_n^T [m]\{\phi\}_r \quad \textbf{(4.33)}$$

- Similarly: $\{\phi\}_n^T [k]\{\phi\}_r = \omega_r^2 \{\phi\}_n^T [m]\{\phi\}_r$ **(4.32)**
- Difference between (4.33) and (4.32): $\left(\omega_n^2 - \omega_r^2\right)\{\phi\}_n^T [m]\{\phi\}_r = 0$
- For $\omega_n^2 \neq \omega_r^2$, which for positive values implies $\omega_n \neq \omega_r$:

$$\{\phi\}_n^T [m]\{\phi\}_r = 0 \xrightarrow{(4.32)} \{\phi\}_n^T [k]\{\phi\}_r = 0$$

orthogonality of natural modes

- Matrices M and K: $[K] \equiv [\Phi]^T [k][\Phi]$ $\quad [M] \equiv [\Phi]^T [m][\Phi]$
  diagonal $\quad K_n = \{\phi\}_n^T [k]\{\phi\}_n \quad M_n = \{\phi\}_n^T [m]\{\phi\}_n \quad K_n = \omega_n^2 M_n$

Dynamics of structures

Normalisation of modes

- Natural modes: vectors for which only relative values are known
- Normalisation of modes:
  - setting the maximum value of a natural mode to unity
  - setting the value corresponding to a characteristic DOF to unity
  - normalisation of natural modes so that M_n are unity (normalisation with respect to the [*m*] matrix)

$$M_n = \{\phi\}_n^T [m] \{\phi\}_n = 1 \qquad [\Phi]^T [m][\Phi] = [I]$$

$$K_n = \{\phi\}_n^T [k] \{\phi\}_n = \omega_n^2 M_n = \omega_n^2 \qquad [K] = [\Phi]^T [k][\Phi] = [\Omega^2]$$

orthonormal natural modes

Dynamics of structures

Modal expansion of displacements

- Any set of *N* independent vectors can be used to express another vector of order *N*

$$\{u\} = \sum_{r=1}^{N} \{\phi\}_r \, q_r = [\Phi]\{q\} \qquad (4.48)$$

q_r - modal coordinates

- Multiplying both sides of (4.48) by $\{\phi\}_n^T [m]$

$$\{\phi\}_n^T [m]\{u\} = \sum_{r=1}^{N} \{\phi\}_n^T [m]\{\phi\}_r \, q_r$$

all terms are equal to zero, excepting those corresponding to *r*=*n*:

$$\{\phi\}_n^T [m]\{u\} = \{\phi\}_n^T [m]\{\phi\}_n \, q_n$$

- Modal coordinates can be determined:

$$q_n = \frac{\{\phi\}_n^T [m]\{u\}}{\{\phi\}_n^T [m]\{\phi\}_n} = \frac{\{\phi\}_n^T [m]\{u\}}{M_n}$$

Dynamics of structures

Solution of equation of motion

- **Equation of motion** — **initial conditions**

$$[m]\{\ddot{u}\}+[k]\{u\}=\{0\} \qquad \{u\}=\{u(0)\} \qquad \{\dot{u}\}=\{\dot{u}(0)\}$$

- **Eigenproblem:**

$$\left([k]-\omega_n^2[m]\right)\{\phi\}_n=\{0\} \Rightarrow \omega_n\ ,\ \phi_n$$

- **Response in mode *n*:** $\{u(t)\}_n=\{\phi\}_n\left(A_n\cos\omega_n t+B_n\sin\omega_n t\right)$
- **General solution ⇒ superposition of individual response in each natural mode:**

$$\{u(t)\}=\sum_{n=1}^{N}\{\phi\}_n\left(A_n\cos\omega_n t+B_n\sin\omega_n t\right) \qquad (4.52)$$

- **Velocity vector:**

$$\{\dot{u}(t)\}=\sum_{n=1}^{N}\{\phi\}_n\,\omega_n\left(-A_n\sin\omega_n t+B_n\cos\omega_n t\right)$$

- **For *t*=0:** $\{u(0)\}=\sum_{n=1}^{N}\{\phi\}_n A_n \qquad \{\dot{u}(0)\}=\sum_{n=1}^{N}\{\phi\}_n\,\omega_n B_n$ **(4.54)**

  a system of *N* linear algebraic equations with unknowns A_n, respectively B_n

Dynamics of structures

Solution of equation of motion

- Using modal expansion, vectors $\{u(0)\}$ and $\{\dot{u}(0)\}$ can be written as

$$\{u(0)\} = \sum_{n=1}^{N} \{\phi\}_n q_n(0) \qquad \{\dot{u}(0)\} = \sum_{n=1}^{N} \{\phi\}_n \dot{q}_n(0) \qquad (4.55)$$

where modal coordinates are given by:

$$q_n(0) = \frac{\{\phi\}_n^T [m]\{u(0)\}}{M_n} \qquad \dot{q}_n(0) = \frac{\{\phi\}_n^T [m]\{\dot{u}(0)\}}{M_n}$$

- Equations (4.54) and (4.55) are equivalent

$$\{u(0)\} = \sum_{n=1}^{N} \{\phi\}_n A_n \qquad \{\dot{u}(0)\} = \sum_{n=1}^{N} \{\phi\}_n \omega_n B_n \quad (4.54) \Rightarrow$$

$$A_n = q_n(0) \qquad B_n = \dot{q}_n(0)/\omega_n$$

- Replacing these expressions in (4.52) $\Rightarrow$

$$\{u(t)\} = \sum_{n=1}^{N} \{\phi\}_n \left(q_n(0)\cos\omega_n t + \frac{\dot{q}_n(0)}{\omega_n} \sin\omega_n t \right)$$

Dynamics of structures

Free vibrations of MDOF systems with damping

- **Equation of motion** $[m]\{\ddot{u}\}+[c]\{\dot{u}\}+[k]\{u\}=\{0\}$
- **Initial conditions:** $\{u\}=\{u(0)\}$ $\{\dot{u}\}=\{\dot{u}(0)\}$
- **Using modal expansion, equation of motion becomes:** $\{u\}=\sum_{r=1}^{N}\{\phi\}_r q_r=[\Phi]\{q\}$

  $$[m][\Phi]\{\ddot{q}\}+[c][\Phi]\{\dot{q}\}+[k][\Phi]\{q\}=\{0\}$$
- **Multiplying to the left by $[\Phi]^T$:**

  $$[M]\{\ddot{q}\}+[C]\{\dot{q}\}+[K]\{q\}=\{0\}$$

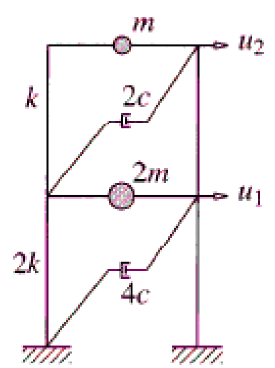
  where:

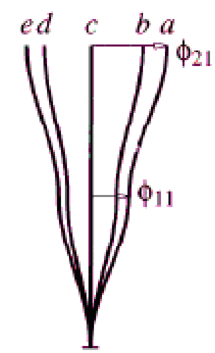
  $$[K]\equiv[\Phi]^T[k][\Phi] \qquad [M]\equiv[\Phi]^T[m][\Phi] \qquad [C]=[\Phi]^T[c][\Phi]$$
- **For a diagonal matrix [*C*] - classical damping:**
  - natural modes of the damped system identical to the ones of the undamped system
  - most engineering structures can be modelled using classical damping

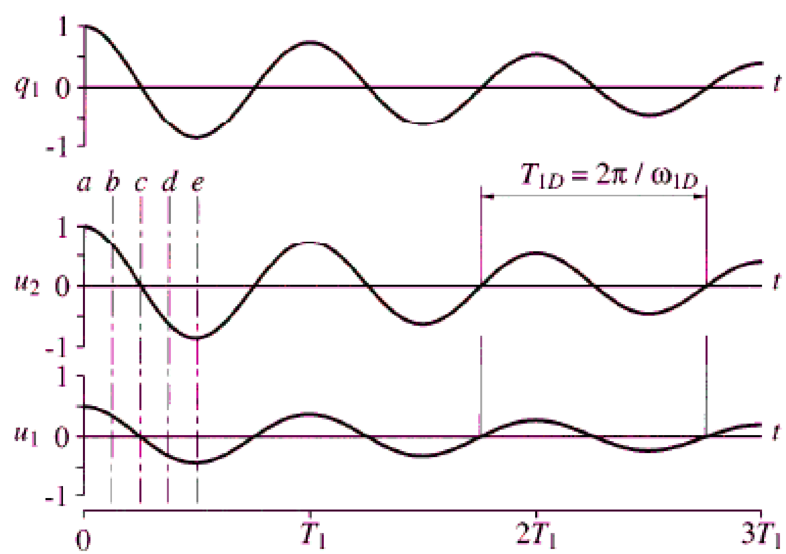
Dynamics of structures

Free vibrations of MDOF systems with damping

- Natural modes of the damped system identical to those of the undamped system - $\{\phi\}_n$
- Displacements similar to those of the undamped system, but amplitudes decrease with time
- Response of each mass is harmonic, similarly to that of a SDOF system







Dynamics of structures

Free vibrations of MDOF systems with damping

- For each natural mode *n*, equation of motion in modal coordinates is

$$M_n\ddot{q}_n + C_n\dot{q}_n + Kq_n = 0$$

- Dividing by M_n one gets: $\ddot{q}_n + 2\xi_n\omega_n\dot{q}_n + \omega_n^2 q_n = 0$

where $\xi_n = \dfrac{C_n}{2M_n\omega_n}$

- The same form as the equation of motion in the case of damped free vibrations of SDOF systems ⇒

$$q_n(t) = e^{-\xi_n\omega_n t}\left[q_n(0)\cos\omega_{nD}t + \frac{\dot{q}(0)+\xi_n\omega_n q_n(0)}{\omega_{nD}}\sin\omega_{nD}t\right] \qquad \omega_{nD} = \omega_n\sqrt{1-\xi_n^2}$$

- Combining modal contributions:

$$\{u(t)\} = \sum_{n=1}^{N}\{\phi\}_n e^{-\xi_n\omega_n t}\left[q_n(0)\cos\omega_{nD}t + \frac{\dot{q}(0)+\xi_n\omega_n q_n(0)}{\omega_{nD}}\sin\omega_{nD}t\right]$$

Dynamics of structures

Modal analysis

- Equation of motions of a MDOF system with damping excited by dynamic forces:

$$[m]\{\ddot{u}\}+[c]\{\dot{u}\}+[k]\{u\}=\{p(t)\} \qquad \textbf{4.82}$$

- Displacements $\{u\}$ can be expanded as: $\{u\}=\sum_{r=1}^{N}\{\phi\}_r q_r=[\Phi]\{q\}$
- Replacing $\{u\}$ in 4.82:

$$\sum_{r=1}^{N}[m]\{\phi\}_r\ddot{q}_r(t)+\sum_{r=1}^{N}[c]\{\phi\}_r\dot{q}_r(t)+\sum_{r=1}^{N}[k]\{\phi\}_r q_r(t)=\{p(t)\} \qquad \textbf{4.84}$$

- Multiplying 4.84 to the left by $\{\phi\}_n^T$ we obtain:

$$\sum_{r=1}^{N}\{\phi\}_n^T[m]\{\phi\}_r\ddot{q}_r(t)+\sum_{r=1}^{N}\{\phi\}_n^T[c]\{\phi\}_r\dot{q}_r(t)+\sum_{r=1}^{N}\{\phi\}_n^T[k]\{\phi\}_r q_r(t)=\{\phi\}_n^T\{p(t)\}$$

- Which, considering orthogonality of natural modes becomes: $M_n\ddot{q}_n(t)+C_n\dot{q}_n(t)+K_n q_n(t)=P_n(t)$ **4.86**

Dynamics of structures

Modal analysis

- Dividing by M_n one gets: $\ddot{q}_n + 2\xi_n\omega_n\dot{q}_n + \omega_n^2 q_n = \dfrac{P_n(t)}{M_n}$
- Solving a system of N differential equations was reduced to solution of N independent equations
- Direct estimation of the damping matrix [*c*] not necessary
- The same form with the equation of motions of a SDOF system ⇒ same solution methods
- Solution: modal coordinate $q_n(t)$ for mode n
- Contribution of mode n to total displacement $\{u(t)\}$:

$$\{u(t)\}_n = \{\phi\}_n q_n(t)$$

- Total displacements:

$$\{u(t)\} = \sum_{n=1}^{N}\{u(t)\}_n = \sum_{n=1}^{N}\{\phi\}_n q_n(t)$$

Dynamics of structures

Modal analysis

- Analysis procedure is called modal analysis and is applicable only to linear systems with classical damping
- Element forces can be obtained using 2 methods:

1. Contributions $r_n(t)$ in n-th mode are obtained from imposing displacements $\{u(t)\}_n$
   Total forces are obtained by superposition of modal contributions

$$r(t) = \sum_{n=1}^{N} r_n(t)$$

2. Equivalent static forces from the n-th mode are determined:

$$\{f(t)\}_n = [k]\{u(t)\}_n = \omega_n^2 [m]\{u(t)\}_n = \omega_n^2 [m]\{\phi\}_n q_n(t)$$

Static analysis $\Rightarrow$ modal contributions $r_n(t)$ from the n-th mode $\Rightarrow$

$$r(t) = \sum_{n=1}^{N} r_n(t)$$

Dynamics of structures

Modal analysis: summary

- Define the structural properties
  - mass [*m*] and stiffness [*k*] matrices
  - critical damping ratio ξ_n
- Determine natural circular frequencies ω_n and natural modes of vibrations $\{\phi\}_n$
- Compute response in each mode following the sequence:
  - set up equation of motion $\ddot{q}_n + 2\xi_n\omega_n\dot{q}_n + \omega_n^2 q_n = \frac{P_n(t)}{M_n}$
  - compute modal displacements $\{u(t)\}_n$
  - compute element forces $r_n(t)$ from the *n*-th mode
- Combine modal contributions to obtain the total response $\{u(t)\} = \sum_{n=1}^{N} \{u(t)\}_n$
- Note: generally it is NOT necessary to consider ALL modes of vibration

•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

•Design response spectra (UBC & IBC)

•Method of analysis – Force based (equivalent lateral force)

•beam, column and joint design issues

Dynamics of structures

Modal analysis of seismic response: summary

- Define numerically ground acceleration
- Define the structural properties
  - mass [m] and stiffness [k] matrices
  - critical damping ratio ξ_n
- Determine ω_n and $\{\phi\}_n$
- Determine modal components $\{s\}_n$ of the distribution of effective seismic forces
- Compute response in each mode following the sequence:
  - static response r_n^{st} of the structure from $\{s\}_n$
  - pseudo-acceleration $A_n(t)$ of n-th mode SDOF system
  - resp. quantities $r_n(t)$ from the n-th mode $r_n(t) = r_n^{st} A_n(t)$
- Combine modal contributions to obtain the total response

$$r(t) = \sum_{n=1}^{N} r_n(t) = \sum_{n=1}^{N} r_n^{st} A_n(t)$$

Dynamics of structures

Effective modal mass

- Modal analysis - equivalent static forces in *n*-th mode $\Rightarrow$ *n*-th mode contributions $r_n(t)$ to the response quantity $r(t)$:

$$\{f(t)\}_n = \{s\}_n A_n(t)$$

$$\{s\}_n = \Gamma_n [m]\{\phi\}_n \qquad s_{jn} = \Gamma_n m_j \phi_{jn}$$

$$\Gamma_n = \frac{L_n}{M_n}$$

$$L_n = \{\phi\}_n^T [m]\{1\} = \sum_{j=1}^{N} m_j \phi_{jn}$$

$$M_n = \{\phi\}_n^T [m]\{\phi\}_n = \sum_{j=1}^{N} m_j \phi_{jn}^2$$

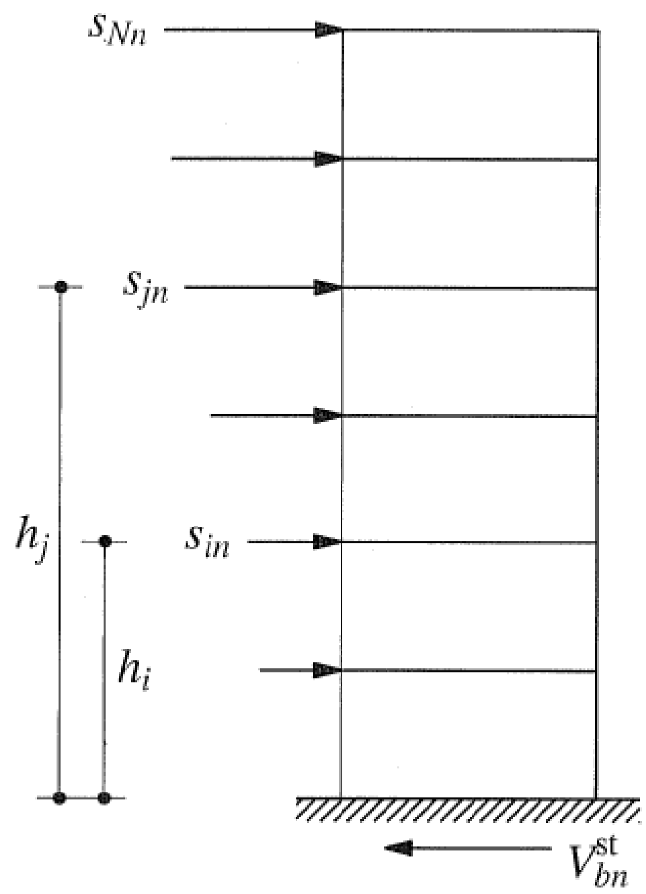
Dynamics of structures

Effective modal mass

- Response quantity $r_n(t)$ can be expressed by: $r_n(t) = r_n^{st} A_n(t)$
- r_n^{st} - modal static response, by applying "forces" $\{s\}_n$

$$\{s\}_n = \Gamma_n [m]\{\phi\}_n \qquad s_{jn} = \Gamma_n m_j \phi_{jn}$$

- Multistorey structures: base shear force V_b



$$V_{bn}^{st} = \sum_{j=1}^{N} s_{jn} = \Gamma_n \sum_{j=1}^{n} m_j \phi_{jn} = \Gamma_n L_n \equiv M_n^*$$

$$M_n^* = \Gamma_n L_n = \left(\sum_{j=1}^{n} m_j \phi_{jn} \right)^2 \Big/ \sum_{j=1}^{n} m_j \phi_{jn}^2$$

Dynamics of structures

Effective modal mass

- Base shear force in *n*-th mode:

$$V_{bn}(t) = V_{bn}^{st} A_n(t)$$

substituting $V_{bn}^{st} = M_n^* \quad \Rightarrow \quad V_{bn}(t) = M_n^* A_n(t)$ 4.113

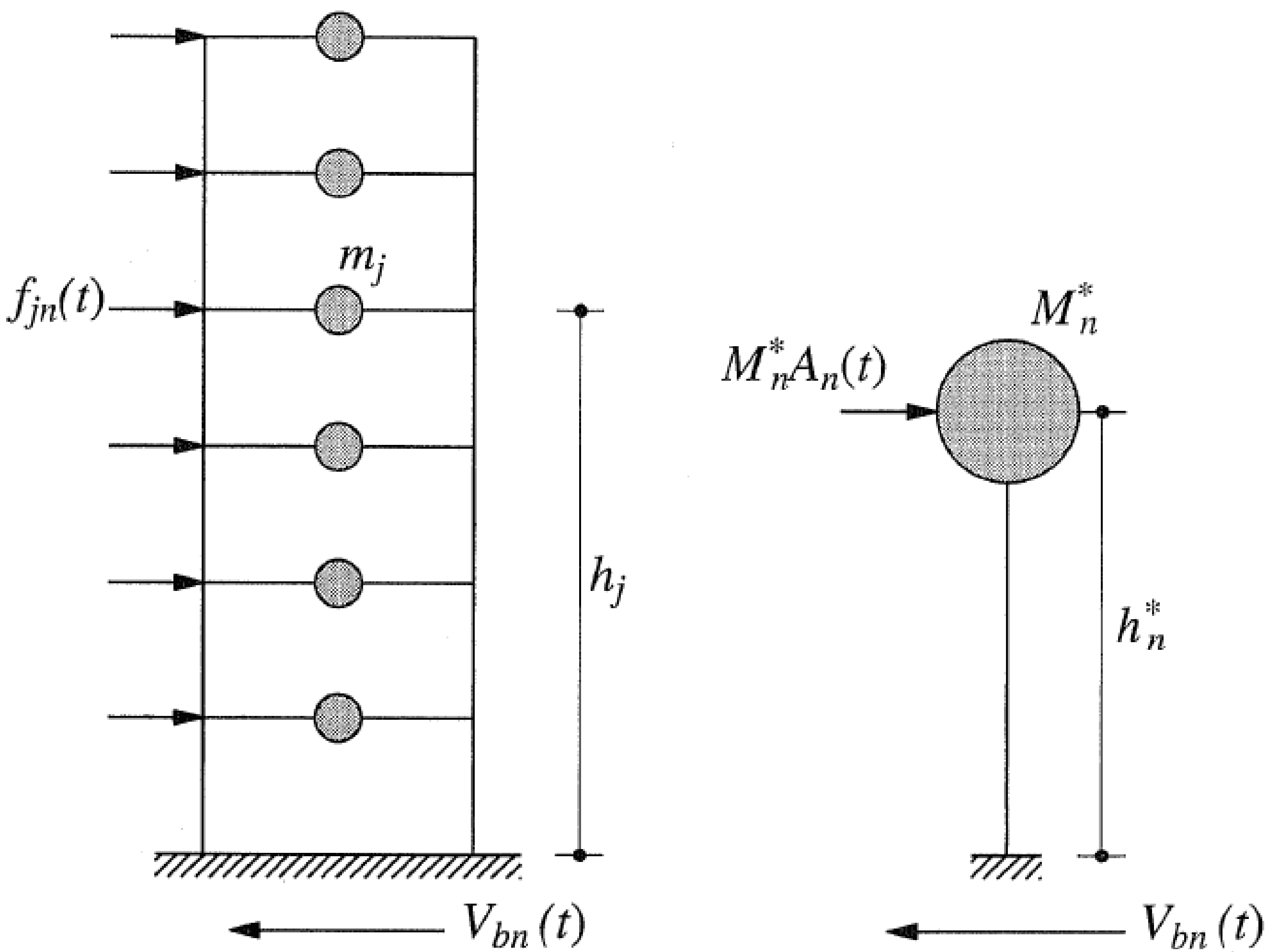
- A SDOF system with mass *m*, natural circular frequency ω_n and critical damping ratio ξ_n

$$V_b(t) = mA_n(t) \qquad 4.114$$

- Comparing 4.113 and 4.114 $\Rightarrow$ M_n^* - effective modal mass
- MDOF: only the portion M_n^* of the total mass of the structure is effective in producing the base shear force
- The sum of effective modal masses over all *N* modes is equal to the total mass of the structure $\sum_{n=1}^{N} M_n^* = \sum_{j=1}^{N} m_j$

Dynamics of structures

Effective modal mass



Dynamics of structures

Spectral analysis

- Modal analysis: time-history response

$$r_n(t) = r_n^{st} A_n(t) \Rightarrow r(t) = \sum_{n=1}^{N} r_n(t) \Rightarrow r_0 = \max_t |r(t)|$$

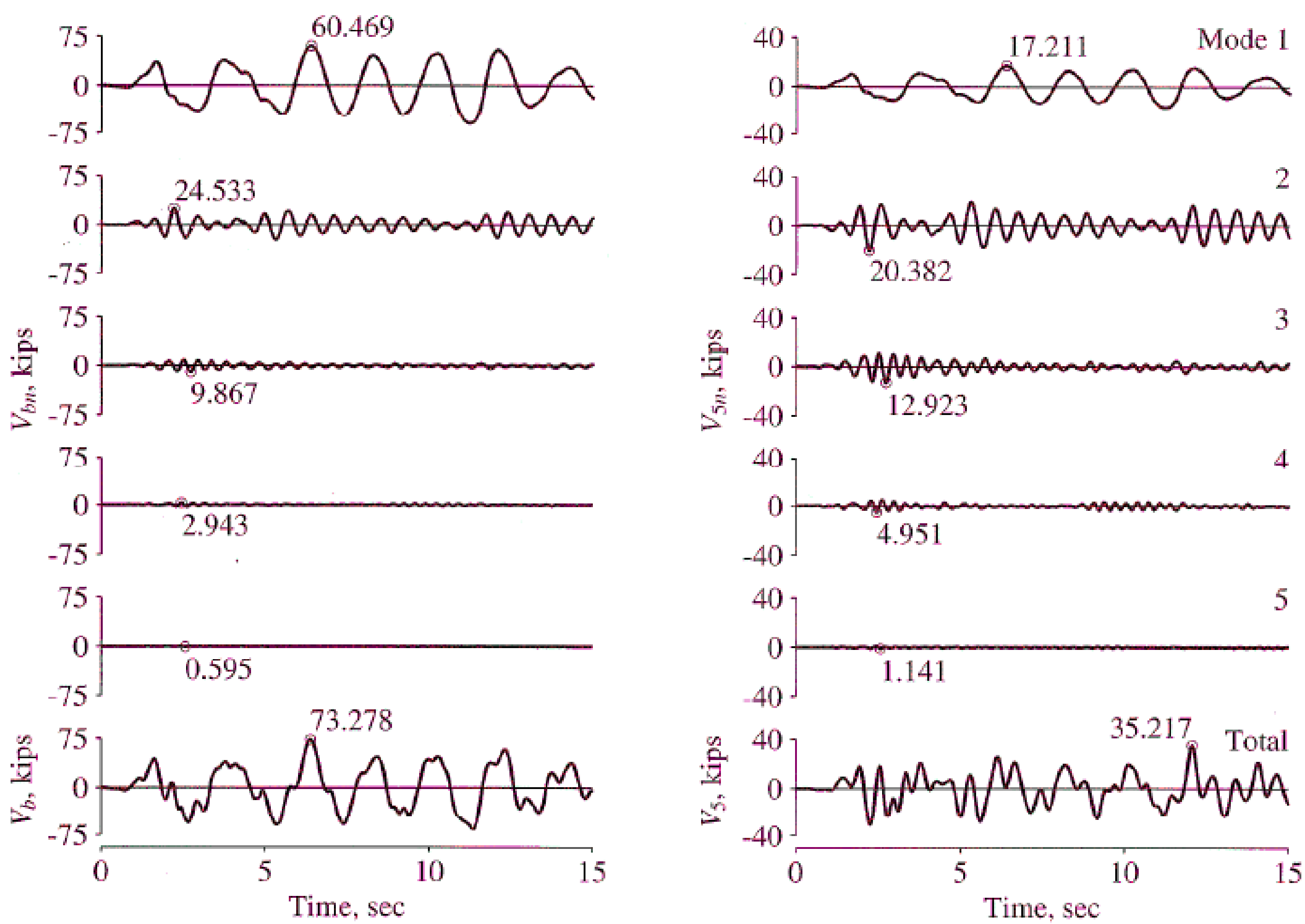
- Design - peak values of forces and displacements
- Spectral analysis: direct determination of peak values of forces and displacements
- Peak response r_{no} of the contribution $r_n(t)$ in the n-th mode to the total response $r(t)$

  $$r_{n0} = r_n^{st} A_n$$

  A_n - spectral pseudo-acceleration

Dynamics of structures

Modal contrib. and total time-history response



- Introduction to earthquake engineering
- Introduction to earthquake seismology
- linear analysis
- nonlinear analysis
- Ductility and nonlinear behavior
- Design response spectra (UBC & IBC)
- Method of analysis – Force based (equivalent lateral force)
- beam, column and joint design issues

Dynamics of structures

Methods for combination of peak modal response

- Absolute sum

$$r_0 = \sum_{n=1}^{N} |r_{n0}|$$

suitable for structures with closely spaced natural modes of vibration

- Square Root of Sum of Squares (SRSS):

$$r_0 = \sqrt{\sum_{n=1}^{N} r_{n0}^2}$$

suitable for structures with distinct modes of vibration

•Introduction to earthquake engineering

•Introduction to earthquake seismology

•linear analysis

•nonlinear analysis

•Ductility and nonlinear behavior

•Design response spectra (UBC & IBC)

•Method of analysis – Force based (equivalent lateral force)

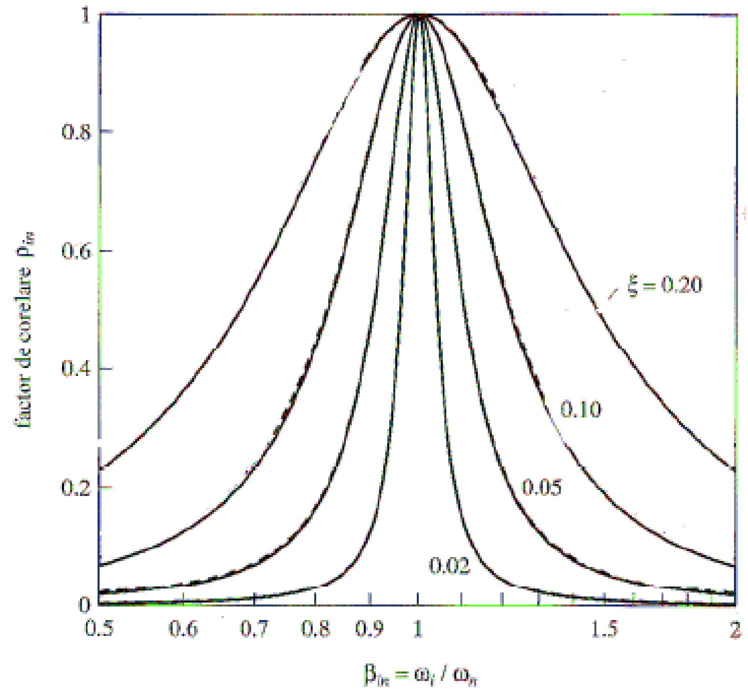
•beam, column and joint design issues

Dynamics of structures

Methods for combination of peak modal response

- Complete quadratic combination (CQC):

$$r_0 = \sqrt{\sum_{i=1}^{N} \sum_{n=1}^{N} \rho_{in} r_{i0} r_{n0}} \qquad r_0 = \sqrt{\sum_{n=1}^{N} r_{n0}^2 + \underbrace{\sum_{i=1}^{N} \sum_{n=1}^{N}}_{i \neq n} \rho_{in} r_{i0} r_{n0}}$$



Dynamics of structures

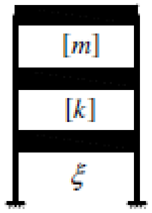
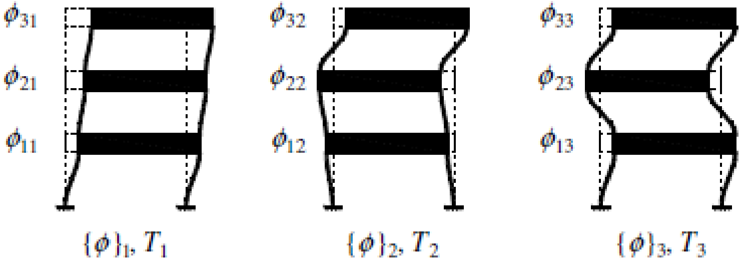
Spectral analysis: summary

- Define structural properties
  - mass [*m*] and stiffness [*k*] matrices
  - critical damping ratio ξ_n
- Determine ω_n ($T_n = 2\pi/\omega_n$) and $\{\phi\}_n$
- Response in *n*-th mode:
  - T_n and $\xi_n \Rightarrow$ pseudo-acceleration A_n from the response spectrum
  - equivalent static forces $\{f\}_n = \{s\}_n A_n$
  - compute response quantity r_n from forces $\{f\}_n$, for each response quantity
- Combine modal contributions r_n to obtain total response using SRSS or CQC combination methods

- Note: generally it is NOT necessary to consider ALL modes of vibration

Dynamics of structures

Spectral analysis: summary

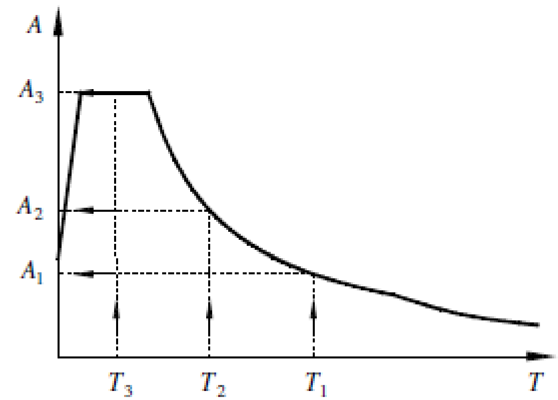
Define properties of the structure: - mass matrix $[m]$ and stiffness matrix $[k]$ - critical damping ratio ξ_n	
Find out natural circular frequencies ω_n (with the corresponding periods $T_n = 2\pi/\omega_n$) and natural modes of vibration $\{\phi\}_n$	

- •Introduction to earthquake engineering
- •Introduction to earthquake seismology
- •linear analysis
- •nonlinear analysis
- •Ductility and nonlinear behavior
- •Design response spectra (UBC & IBC)
- •Method of analysis – Force based (equivalent lateral force)
- •beam, column and joint design issues

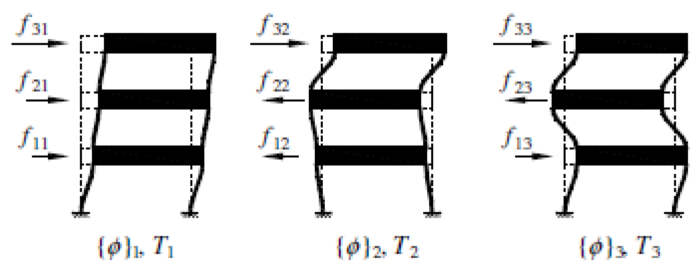
Dynamics of structures

For each mode of vibration find out:

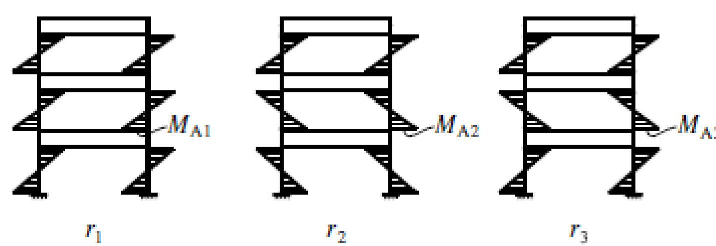
Pseudo accelerations A_n from the response spectrum corresponding to periods of vibration T_n



Equivalent static forces $\{f\}_n$



Response r_n due to forces $\{f\}_n$, for each required response quantity (forces, displacements, etc.



- •Introduction to earthquake engineering
- •Introduction to earthquake seismology
- •linear analysis
- •nonlinear analysis
- •Ductility and nonlinear behavior
- •Design response spectra (UBC & IBC)
- •Method of analysis – Force based (equivalent lateral force)
- •beam, column and joint design issues

Dynamics of structures

Compute the total response r by combining modal contributions r_n (e.g. using the SRSS method)