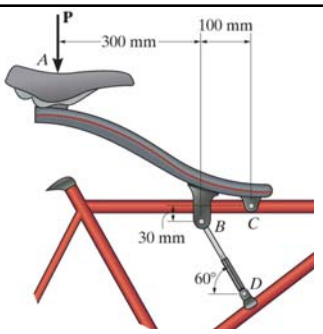
MECHANICS OF MATERIALS

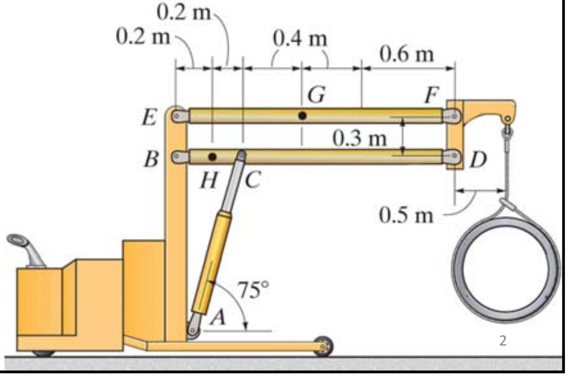
CHAPTER ONE
INTRODUCTION
CONCEPT OF STRESS

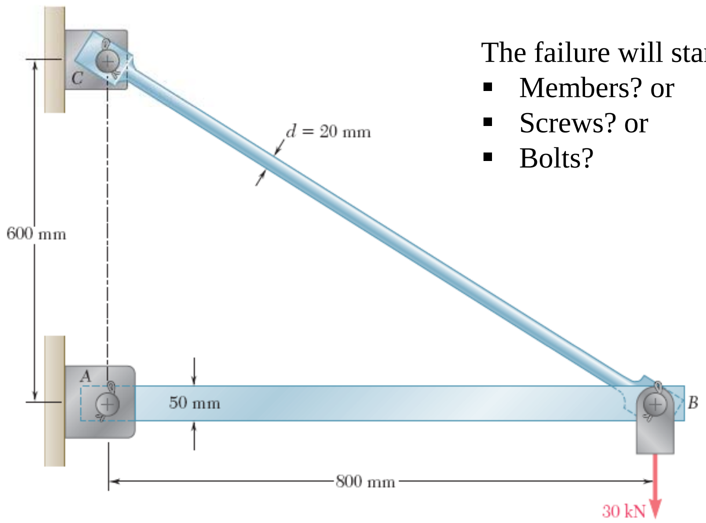
Prepared by : Dr. Mahmoud Rababah

1.1 INTRODUCTION

- Objective of the mechanics of materials is to analyse and design a given structure involving determination of stress and deformation.





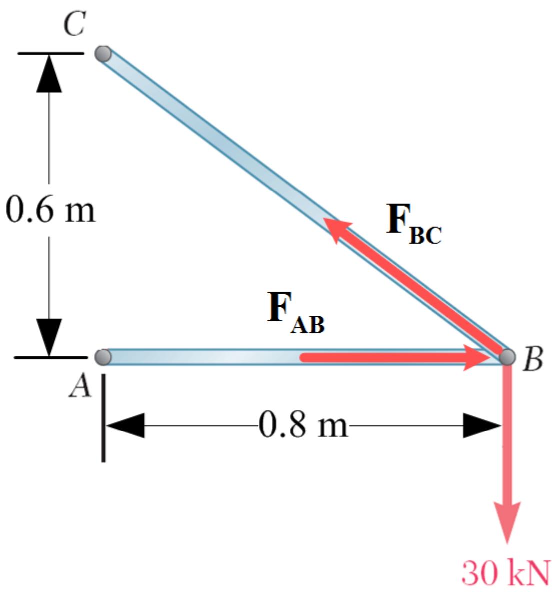


Can the system withstand the load or will break down?

The failure will start in

- Members? or
- Screws? or
- Bolts?

3



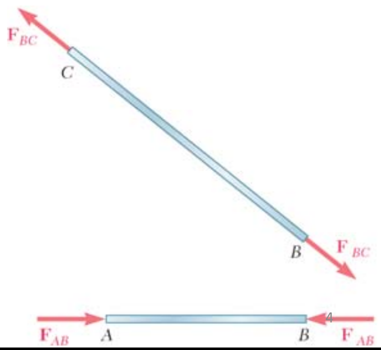
$$\sum \mathbf{F}_x = 0$$

$$\sum \mathbf{F}_y = 0$$

Then, we obtain

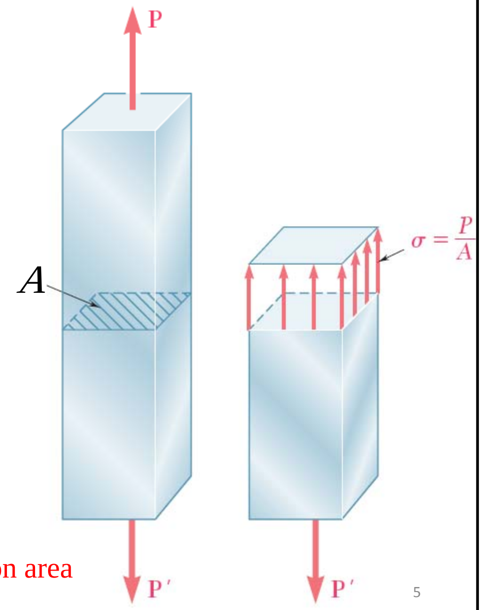
$$\mathbf{F}_{AB} = 40 \text{ kN}$$

$$\mathbf{F}_{BC} = 50 \text{ kN}$$



1.3 STRESS IN THE MEMBERS OF A STRUCTURE

- can the system withstand the force?
- Will the system break down?
- Its ability to withstand the depends on
  1- its material (Steel is stronger than Aluminum)
  2- the cross-section of the rod



Stress is the intensity of the force distributed over a given area

$$\sigma = \frac{\mathbf{P}}{A}$$

stress (N/m^2 = Pa)

force (N)

cross-section area (m^2)

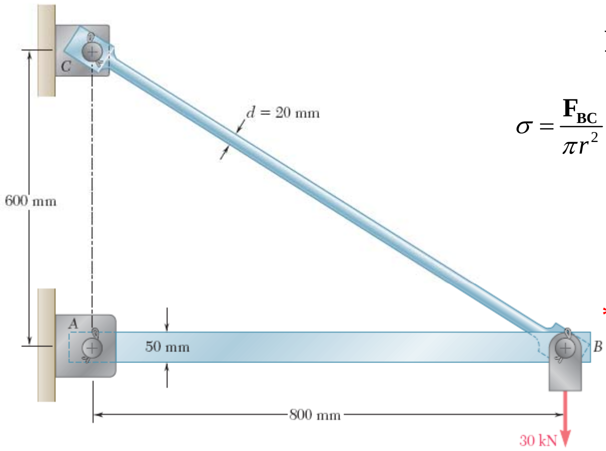
STRESS UNITS

$$1 \text{ kPa} = 10^3 \text{ Pa} = 10^3 \text{ N/m}^2$$

$$1 \text{ MPa} = 10^6 \text{ Pa} = 10^6 \text{ N/m}^2$$

$$1 \text{ GPa} = 10^9 \text{ Pa} = 10^9 \text{ N/m}^2$$

1.4 ANALYSIS AND DESIGN



Assume rod BC is made of steel with maximum allowable stress $\sigma_{all} = 165 \text{ MPa}$

$$\mathbf{F_{BC}} = 50\text{kN}$$

$$\sigma = \frac{\mathbf{F_{BC}}}{\pi r^2} = \frac{50 \text{ kN}}{\pi(10\times10^{-3})^2} = 159 \text{ MPa}$$

$$\sigma < \sigma_{all}$$

*Thus, rod *BC* can withstand the load without breaking down

- ❑ Same concept of stress is used in design.

Example:

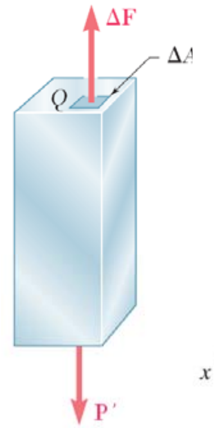
What will be the suitable diameter for rod *BC* without exceeding 100 MPa stress.

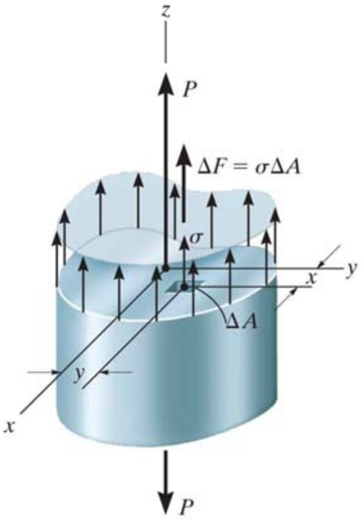
1.5 AXIAL LOADING, NORMAL STRESS

The stress $\sigma = \dfrac{\mathbf{P}}{A}$ is the average stress on the cross – section, the stress at point Q is

$$\sigma = \lim_{A \to 0} \frac{\Delta \mathbf{F}}{\Delta A}$$

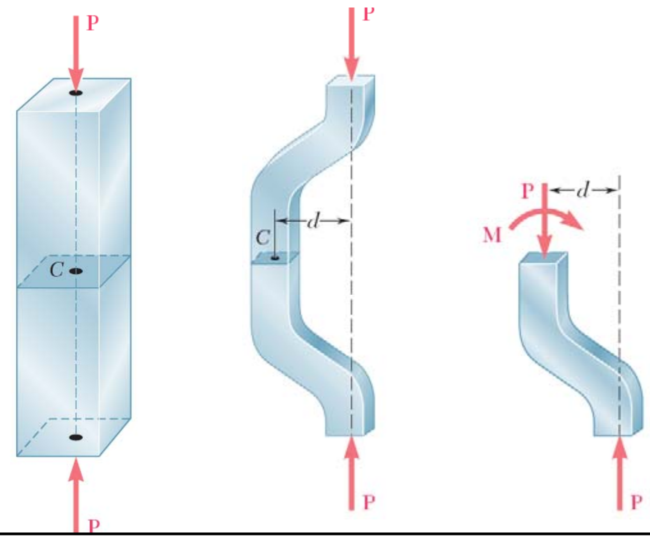
$$P = \int dF = \int_A \sigma dA$$





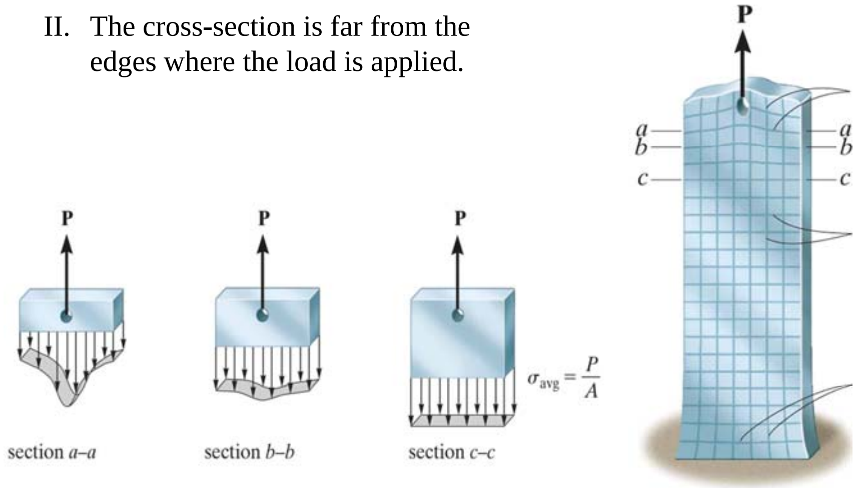
9

- ❑ The stress is considered uniformly distributed if:
  - I. The line of action of the concentrated load passes through the centroid of the cross-section.



10

II. The cross-section is far from the edges where the load is applied.

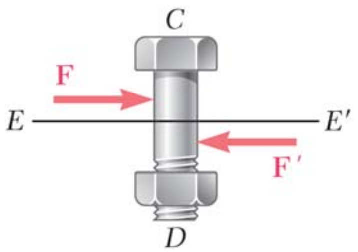


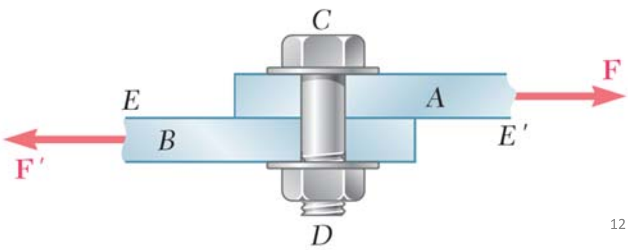
11

1.6 SHEARING STRESS

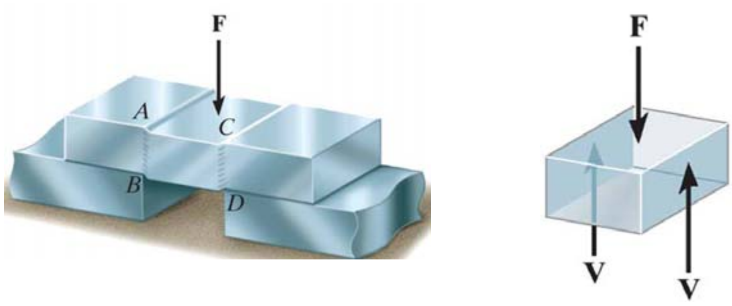
- Transverse load is acting perpendicular to the rod (not in the normal direction).
- The load cause shear stress.

$$\tau_{ave} = \frac{\mathbf{P}}{A} = \frac{\mathbf{F}}{A}$$





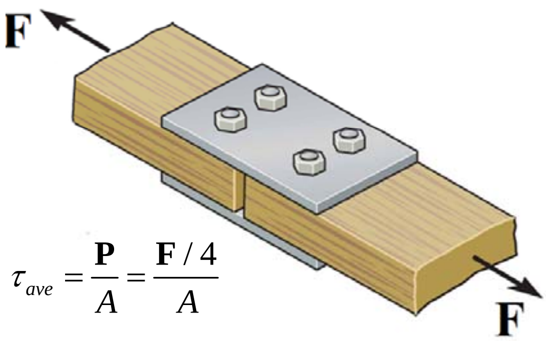
12



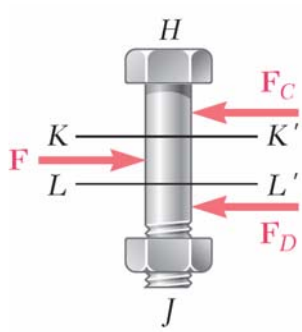
$$\tau_{ave} = \frac{\mathbf{V}}{A} = \frac{\mathbf{F}/2}{A}$$

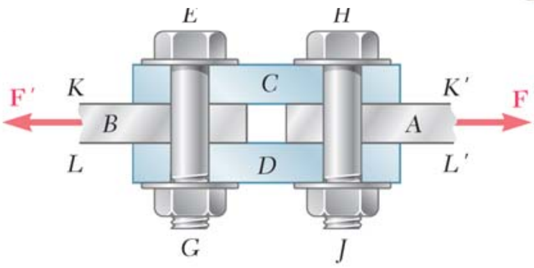
13

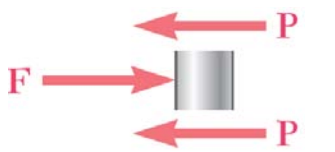
MORE EXAMPLES ON SHEAR



$$\tau_{ave} = \frac{\mathbf{P}}{A} = \frac{\mathbf{F}/4}{A}$$

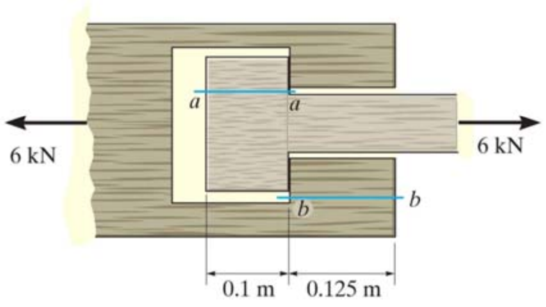






14

Example: given width w = 150 mm. Find the average shear stress along sections *a-a* and *b-b*.



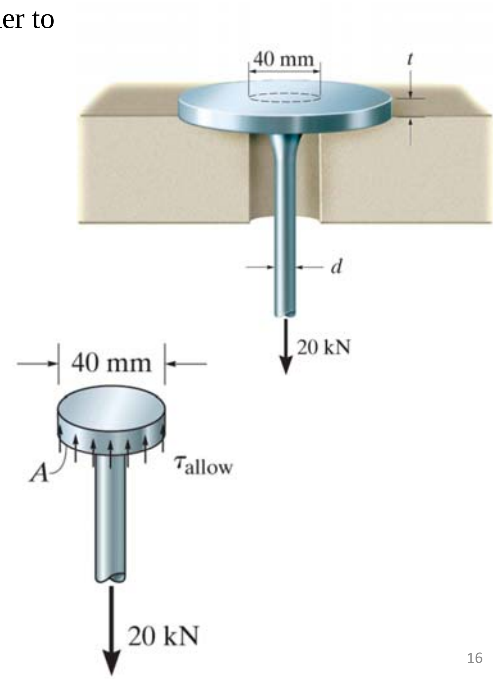
Solution:

15

Example: Find d and t in order to support the 20kN, given

$\sigma_{all} = 60$ MPa

$\tau_{all} = 35$ MPa



16

Example:

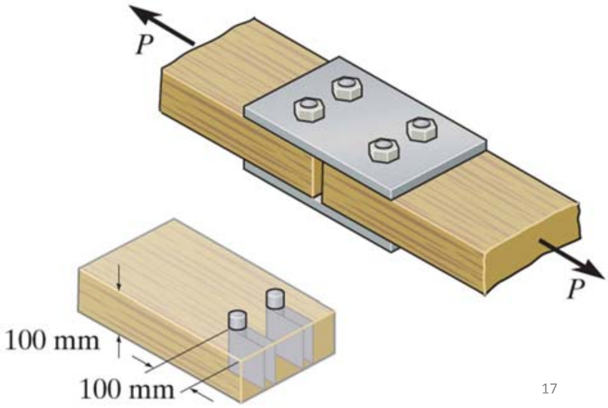
$d = 6$ mm

$P = 9$ kN

Find

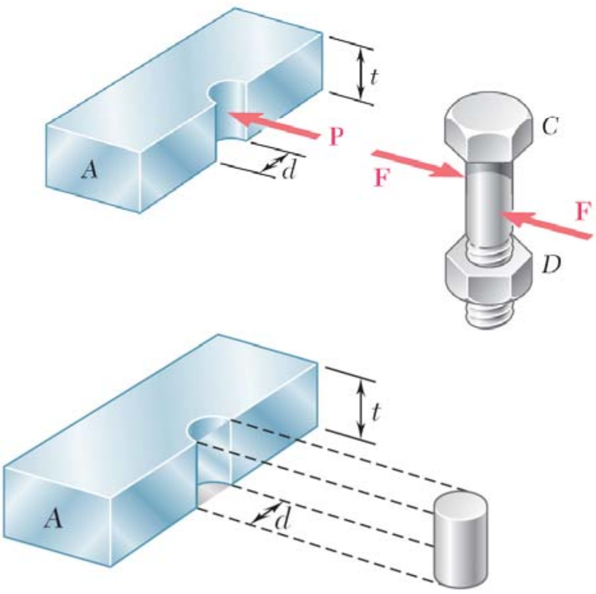
τ_{ave} in pins

τ_{ave} in the shadow planes (tear out)



17

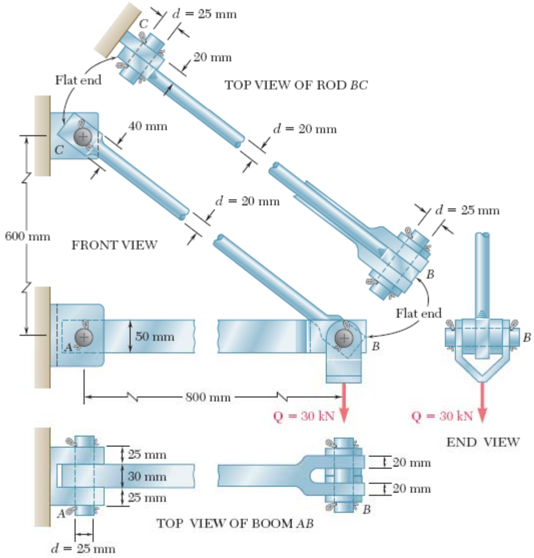
SEC 1.7 BEARING STRESS IN CONNECTIONS



$$\sigma_b = \frac{\mathbf{P}}{td}$$

18

1.8 APPLICATION TO THE ANALYSIS AND DESIGN OF SIMPLE STRUCTURES



$$A_{BC\_B} = A_{BC\_C} = (20\text{ mm})(40\text{ mm} - 25\text{ mm})$$
$$= 300\times10^{-6}\text{ m}^2$$
$$\sigma_{BC-B} = \frac{\mathbf{F_{BC}}}{A_{BC-B}} = \frac{50\times10^3}{300\times10^{-6}} = 167\text{ MPa}$$
$$\sigma_{BC-C} = \sigma_{BC-B} = 167\text{ MPa}$$
$$\sigma_{BC-mid} = \frac{\mathbf{F_{BC}}}{\pi r^2} = \frac{50\text{ kN}}{\pi r^2} = 159\text{ MPa}$$

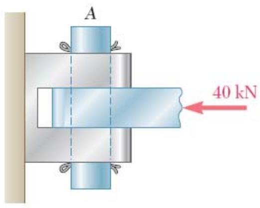
$$\therefore\ \sigma_{BC} = 167\text{ MPa}$$

19

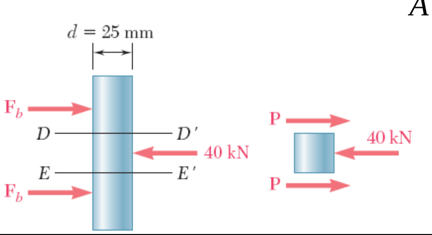
$$\sigma_{AB} = \frac{\mathbf{F_{AB}}}{A_{AB}} = \frac{-40\times10^3}{30\times50\times10^{-6}} = -26.7\text{ MPa}$$

Note that there is no stress on the joints A and B as the rod is under compression.

Shearing stress (Pin A)



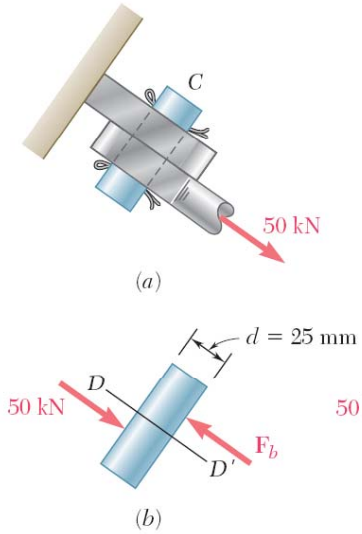
(a)



$$A = \pi r^2 = \pi(12.5\times10^{-3})^2 = 491\times10^{-6}\text{ m}^2$$
$$\mathbf{P} = \frac{\mathbf{F_{AB}}}{2} = 20\text{ kN}$$
$$\tau_A = \frac{\mathbf{P}}{A} = 40.7\text{ MPa}$$

20

Shearing stress (Pin C)



$$A = \pi r^2 = \pi(12.5\times10^{-3})^2 = 491\times10^{-6}\ \text{m}^2$$

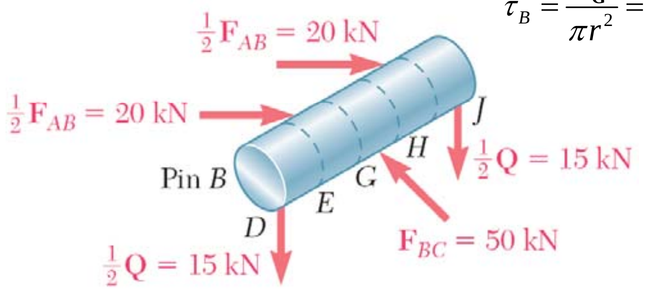
$$\mathbf{P} = \mathbf{F_{BC}} = 50\ \text{kN}$$

$$\tau_C = \frac{\mathbf{P}}{A} = 102\ \text{MPa}$$

21

Shearing stress (Pin B)

$$\tau_B = \frac{\mathbf{P_G}}{\pi r^2} = \frac{25\times10^3}{491\times10^{-6}} = 50.9\ \text{MPa}$$



Bearing stress at point A

1- on the rod

$$\sigma_b = \frac{\mathbf{P}}{td} = \frac{40\times10^3}{30\times25\times10^{-6}} = 53.3\ \text{MPa}$$

2 – on the brackets

$$\sigma_b = \frac{\mathbf{P}}{td} = \frac{40\times10^3}{2\times25\times25\times10^{-6}} = 32.0\ \text{MPa}$$

22

Example : (EDC is rigid)

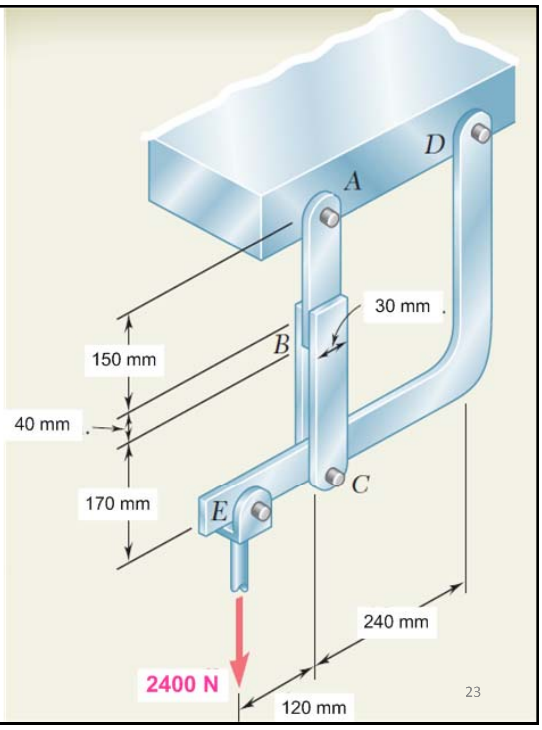
Given :

$t_{AB} = 9$ mm $t_{BC} = 6$ mm each

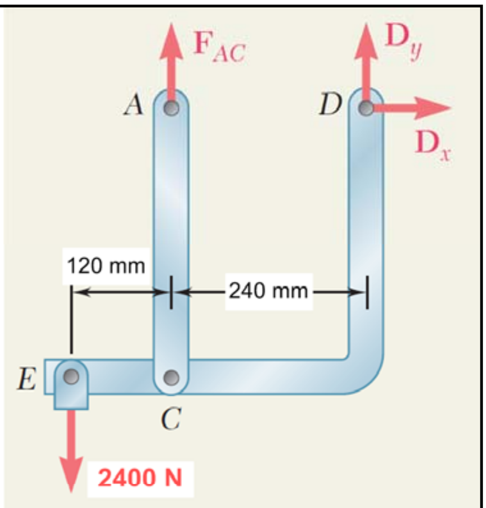
$d_A = 9$ mm $d_C = 6$ mm

Find :

1 – shearing stress at pin *A*.
2 – shearing stress at pin *C*.
3 – the normal stress at link *ABC*.
4 – shearing stress at *B*.
5 – bearing stress in the link at *C*.

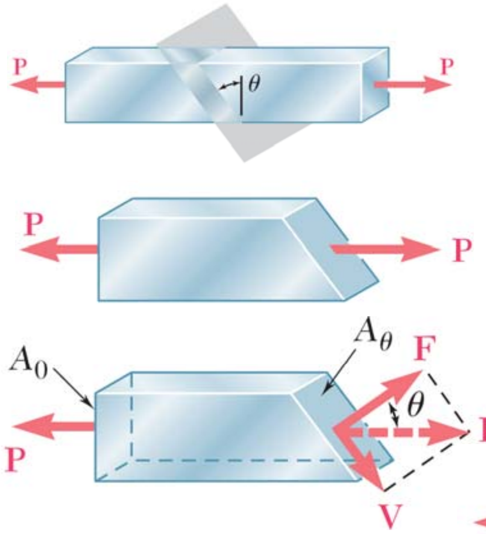


23



24

1.11. STRESS ON AN OBLIQUE PLANE UNDER AXIAL LOADING



$$\sigma = \frac{\mathbf{F}}{A_\theta} \qquad \tau = \frac{\mathbf{V}}{A_\theta}$$

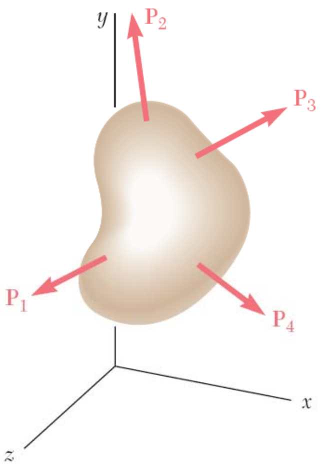
$$A_\theta = \frac{A_0}{\cos\theta}$$

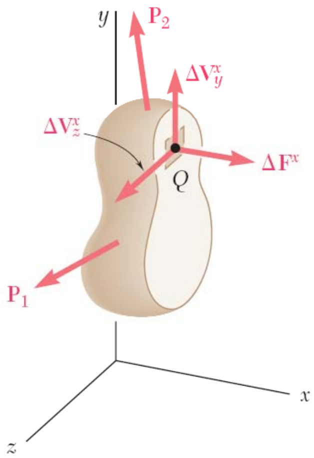
$$\sigma = \frac{P\cos\theta}{A_0/\cos\theta} \qquad \tau = \frac{P\sin\theta}{A_0/\cos\theta}$$

$$\sigma = \frac{P}{A_0}\cos^2\theta \qquad \tau = \frac{P}{A_0}\sin\theta\cos\theta$$

25

1.12. STRESS UNDER GENERAL LOADING CONDITIONS; COMPONENTS OF STRESS

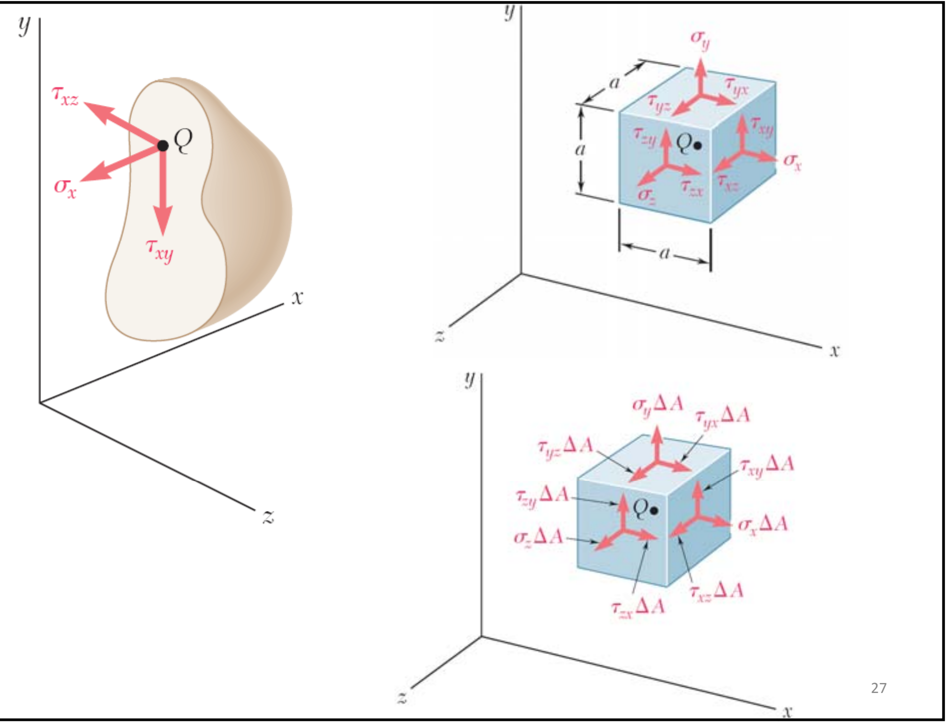




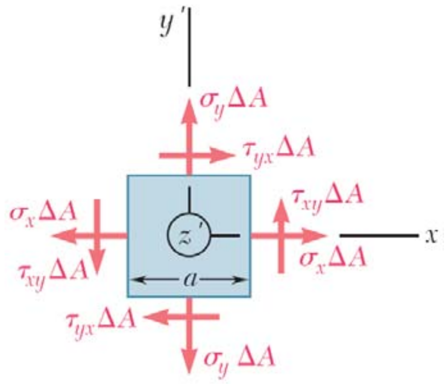
$$\sigma_x = \lim_{\Delta A \to 0} \frac{\Delta F^x}{\Delta A}$$

$$\tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta V_y^x}{\Delta A} \qquad \tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta V_z^x}{\Delta A}$$

26



27



$$\sum M_z = 0; \qquad (\tau_{xy}\Delta A)a - (\tau_{yx}\Delta A)a = 0$$

$$\tau_{xy} = \tau_{yx}$$

also

$$\tau_{xz} = \tau_{zx} \qquad \tau_{yz} = \tau_{zy}$$

28

1.13 DESIGN CONSIDERATION

1- Determination of the ultimate strength of a material.

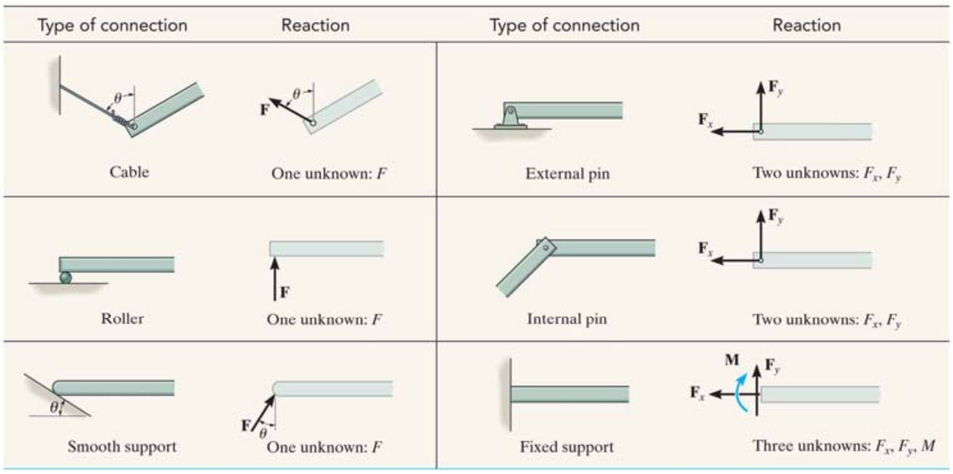
$$\sigma_U = \frac{P_U}{A}$$

2- Allowable stress; factor of safety

$$\text{Factor of safety} = \text{F.S} = \frac{\text{Ultimate stress}}{\text{Allowable stress}}$$

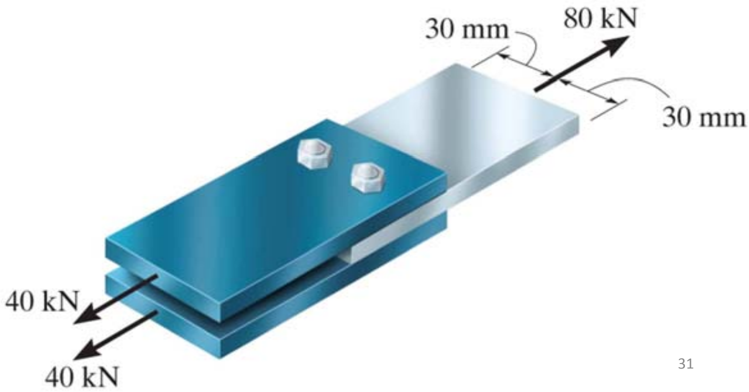
29

TIPS (SUPPORT REACTIONS)



30

Example : Determine the required diameter of the bolts if the failure shear stress is $\tau_{Fail} = 350$ MPa. use a factor of safety F.S = 2.5.



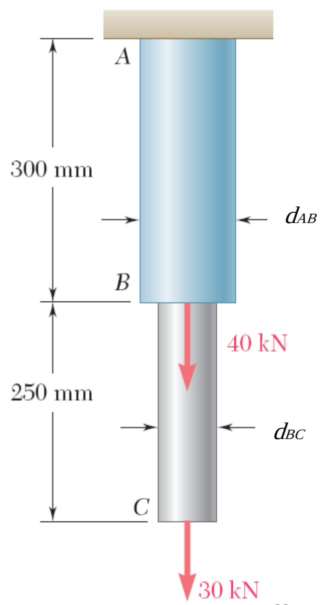
31

Example : Given

$(\sigma_{AB})_{all} = 175$ MPa

$(\sigma_{BC})_{all} = 150$ MPa

Find d_{AB} and d_{BC}



32

END OF CHAPTER ONE

33

MECHANICS OF MATERIALS

CHAPTER TWO
STRESS AND STRAIN-AXIAL LOADING

Prepared by : Dr. Mahmoud Rababah

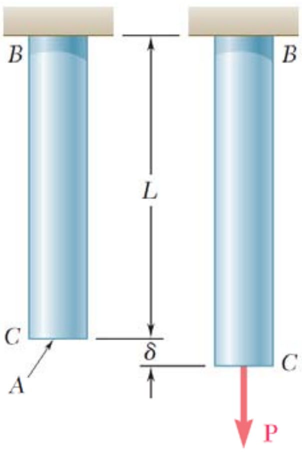
2.2 NORMAL STRAIN UNDER AXIAL LOADING

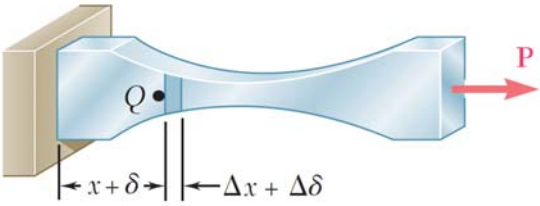
- Normal strain under axial loading is the deformation per unit length.

$$\varepsilon = \frac{\delta}{L}$$

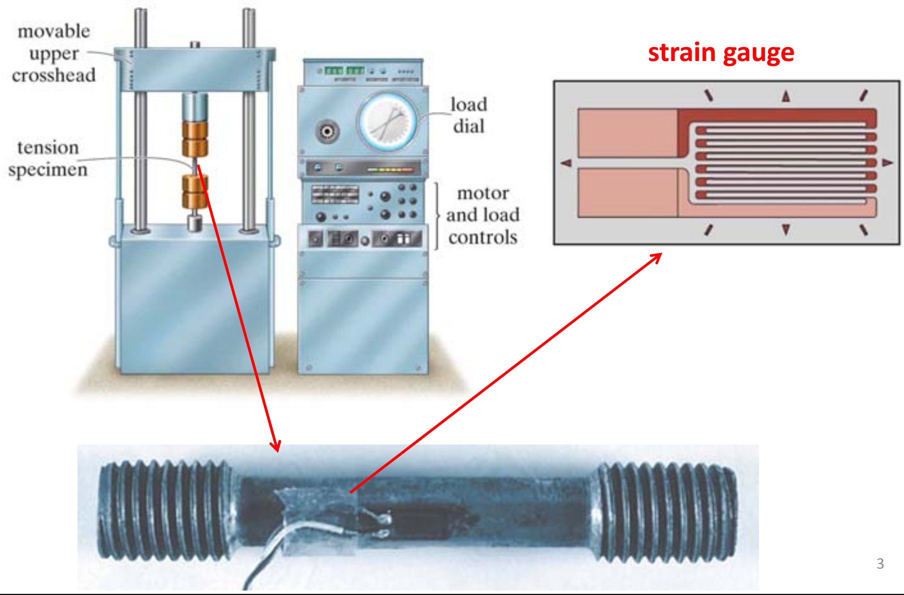
- In case of non uniform cross-sections, the normal stress will vary along the member. Thus, we consider elements of small length
- After deformation, element Δx will increase by Δδ. Thus

$$\varepsilon = \frac{\Delta\delta}{\Delta x} \text{.....for infinite elements, } \quad \varepsilon = \frac{d\delta}{dx}$$

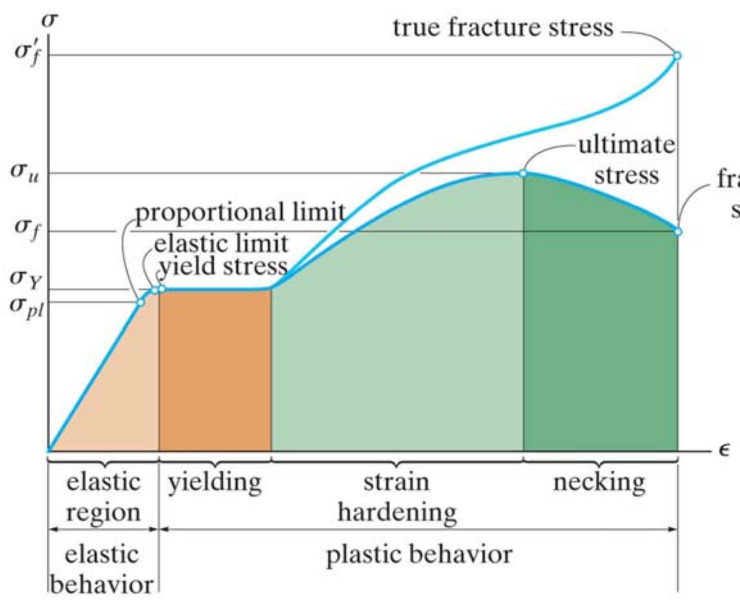




2.3 STRESS-STRAIN DIAGRAM



3



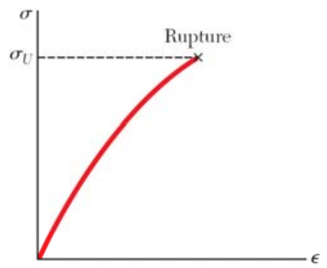
$$\sigma = \frac{\mathbf{P}}{A_0}$$

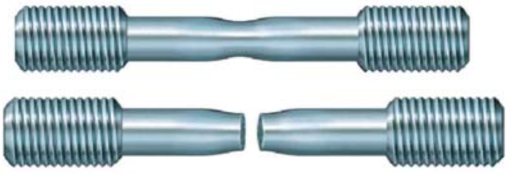
$$\varepsilon = \frac{L - L_0}{L_0}$$

4

Tension failure of
a brittle material

Failure occurs due to internal defects that initiate a crack perpendicular to the normal stress.





Failure of a
ductile material

Failure occurs by slippage of the material along oblique surfaces and is due primarily to shearing stresses (form a cone shape of angle 45°)

5

2.5 HOOK'S LAW (MODULUS OF ELASTICITY)

- In the elastic region a linear relationship between stress and strain is existed.

$$\sigma = E\varepsilon \qquad \text{hook's law}$$

2.6 ELASTIC VERSUS PLASTIC BEHAVIOUR OF A MATERIAL

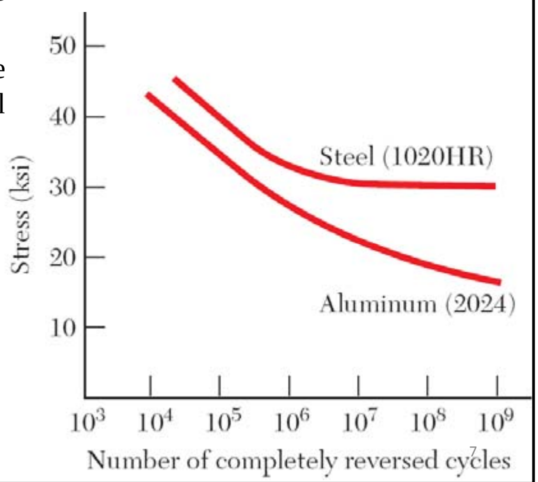
- Elastic region:
  - The deformation is recovered after releasing the load.
  - Percentage of deformation is small.

- Plastic region:
  - the deformation is permanent.
  - Percentage of deformation is magnificent.

6

2.7 REPEATED LOADING (FATIGUE)

- Local plastic deformation occurs before reaching the yielding point.
- Endurance limit: is the stress for which failure will never occur.



2.8 DEFORMATION OF MEMBERS UNDER AXIAL LOADING

$$\sigma = E\varepsilon$$

$$\frac{\mathbf{P}}{A} = E \times \frac{\delta}{L}$$

$$\delta = \frac{\mathbf{P}L}{AE}$$

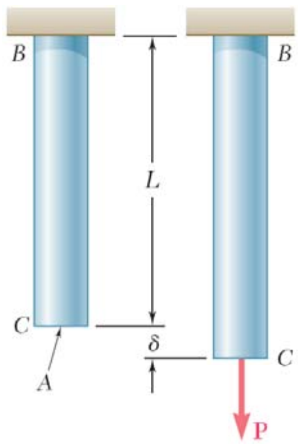
For multi – sections

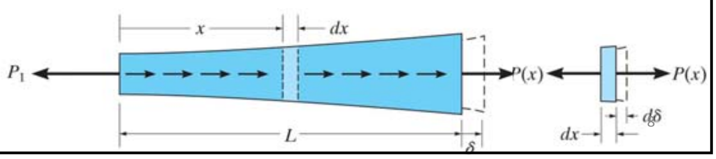
$$\delta = \sum_i \frac{\mathbf{P}_i L_i}{A_i E_i}$$

For variable cross – section

$$d\delta = \varepsilon dx = \frac{\mathbf{P}(x)}{A(x)E} dx$$

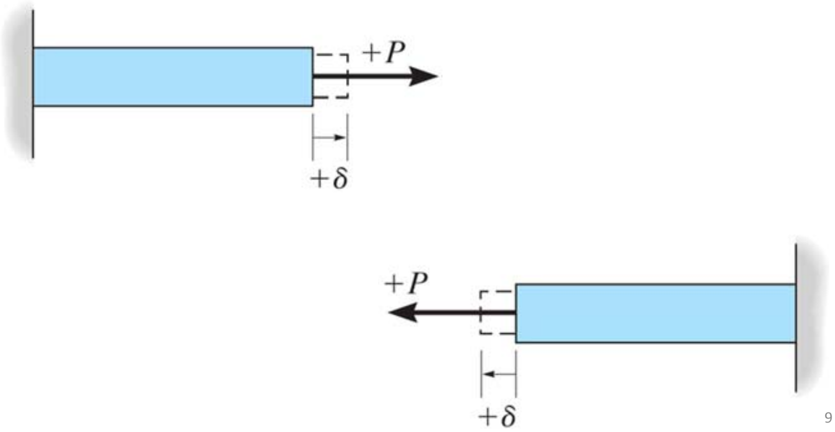
$$\delta = \int_0^L \frac{\mathbf{P}(x)}{A(x)E} dx$$





SIGN CONVENTION

Regardless of the direction, the deformation δ is positive if the length increased and negative otherwise



EXAMPLES

Example :

$E_{AB} = 70$ GPa

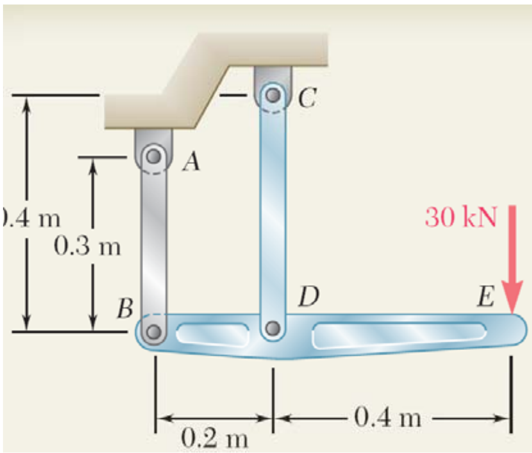
$A_{AB} = 500\ \text{mm}^2$

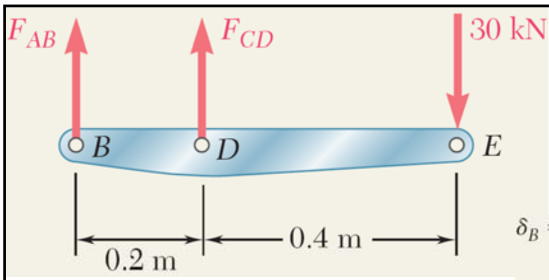
$E_{CD} = 200$ GPa

$A_{CD} = 600\ \text{mm}^2$

find

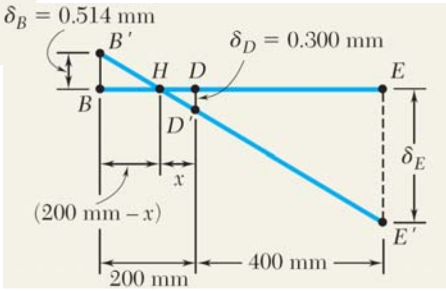
δ_B, δ_D and δ_E





$\sum \mathbf{M_B} = 0 \quad \rightarrow \quad \mathbf{F_{CD}} = 90 \text{ kN (Tension)}$

$\sum \mathbf{M_D} = 0 \quad \rightarrow \quad \mathbf{F_{AB}} = -60 \text{ kN (compression)}$



$$\delta_B = \frac{\mathbf{F_{AB}} L_{AB}}{A_{AB} E_{AB}} = \frac{-60\times10^3\times0.3}{500\times10^{-6}\times70\times10^9} = -514\times10^{-6}\ \text{m}$$

$$\delta_D = \frac{\mathbf{F_{CD}} L_{CD}}{A_{CD} E_{CD}} = \frac{90\times10^3\times0.4}{600\times10^{-6}\times200\times10^9} = 300\times10^{-6}\ \text{m}$$

$x = 73.7$ mm

$$\frac{\delta_E}{\delta_D} = \frac{400 + x}{x}$$

$\delta_E = 1.928$ mm

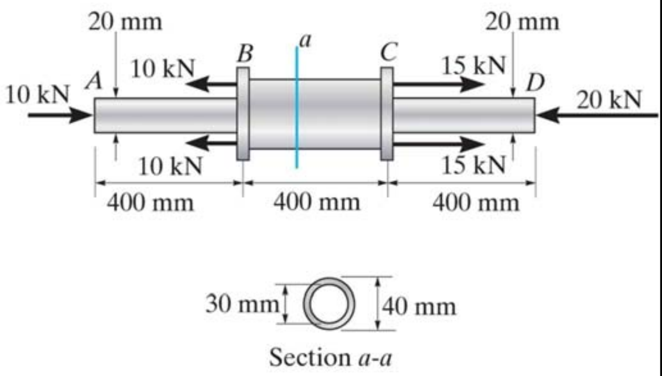
11

Example :

$E = 200$ GPa

Find

$\delta_{D/A}$



12

Example:

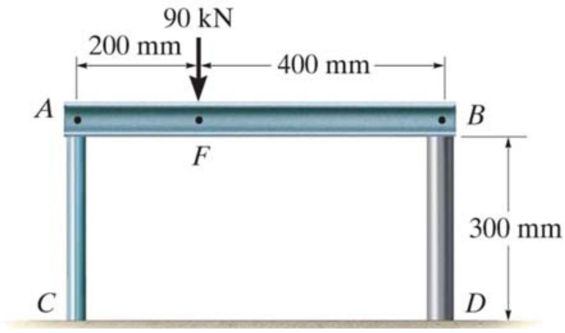
$d_{AC} = 20$ mm

$E_{AC} = 200$ GPa

$d_{BD} = 40$ mm

$E_{BD} = 70$ GPa

find δ_F



Solution:

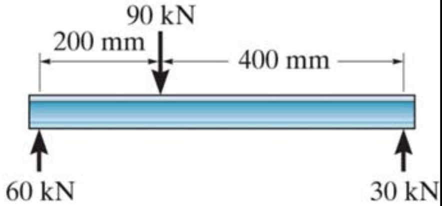
$$\delta_C = \frac{\mathbf{F_{AC}} L_{AC}}{A_{AC} E_{AC}} = \frac{-60\times10^3\times0.3}{\frac{\pi}{4}\times(0.02)^2\times200\times10^9}$$

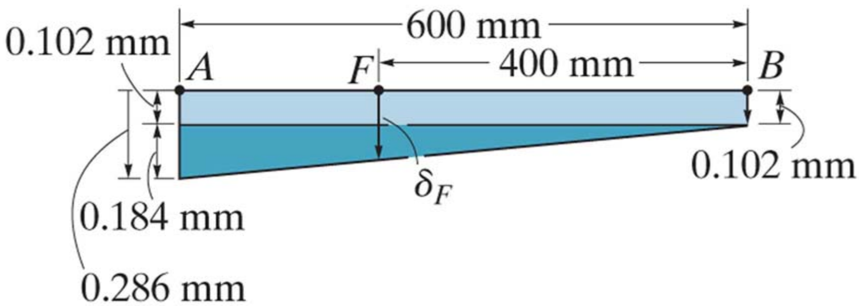
$$= -286\times10^{-6}\ \text{m}$$

$$\delta_B = \frac{\mathbf{F_{BD}} L_{BD}}{A_{BD} E_{BD}} = \frac{-30\times10^3\times0.3}{\frac{\pi}{4}\times(0.04)^2\times70\times10^9}$$

$$= -102\times10^{-6}\ \text{m}$$

FBD





$$\delta_F = 0.102\ \text{mm} + 0.184\ \text{mm}\left(\frac{400\ \text{mm}}{600\ \text{mm}}\right) = 0.225\ \text{mm}$$

Example :

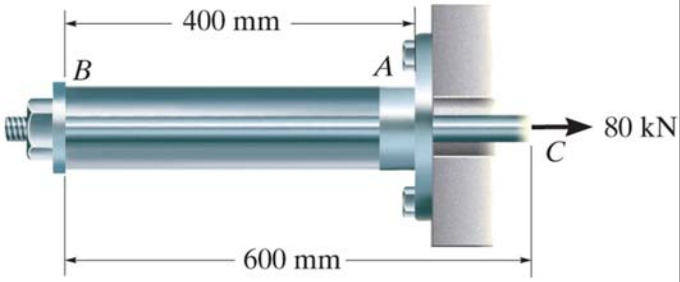
$A_{AB} = 400 \text{ mm}^2$

$E_{AB} = 70 \text{ GPa}$

$d_{BC} = 10 \text{ mm}$

$E_{BC} = 200 \text{ GPa}$

Find δ_C



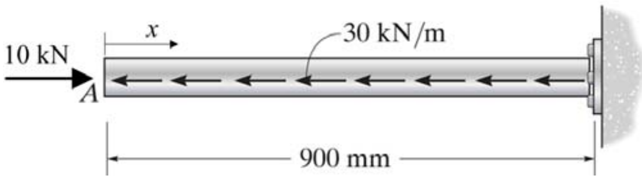
15

Example :

$E = 200 \text{ GPa}$

$A = 100 \text{ mm}^2$

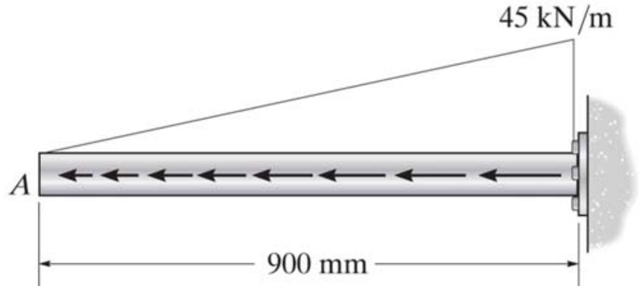
find δ_A



16

Example :

find δ_A



Example :

$w = 10$ kN/m

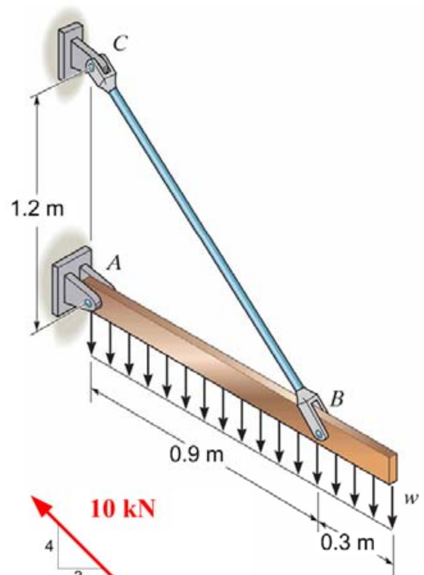
$d_{BC} = 13$ mm

$d_B = d_C = 10$ mm

$\sigma_y = 250$ MPa

$\tau_y = 125$ MPa

what is the factor of safety F.S



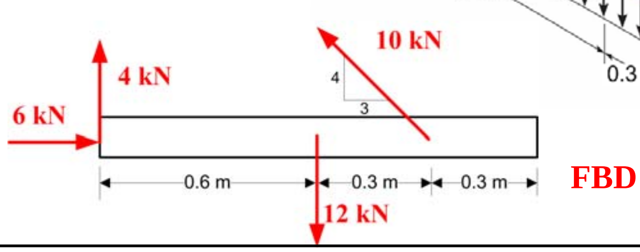
Solution :

$\sum \mathbf{M_B} = 0$

$\mathbf{F_{AC}} = 4$ kN

$\sum \mathbf{F_y} = 0$

$\mathbf{F_{BC}} = 10$ kN



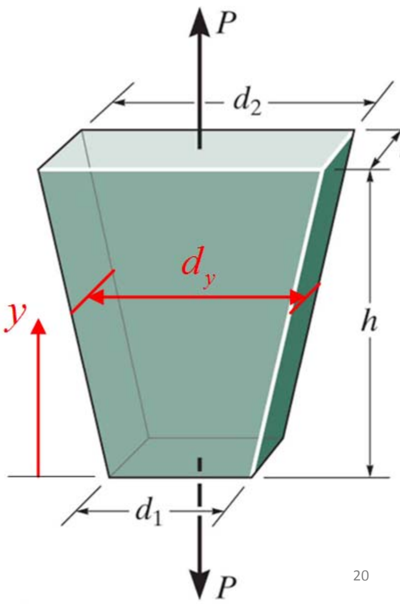
Example :

$$d_y = d_1 + \frac{(d_2 - d_1)}{h} y$$

$$A(y) = t d_y$$

$$\delta = \int_0^h \frac{\mathbf{P}}{A(y)E} dy$$

$$\delta = \frac{\mathbf{P}h}{t(d_2 - d_1)} \ln\left(d_1 + \frac{(d_2 - d_1)}{h} y \right)\Bigg]_0^h$$



2.9 STATICALLY INDETERMINATE PROBLEMS

- Number of unknowns are more than number of equilibrium equations.
- We use the compatibility conditions to solve the problem.

$$\mathbf{F_A} + \mathbf{F_B} - \mathbf{P} = 0 \qquad (1)$$

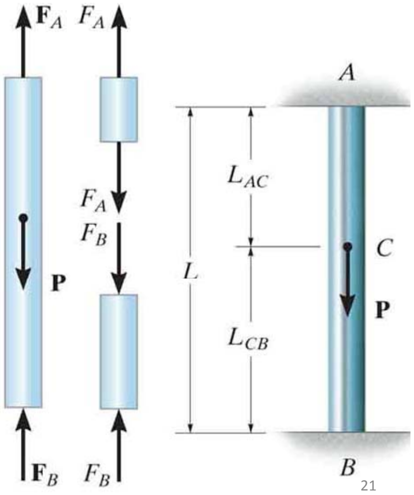
Compatibility condition

$$\delta_{A/B} = 0$$

$$\frac{\mathbf{F_A} L_{AC}}{AE} - \frac{\mathbf{F_B} L_{CB}}{AE} = 0 \qquad (2)$$

From Eq. 1 and Eq. 2, we get

$$\mathbf{F_A} = \left(\frac{L_{BC}}{L}\right)\mathbf{P} \quad \text{and} \quad \mathbf{F_B} = \left(\frac{L_{AC}}{L}\right)\mathbf{P}$$



21

Example :

Find $\mathbf{P}_1$ and $\mathbf{P}_2$

Solution :

$$\mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P} = 0 \qquad (1)$$

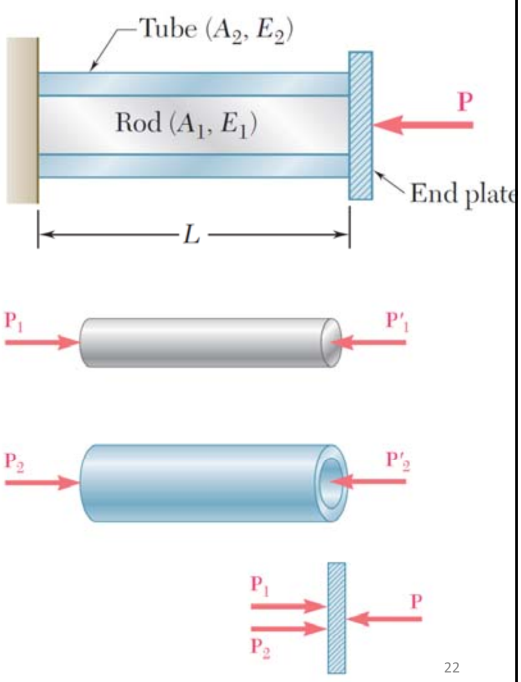
$$\delta_1 = \delta_2$$

$$\frac{\mathbf{P}_1}{A_1E_1} = \frac{\mathbf{P}_2}{A_2E_2} \qquad (2)$$

then we get

$$\mathbf{P}_1 = \frac{A_1E_1}{A_1E_1 + A_2E_2}\mathbf{P}$$

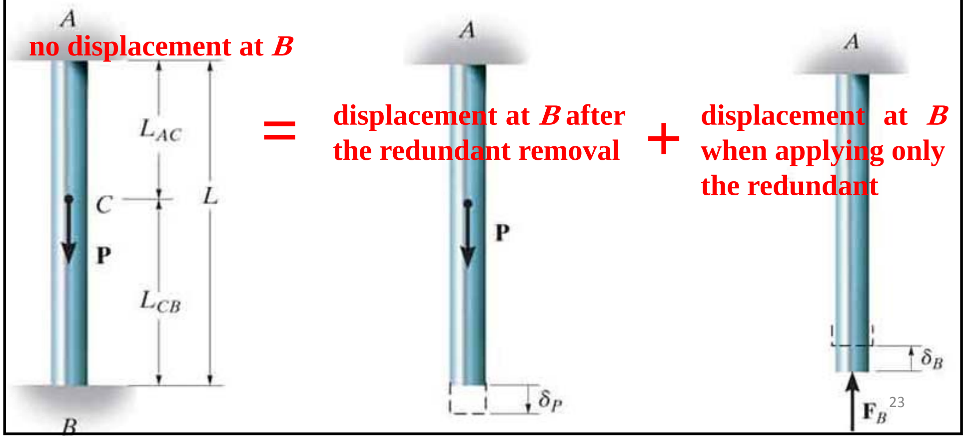
$$\mathbf{P}_2 = \frac{A_2E_2}{A_1E_1 + A_2E_2}\mathbf{P}$$



22

SUPERPOSITION METHOD

- The reaction force to be removed is called *redundant*
- The redundant force is removed and the deformation is calculated
- The deformation of the redundant is considered in separate.



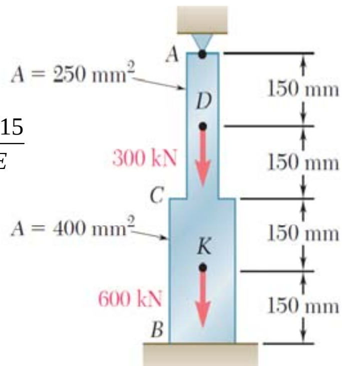
23

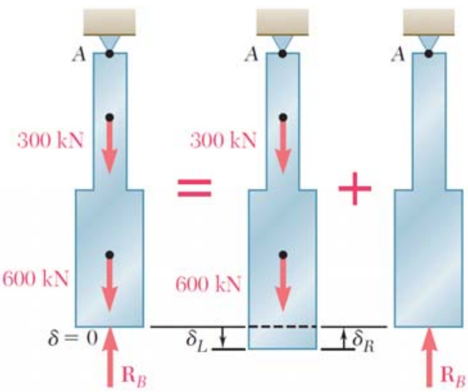
Example: $E = 200\text{GPa}$

Find $\mathbf{R_A}$ and $\mathbf{R_B}$

$$\delta_L = \sum_{i=1}^{4} \frac{\mathbf{P}_i L_i}{A_i E} = \left(0 + \frac{600\times10^3}{400\times10^{-6}} + \frac{600\times10^3}{250\times10^{-6}} + \frac{900\times10^3}{250\times10^{-6}}\right) \times \frac{0.15}{E}$$

$$\delta_L = \frac{1.125\times10^9}{E} = 5.63 \text{ mm}$$





$$\delta_R = \sum_{i=1}^{2} \frac{\mathbf{P_i} L_i}{A_i E} = \left(\frac{-\mathbf{R_B}}{250\times10^{-6}} + \frac{-\mathbf{R_B}}{400\times10^{-6}}\right) \times \frac{0.3}{E}$$

$$\delta_R = \frac{-1.95\times10^3 \mathbf{R_B}}{E}$$

$$\delta_L + \delta_R = 0$$

$$\mathbf{R_B} = 577 \text{ kN}$$

$$\mathbf{R_A} = 900 \text{ kN} - 577 \text{ kN} = 323 \text{ kN}$$

24

Example:

$E = 200$ GPa

Find $\mathbf{R_B}$

Solution:

$$\delta = \delta_L + \delta_R = 4.5\times10^{-3}$$

$$\delta_L = \frac{1.125\times10^{9}}{E} = 5.63 \text{ mm}$$

$$\delta_R = \frac{-1.95\times10^{3}\mathbf{R_B}}{E}$$

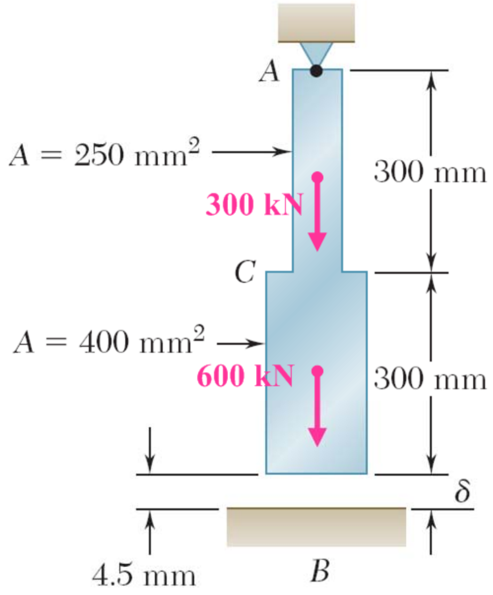
$\mathbf{R_B} = 115.4$ kN

$\mathbf{R_A} = 900 \text{ kN} - \mathbf{R_B} = 785$ kN

Note: if $\delta_L < 4.5\times10^{-3}$ then

$\mathbf{R_A} = 900$ kN

$\mathbf{R_B} = 0$



25

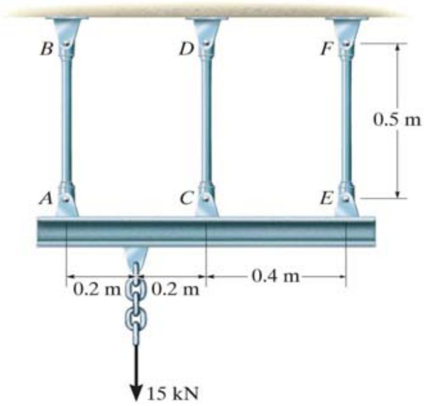
Example:

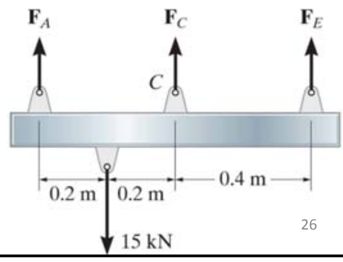
Find $\mathbf{R_A}$, $\mathbf{R_C}$ and $\mathbf{R_E}$ given

$A_{AB} = A_{EF} = 50 \text{ mm}^2$

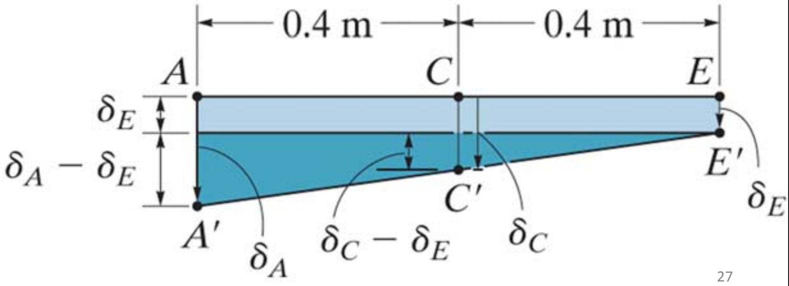
$A_{CD} = 30 \text{ mm}^2$

all made of steel





26



27

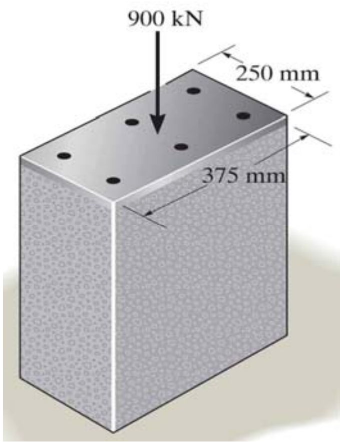
Example:

$d_{st} = 20$ mm

$E_{st} = 200$ GPa

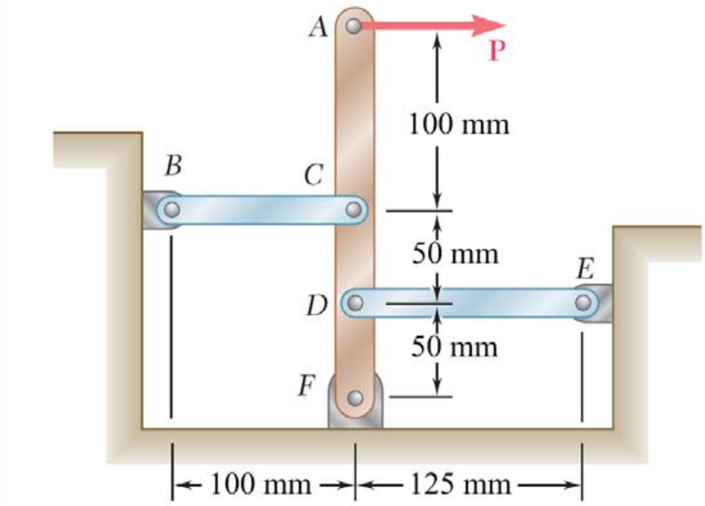
$E_c = 25$ GPa

Find σ_{st} and σ_c



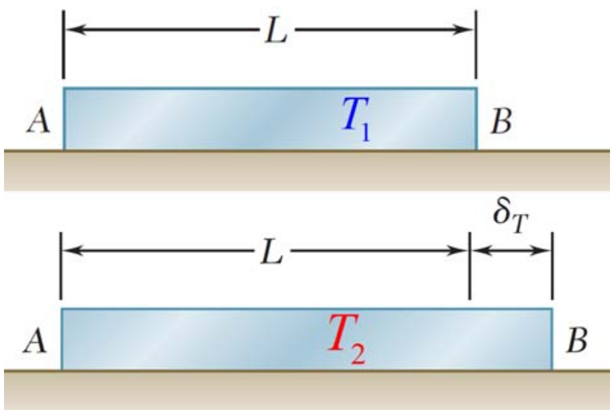
28

Problem: (*ACDF* is rigid).
Given E = 100 GPa, find:
1- the forces acting on members *BC* and *DE*,
2- the deflection at point *A*



29

2.10 PROBLEMS INVOLVING TEMPERATURE CHANGES



thermal expansion coefficient

$$\delta_T = \alpha(\Delta T)L$$

thermal deformation

$$\varepsilon_T = \alpha(\Delta T)$$

thermal strain

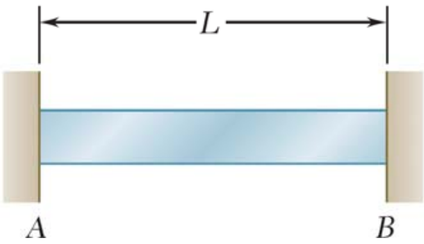
30

THERMAL FORCES

$$\delta_T + \delta_L = 0$$

$$\alpha(\Delta T)L - \frac{\mathbf{P}L}{AE} = 0$$

$$\mathbf{P} = \alpha(\Delta T)AE$$



31

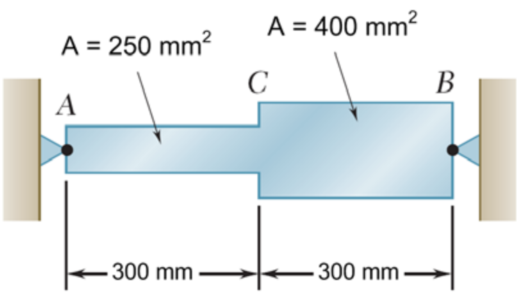
Example :

$E = 70$ GPa

$T_1 = 20\ ^0$C

$T_2 = 80\ ^0$C

$\alpha = 21\times10^{-6} / {}^0$C



32

Example :

$E_{al} = 73.1$ GPa

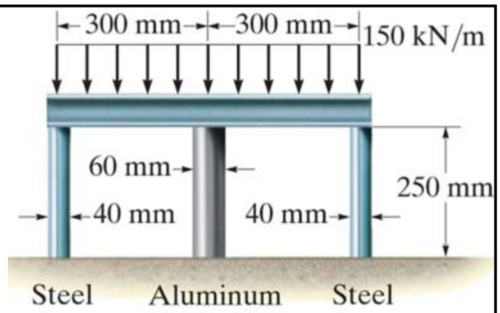
$E_{st} = 200$ GPa

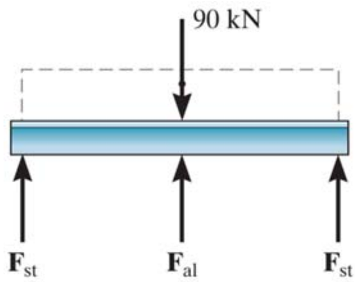
$\alpha_{st} = 12\times10^{-6} / {}^0\mathrm{C}$

$\alpha_{al} = 23\times10^{-6} / {}^0\mathrm{C}$

$T_1 = 20\ {}^0\mathrm{C}$

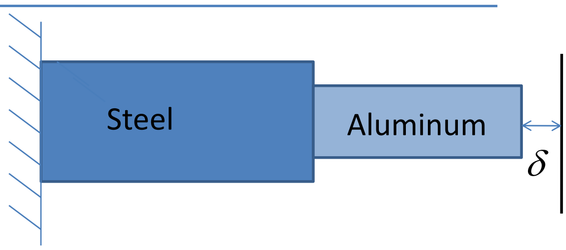
$T_2 = 80\ {}^0\mathrm{C}$





33

Example :



note:

$if \quad (\delta_{st})_T + (\delta_{al})_T < \delta$

$F = 0; \ \sigma_T = 0$

else

$(\delta_{st})_T + (\delta_{st})_F + (\delta_{al})_T + (\delta_{al})_F = \delta$

34

Example :

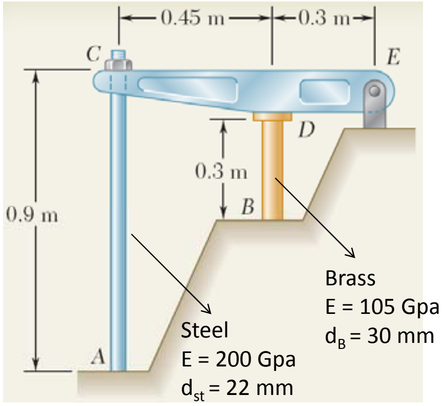
$T_1 = 20\ ^0\text{C}$

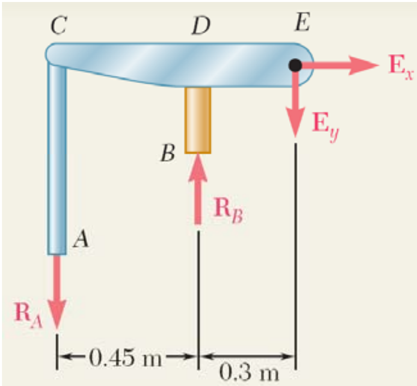
$T_2 = 50\ ^0\text{C}$

only for brass

$\alpha_{st} = 12\times10^{-6}\ /\ ^0\text{C}$

$\alpha_B = 18.8\times10^{-6}\ /\ ^0\text{C}$





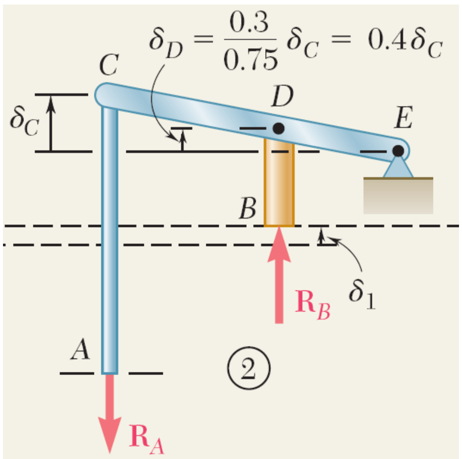
Solution :

$\sum M_E = 0$

$0.75\mathbf{R_A} - 0.3\mathbf{R_B} = 0$

$\mathbf{R_A} = 0.4\mathbf{R_B}$ (1)

35



$\delta_D = 0.4\delta_C = 0.4\times\frac{\mathbf{R_A}L_{AC}}{A_{AC}E_{st}} = 4.74\times10^{-9}\mathbf{R_A}$

$\delta_D = (\delta_D)_T - (\delta_D)_F$

$\alpha_B \Delta T L_{BD} - \frac{\mathbf{R_B}L_{BD}}{A_{BD}E_B} = 4.74\times10^{-9}\mathbf{R_A}$ (2)

get

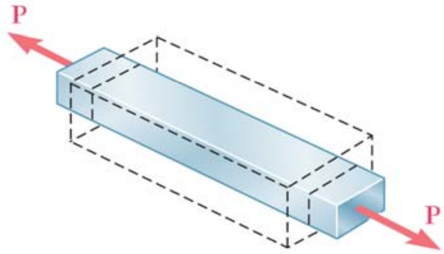
$\mathbf{R_A} = 11.4$ kN

$\mathbf{R_B} = 28.5$ kN

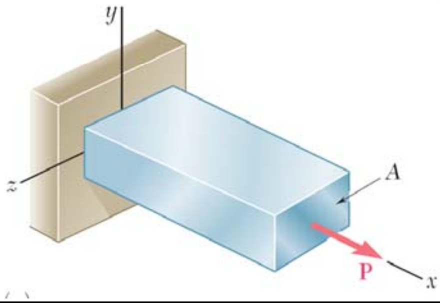
$\sigma_B = \frac{\mathbf{R_B}}{A_{BD}} = 40.3$ MPa

36

2.11 POISSON'S RATIO



$$v = -\frac{\text{lateral strain}}{\text{axial strain}}$$



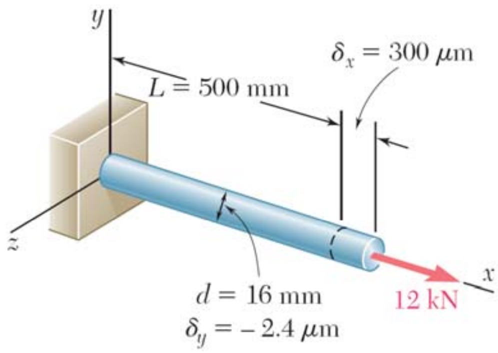
$$v = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}$$

$$\varepsilon_x = \frac{\sigma_x}{E}. \text{ Thus, } \varepsilon_y = \varepsilon_z = -\frac{v\sigma_x}{E}$$

37

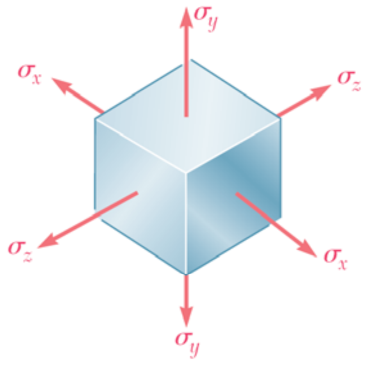
Example:

Find v and E



38

2.12 MULTI-AXIAL LOADING
(GENERALIZED HOOK'S LAW)



$$\varepsilon_x = +\frac{\sigma_x}{E} - v\frac{\sigma_y}{E} - v\frac{\sigma_z}{E}$$

$$\varepsilon_y = -v\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - v\frac{\sigma_z}{E}$$

$$\varepsilon_z = -v\frac{\sigma_x}{E} - v\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

39

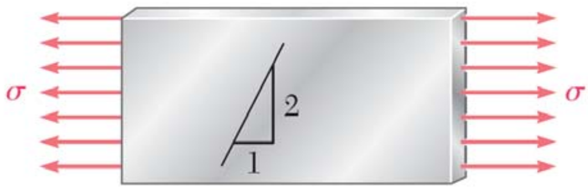
Example :

$\sigma = 125$ MPa

$E = 75$ GPa

$v = 0.33$

Find slope of the line



40

Example :

$t = 18$ mm

$\sigma_x = 85$ MPa

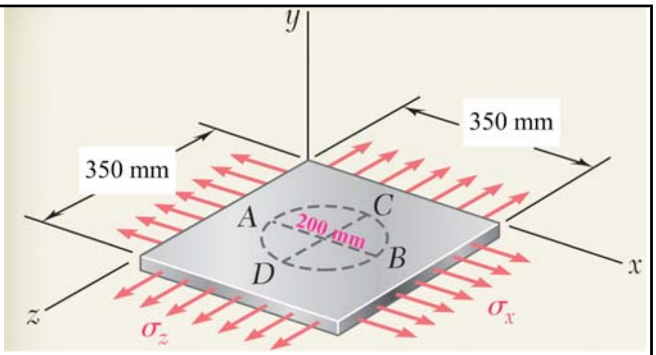
$\sigma_z = 150$ MPa

$E = 70$ GPa

$v = \frac{1}{3}$

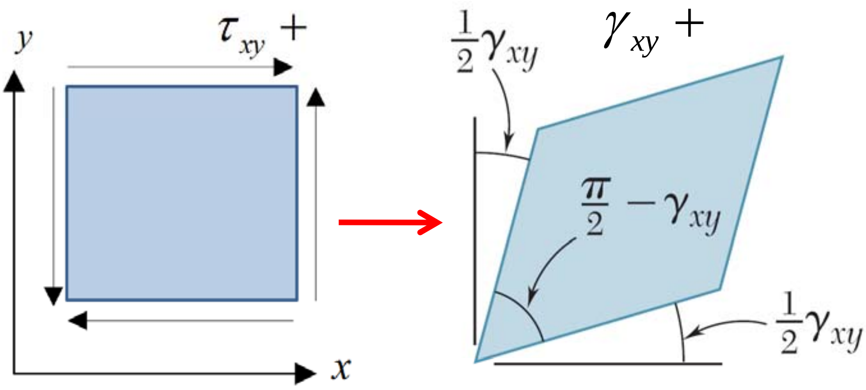
Find

L_{AB}, L_{CD}, t, V .



41

2.14 SHEARING STRAIN



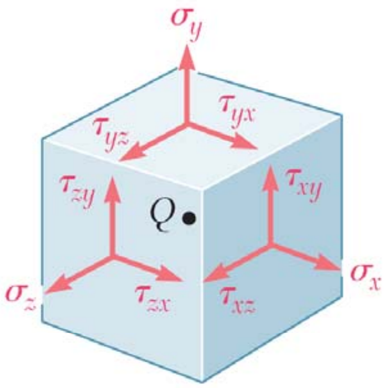
$$\tau_{xy} = G\gamma_{xy}$$

Shear stress

Modulus of rigidity

Shear strain (rad)

42



Also

$$\tau_{yz} = G\gamma_{yz}$$

$$\tau_{xz} = G\gamma_{xz}$$

recall from chapter 1 that

$$\tau_{xy} = \tau_{yx}$$

$$\tau_{xz} = \tau_{zx}$$

$$\tau_{yz} = \tau_{zy}$$

43

Example:

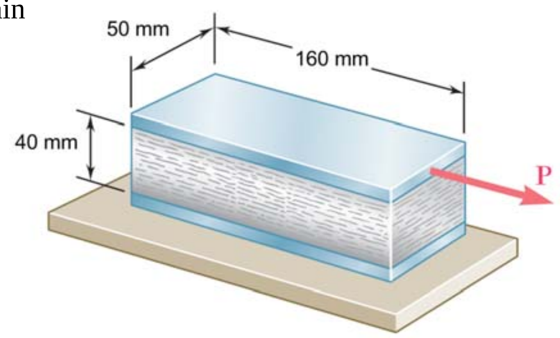
G = 600 MPa

The upper plate is **rigid** and moved 0.8 mm

Determine:

1- The average shearing strain

2- The force **P**



44

2.15 RELATIONSHIP AMONG *v*, E AND G

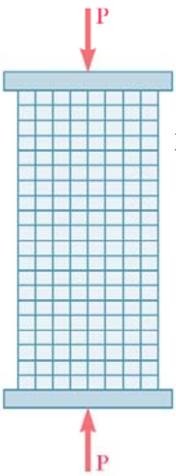
modulus of elasticity

modulus of rigidity

$$G = \frac{E}{2(1+v)}$$

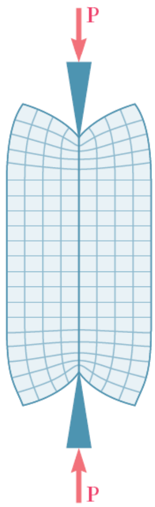
poisson's ratio

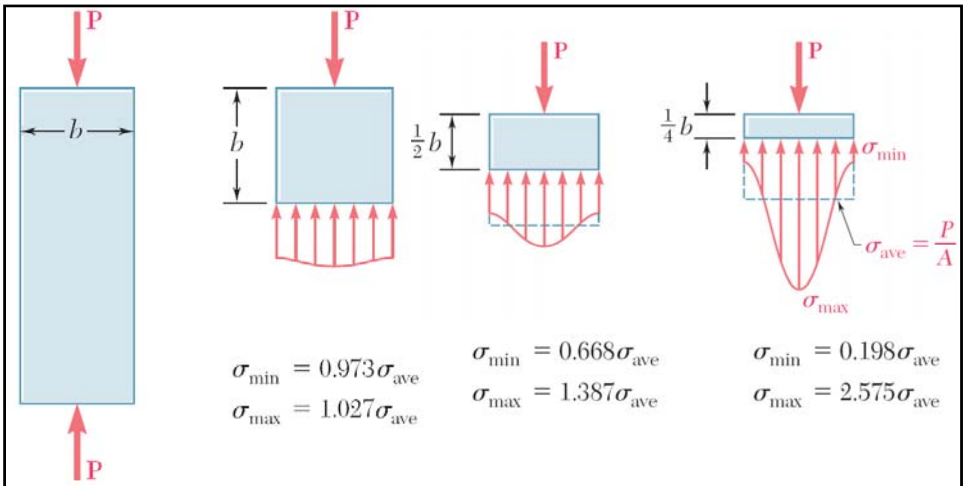
2.17 STRESS AND STRAIN DISTRIBUTION UNDER AXIAL LOADING (SAINT-VENANT'S PRINCIPLE)



Axial load applied
by rigid plate

Concentrated axial
load applied directly





- **Saint-Venant's Principle**: The stress distribution is uniform far from the vicinity of the applied loads.
- **Far** means away by at least a magnitude equal to the largest dimension of the cross-section.

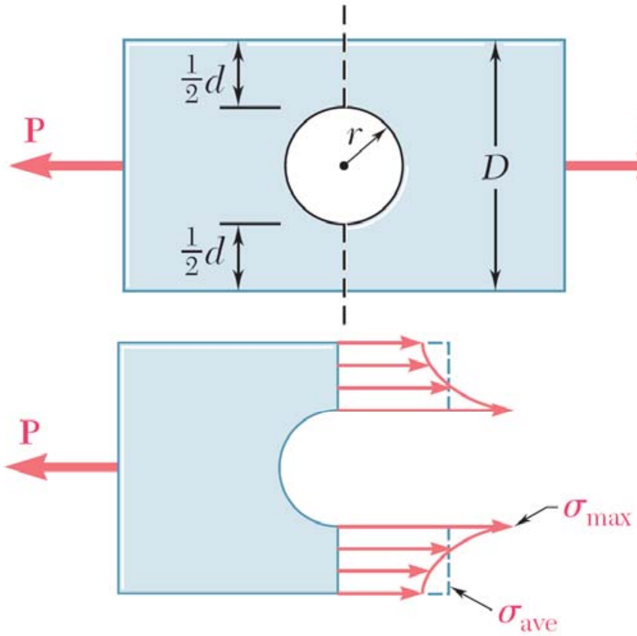
47

2.18 STRESS CONCENTRATION

- Stress concentration is independent of the size of the piece.
- It depends upon the relation between the geometric parameters.
- Designers are only interested in maximum value of stress.

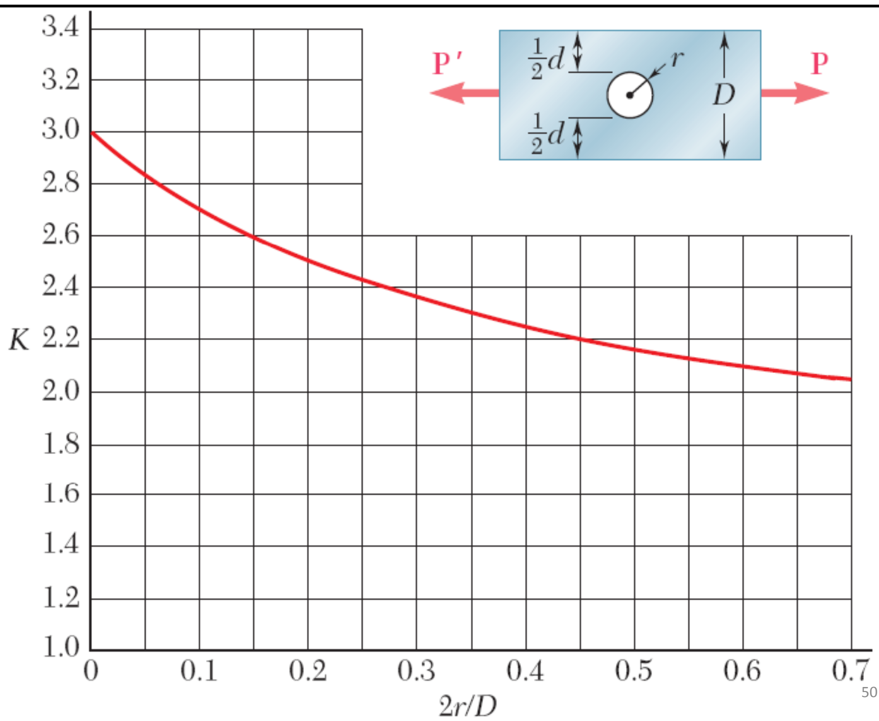
- Fillet is used to reduce the stress concentration.
- For brittle materials: crack will be initiated at the place of the stress concentration and will propagate until failure.
- For ductile materials: stress concentration will cause local plastic deformation.

48

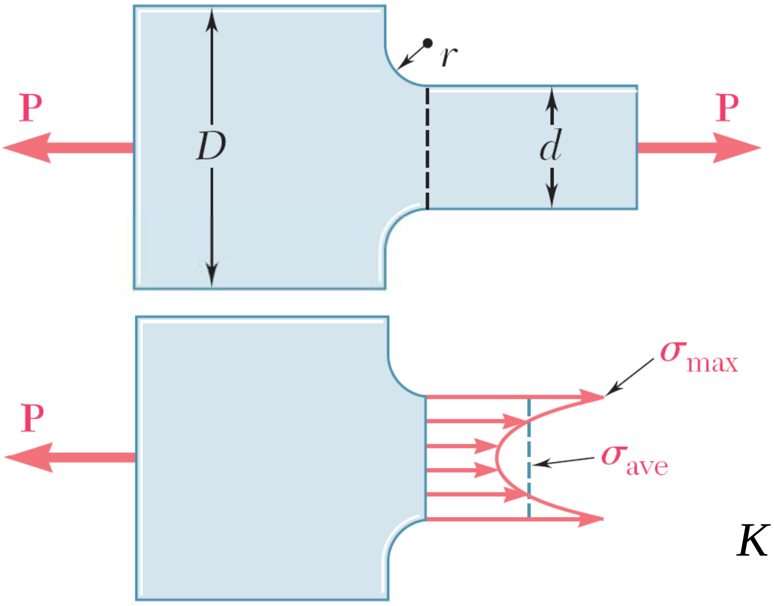


$$K = \frac{\sigma_{max}}{\sigma_{ave}}$$

49

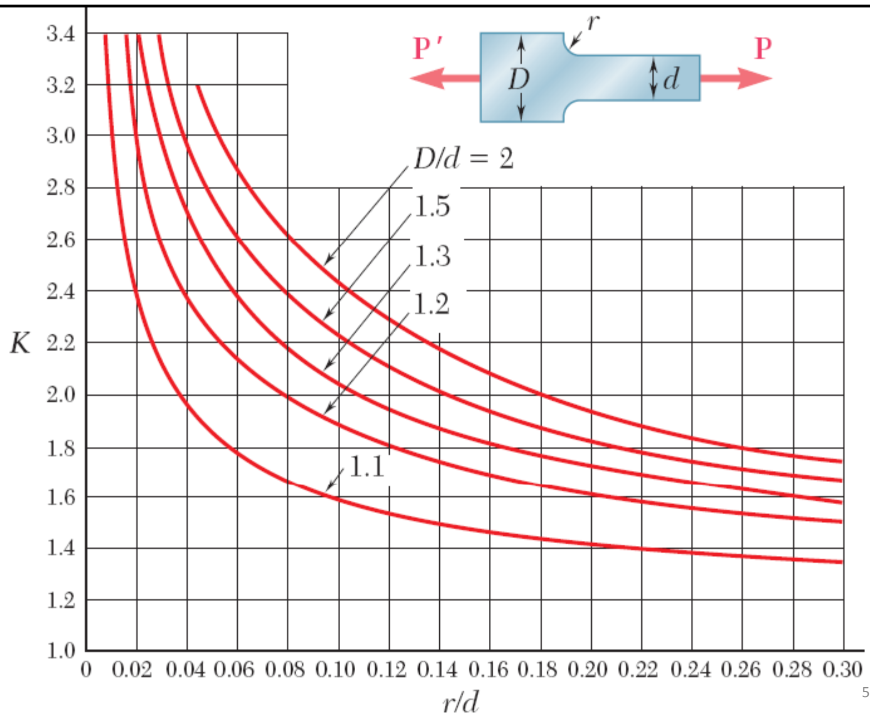


50



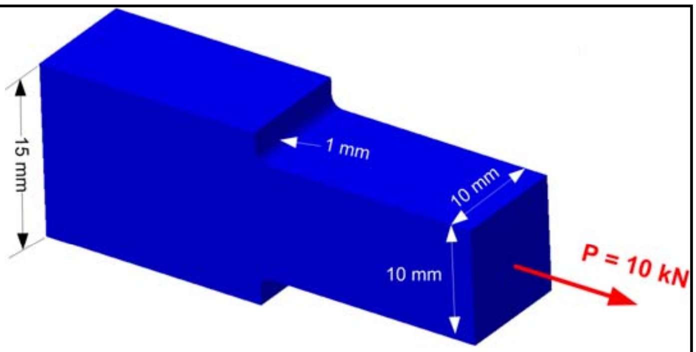
$$K = \frac{\sigma_{max}}{\sigma_{ave}}$$

51



52

Example : Given
$\sigma_y = 300$ MPa
Find F.S



END OF CHAPTER TWO

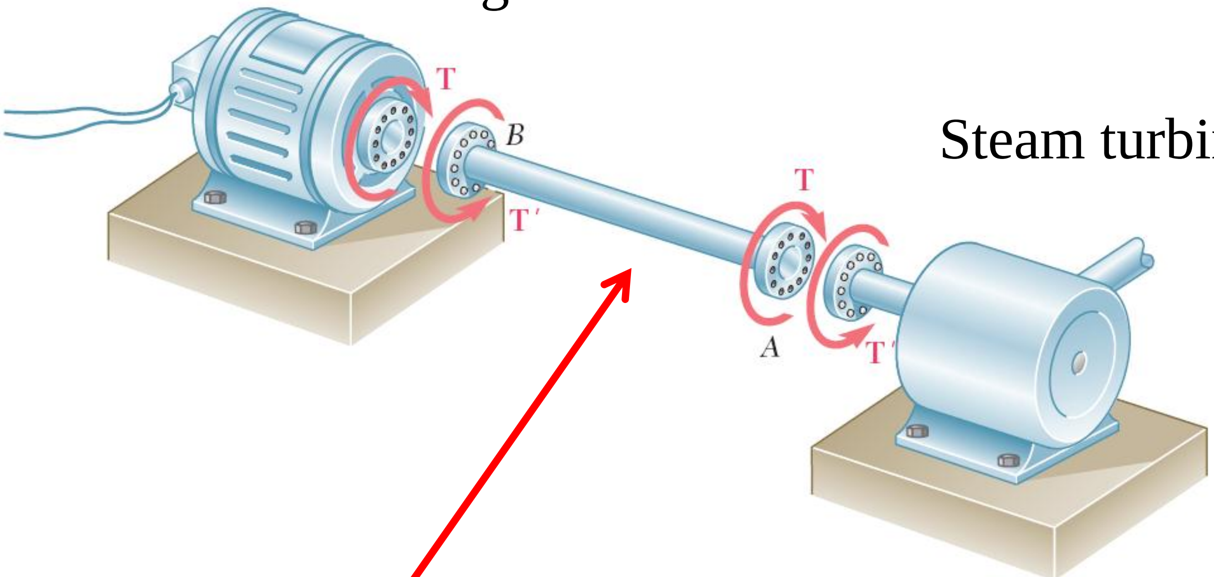
MECHANICS OF MATERIALS

CHAPTER THREE
TORSION

Prepared by : Dr. Mahmoud Rababah

3.1 INTRODUCTION

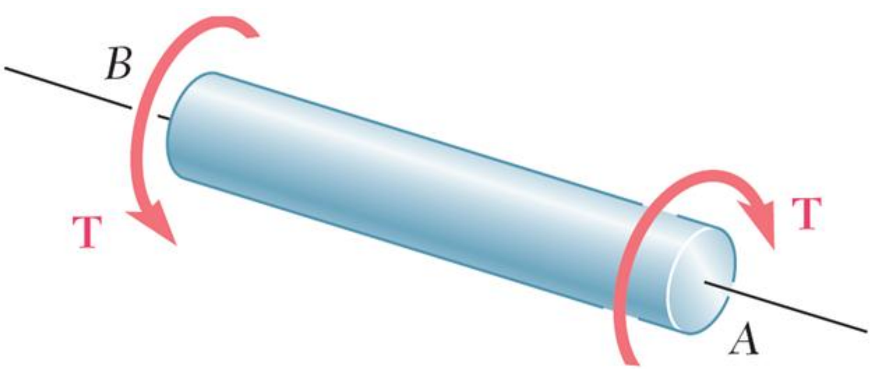
Electric generator

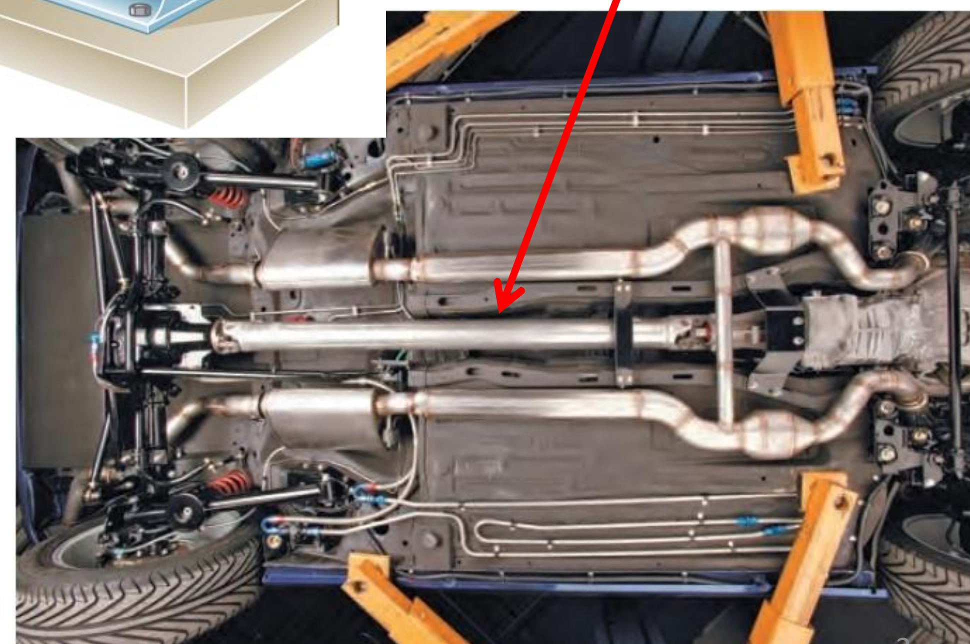


Steam turbine

Transmission shaft

Transmission shaft



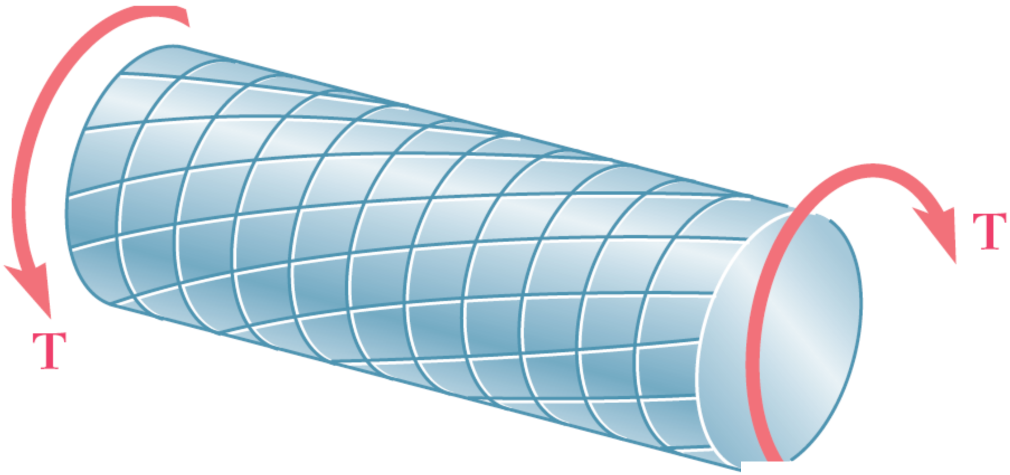


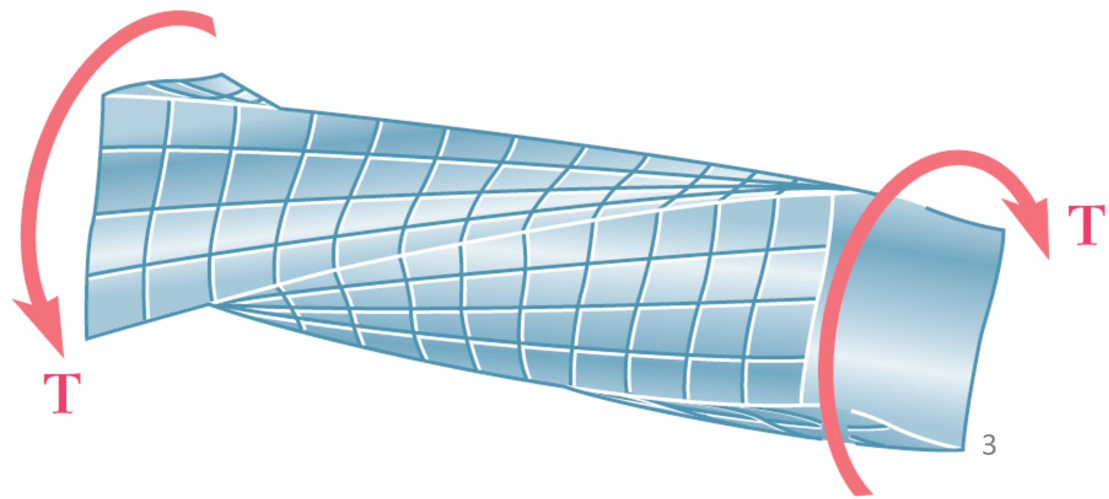
3.2 - 3.3 DEFORMATION IN A CIRCULAR SHAFT

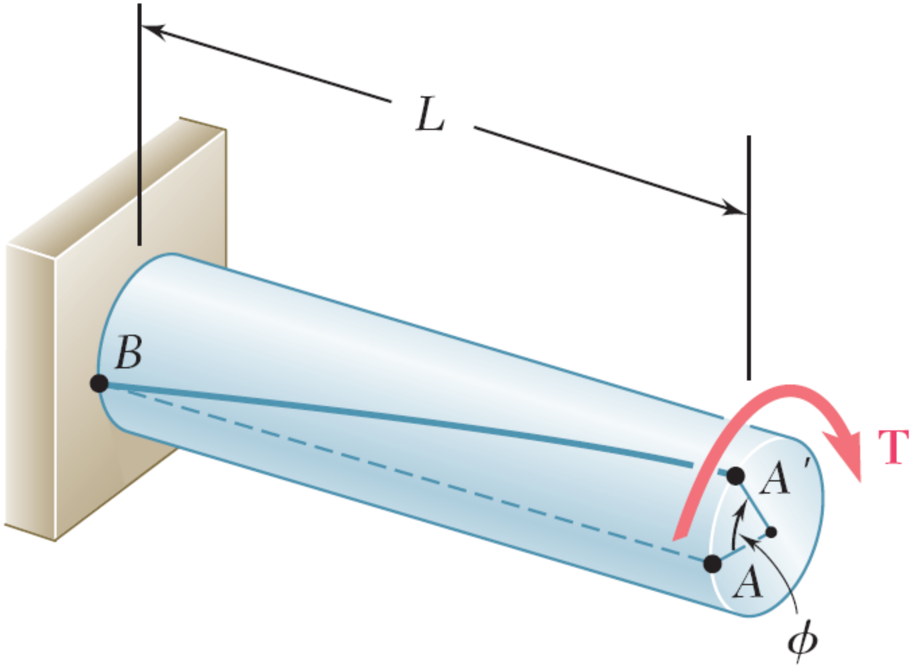
Every cross-section remains plane and undistorted

i.e. **each cross-section rotates as a rigid disk**.

This is only applicable for circular cross-sections.







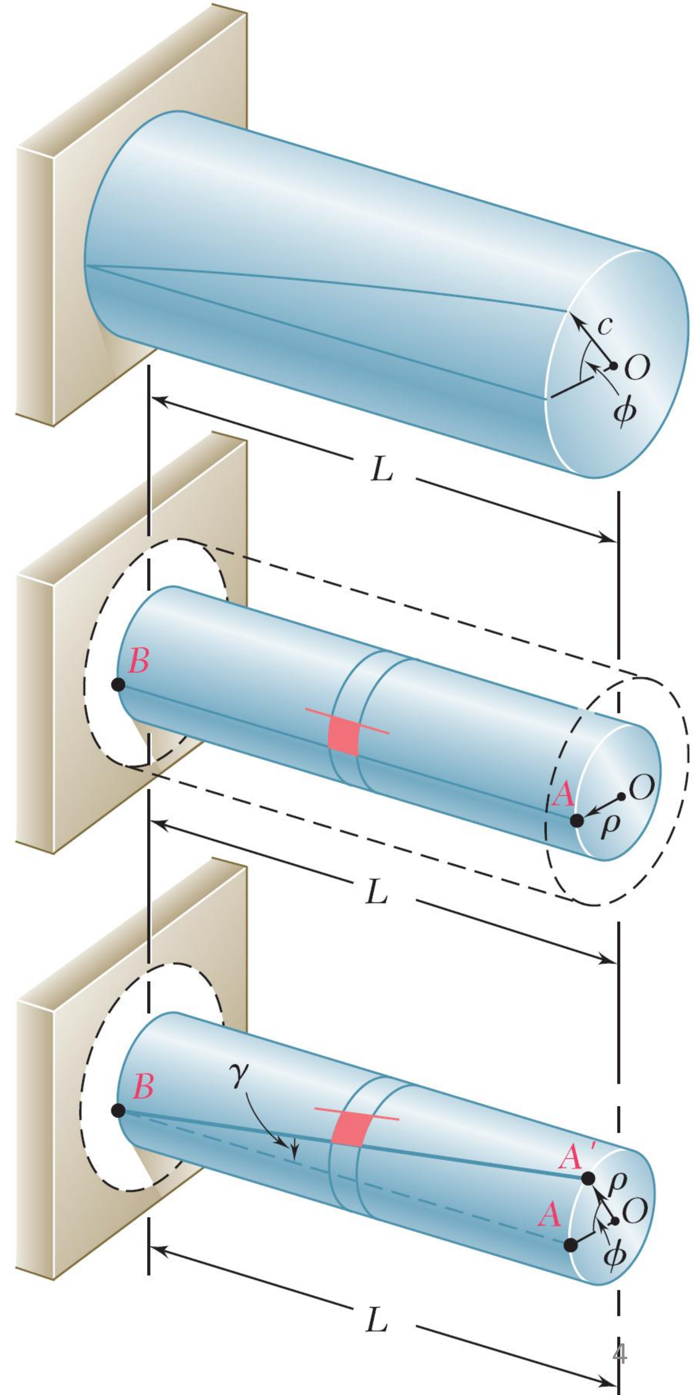
ϕ : the angle of twist

it is proportional to **T** and L

For small shear strain γ

$$\gamma L = \rho\phi$$

$$\gamma = \frac{\rho\phi}{L}$$

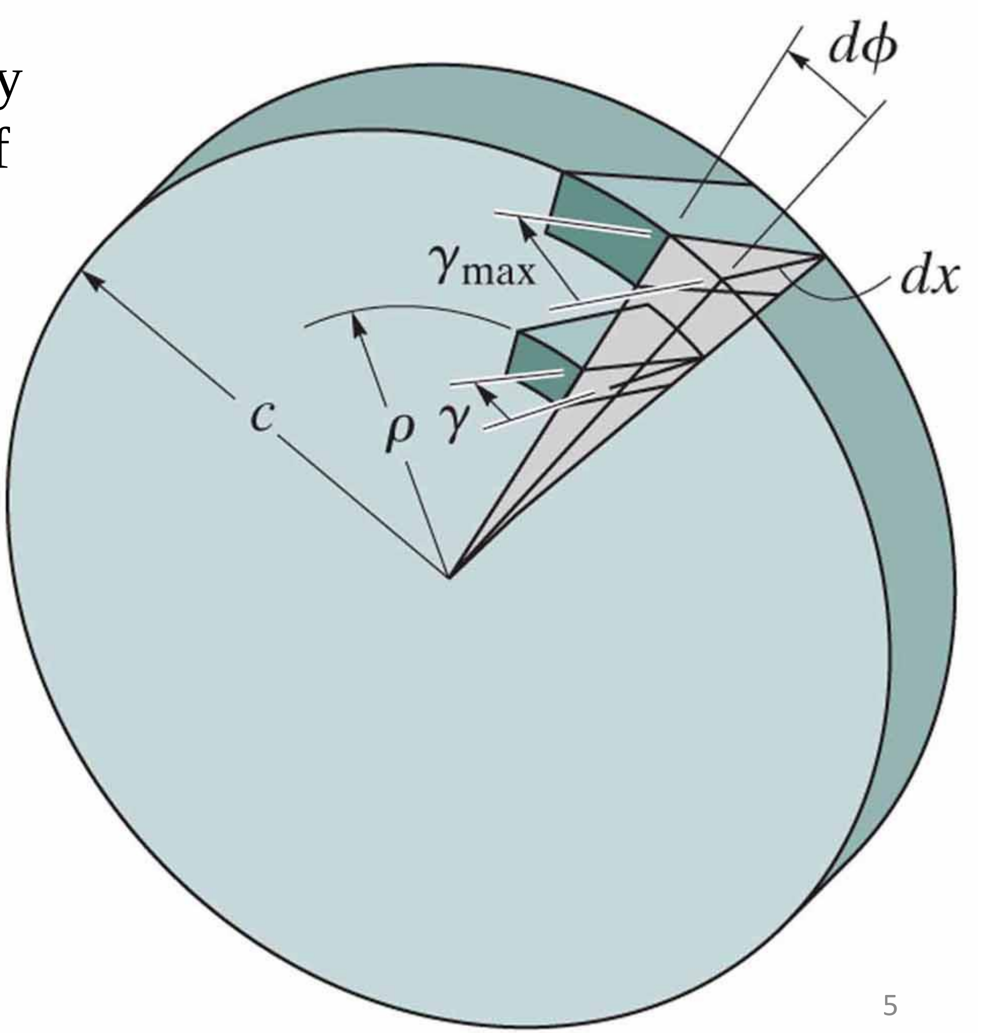


SHEARING STRAIN ALONG THE RADIAL DIRECTION

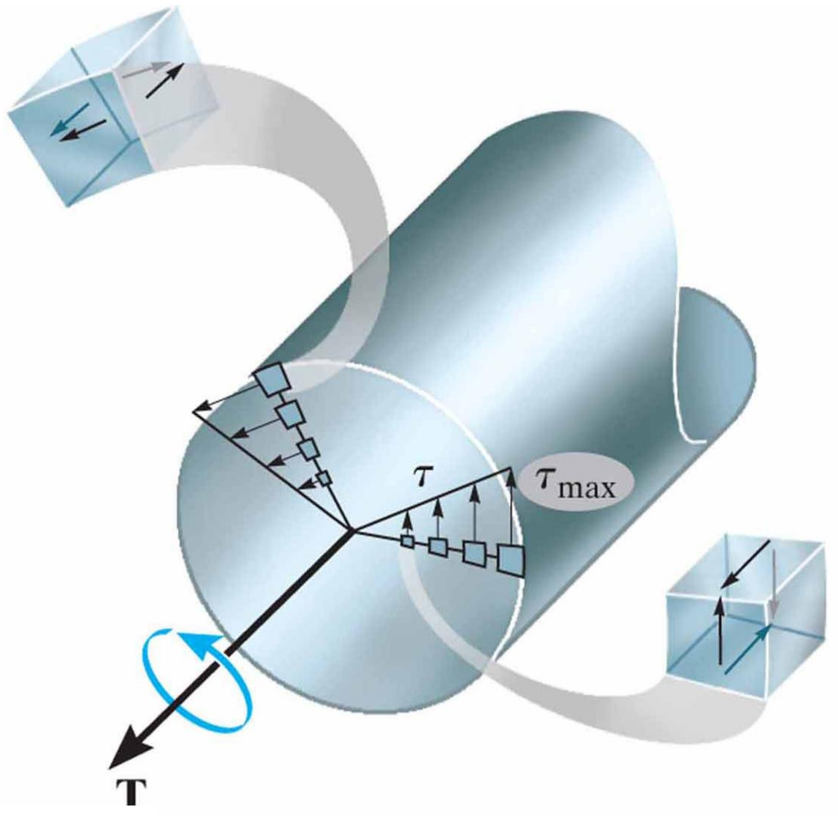
The shearing strain varies linearly with the distance from the axis of the shaft

$$\gamma = \frac{\rho}{c}\gamma_{\max}$$

Shaft radius



3.4 STRESSES IN THE ELASTIC RANGE



Hook's law is applied

Thus, linear variation in shearing strain leads to linear variation in shearing stress.

$$\tau = G\gamma$$

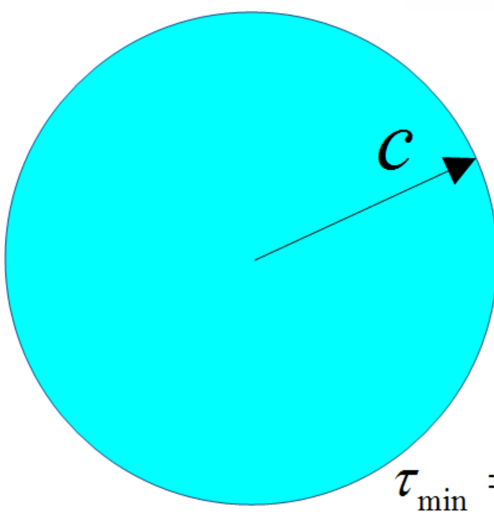
$$\tau_{\max} = G\gamma_{\max}$$

but

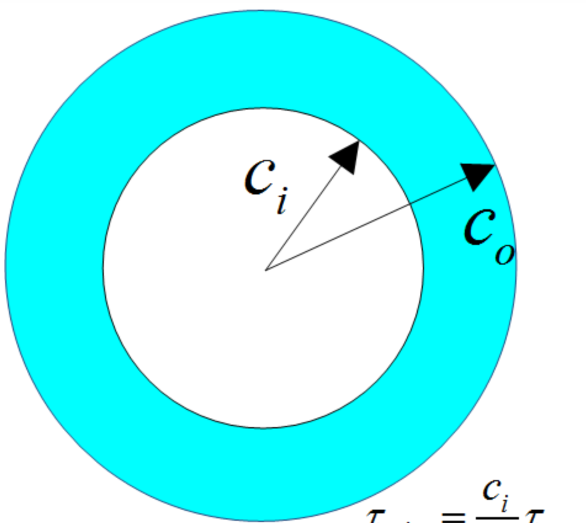
$$\gamma = \frac{\rho}{c}\gamma_{\max}$$

thus

$$\tau = \frac{\rho}{c}\tau_{\max}$$



$$\tau_{\min} = 0$$



$$\tau_{\min} = \frac{c_i}{c_0}\tau_{\max}$$

THE TORSION FORMULA

$$\mathbf{T} = \int_A \rho(\tau dA) = \int_A \rho\left(\frac{\rho}{c}\right)\tau_{\max} dA$$

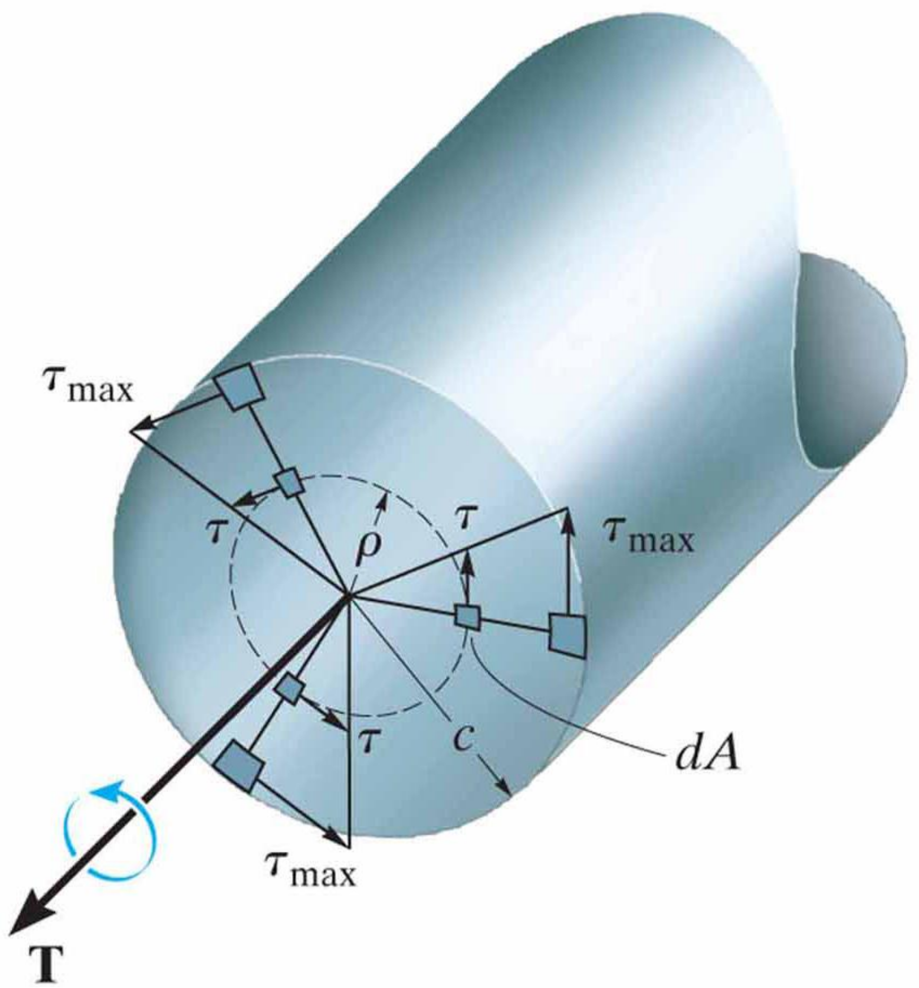
$$\mathbf{T} = \frac{\tau_{\max}}{c}\int_A \rho^2 dA$$

but

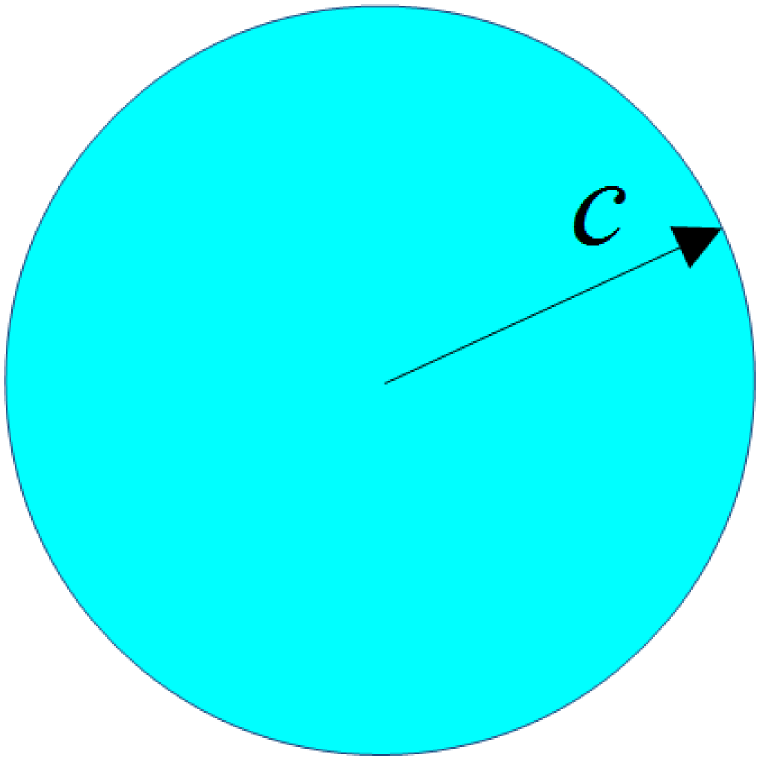
$$J = \int_A \rho^2 dA \quad \text{(polar moment of inertia)}$$

$$\tau_{\max} = \frac{\mathbf{T}\cdot c}{J} \qquad \text{(Torsion formula)}$$

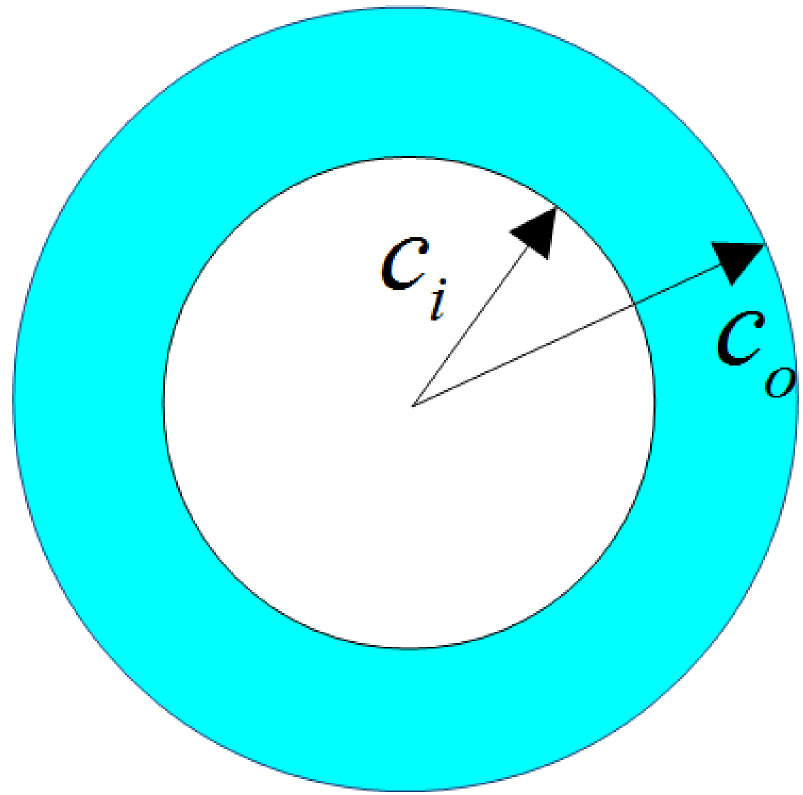
$$\tau = \frac{\mathbf{T}\cdot \rho}{J}$$



POLAR MOMENT OF INERTIA



$$J = \frac{\pi}{2} c^4$$

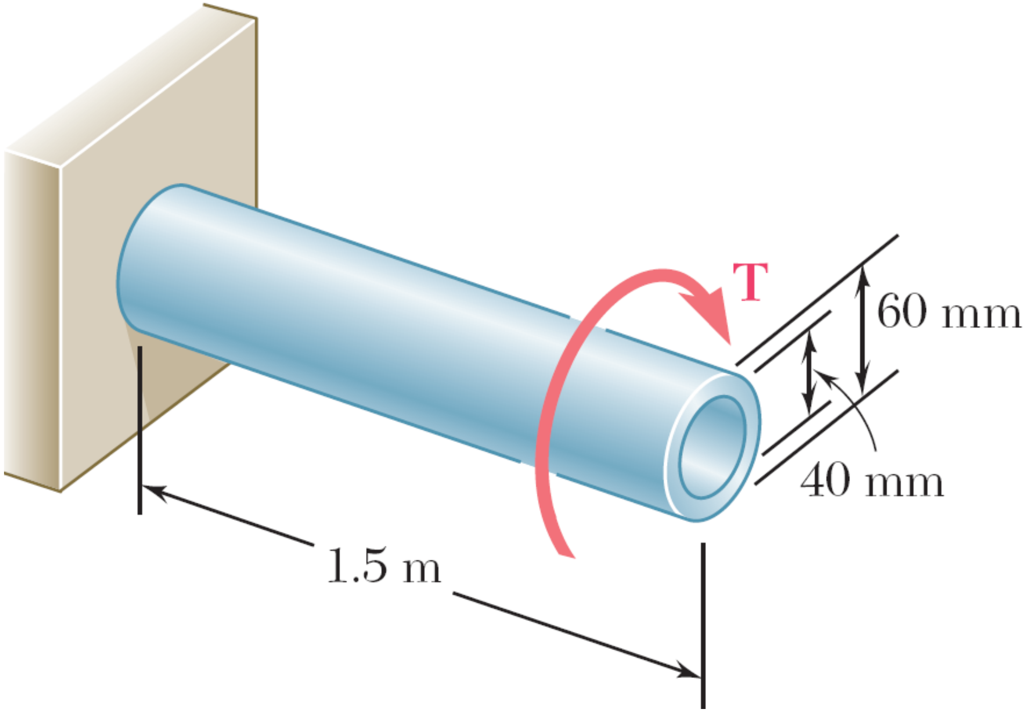


$$J = \frac{\pi}{2} \left(c_o{}^4 - c_i{}^4 \right)$$

Example :

$\tau_{\max} = 120 \text{ MPa}$

Find T



Solution :

$$\mathbf{T} = \frac{\tau_{\max} \cdot J}{c_o} = \frac{120 \times 10^6 \times \frac{\pi}{2}\left((0.03)^4 - (0.02)^4\right)}{0.03} = 4.08 \text{ kN.m}$$

Example:

$(d_{BC})_{inner} = 90 \text{ mm} \rightarrow C_1 = 0.045 \text{ m}$

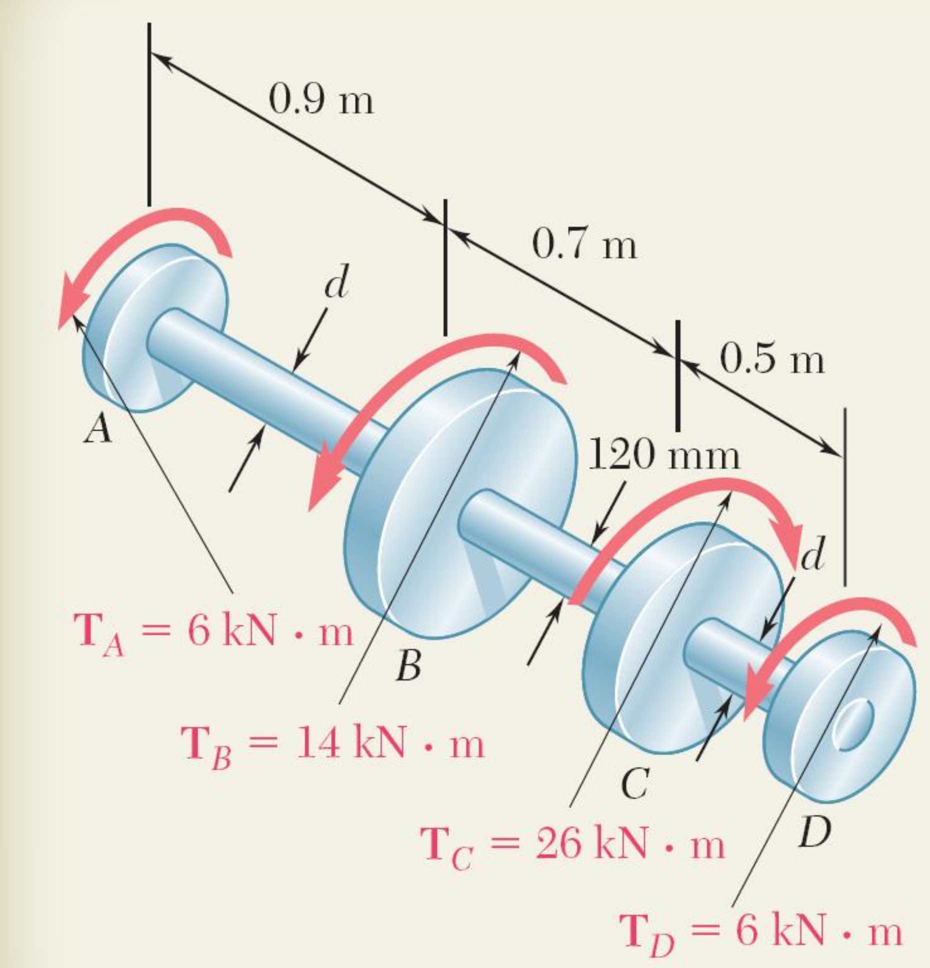
$(d_{BC})_{outer} = 120 \text{ mm} \rightarrow C_2 = 0.06 \text{ m}$

shafts AB and CD are solids of diameter d

Find:

1- $\tau_{\max}$ and $\tau_{\min}$ in shaft BC.

2- diameter d for $\tau_{all} = 65$ MPa.



Solution:

$\sum \mathbf{M} = 0$

$T_{AB} = 6$ kN

$T_{BC} = 20$ kN

$T_{CD} = 6$ kN

$$1 - \tau_{\max} = \frac{\mathbf{T_{BC}} \cdot C_2}{J_{BC}} = \frac{20 \times 10^3 \times 0.06}{\frac{\pi}{2}\left((0.06)^4 - (0.045)^4\right)} = 86.2 \text{ MPa}$$

$$\tau_{\min} = \frac{C_1}{C_2}\tau_{\max} = 64.7 \text{ MPa}$$

$$2 - \tau_{all} = 65 \text{ MPa} = \frac{\mathbf{T_{AB}} \times (d/2)}{J_{AB}}$$

$$d = 77.8 \text{ mm}$$

Example :

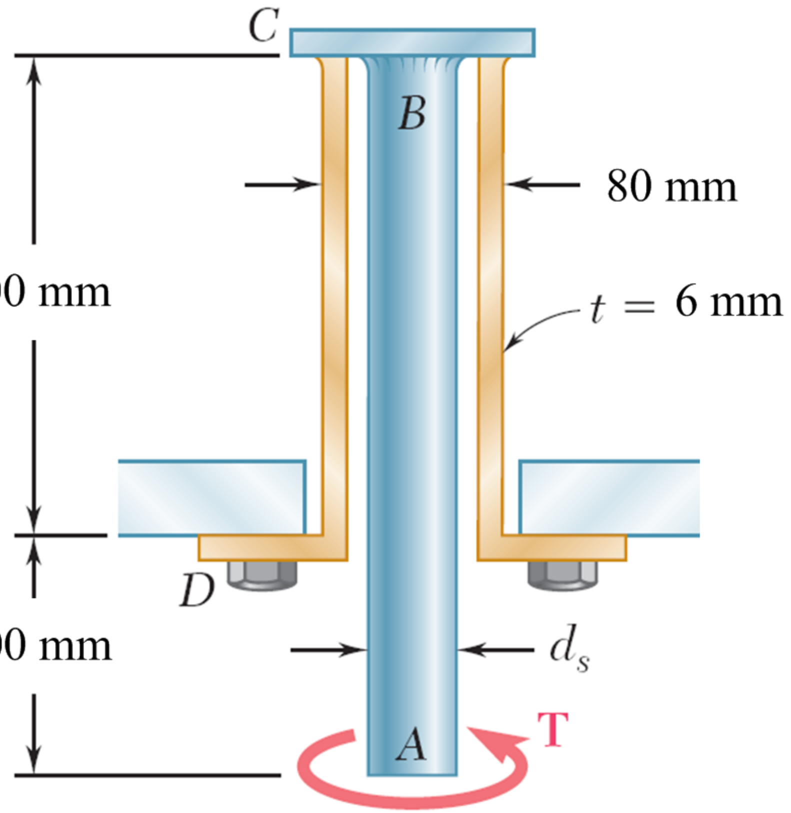
$$(\tau_{all})_{AB} = 80 \text{ MPa}$$

$$(\tau_{all})_{CD} = 50 \text{ MPa}$$

Find

$1-\mathbf{T}_{\mathbf{max}}$ for not exceeding the maximum shear stress in sleeve *CD*

$2-d_s$



Solution :

$$1-\tau_{CD} = \frac{\mathbf{T}\cdot C_{CD}}{J_{CD}}$$

$$\mathbf{T} = \frac{50\times10^{6}\times\frac{\pi}{32}\left((0.08)^{4}-(0.068)^{4}\right)}{0.04} = 2.4 \text{ kN.m}$$

$$2-\tau_{AB} = \frac{\mathbf{T}\cdot C_{AB}}{J_{AB}}$$

$$80\times10^{6} = \frac{2.4\times10^{3}\times(d_s/2)}{\frac{\pi}{32}d_s^{4}} \rightarrow d_s = 53.5 \text{ mm}$$

Example :

$\mathbf{T} = 1000$ N.m

$d_{AB} = 56$ mm and $d_{CD} = 42$ mm

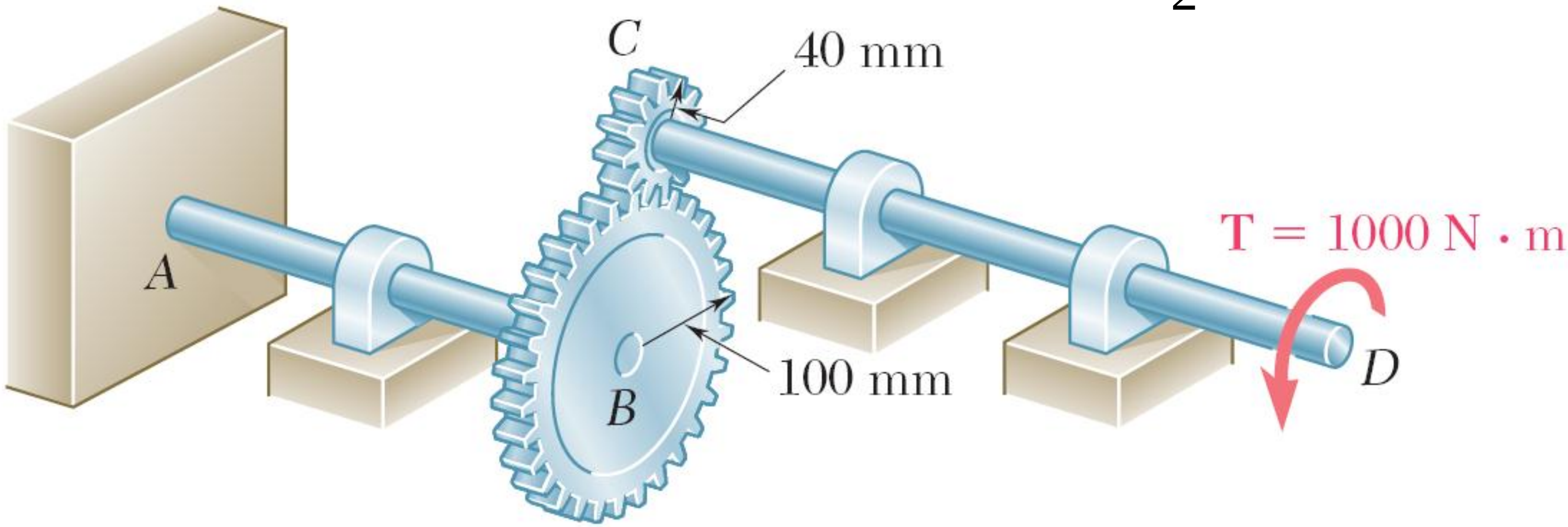
Determine the maximum shearing stress in both shafts.

Solution :

$$\frac{\mathbf{T_{AB}}}{r_{AB}} = \frac{\mathbf{T_{CD}}}{r_{CD}} \rightarrow T_{AB} = 2500 \text{ N.m}$$

$$\tau_{AB} = \frac{\mathbf{T_{AB}} \cdot C_{AB}}{J_{AB}} = \frac{2500 \times 0.028}{\frac{\pi}{2}(0.028)^4} = 72.5 \text{ MPa}$$

$$\tau_{CD} = \frac{\mathbf{T_{CD}} \cdot C_{CD}}{J_{CD}} = \frac{1000 \times 0.021}{\frac{\pi}{2}(0.021)^4} = 68.7 \text{ MPa}$$



Example :

$\mathbf{T} = 100$ N.m

$d_{AB} = 21$ mm, $d_{CD} = 30$ mm and $d_{EF} = 40$ mm

Determine the maximum shearing stress in the three shafts.

Solution

$$\frac{\mathbf{T_{AB}}}{r_{AB}} = \frac{\mathbf{T_{CD}}}{r_{CD1}} \quad \rightarrow \quad \mathbf{T_{CD}} = 240 \text{ N.m}$$

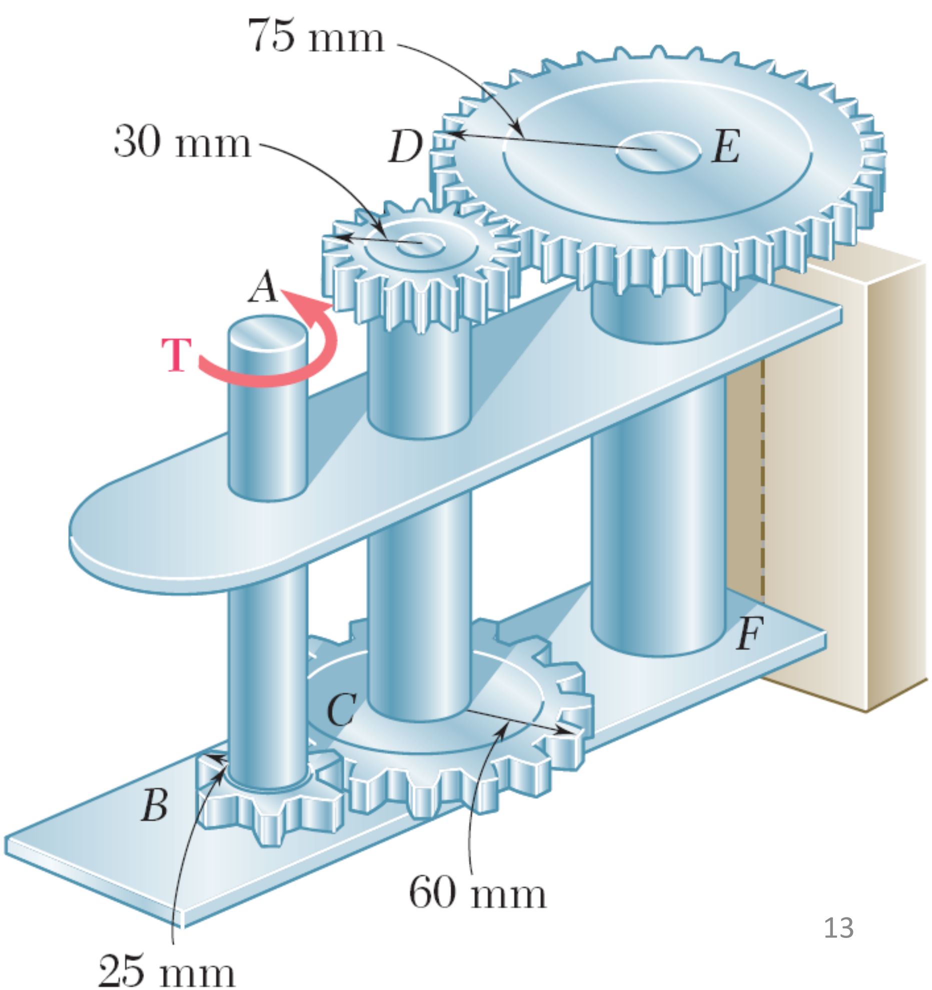
$$\frac{\mathbf{T_{CD}}}{r_{CD2}} = \frac{\mathbf{T_{EF}}}{r_{EF}} = \quad \rightarrow \quad \mathbf{T_{EF}} = 600 \text{ N.m}$$

note: $r_{CD1} = 60$ mm and $r_{CD2} = 30$ mm

$$\tau_{AB} = \frac{\mathbf{T_{AB}} \cdot C_{AB}}{J_{AB}} = \frac{100 \times 0.0105}{\frac{\pi}{2}(0.0105)^4} = 55 \text{ MPa}$$

$$\tau_{CD} = \frac{\mathbf{T_{CD}} \cdot C_{CD}}{J_{CD}} = \frac{240 \times 0.015}{\frac{\pi}{2}(0.015)^4} = 45.3 \text{ MPa}$$

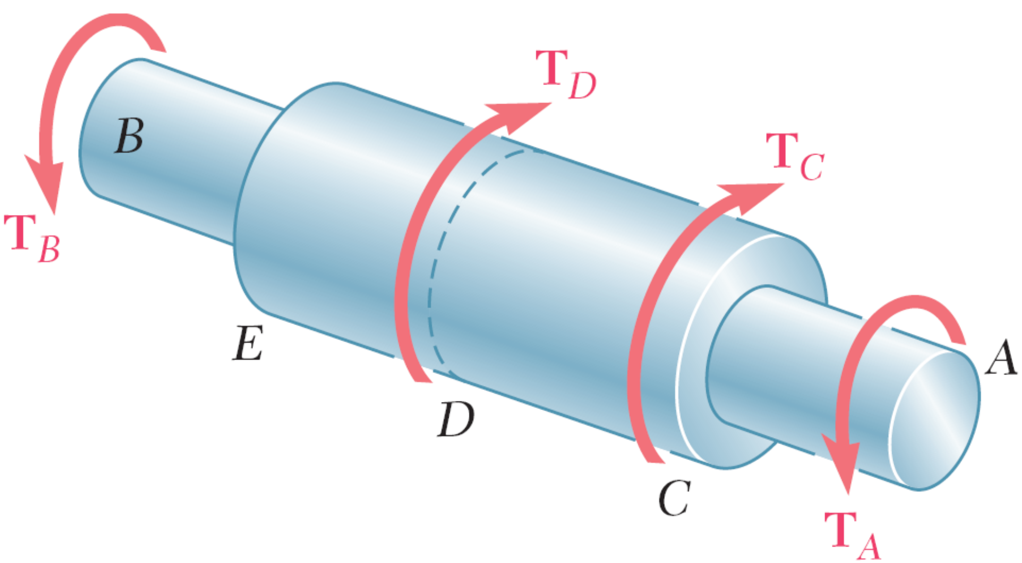
$$\tau_{EF} = \frac{\mathbf{T_{EF}} \cdot C_{EF}}{J_{EF}} = \frac{600 \times 0.02}{\frac{\pi}{2}(0.02)^4} = 47.7 \text{ MPa}$$



3.5 ANGLE OF TWIST IN THE ELASTIC RANGE

$$\gamma_{\max} = \frac{C}{L}\phi = \frac{\tau_{\max}}{G} = \frac{\mathbf{T}\cdot C}{J}\left(\frac{1}{G}\right)$$

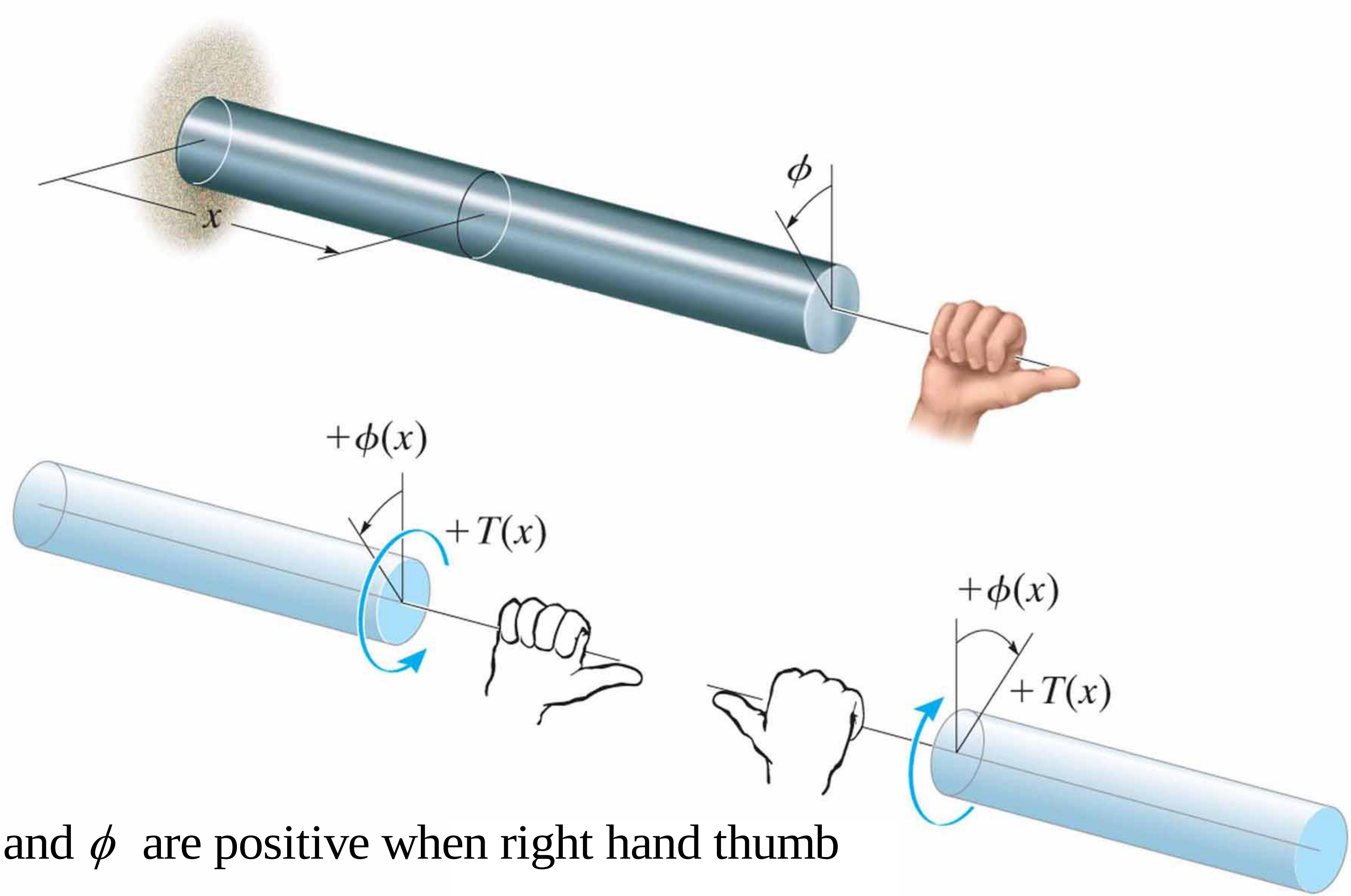
$$\phi = \frac{\mathbf{T}L}{JG} \quad \text{(analogus to } \delta = \frac{\mathbf{P}L}{AE}\text{)}$$



For multi-sections

$$\phi = \sum_i \frac{\mathbf{T}_i L_i}{J_i G_i}$$

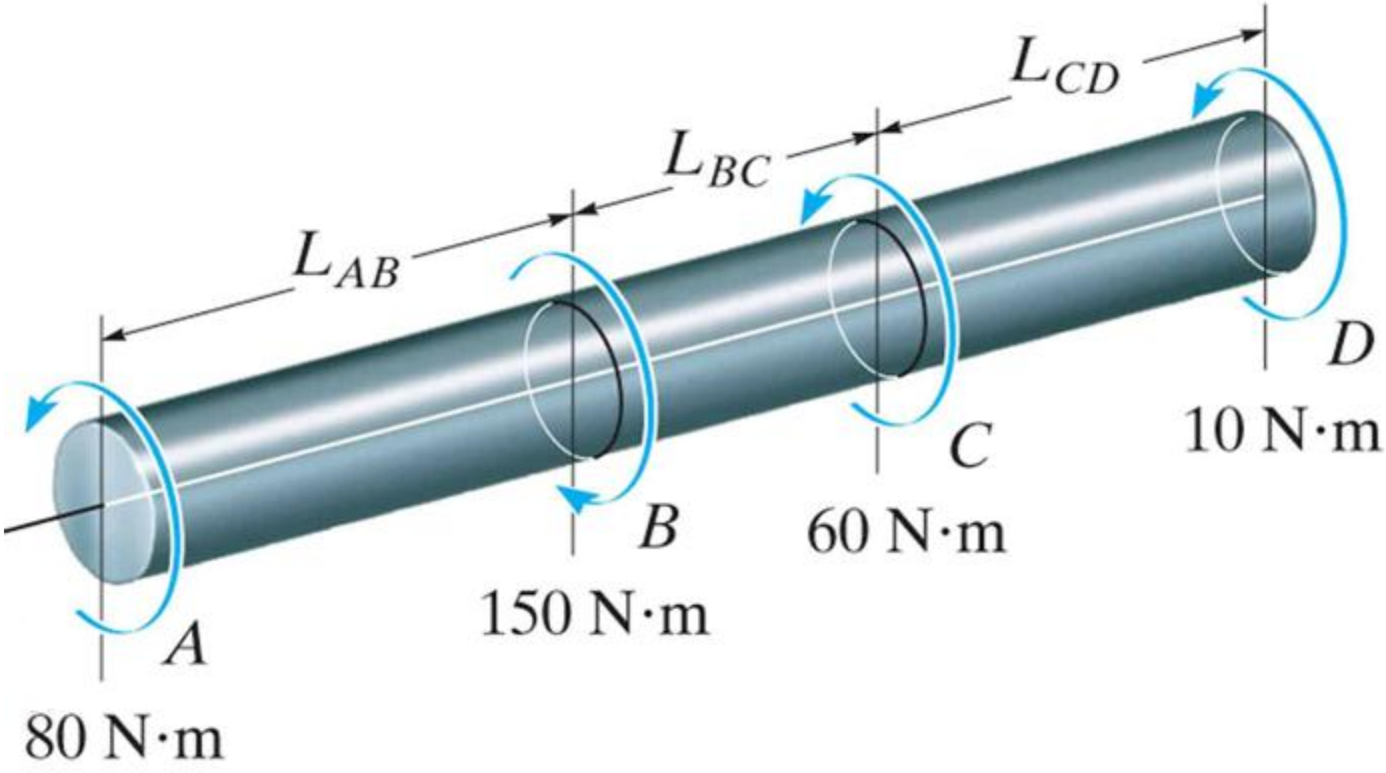
SIGN CONVENTION



T and ϕ are positive when right hand thumb is outward of the surface

Example:

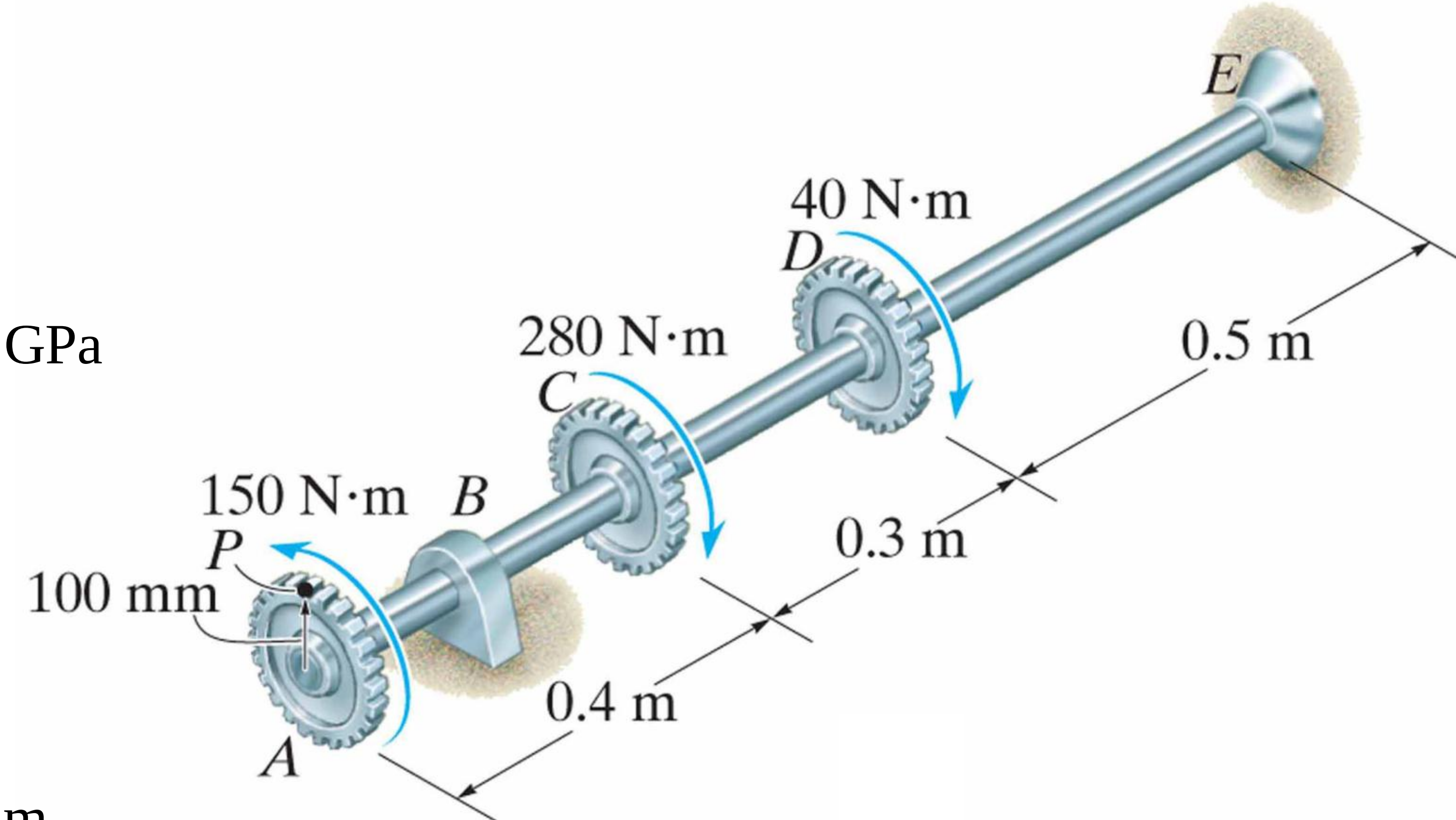
$$\phi_{A/D} = \frac{80 \times L_{AB}}{JG} + \frac{-70 \times L_{BC}}{JG} + \frac{-10 \times L_{CD}}{JG}$$



Example :

$d = 14$ mm

$G = 80 \times 10^9$ GPa



Solution :

$\mathbf{T_{AC}} = 150$ N.m

$\mathbf{T_{CD}} = -130$ N.m

$\mathbf{T_{DE}} = -170$ N.m

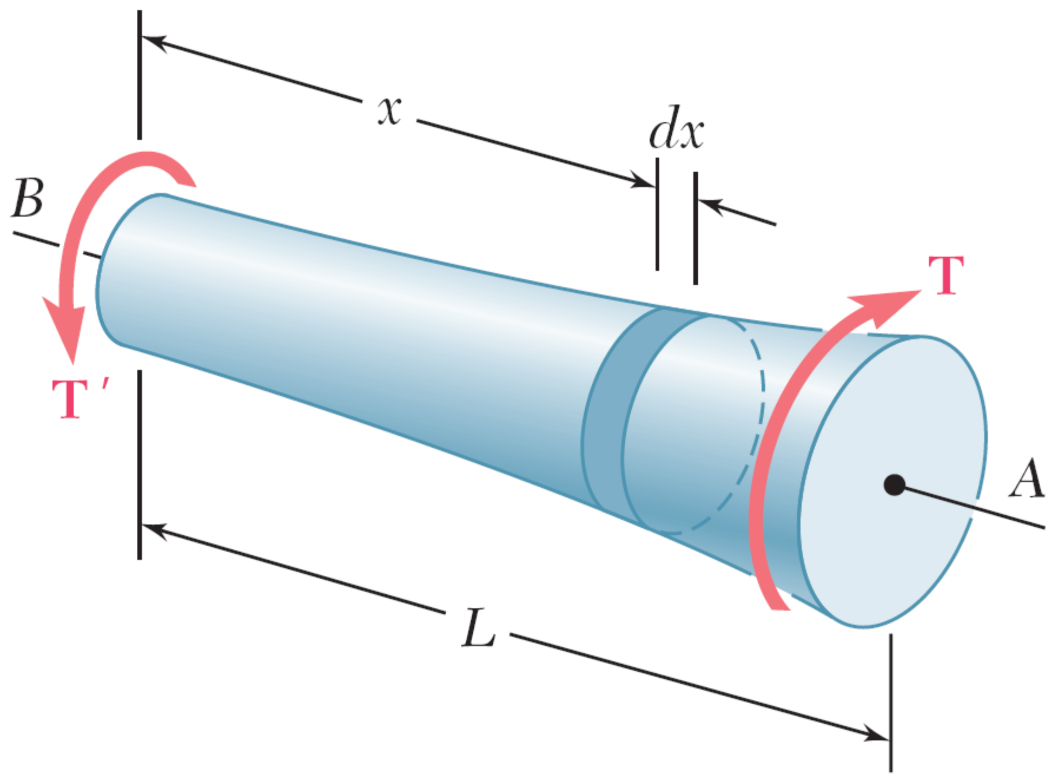
$$J = \frac{\pi}{2}(0.007)^4$$

$$\phi_A = \sum \frac{\mathbf{T}_i L_i}{JG} = \frac{150 \times 0.4}{JG} + \frac{-130 \times 0.3}{JG} + \frac{-170 \times 0.5}{JG} = -0.2121 \text{ rad}$$

ANGLE OF TWIST FOR VARIABLE CROSS-SECTION

$$d\phi = \frac{\mathbf{T}}{JG}dx$$

$$\phi = \int_0^L \frac{\mathbf{T}(x)}{J(x)G}dx$$



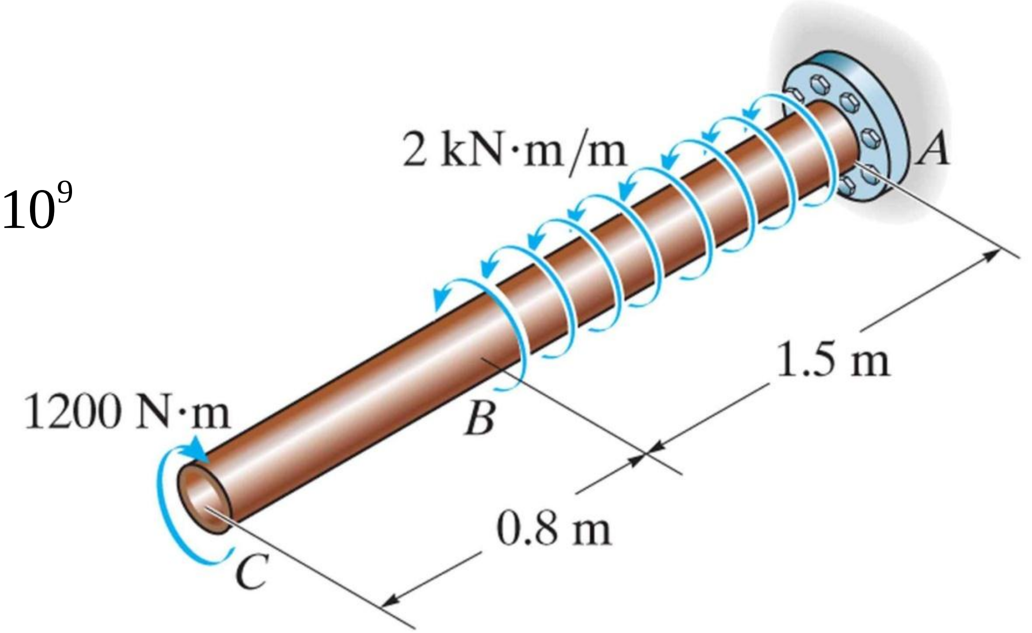
Example :

$d = 60$ mm (solid shaft), $G = 75\times10^9$

Find

$1-\tau_{max}$

$2-\phi_C$



Solution :

$$1-\tau_{max} = \frac{\mathbf{T_{max}}\cdot C}{J} = \frac{1800\times0.03}{\frac{\pi}{2}(0.03)^4} = 42.4 \text{ MPa}$$

$$\phi_C = \phi_{C/B} + \phi_{B/A}$$

$$\mathbf{T_{AB}} = -1200 + 2000x \text{ N.m}$$

$$\mathbf{T_{BC}} = -1200 \text{ N.m}$$

$$\phi_C = -\frac{1200\times0.8}{\frac{\pi}{2}(0.03)^4\times75\times10^9} + \int_0^{1.5}\frac{-1200+2000x}{\frac{\pi}{2}(0.03)^4\times75\times10^9}dx$$

$$\phi_C = -5.344\times10^{-3} \text{ rad}$$

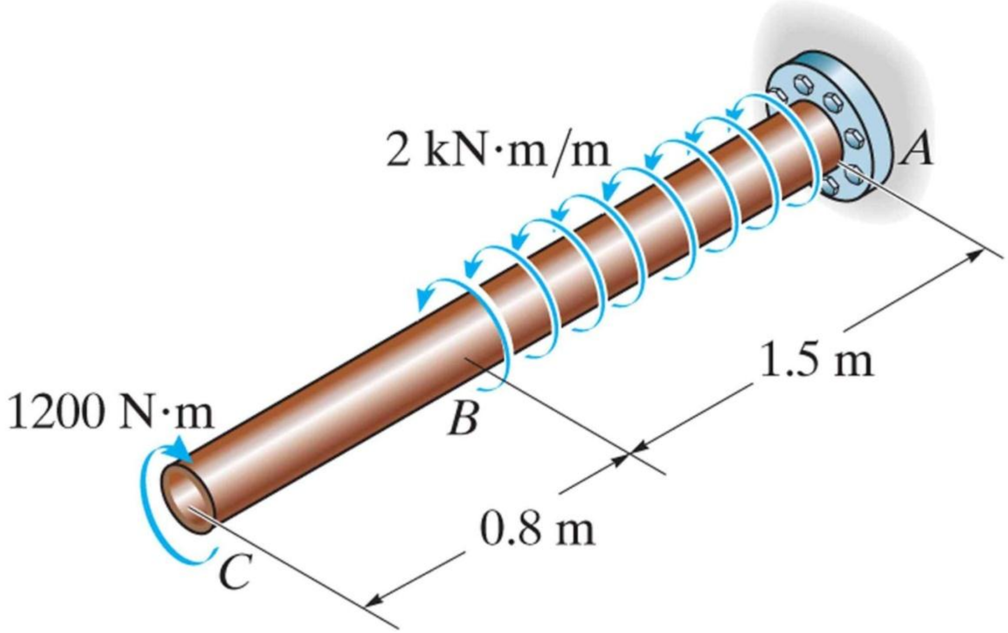
or apply superposition

$$\phi_C = \phi_{C1} + \phi_{C2}$$

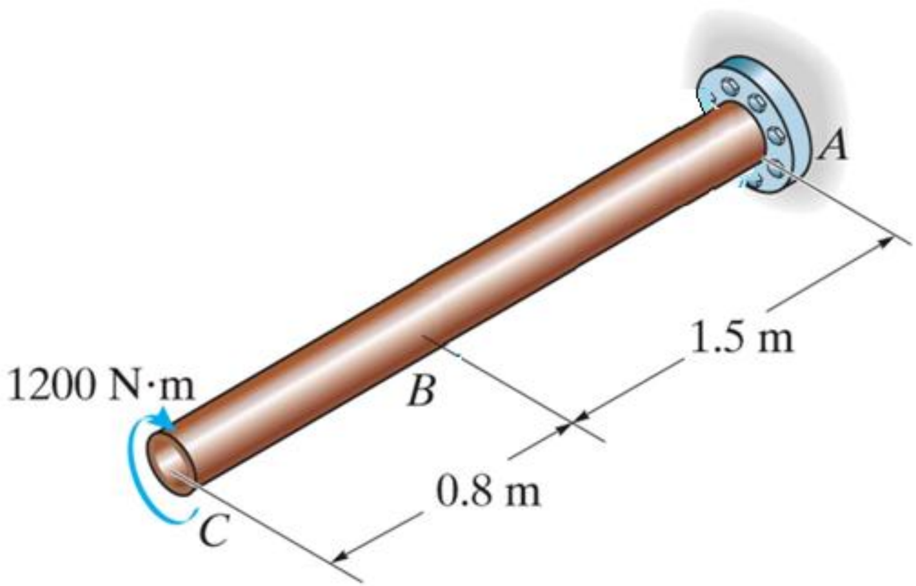
$$\phi_{C1} = -\frac{1200 \times 2.3}{\frac{\pi}{2}(0.03)^4 \times 75 \times 10^9}$$

$$\phi_{C2} = \phi_{B2} = \int_0^{1.5} \frac{2000x}{\frac{\pi}{2}(0.03)^4 \times 75 \times 10^9} dx$$

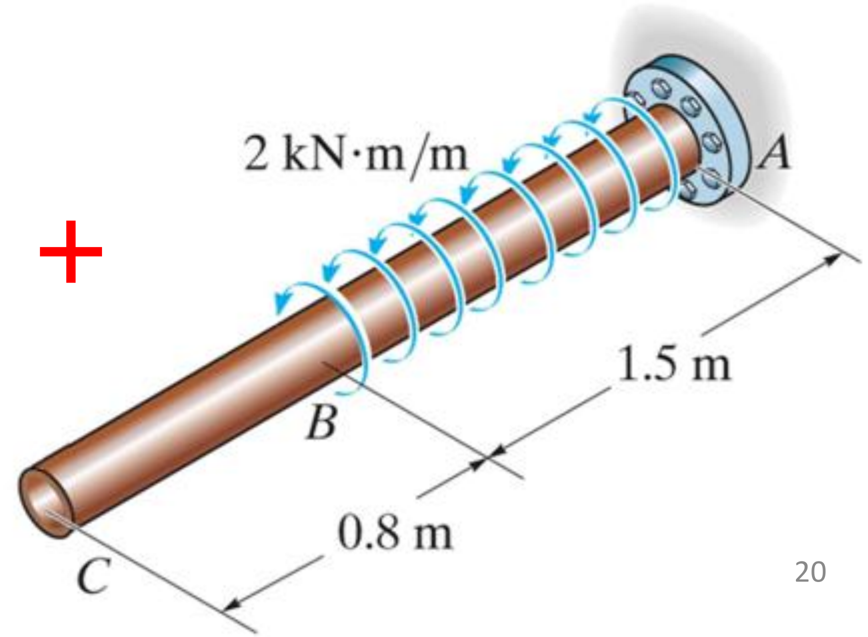
$$\phi_C = -5.344 \times 10^{-3} \text{ rad}$$



=



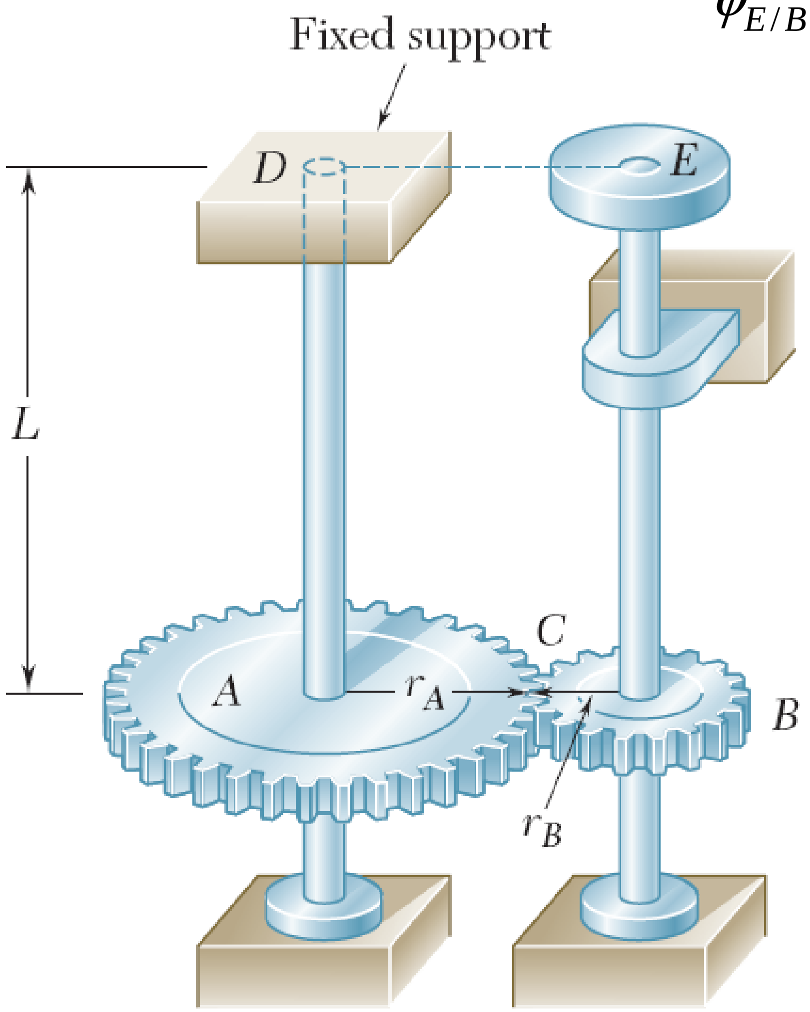
+

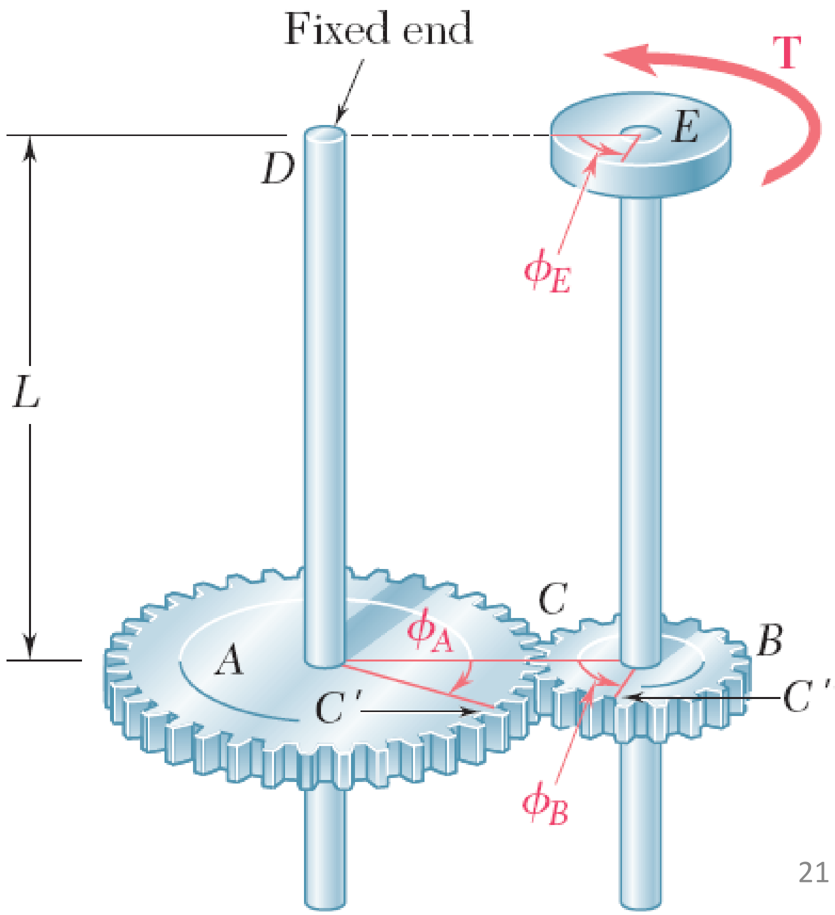


RELATIVE TWIST ANGLES

$$\phi_E = \phi_B + \phi_{E/B}$$

$$\phi_{E/B} = \frac{TL}{JG}$$



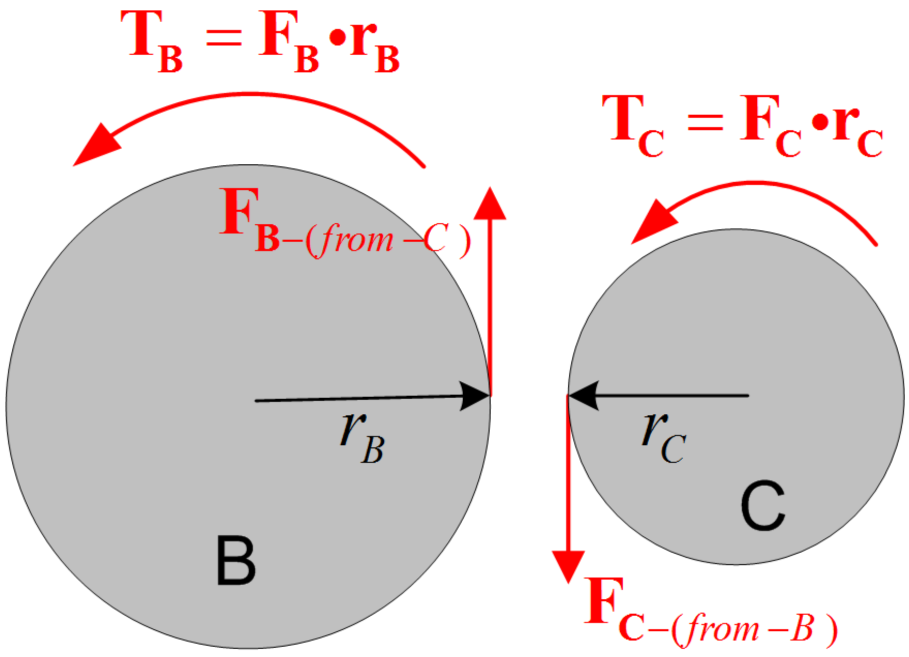


Example :

$d = 20 \text{ mm}$

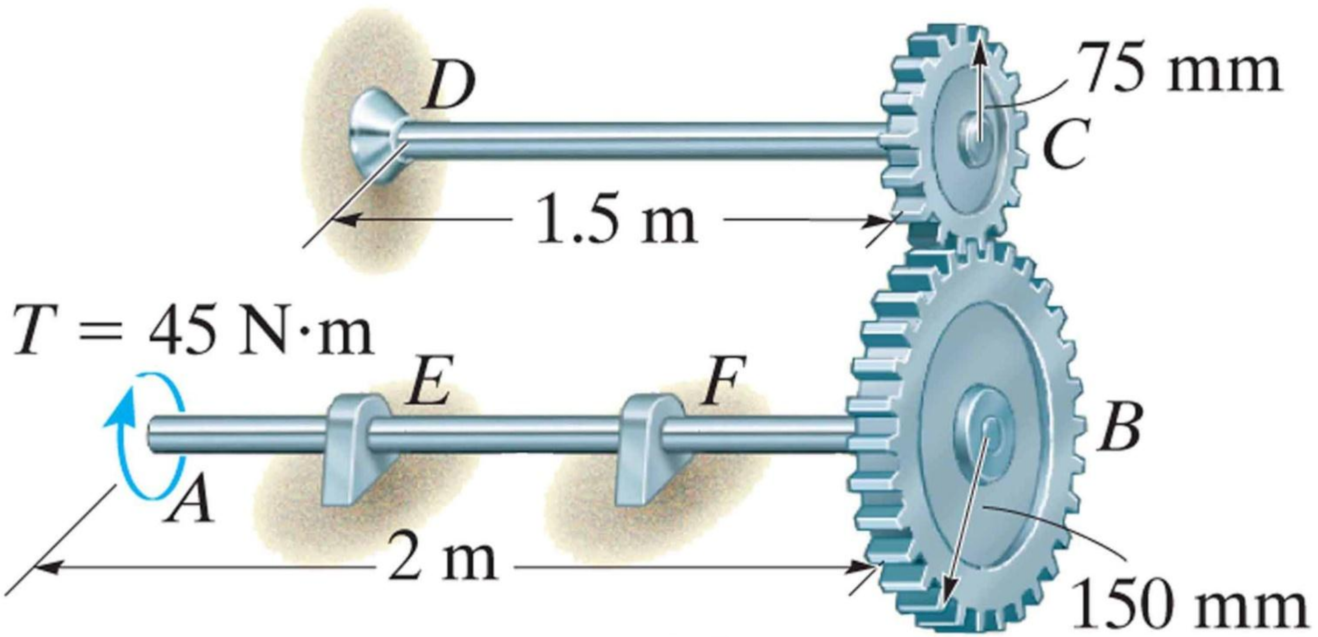
$G = 80 \text{ GPa}$

$\mathbf{T_{AB}} = +45 \text{ N.m}$



Solution :

$$\frac{\mathbf{T_{AB}}}{r_{AB}} = \frac{\mathbf{T_{CD}}}{r_{CD}} \rightarrow \mathbf{T_{CD}} = +22.5 \text{ N.m}$$



$$\phi_C = \frac{\mathbf{T_{CD}} L_{CD}}{JG} = \frac{22.5 \times 1.5}{\frac{\pi}{2}(0.01)^4 \times 80 \times 10^9} = +0.0269 \text{ rad}$$

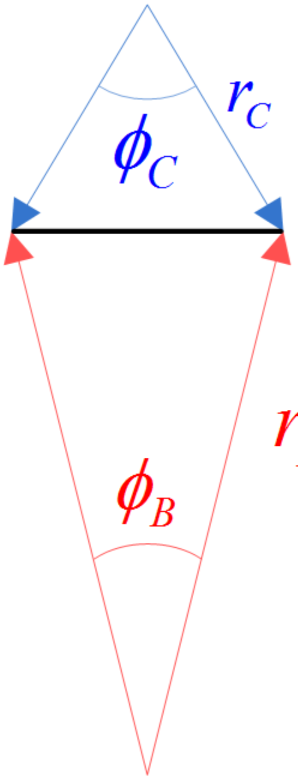
$$r_C \phi_C = r_B \phi_B$$

$$\phi_B = 0.0134 \text{ rad}$$

$$\phi_{A/B} = \frac{\mathbf{T_{AB}} L_{AB}}{JG} = \frac{45 \times 2}{\frac{\pi}{2}(0.01)^4 \times 80 \times 10^9} = +0.0716 \text{ rad}$$

$$\phi_A = \phi_{A/B} + \phi_B = 0.085 \text{ rad}$$

ENGAGED TWIST ANGLES RELATIONS



Example:

AB is steel

$G_{steel} = 77 \text{ GPa}$

$(\tau_{all})_{steel} = 80 \text{ MPa}$

CD is Brass

$G_{brass} = 38 \text{ GPa}$

$(\tau_{all})_{Brass} = 50 \text{ MPa}$

Find $\mathbf{T}_{\mathbf{max}}$ and ϕ_A

$$\phi_A = \frac{\mathbf{T}_{\mathbf{max}} L_{AB}}{J_{steel} G_{steel}} + \frac{\mathbf{T}_{\mathbf{max}} L_{CD}}{J_{brass} G_{brass}}$$

$$\phi_A = \frac{1005 \times 0.3}{\frac{\pi}{2}(0.02)^4 \times 77 \times 10^9} + \frac{1005 \times 0.2}{\frac{\pi}{2}\left((0.04)^4 - (0.034)^4\right) \times 38 \times 10^9}$$

$$\phi_A = 0.01833 \text{ rad}$$

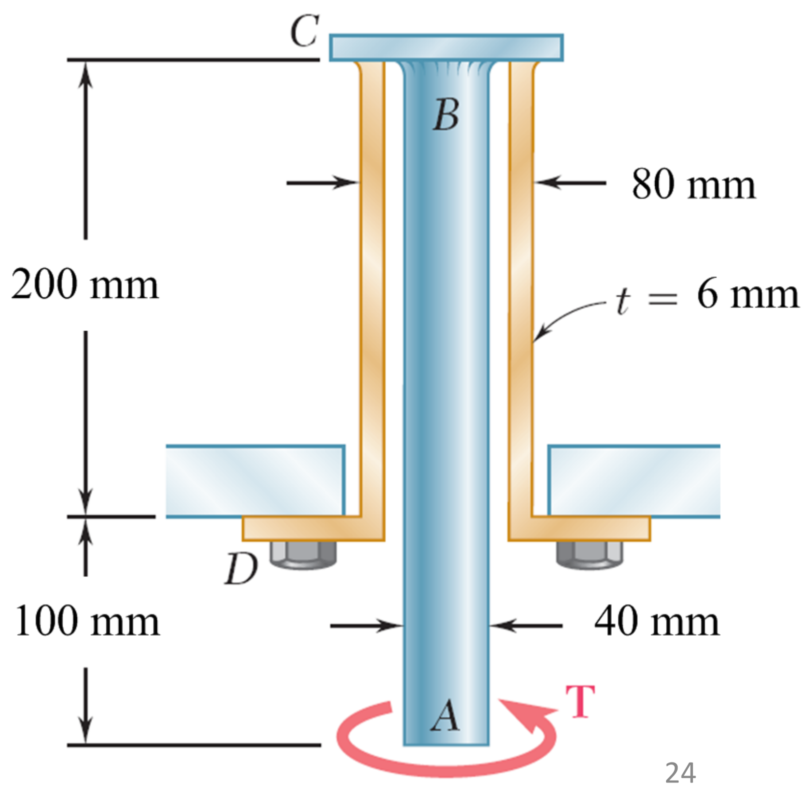
Solution:

$$(\tau_{all})_{steel} = \frac{\mathbf{T} \cdot C}{J}$$

$$80 \times 10^6 = \frac{\mathbf{T}_{\mathbf{max}} \times 0.02}{\frac{\pi}{2}(0.02)^4} \rightarrow T_{max} = 1.005 \text{ kN.m}$$

$$(\tau_{all})_{Brass} = \frac{\mathbf{T} \cdot C}{J}$$

$$50 \times 10^6 = \frac{\mathbf{T}_{\mathbf{max}} \times 0.04}{\frac{\pi}{2}\left((0.04)^4 - (0.034)^4\right)} \rightarrow \mathbf{T}_{\mathbf{max}} = 2.4 \text{ kN.m}$$



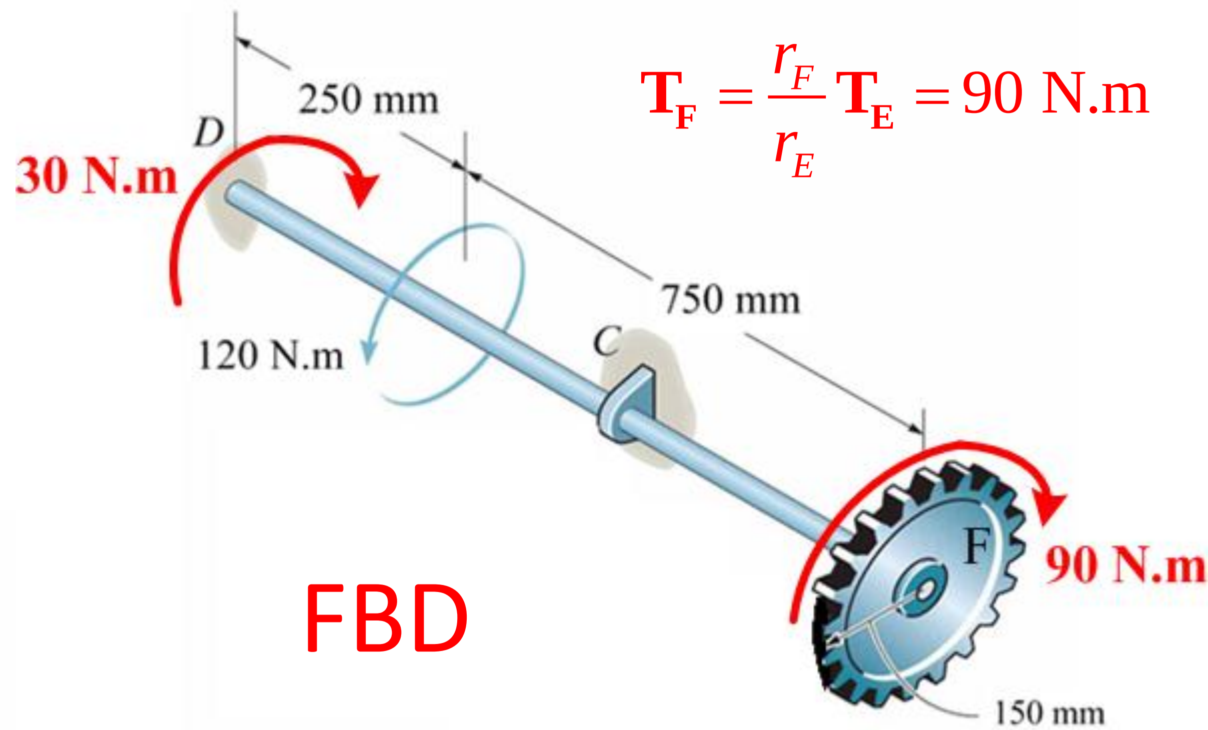
Example :

$d = 25$ mm

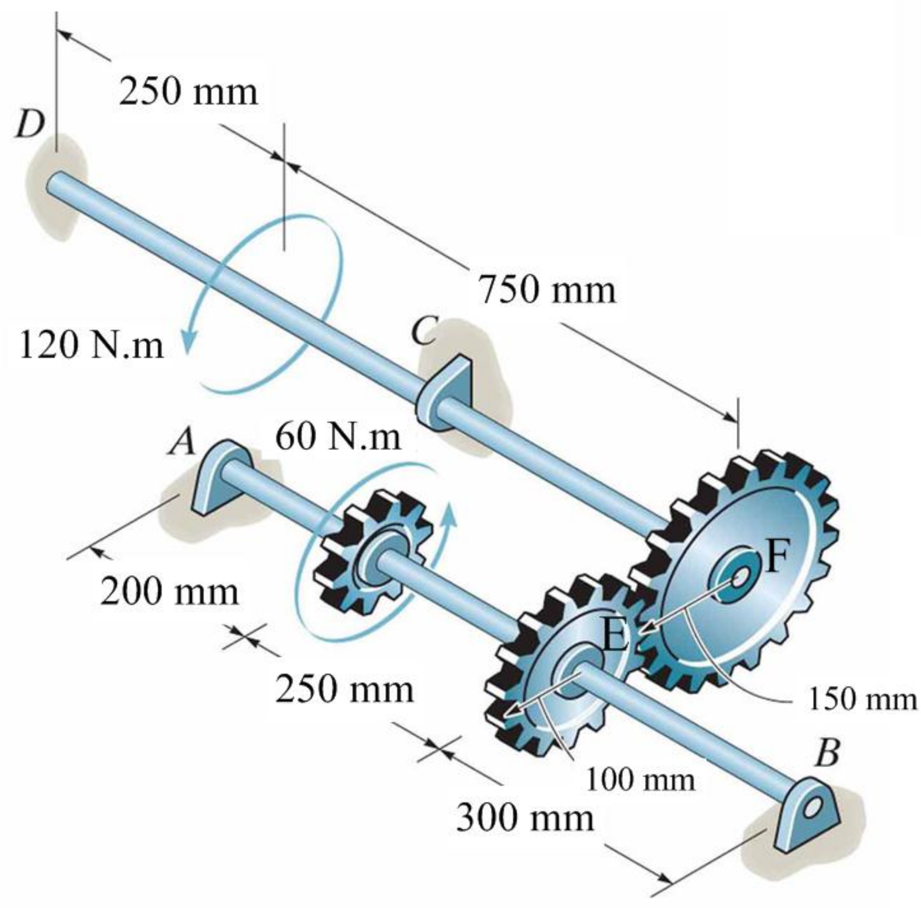
$G = 75$ GPa

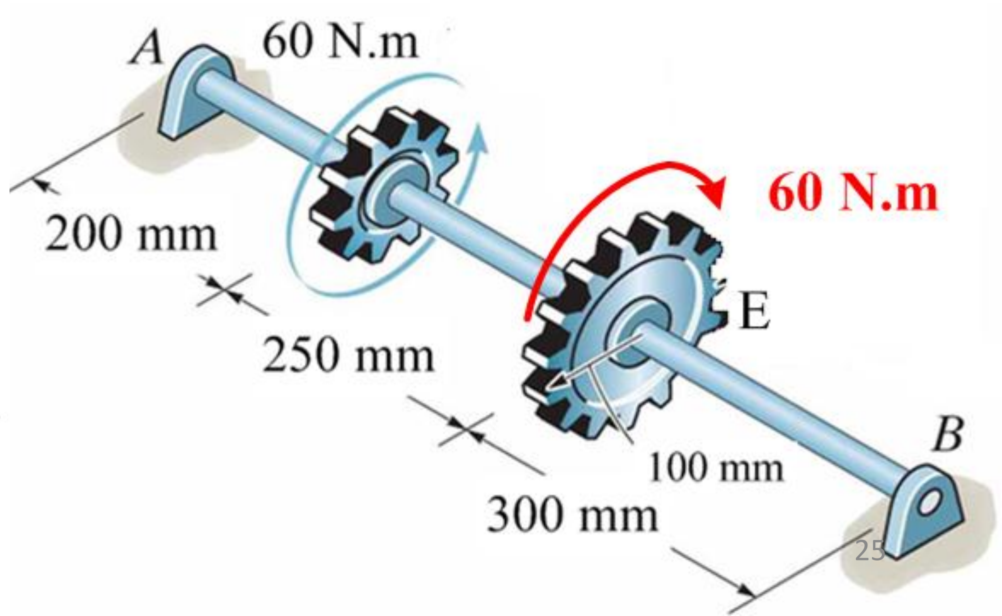
Find ϕ_A

$$\mathbf{T_F} = \frac{r_F}{r_E}\mathbf{T_E} = 90\ \text{N.m}$$



FBD





Solution :

$$\phi_F = \frac{30\times 0.25}{\frac{\pi}{2}(0.0125)^4\times 75\times 10^9} + \frac{-90\times 0.75}{\frac{\pi}{2}(0.0125)^4\times 75\times 10^9} = -20.86\times 10^{-3}\ \text{rad}$$

$$\phi_F r_F = \phi_E r_E$$

$$\phi_E = -0.03129\ \text{rad}$$

$$\phi_{A/E} = \frac{\mathbf{T}L}{JG} = \frac{-60\times 0.25}{\frac{\pi}{2}(0.0125)^4\times 75\times 10^9} = -5.215\times 10^{-3}\ \text{rad}$$

$$\phi_A = \phi_E + \phi_{A/E} = -2.09\ \text{deg.}$$

Example :

$d_{AC} = 60$ mm

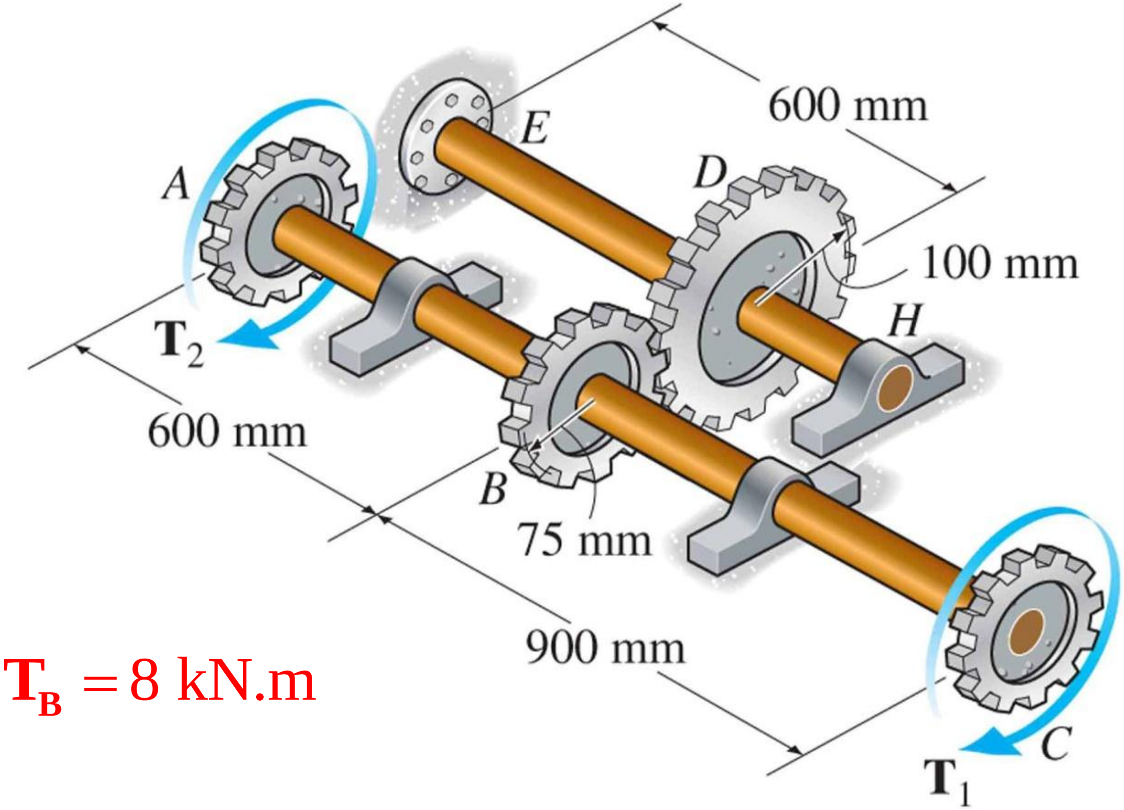
$d_{EH} = 80$ mm

$\mathbf{T_1} = 2$ kN.m, $\mathbf{T_2} = 4$ kN.m

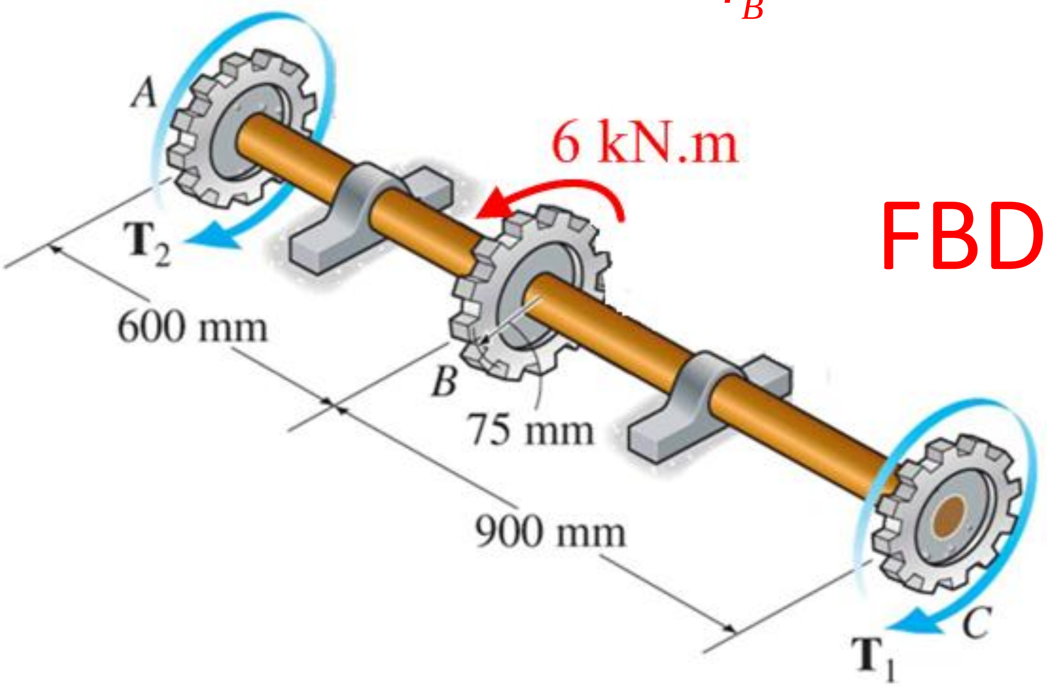
$G = 75$ GPa

Find

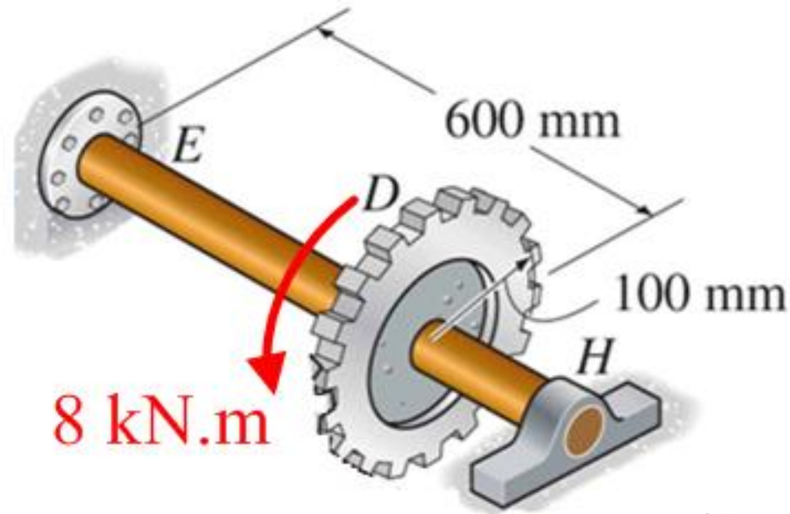
ϕ_A and ϕ_C



$$\mathbf{T_D} = \frac{r_D}{r_B}\mathbf{T_B} = 8 \text{ kN.m}$$



FBD



Solution :

$$\phi_D = \frac{T_D L_{ED}}{J_{ED} G} = \frac{8 \times 10^3 \times 0.6}{\frac{\pi}{32}(0.08)^4 \times 75 \times 10^9} = +0.0159 \text{ rad}$$

$$\phi_B = \frac{r_D}{r_B}\phi_D = \frac{100}{75} \times 0.0159 = +0.0212 \text{ rad}$$

$$\phi_A = \phi_B + \phi_{A/B}$$

$$\phi_{A/B} = \frac{T_2 L_{AB}}{J_{AB} G} = \frac{4 \times 10^3 \times 0.6}{\frac{\pi}{32}(0.06)^4 \times 75 \times 10^9} = +0.025 \text{ rad}$$

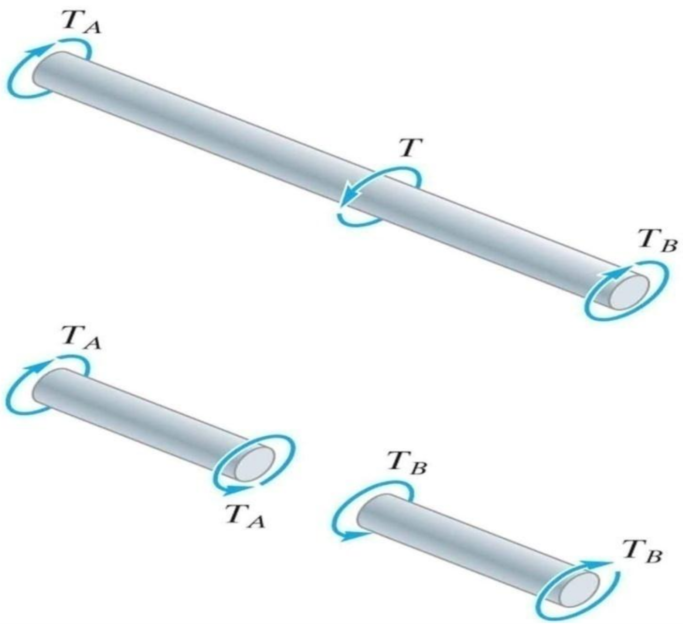
$$\phi_A = 0.025 + 0.0212 = 0.0462 \text{ rad}$$

$$\phi_C = \phi_B + \phi_{C/B}$$

$$\phi_{C/B} = \frac{T_1 L_{BC}}{J_{BC} G} = \frac{-2 \times 10^3 \times 0.9}{\frac{\pi}{32}(0.06)^4 \times 75 \times 10^9} = -0.01886 \text{ rad}$$

$$\phi_C = -0.01886 - 0.0212 = -0.04 \text{ rad}$$

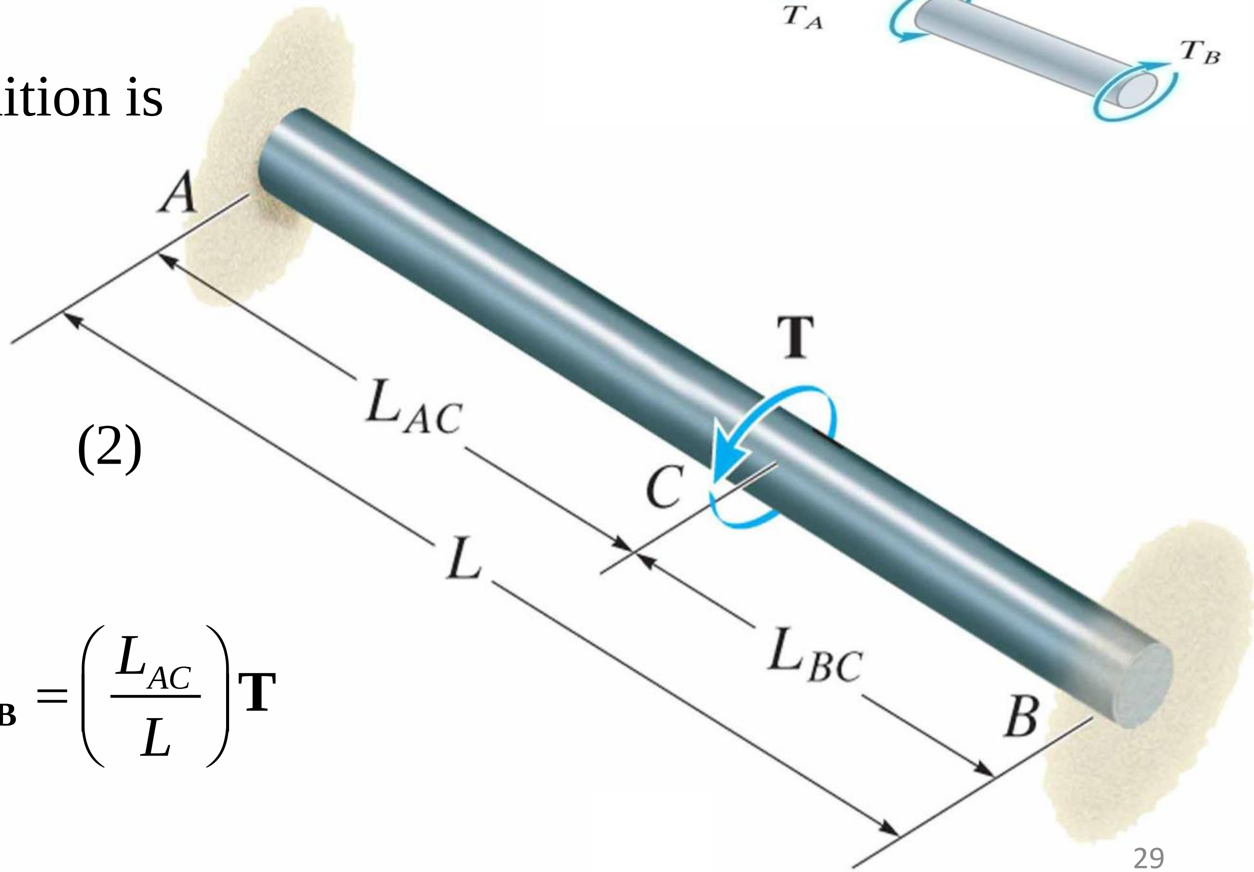
3.6 STATICALLY INDETERMINATE SHAFTS



$$\sum M_x = 0 \;\; \rightarrow \;\; \mathbf{T} - \mathbf{T_A} - \mathbf{T_B} = 0 \qquad (1)$$

one equation and two unknowns

the compatibility condition is



$$\phi_{A/B} = 0$$

$$\frac{\mathbf{T_A} L_{AC}}{JG} - \frac{\mathbf{T_B} L_{BC}}{JG} = 0 \qquad (2)$$

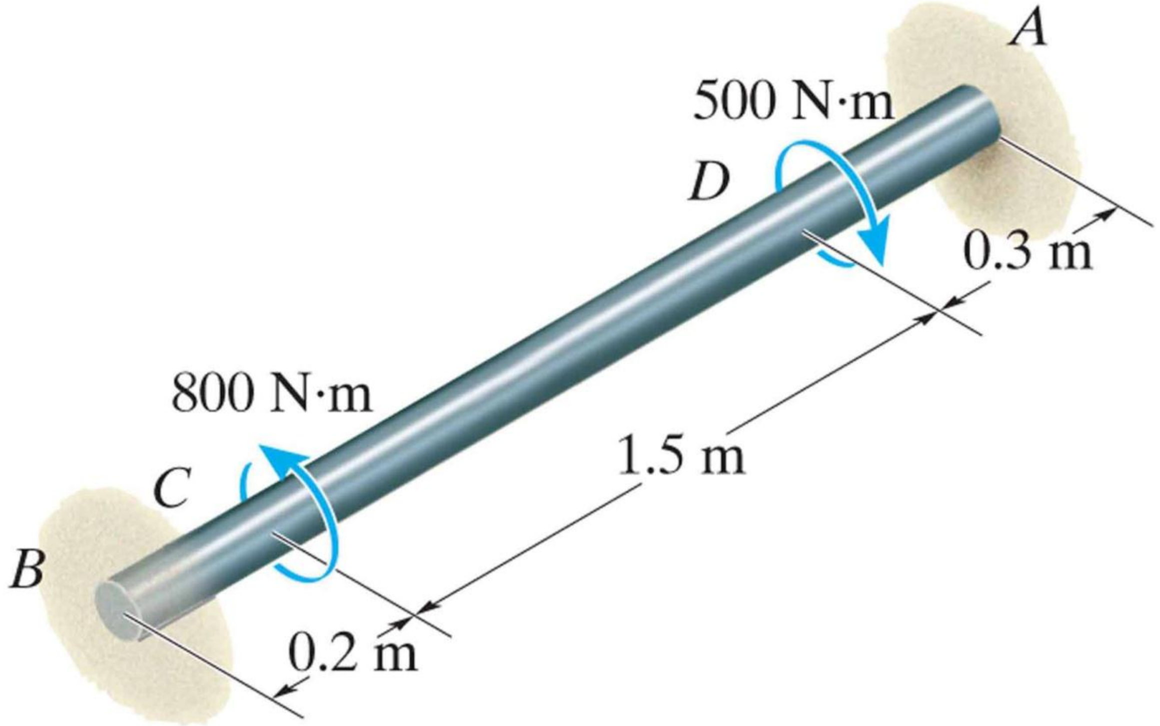
From Eqs. 1 and 2

$$\mathbf{T_A} = \left(\frac{L_{BC}}{L}\right)\mathbf{T} \quad \text{and} \quad \mathbf{T_B} = \left(\frac{L_{AC}}{L}\right)\mathbf{T}$$

Example:

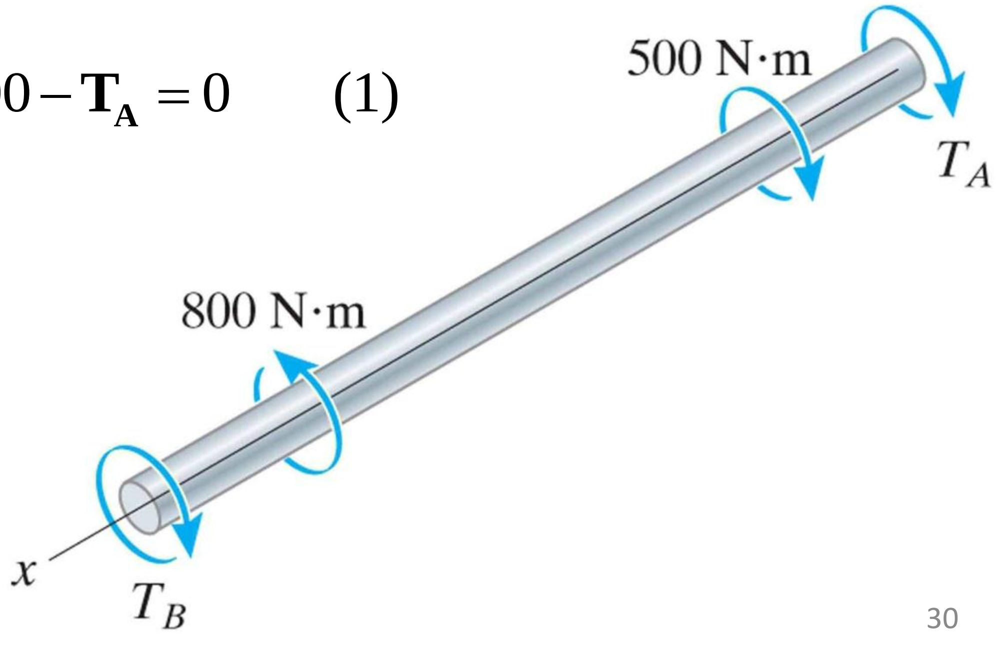
$d = 20$ mm

Find $\mathbf{T_A}$ and $\mathbf{T_B}$



Solution :

$$\sum \mathbf{M_x} = 0 \quad \rightarrow \quad -\mathbf{T_B} + 800 - 500 - \mathbf{T_A} = 0 \qquad (1)$$

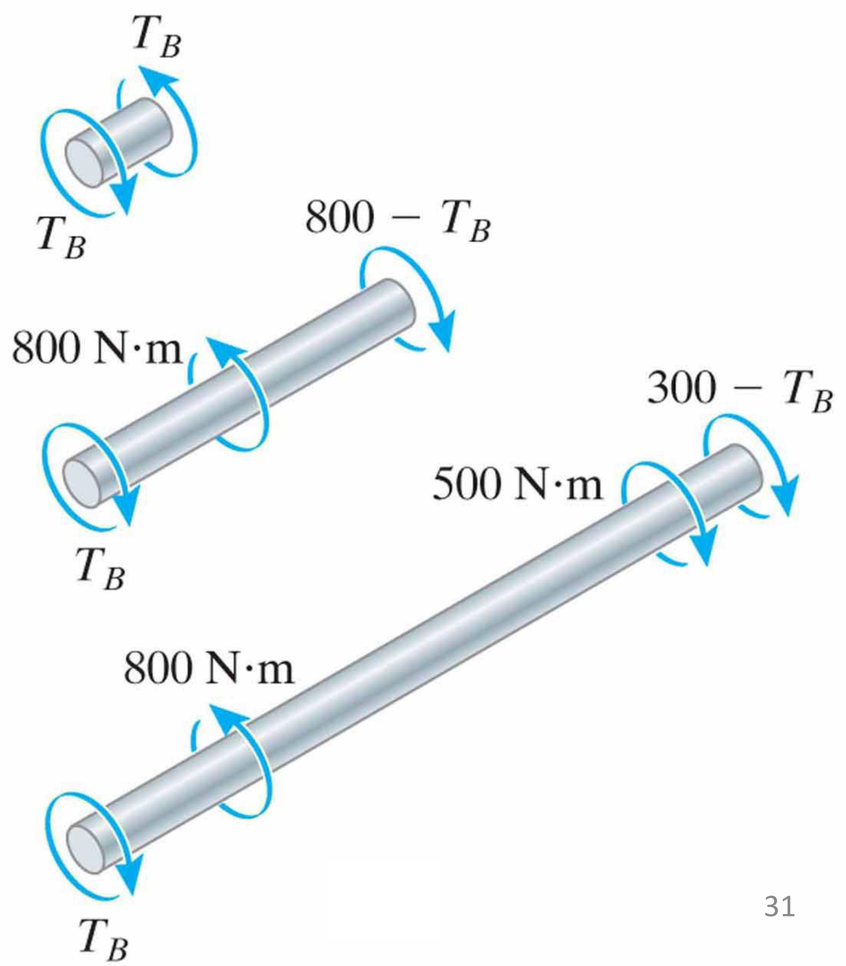


$$\phi_{A/B} = 0$$

$$\frac{-\mathbf{T_B} \times 0.2}{JG} + \frac{(800 - \mathbf{T_B}) \times 1.5}{JG} + \frac{(300 - \mathbf{T_B}) \times 0.3}{JG} = 0 \qquad (2)$$

$$\mathbf{T_B} = 645 \text{ N.m}$$

$$\mathbf{T_A} = -345 \text{ N.m}$$

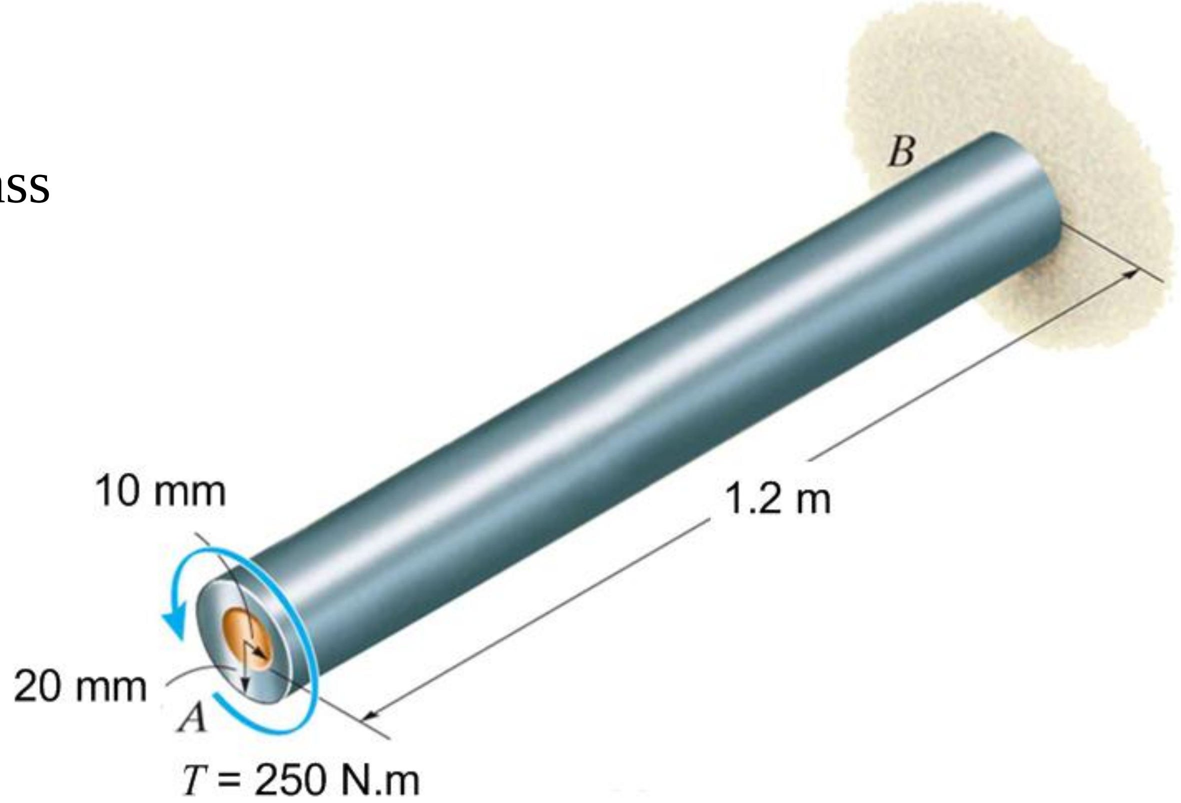


Example:

Tube of steel and core of brass

T = 250 N.m

G_{st} = 80 GPa, G_{br} = 36 GPa.



Solution :

$$-\mathbf{T_{st}} - \mathbf{T_{br}} + 250 = 0 \qquad (1)$$

$$\phi_{st} = \phi_{br} \qquad (2)$$

$$\frac{\mathbf{T_{st}} L}{\frac{\pi}{2}\left((0.02)^4 - (0.01)^4\right) \times 80 \times 10^9} = \frac{\mathbf{T_{br}} L}{\frac{\pi}{2}(0.01)^4 \times 36 \times 10^9}$$

From Eq. 1 and 2, we get

$\mathbf{T_{st}} = 242.72$ N.m

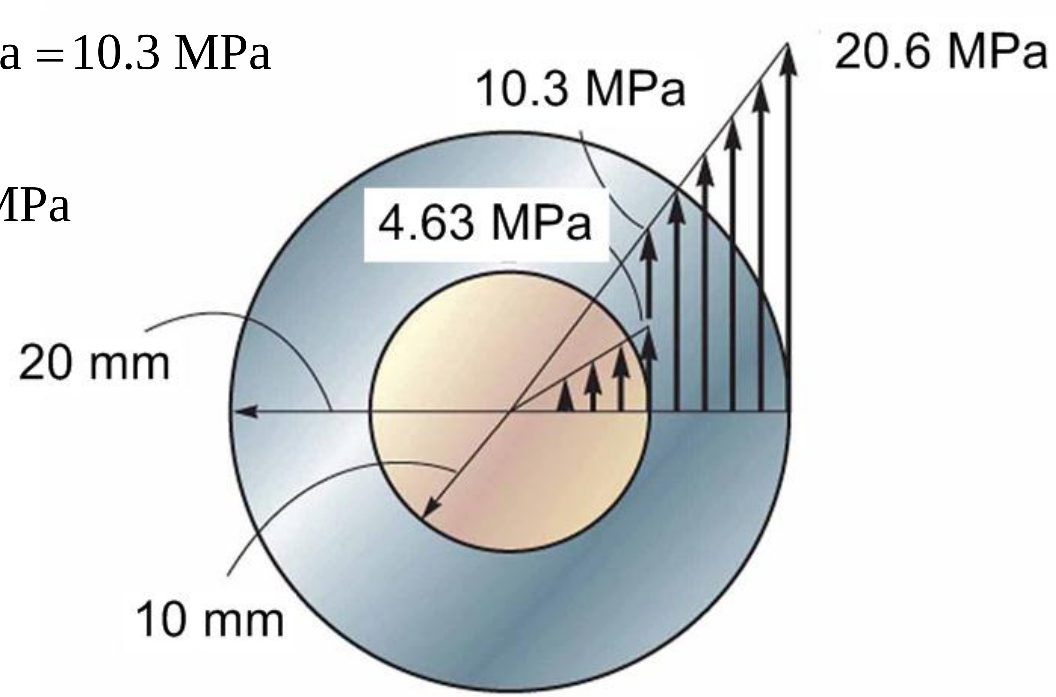
$\mathbf{T_{br}} = 7.28$ N.m

$$\left(\tau_{st}\right)_{\max} = \frac{\mathbf{T_{st}} \times 0.02}{\frac{\pi}{2}\left((0.02)^4 - (0.01)^4\right) \times 80 \times 10^9} = 20.6 \text{ MPa}$$

$$\left(\tau_{st}\right)_{\min} = \frac{C_1}{C_2}\left(\tau_{st}\right)_{\max} = \frac{0.01}{0.02} \times 20.6 \text{ MPa} = 10.3 \text{ MPa}$$

$$\left(\tau_{br}\right)_{\max} = \frac{T_{br} \times 0.01}{\frac{\pi}{2}(0.01)^4 \times 36 \times 10^9} = 4.63 \text{ MPa}$$

$$\left(\tau_{br}\right)_{\min} = 0$$



Shear–stress distribution

3.7 DESIGN OF TRANSMISSION SHAFTS

$$P = \mathbf{T} \cdot w = \mathbf{T} \cdot 2\pi f$$

Power in Watt (N.m/s)

Angular velocity in rad/sec

Frequency in Hz (1/s)

Example:

a solid shaft is used to transmit 3750 W at angular 175 rpm. If allowable shear is 100 MPa, find the diameter of the shaft.

Solution :

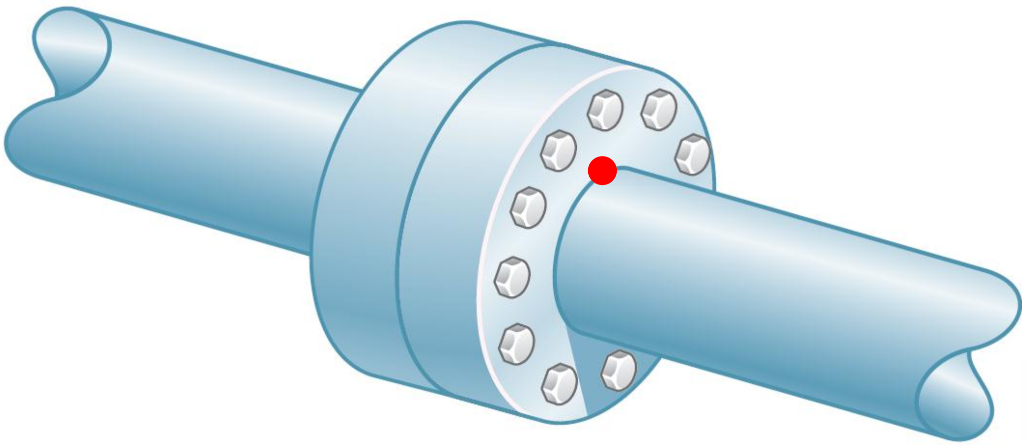
$$w = 175 \times \frac{2\pi}{60} = 18.33 \text{ rad/sec}$$

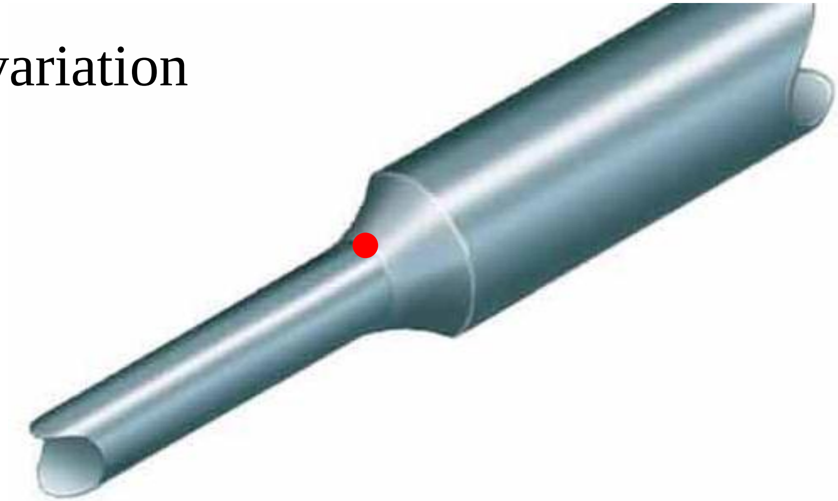
$$P = T \cdot w \quad \rightarrow \quad T = 204.6 \text{ N.m}$$

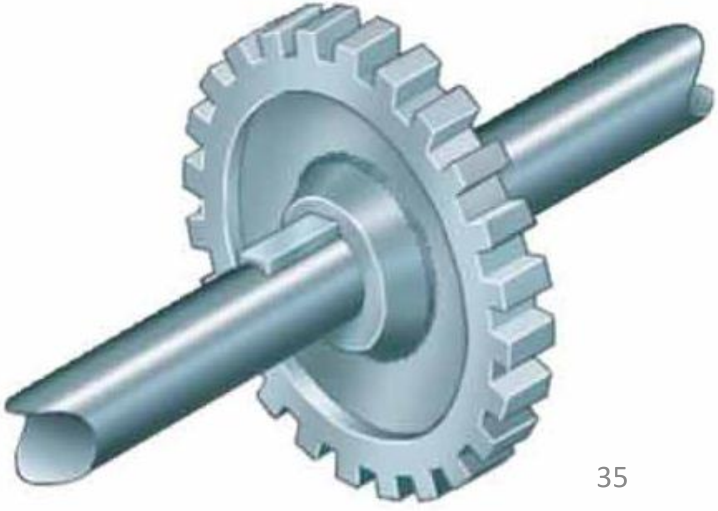
$$\tau_{all} = \frac{T \cdot C}{J} \quad \rightarrow \quad C = 10.92 \text{ mm } (d = 22 \text{ mm})$$

3.8 STRESS CONCENTRATION IN CIRCULAR SHAFTS

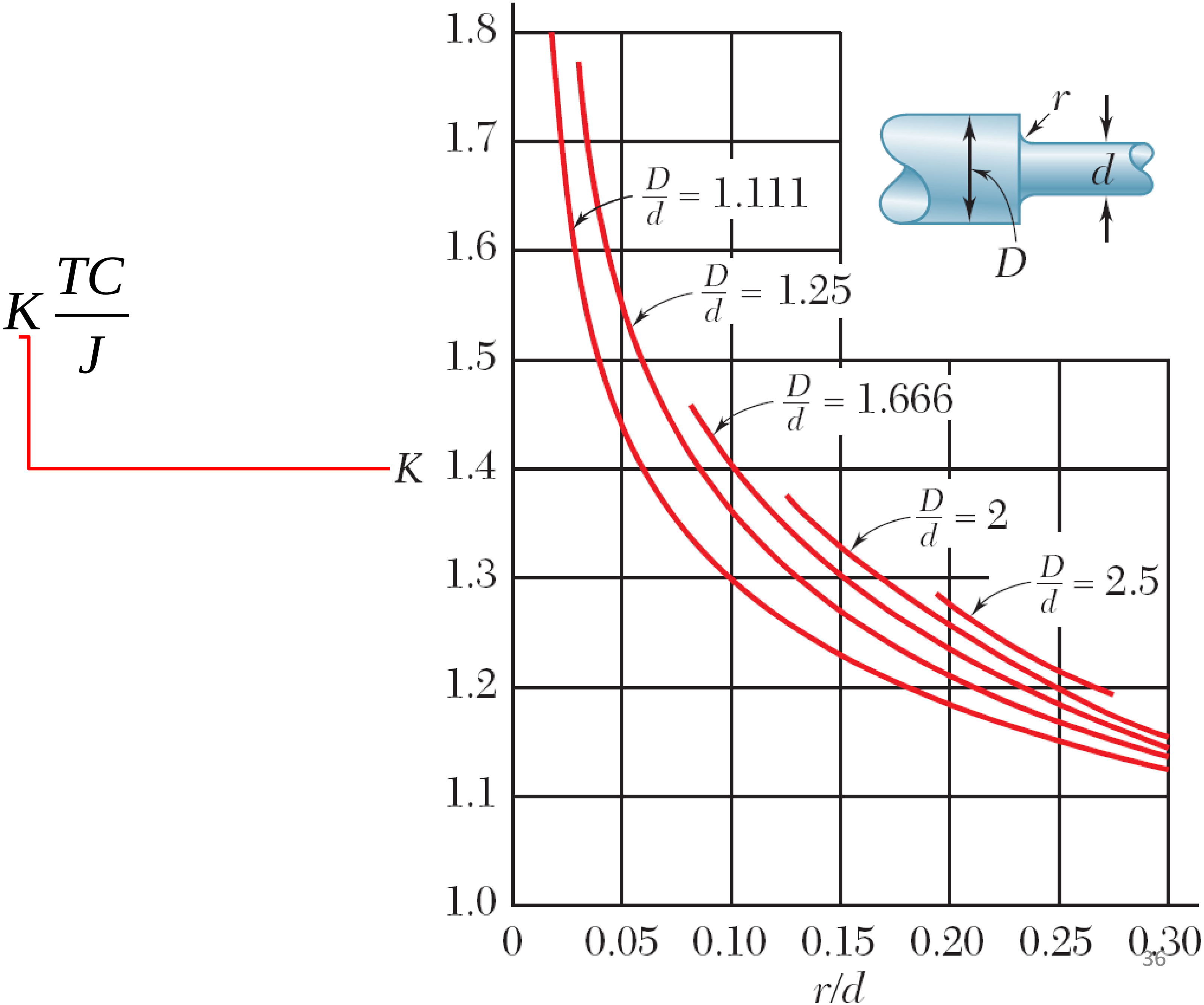
Stress concentration occurs when variation in the cross-section occurs



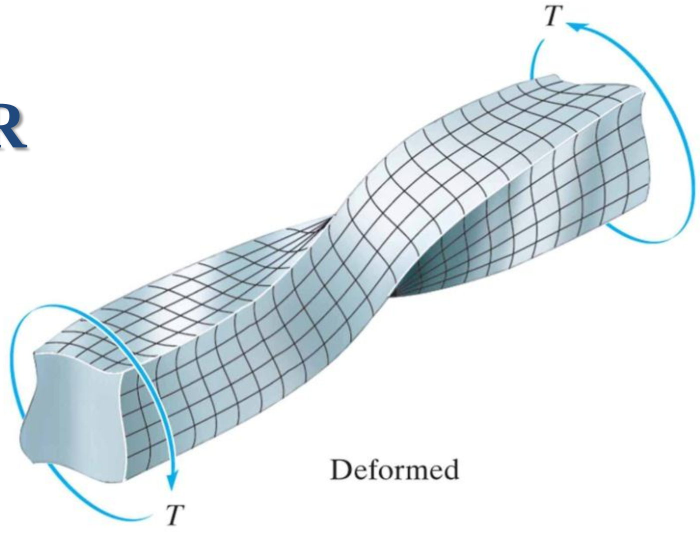


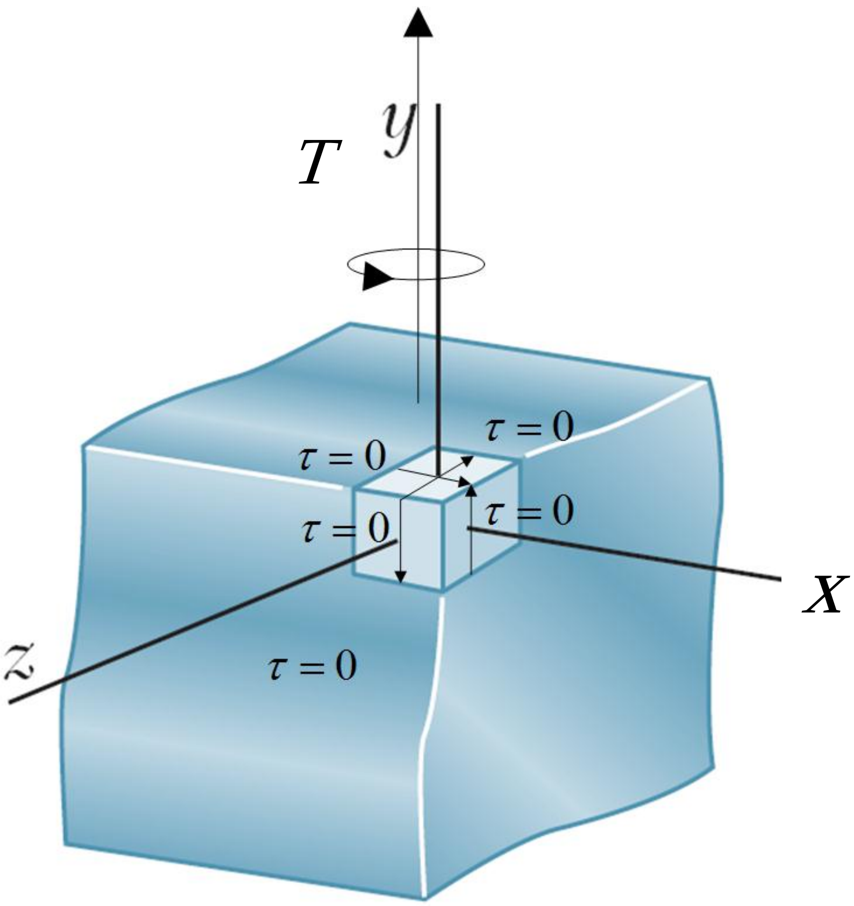


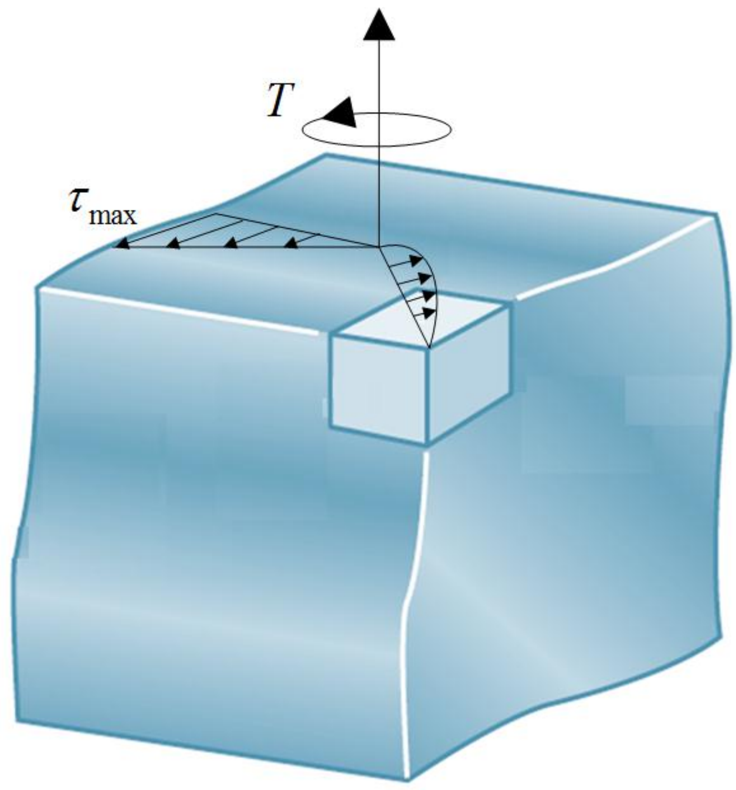
$$\tau = K \frac{TC}{J}$$

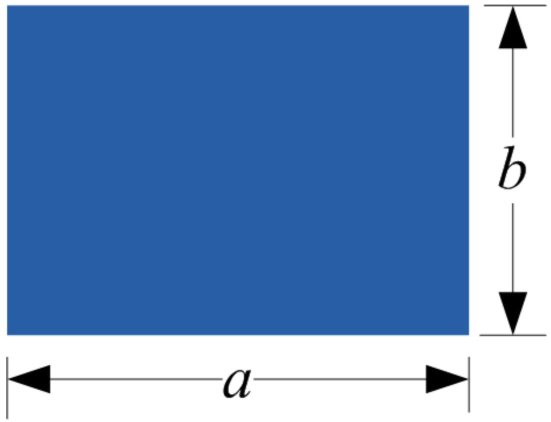


3.12 TORSION OF NON-CIRCULAR MEMBERS









From elasticity theory

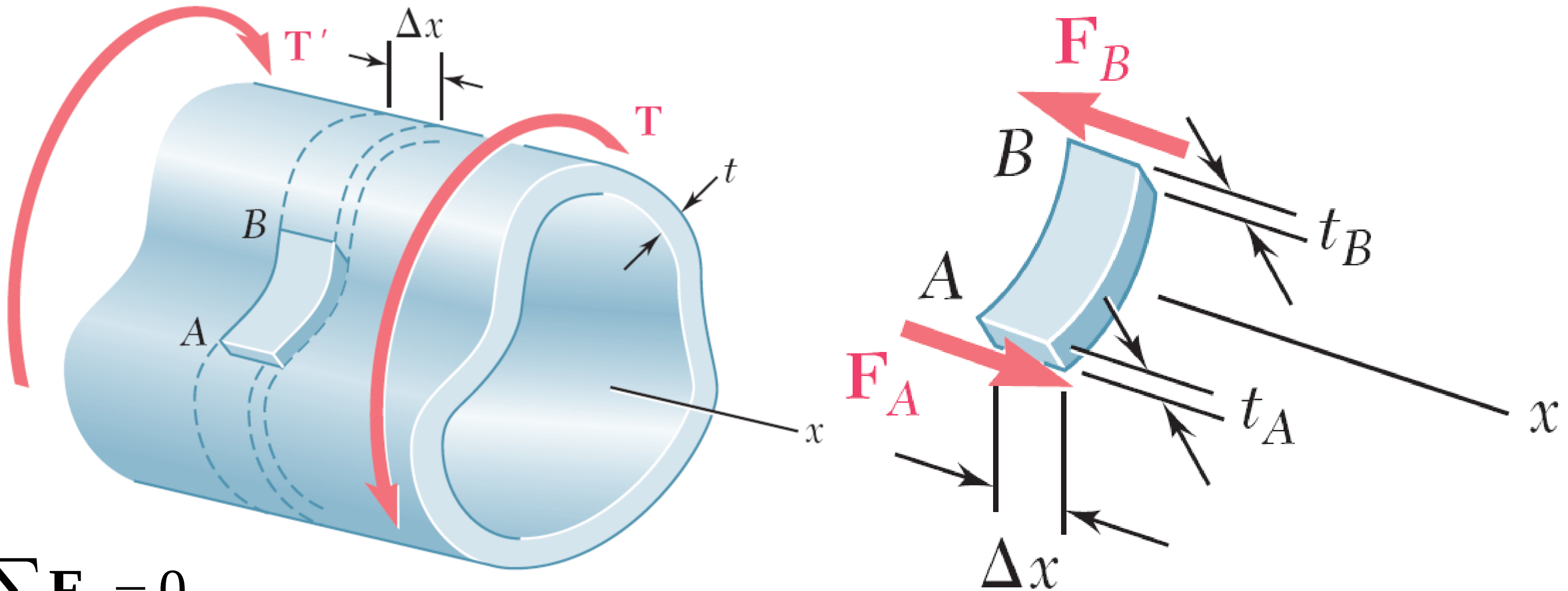
$$\tau_{\max} = \frac{\mathbf{T}}{c_1 a b^2}$$

$$\phi = \frac{\mathbf{T}L}{c_2 a b^3 G}$$

TABLE Coefficients for Rectangular Bars in Torsion

a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

3.15 THIN WALLED HOLLOW SHAFTS



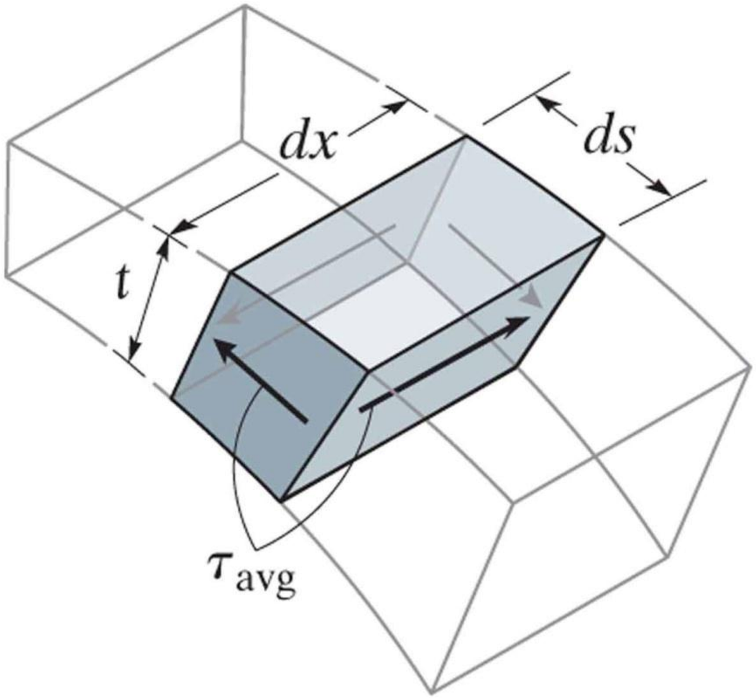
$$\sum \mathbf{F_x} = 0$$

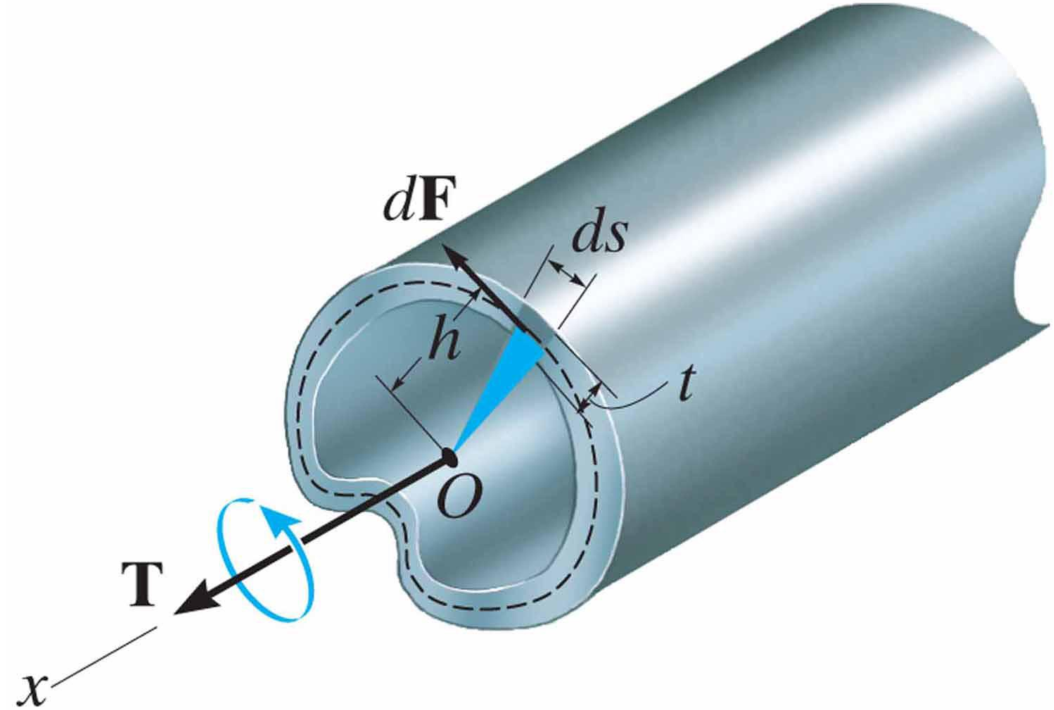
$$\mathbf{F_A} = \mathbf{F_B}$$

$$\tau_A (t_A \Delta x) = \tau_B (t_B \Delta x)$$

Thus,

$$\tau t = q = \text{constant (Shear flow)}$$





$$d\mathbf{T} = h(dF) = h(\tau_{ave} t ds)$$

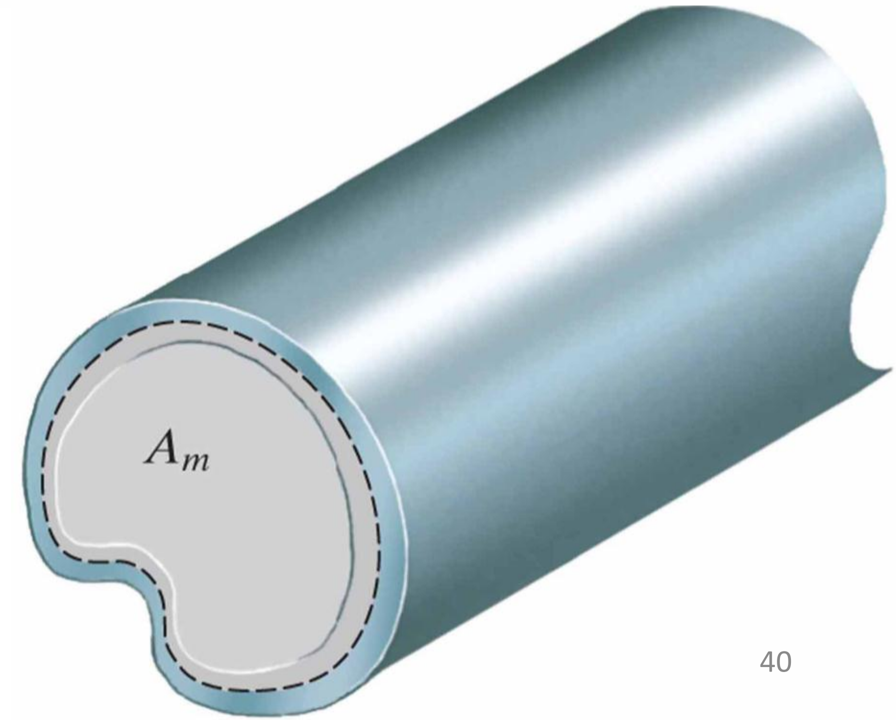
$$\mathbf{T} = \oint h\tau_{ave} t ds = q\oint h ds$$

but

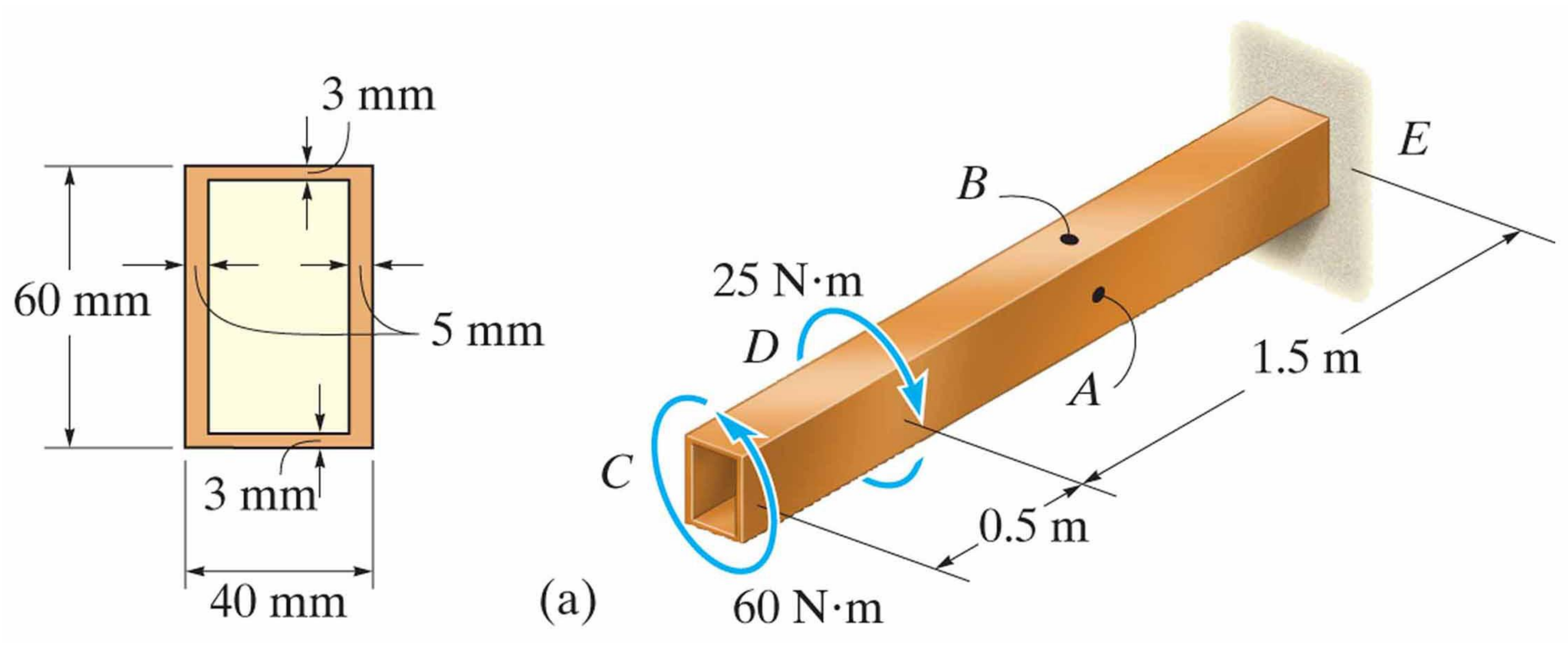
$$dA_m = \frac{1}{2} h ds$$

$$\mathbf{T} = 2\tau_{ave} t A_m$$

$$\tau_{ave} = \frac{T}{2tA_m}$$



Example:



$$A_m = 0.035 \times 0.057 = 0.002 \text{ m}^2$$

$$\tau_A = \frac{\mathbf{T}}{2t_A A_m} = 1.75 \text{ MPa}$$

$$\tau_B = \frac{\mathbf{T}}{2t_B A_m} = 2.92 \text{ MPa}$$

END OF CHAPTER THREE

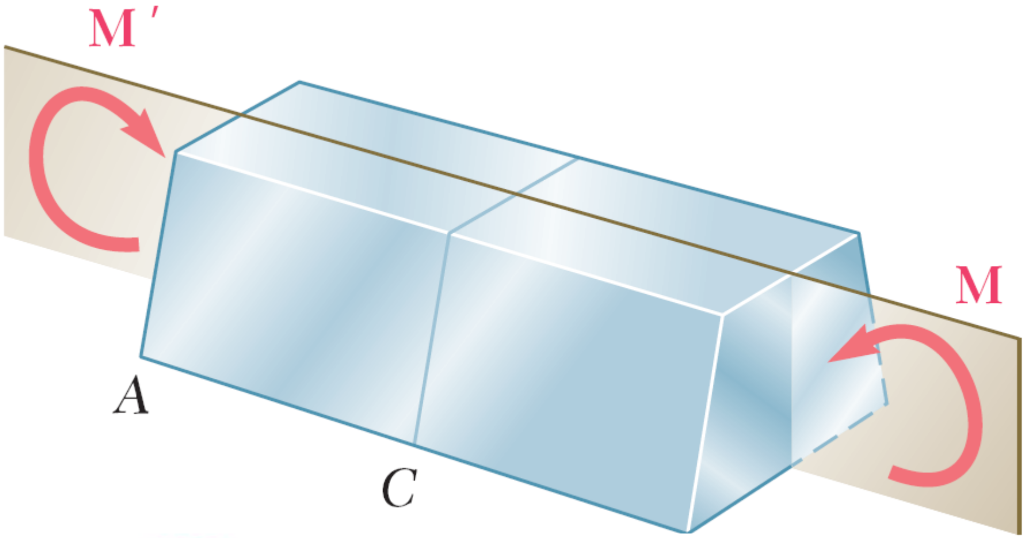
MECHANICS OF MATERIALS

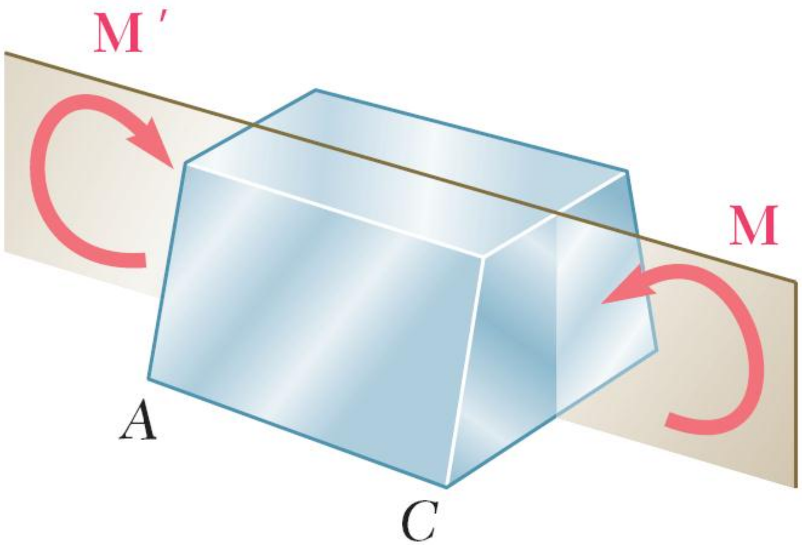
CHAPTER FOUR

PURE BENDING

Prepared by : Dr. Mahmoud Rababah

4.2 SYMMETRIC MEMBER IN PURE BENDING

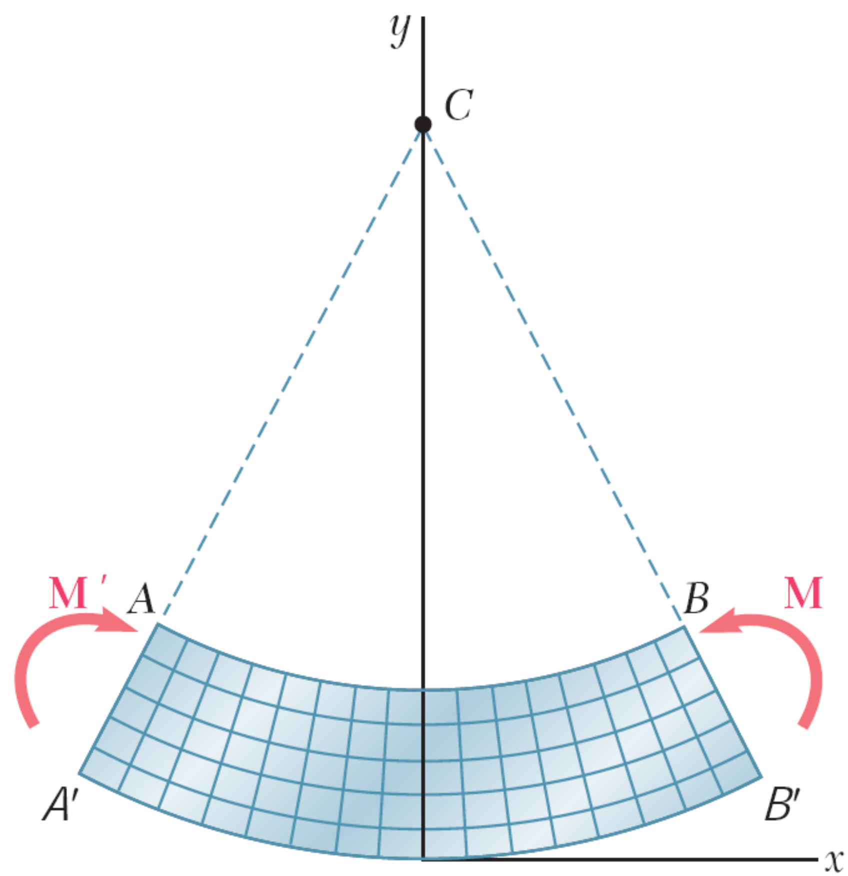


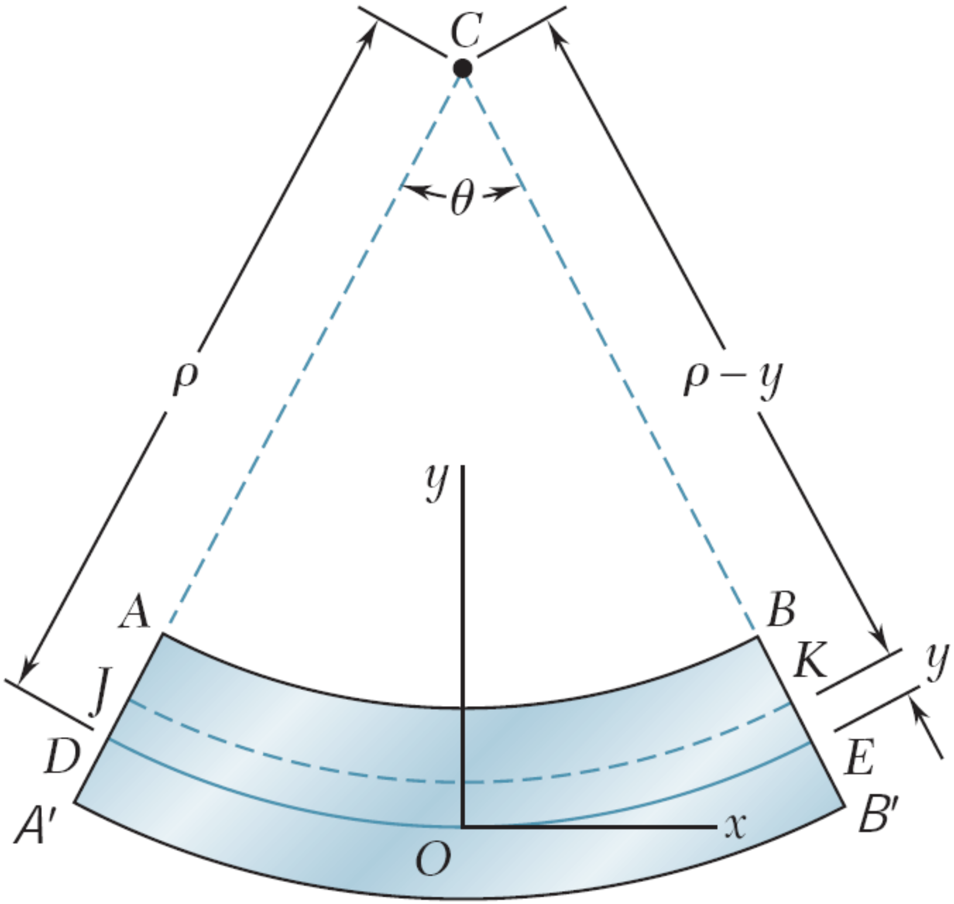


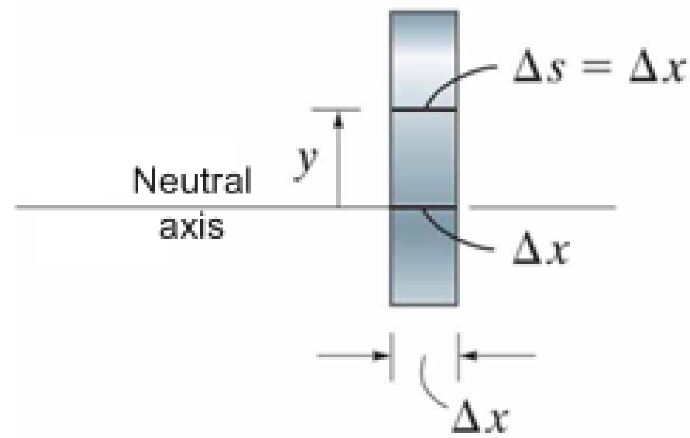
Any section will have same magnitude of moment with no other forces acting (Pure bending)

4.3 DEFORMATION IN A SYMMETRIC MEMBER IN PURE BENDING

- Line *AB* will be transformed to circular arc centered at C.
- Any cross-section perpendicular to the axis of the member remains plane.
- Line AB decreased in length and line A'B' increase in length; causing compression on the upper surface and tension on the lower surface.
- There should be a surface in between where no tension or compression occurs; this called *the neutral surface*.

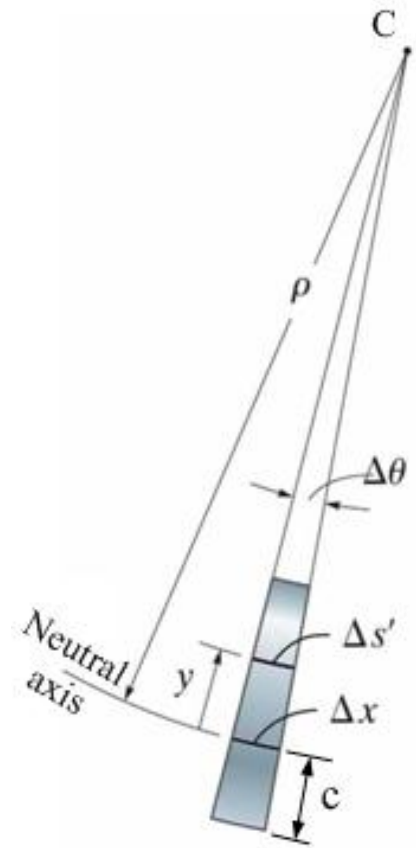






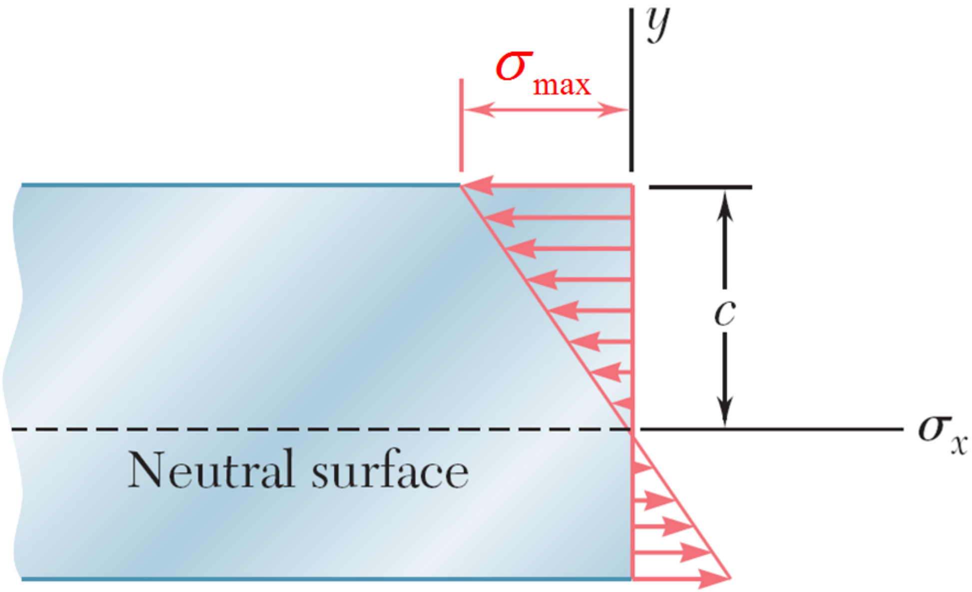
$$\varepsilon = \frac{\Delta s' - \Delta x}{\Delta x} = \frac{(\rho - y)\Delta\theta - \rho\Delta\theta}{\rho\Delta\theta} = -\frac{y}{\rho}$$

$$\varepsilon_{\max} = \frac{c}{\rho}, \quad \text{then} \;\; \varepsilon = -\left(\frac{y}{c}\right)\varepsilon_{\max}$$



4.4 STRESSES AND DEFORMATIONS IN THE ELASTIC RANGE

From hook's law: linear variation of normal strain leads to linear variation in normal stress



$$\sigma = -\left(\frac{y}{c}\right)\sigma_{\max}$$

$$\sum \mathbf{F_x} = 0$$

$$\int_A dF = \int_A \sigma dA = \int_A -\left(\frac{y}{c}\right)\sigma_{\max} dA$$

thus,

$$\int_A y dA = 0$$

The neutral axis is the horizontal centroidal axis

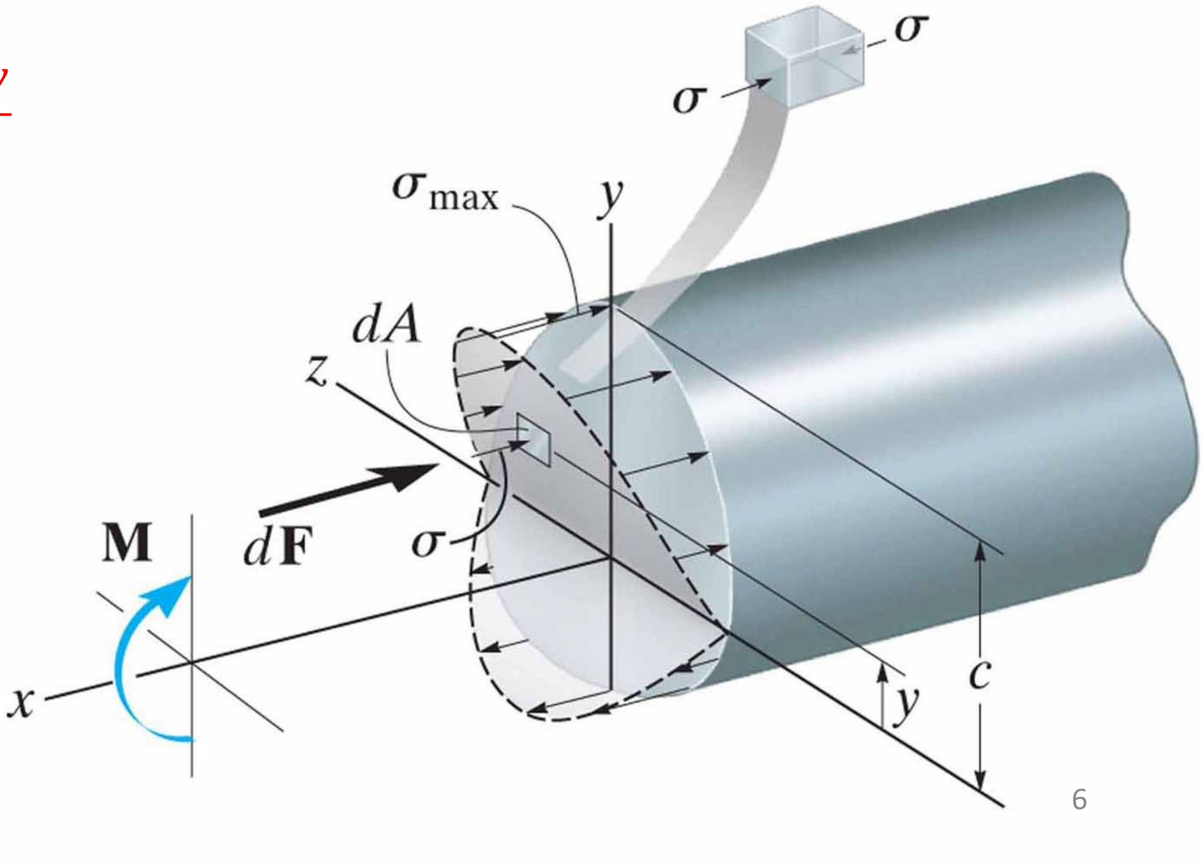
FLEXURE FORMULA

$$\mathbf{M} = \int_A y\,dF = \int_A y\sigma\,dA = \int_A y\left(\frac{y}{c}\sigma_{\max}\right)dA$$

$$\mathbf{M} = \frac{\sigma_{\max}}{c}\int_A y^2\,dA$$ → Moment of inertia (I)

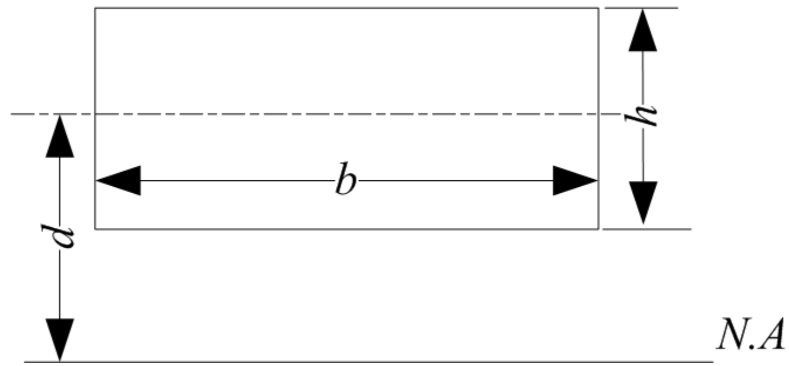
$$\sigma_{\max} = \frac{\mathbf{M}c}{I} \quad \text{and} \quad \sigma = \frac{-\mathbf{M}y}{I}$$

$$\text{curveture} = \frac{1}{\rho} = \frac{\mathbf{M}}{EI}$$



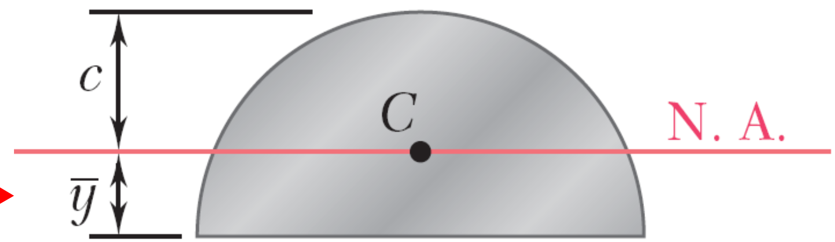
in general

$$I = \frac{1}{12} bh^3 + (bh)d^2$$

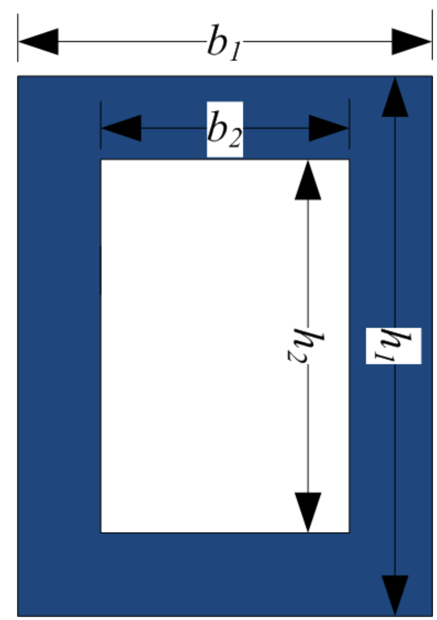


$$\bar{y} = \frac{4r}{3\pi}$$

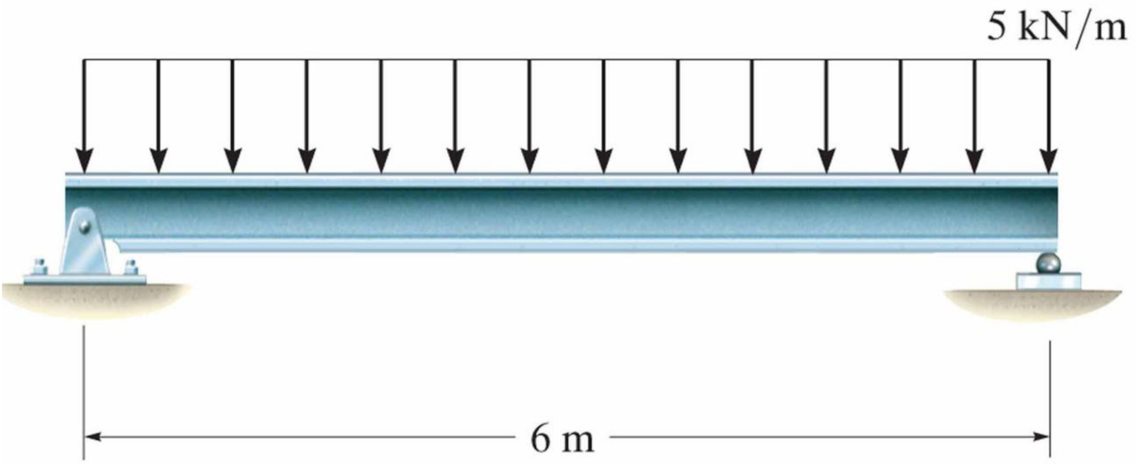
$$I = \frac{\pi}{8} r^4 + \frac{\pi}{2} r^2 \left(\frac{4r}{3\pi} \right)^2$$



$$I = \frac{1}{12} b_1 h_1^3 - \frac{1}{12} b_2 h_2^3$$



Example: Draw the stress distribution over the cross-section.



Solution :

$$\mathbf{V} = \int w dx = -5x + c_1$$

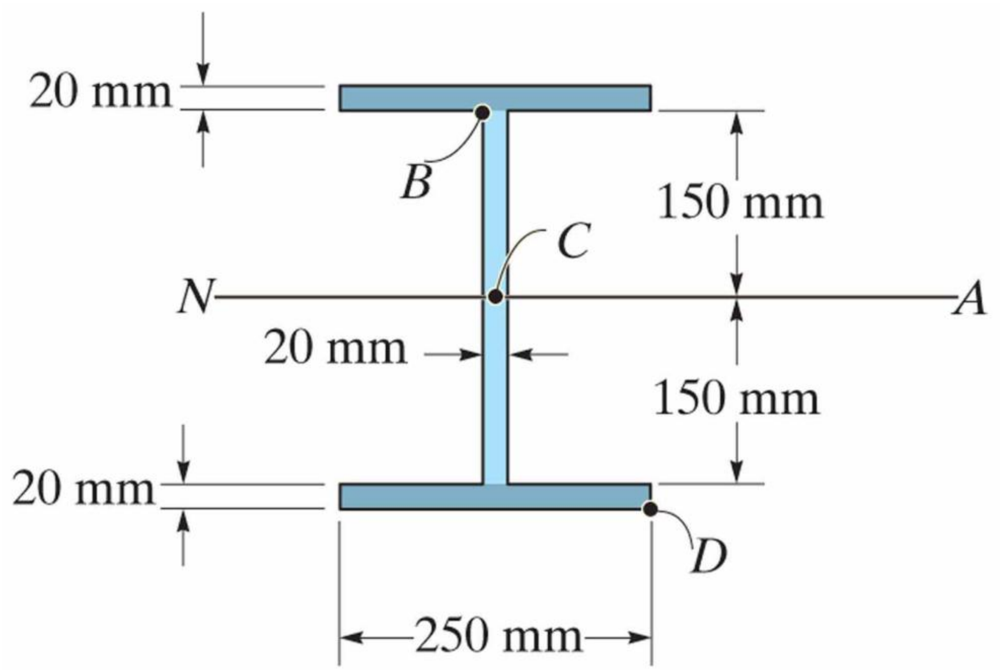
$$\mathbf{V}\,\Big|_{x=0} = 15 \;\rightarrow\; c_1 = 15$$

$$\mathbf{V} = -5x + 15$$

$$\mathbf{M} = \int \mathbf{V} dx = -2.5x^2 + 15x + c_2$$

$$\mathbf{M}\,\Big|_{x=0} = 0 \;\rightarrow\; c_2 = 0$$

$$\mathbf{M}_{\mathbf{max}} = \mathbf{M}\,\Big|_{x=3} = 22.5 \text{ kN.m}$$

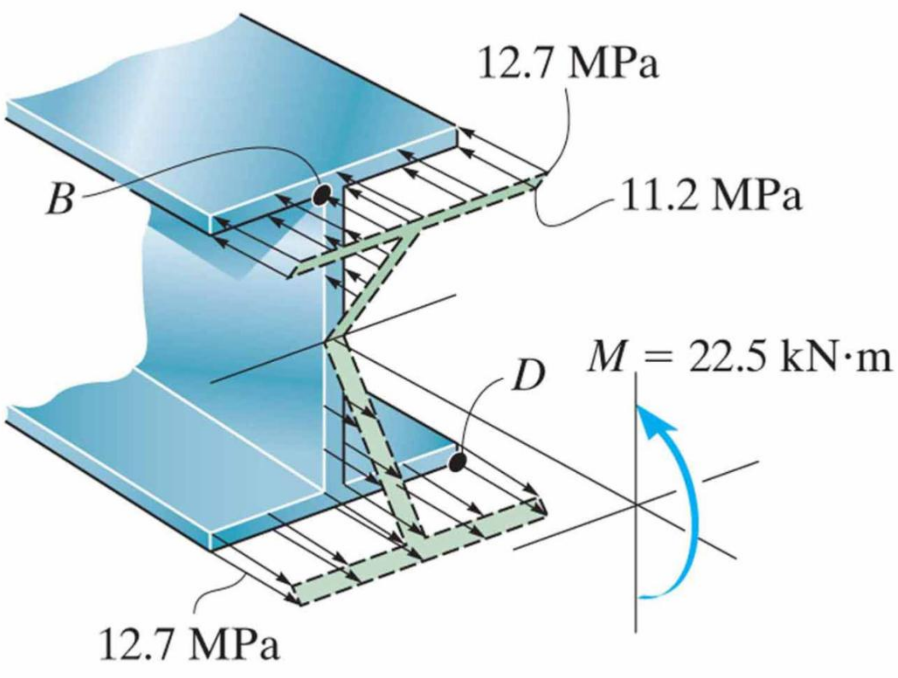


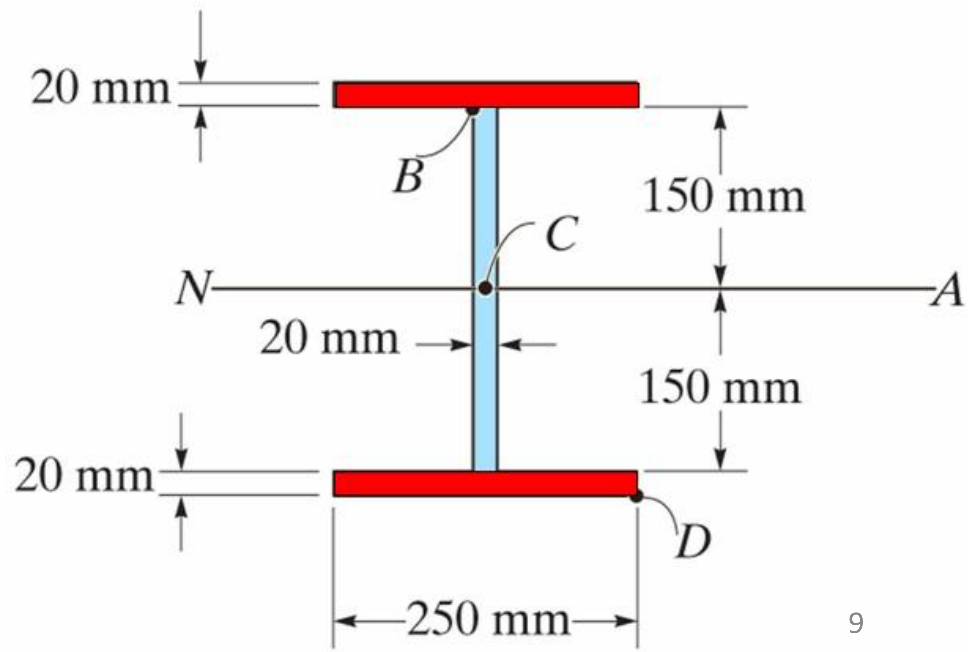
$$I_1 = \frac{1}{12}\times 0.25\times(0.02)^3 + 0.25\times 0.02\times(0.16)^2$$

$$I_2 = \frac{1}{12}\times 0.02\times(0.3)^3$$

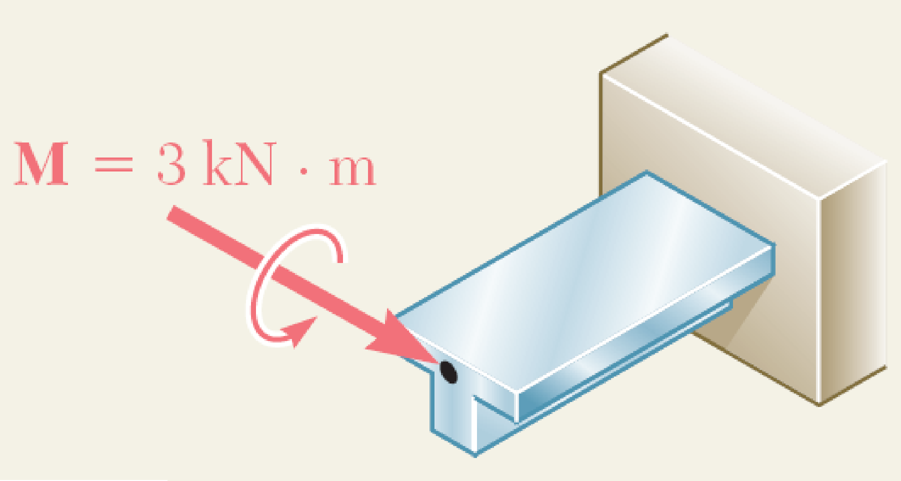
$$I = 2I_1 + I_2 = 301.3\times 10^{-6}\ \text{m}^4$$

$$\sigma_{\max} = \frac{\mathbf{M}_{\mathbf{max}}\cdot c}{I} = \frac{22.5\times 10^3\times 0.17}{301.3\times 10^{-6}} = 12.7\ \text{MPa}$$



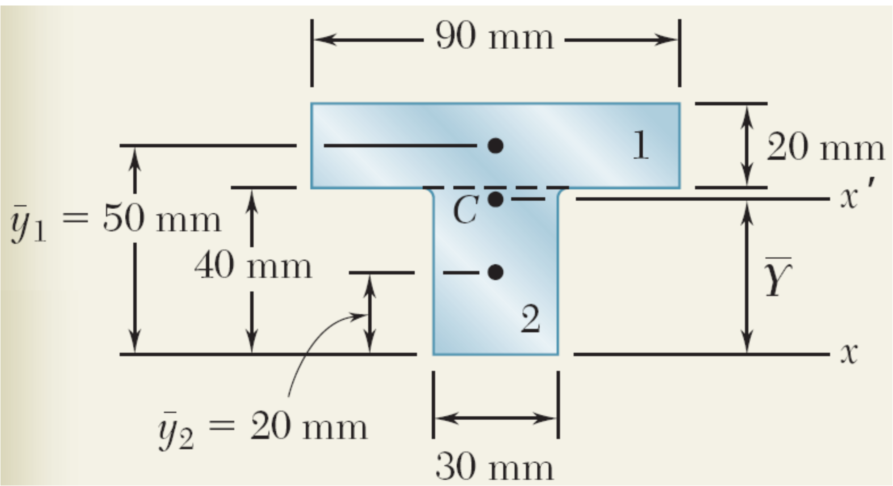


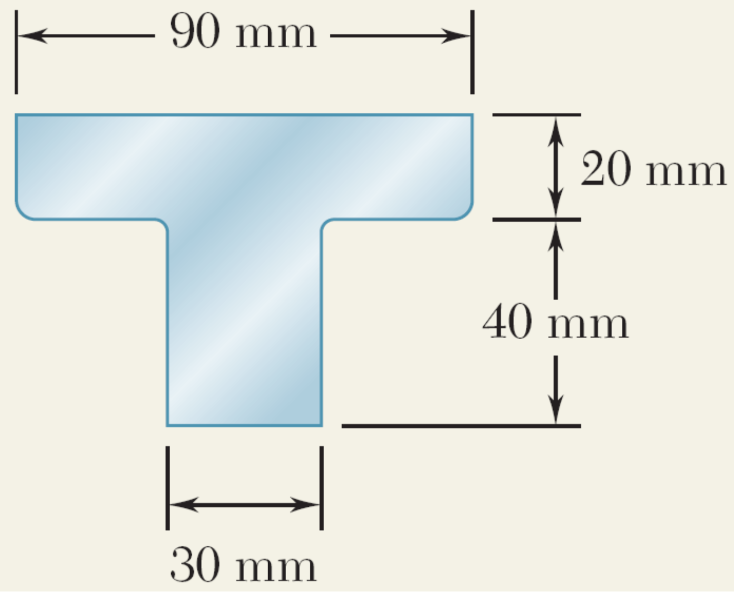
Example: Find maximum tensile and compressive stresses.



Solution :

$$\bar{y} = \frac{A_1\bar{y}_1 + A_2\bar{y}_2}{A_1 + A_2} = 38 \text{ mm}$$





$$I_1 = \frac{1}{12}\times 90\times(20)^3 + 90\times 20\times(12)^2 \text{ mm}^4$$

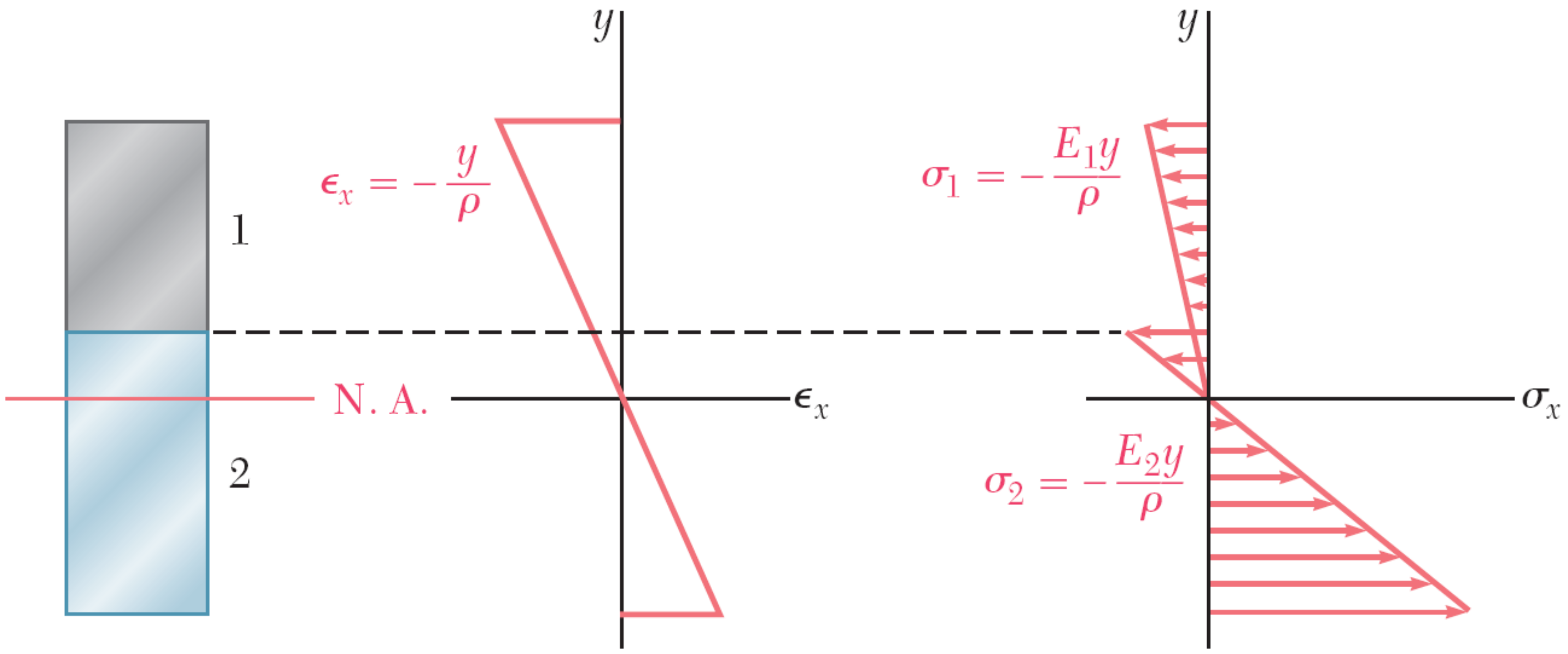
$$I_2 = \frac{1}{12}\times 30\times(40)^3 + 30\times 40\times(18)^2 \text{ mm}^4$$

$$I = I_1 + I_2 = 868\times 10^{-9} \text{ m}^4$$

$$(\sigma_t)_{\max} = \frac{3\times 10^3\times 22\times 10^{-3}}{868\times 10^{-9}} = 76 \text{ MPa}$$

$$(\sigma_c)_{\max} = \frac{3\times 10^3\times 38\times 10^{-3}}{868\times 10^{-9}} = 131.3 \text{ MPa}$$

4.6 BENDING OF MEMBERS MADE OF SEVERAL MATERIALS



Use a method called transformed section method.

Procedure

Assume $E_1 > E_2$,

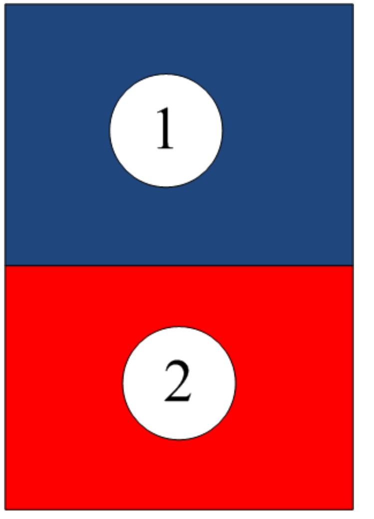
1- $$n = \frac{E_1}{E_2}$$

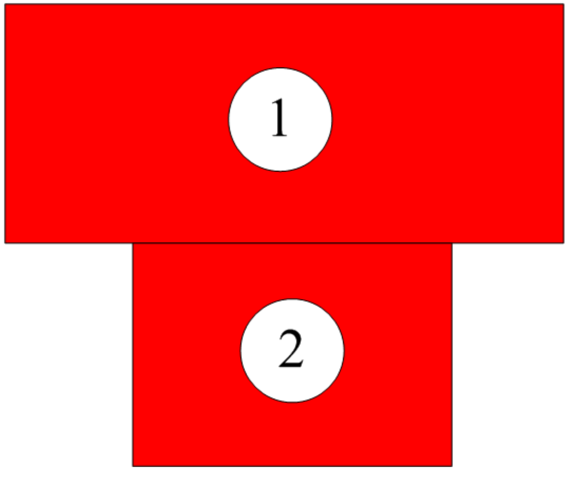
2- Multiply the width of material 1 by *n.*

3- Now consider all the section as made of material 2.

4- Find *I* and then the stresses at any point on the section.

5- the stress at any point located on material 1 should by multiplied by *n.*



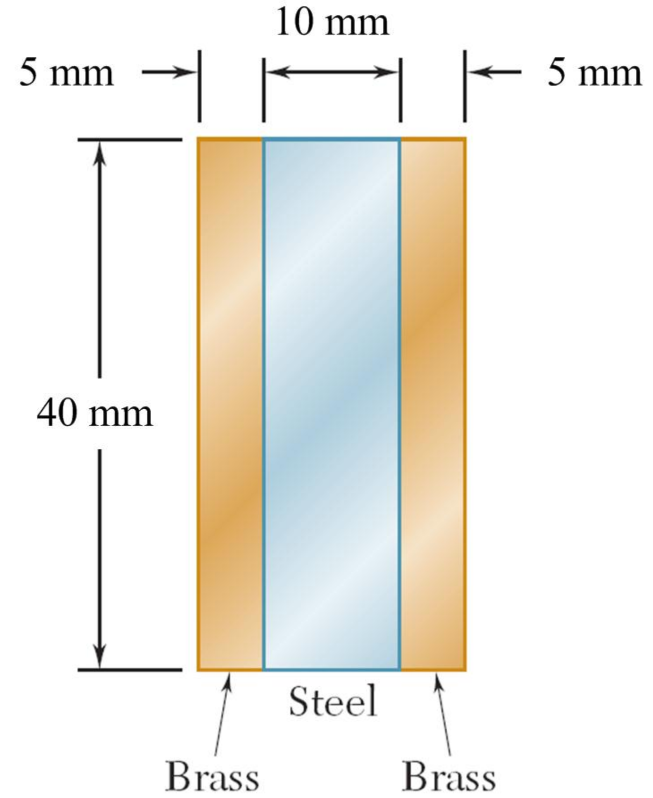


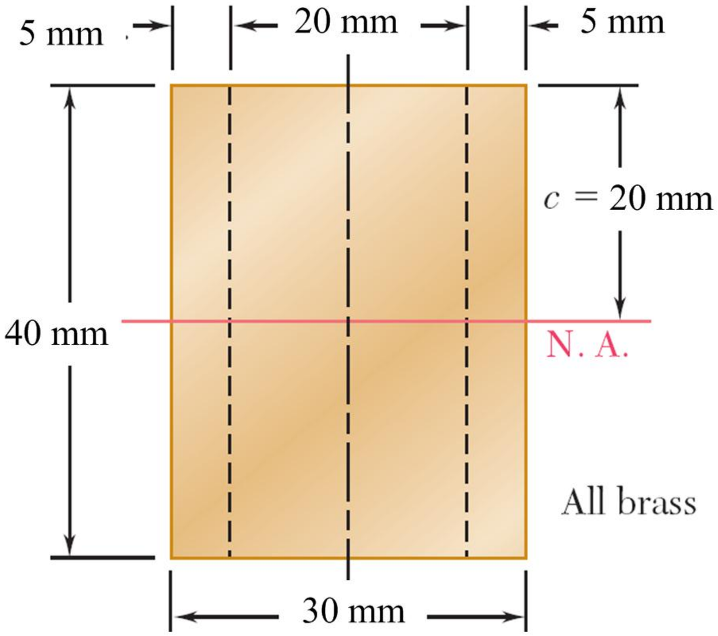
Example: find maximum stress in brass and steel

$\mathbf{M} = 2$ kN.m

$E_{br} = 100$ GPa

$E_{st} = 200$ GPa





$$n = \frac{200}{100} = 2$$

$$I = \frac{1}{12} \times 0.03 \times (0.04)^3 = 160 \times 10^{-9} \text{ m}^4$$

$$c = 0.02$$

$$\sigma_{br} = \frac{\mathbf{M} \cdot c}{I} = \frac{2 \times 10^3 \times 0.02}{160 \times 10^{-9}} = 250 \text{ MPa}$$

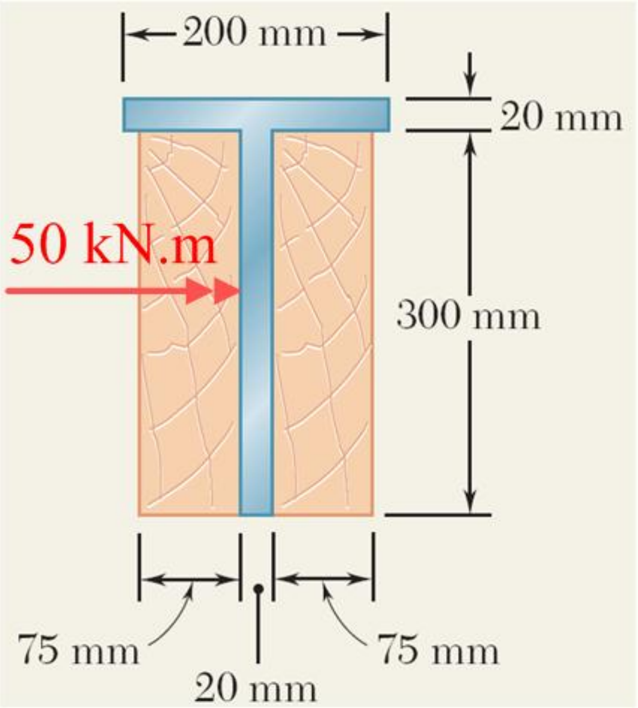
$$\sigma_{st} = n \frac{\mathbf{M} \cdot c}{I} = 2 \times \frac{2 \times 10^3 \times 0.02}{160 \times 10^{-9}} = 500 \text{ MPa}$$

Example: Find the maximum tensile and compressive stress in both steel and wood

$$\mathbf{M} = 50 \text{ kN.m}$$

$$E_w = 12.5 \text{ GPa}$$

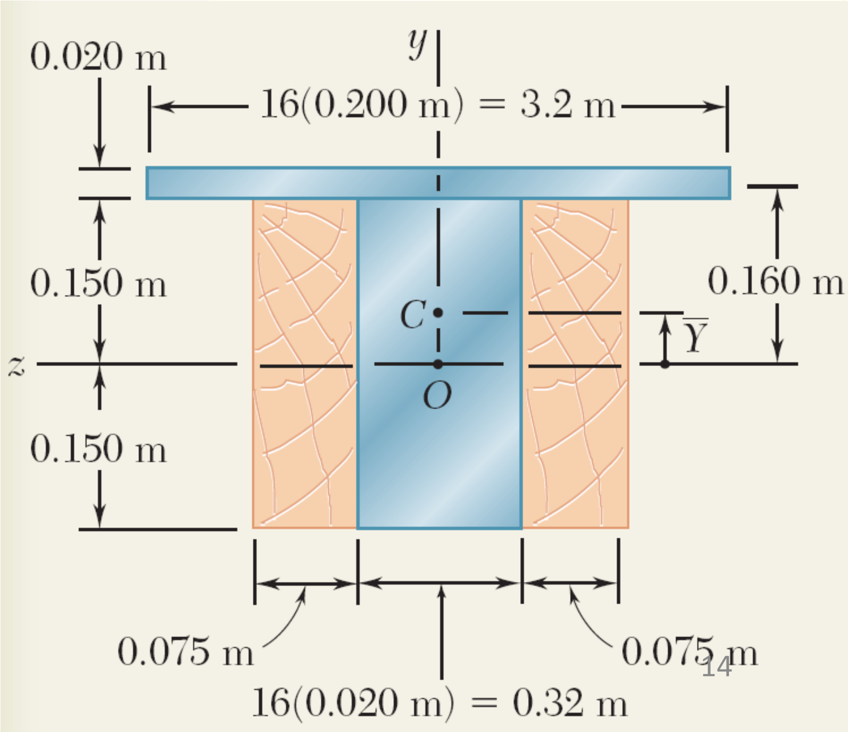
$$E_{st} = 200 \text{ GPa}$$



Solution :

$$n = \frac{200}{12.5} = 16$$

the transformed section will be



locate the neutral axis:

$$\bar{y} = \frac{0.16 \times 3.2 \times 0.02}{3.2 \times 0.02 + 0.47 \times 0.3} = 0.05 \text{ m}$$

$$I_1 = \frac{1}{12} \times 0.47 \times (0.3)^2 + 0.47 \times 0.3 \times (0.05)^2$$

$$I_2 = \frac{1}{12} \times 3.2 \times (0.02)^2 + 3.2 \times 0.02 \times (0.11)^2$$

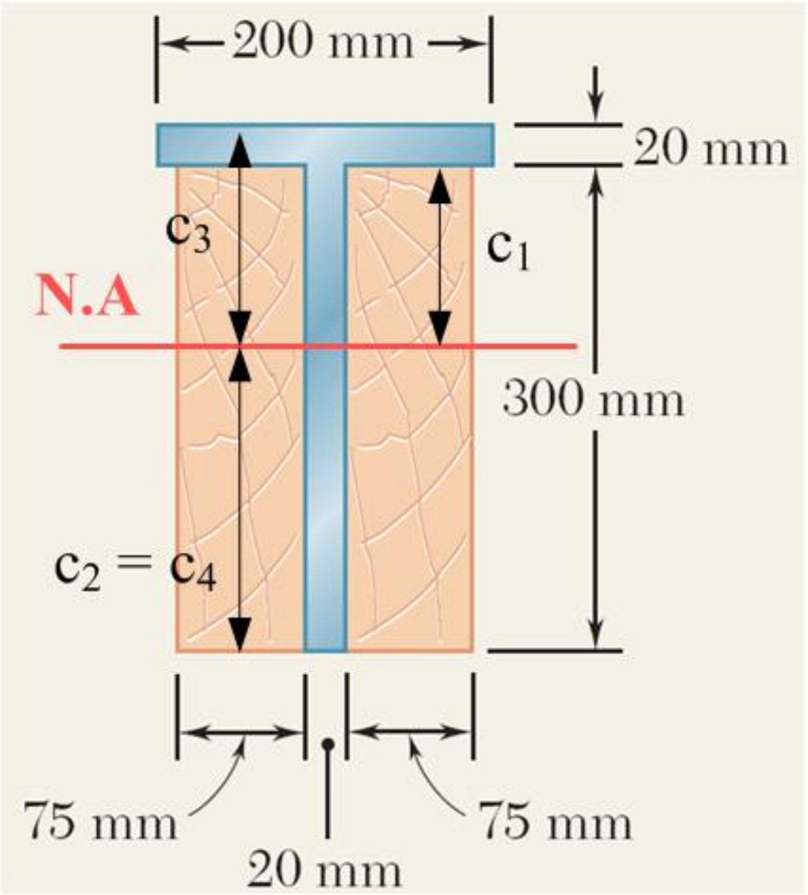
$$I = I_1 + I_2 = 2.19 \times 10^{-3} \text{ m}^4$$

$$(\sigma_w)_t = \frac{\mathbf{M} \cdot c_1}{I} = \frac{50 \times 10^3 \times 0.1}{2.19 \times 10^{-3}} = 2.29 \text{ MPa}$$

$$(\sigma_w)_c = \frac{\mathbf{M} \cdot c_2}{I} = \frac{50 \times 10^3 \times 0.2}{2.19 \times 10^{-3}} = 4.57 \text{ MPa}$$

$$(\sigma_s)_t = \frac{\mathbf{M} \cdot c_3}{I} = 16 \times \frac{50 \times 10^3 \times 0.12}{2.19 \times 10^{-3}} = 43.8 \text{ MPa}$$

$$(\sigma_s)_c = \frac{\mathbf{M} \cdot c_4}{I} = 16 \times \frac{50 \times 10^3 \times 0.2}{2.19 \times 10^{-3}} = 73.12 \text{ MPa}$$



REINFORCED CONCRETE BEAMS

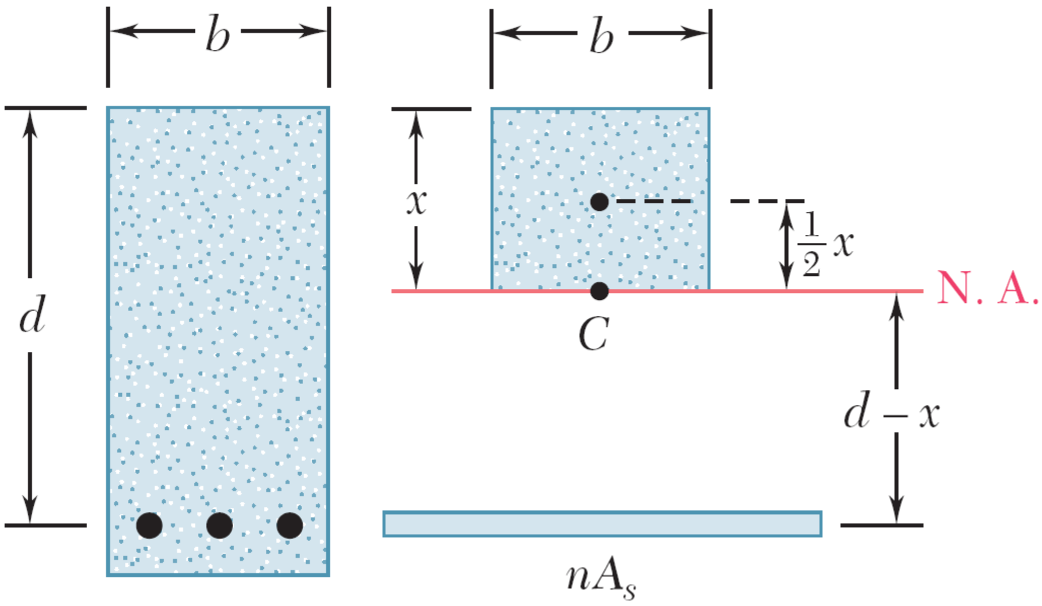
Procedure:

1- $n = \dfrac{E_{st}}{E_c}$

2- Replace the steel bars by equivalent area of nA_s.

3- Consider the concrete only in compression.

4- from the first moment of inertia, apply the following equation to locate the neutral axis and find the effective portion of concrete



$$bx \cdot \frac{x}{2} - nA_s(d-x) = 0$$

or

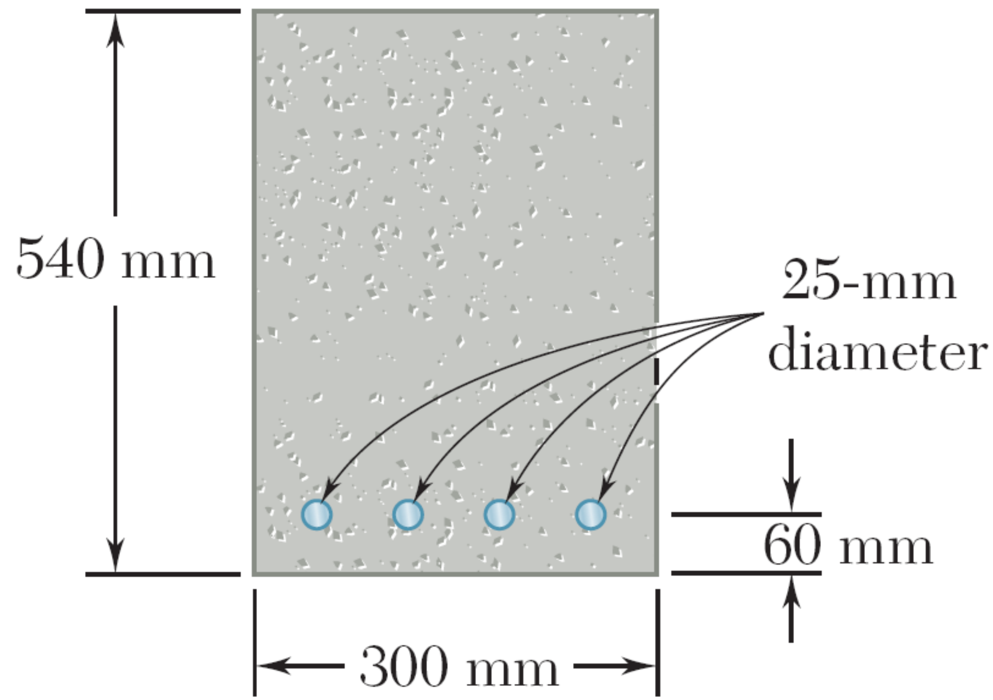
$$\frac{1}{2}bx^2 + nA_s x - nA_s d = 0$$

5- Follow the same procedure explained before on the transformed section to get the stresses

Example: $\mathbf{M} = 175$ kN.m

$E_c = 25$ GPa

$E_{st} = 200$ GPa



Solution :

$A_s = 4\pi r^2 = 1.9635\times10^{-3}\ \text{m}^2$

apply the equation

$$\frac{1}{2}bx^2 + nA_s x - nA_s d = 0$$

$x = 178$ mm

$$I = \frac{1}{3}bh^3 + nA_s(d-x)^2$$

$$I = \frac{1}{3}\times0.3\times(0.178)^3 + 8\times1.9635\times10^{-3}\times(0.48-0.178)^2 = 4.6\times10^{-3}\ \text{m}^4$$

$$\sigma_s = n\times\frac{\mathbf{M}(d-x)}{I} = 8\times\frac{175\times10^3\times(0.48-0.178)}{4.6\times10^{-3}} = 91.9\ \text{MPa (Tension)}$$

$$\sigma_c = \frac{\mathbf{M}\cdot x}{I} = \frac{175\times10^3\times0.178}{4.6\times10^{-3}} = 6.77\ \text{MPa (Compression)}$$

Problem:

$\mathbf{M} = 1$kN.m

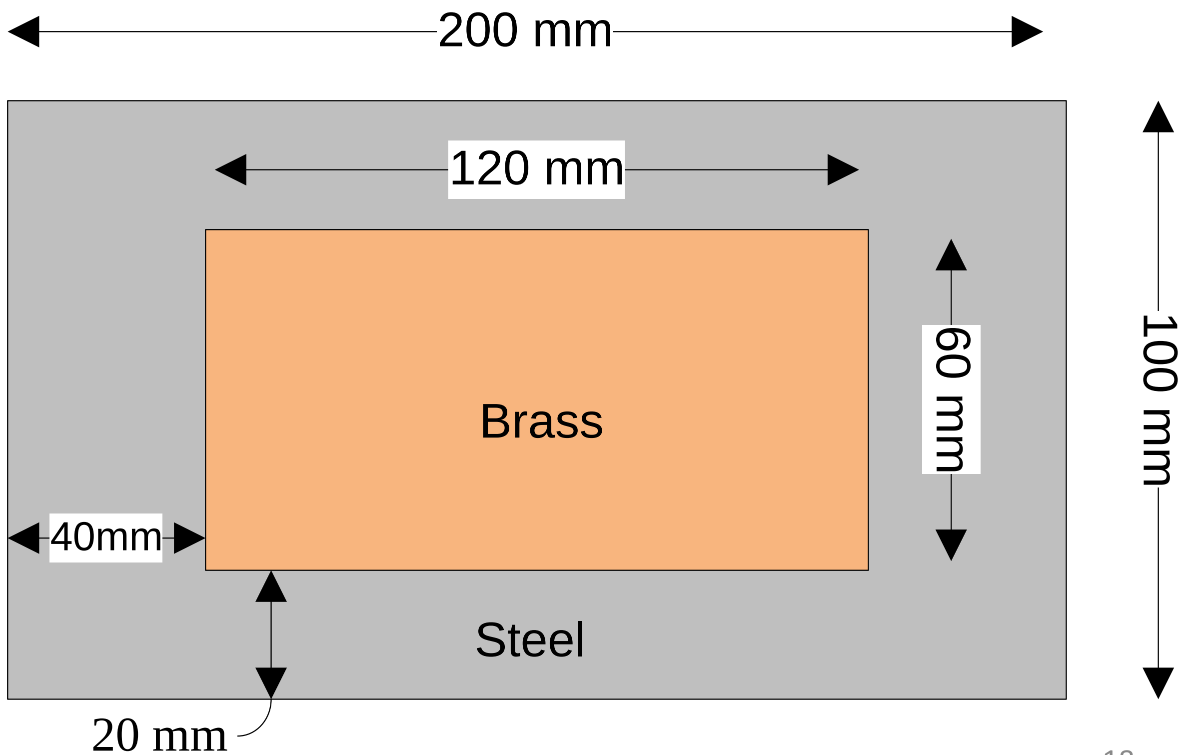
$E_b = 100$GPa

$E_s = 200$ Gpa

Find

1- The maximum stress in brass.

2- The maximum stress an steel.



Problem:

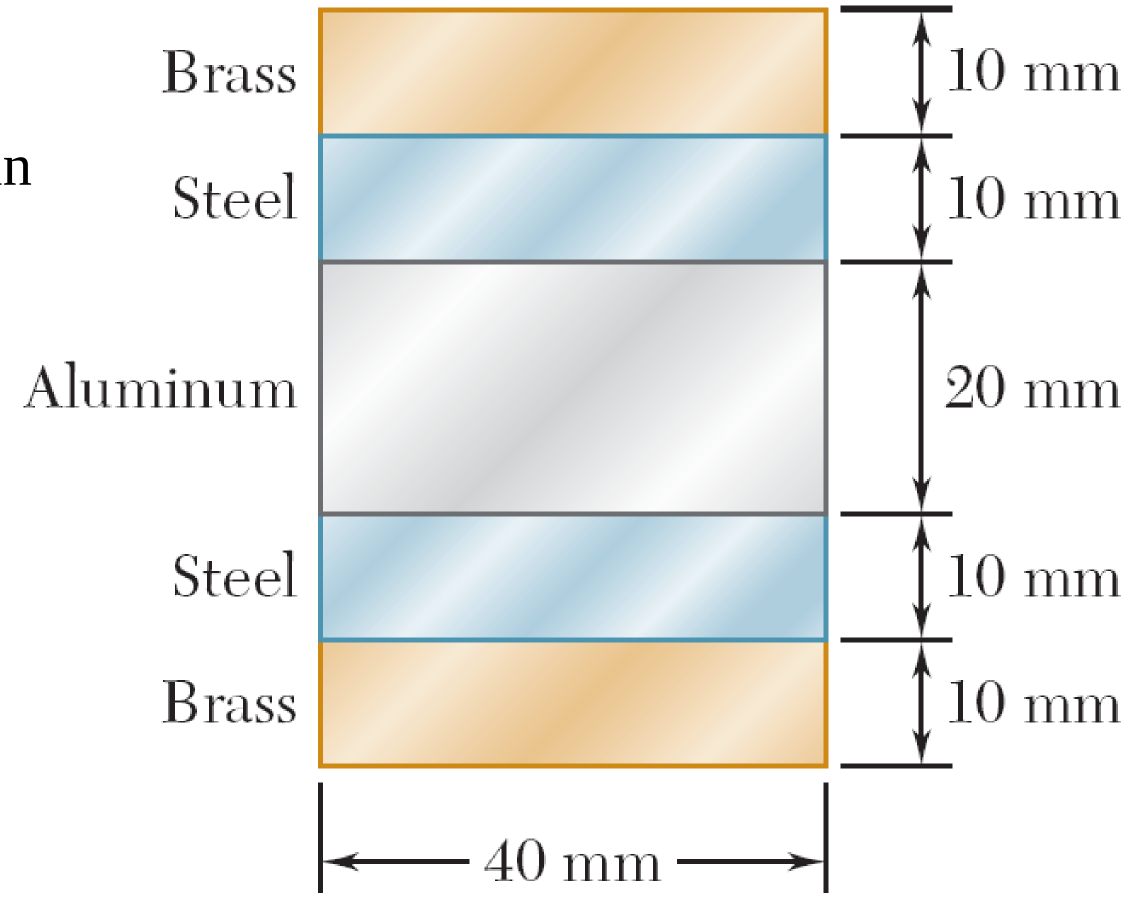
$E_s = 200$ GPa

$E_b = 100$ GPa

$E_{al} = 70$ GPa

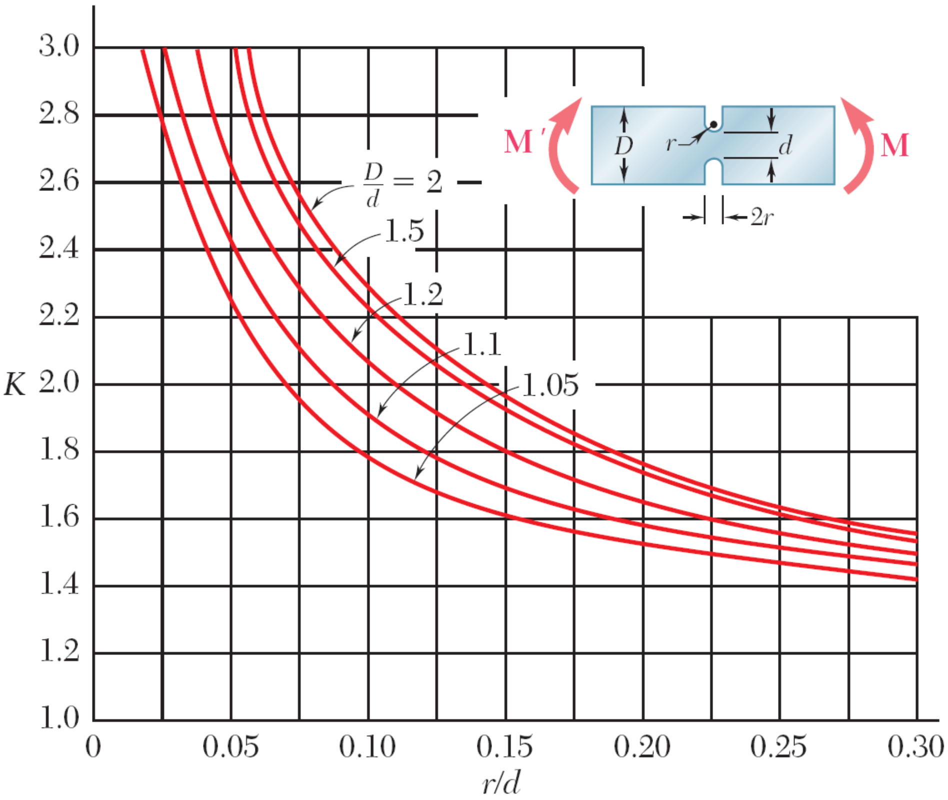
$\mathbf{M} = 2$ kN.m

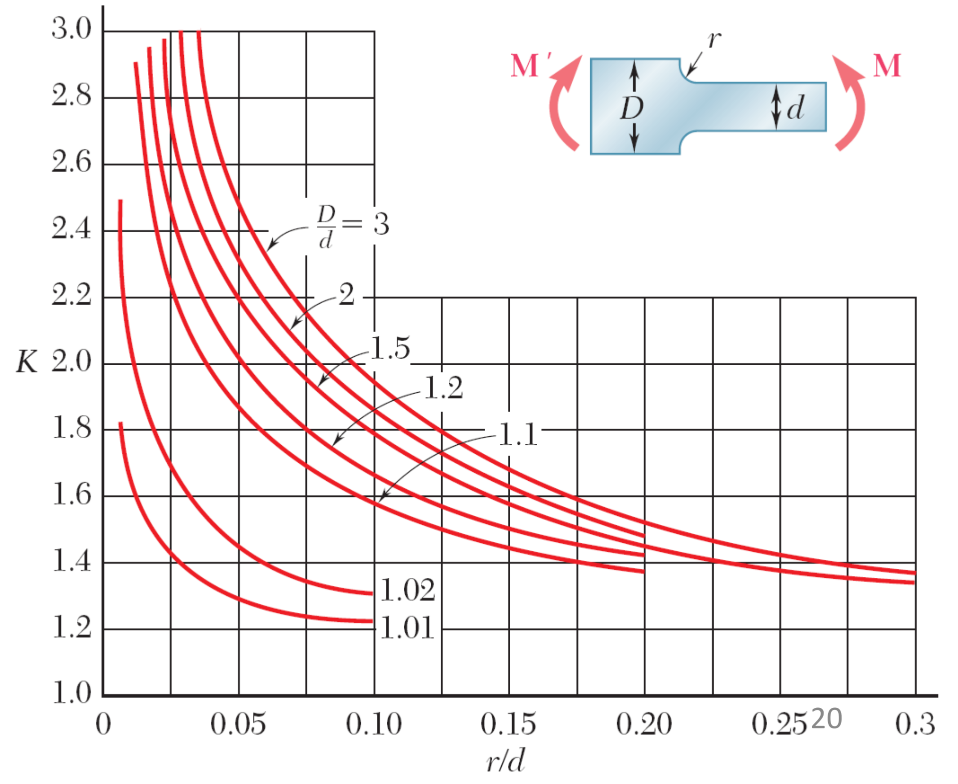
Find the maximum stresses in steel , aluminum and brass



4.7 STRESS CONCENTRATION

$$\sigma_{max} = K\frac{\mathbf{M}\cdot C}{I}$$





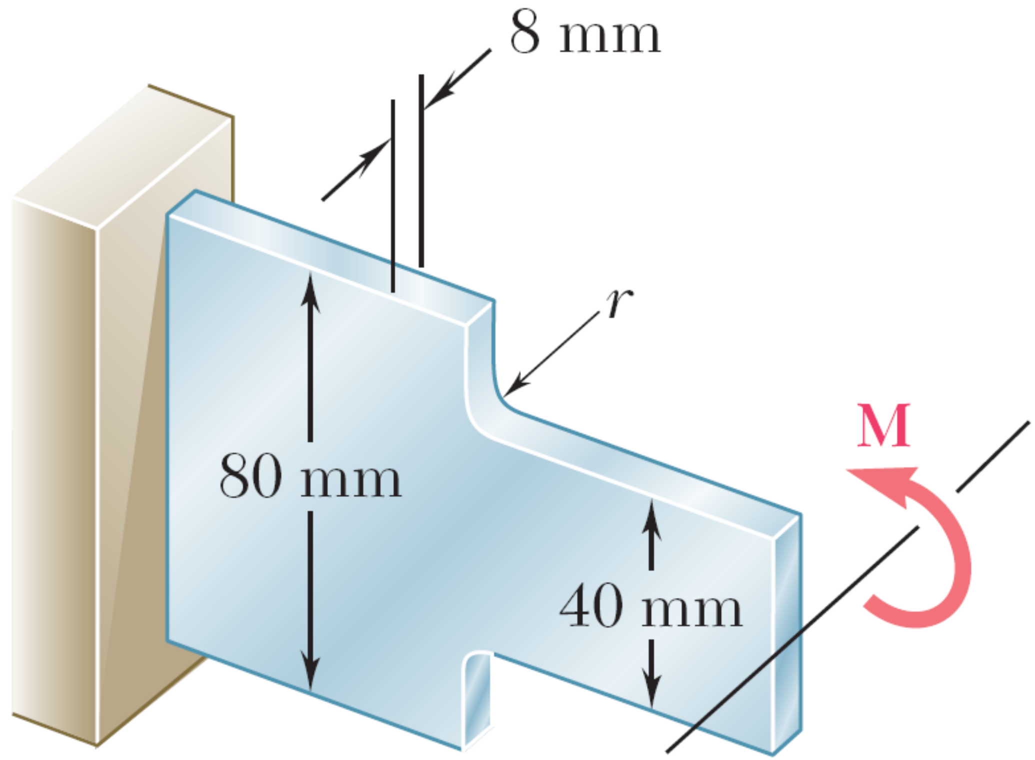
Example :

$\mathbf{M} = 250$ N.m

$r = 4$ mm

Find

$\sigma_{\max}$



Solution :

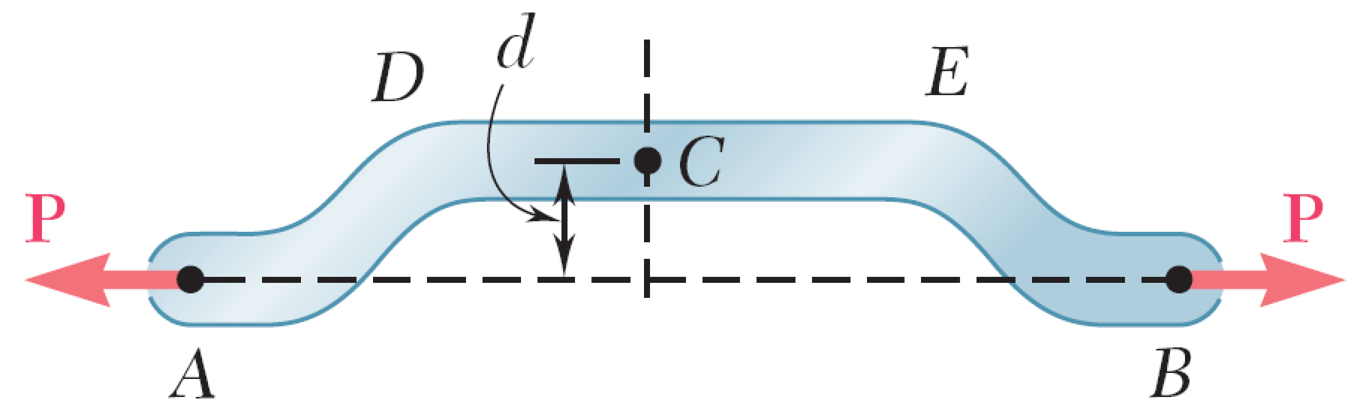
from the previous figures

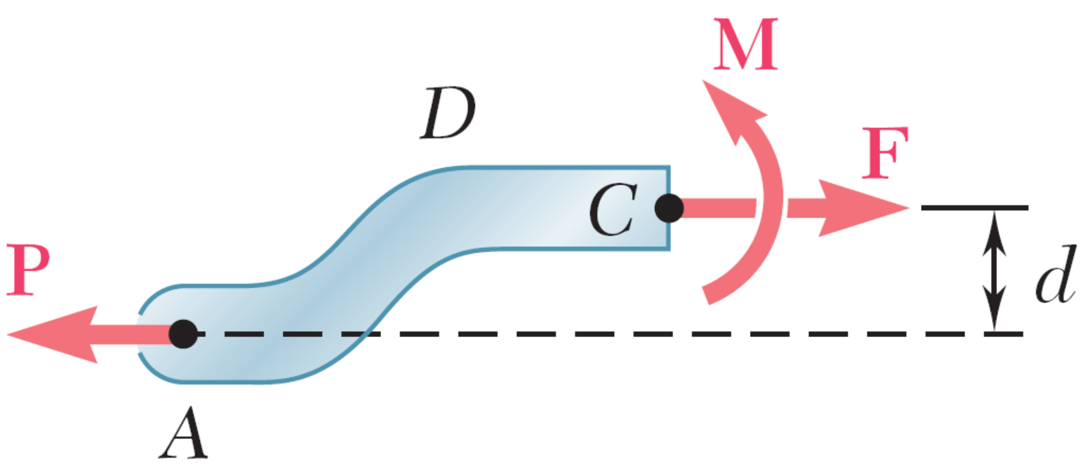
$k = 1.87$

$$I = \frac{1}{12} bh^3 = \frac{1}{12} \times (0.008) \times (0.04)^3 = 42.67 \times 10^{-9}$$

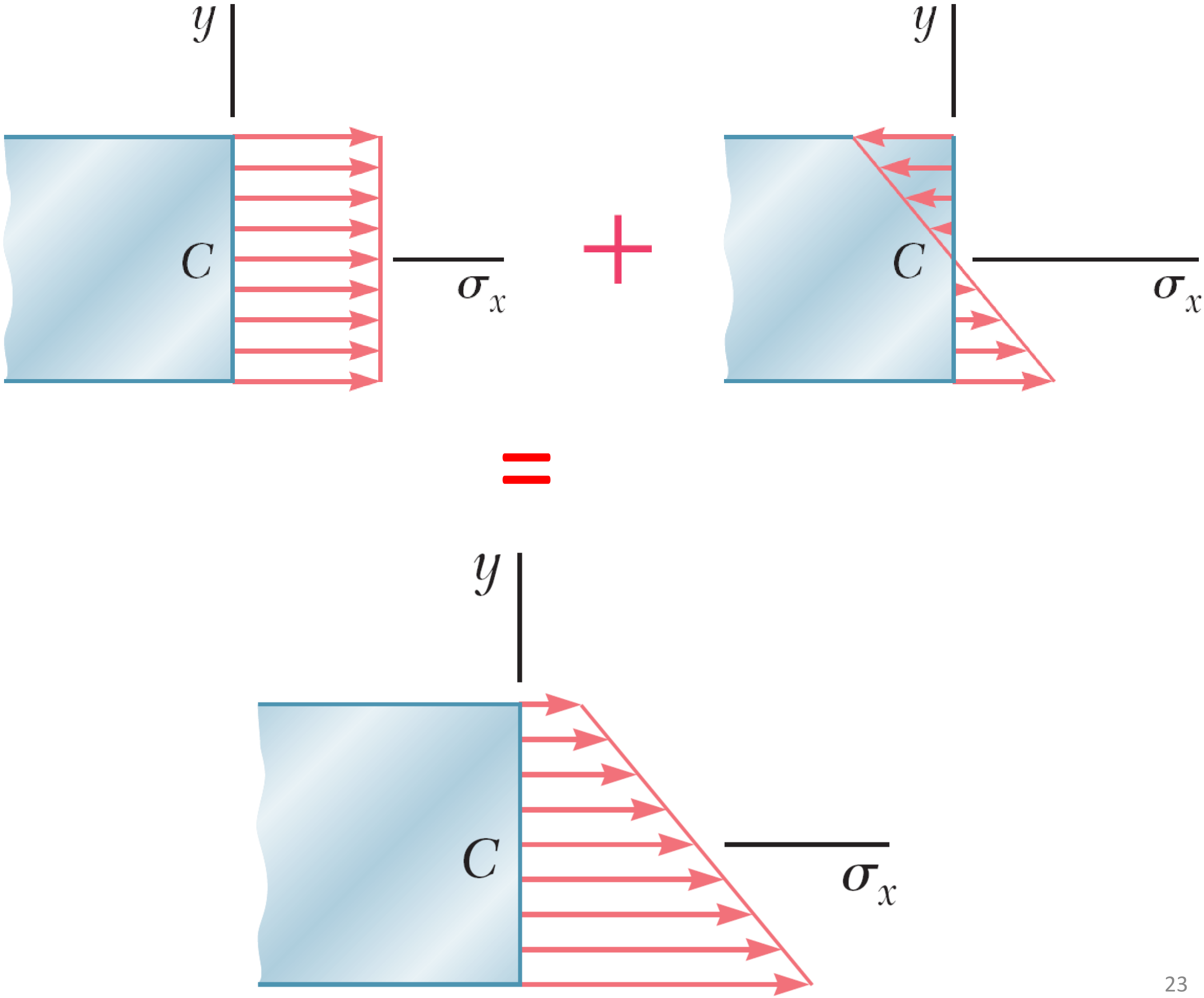
$$\sigma_{\max} = K \frac{\mathbf{M} \cdot C}{I} = 1.87 \times \frac{250 \times 0.02}{42.67 \times 10^{-9}} = 219.1 \text{ MPa}$$

4.12 ECCENTRIC AXIAL LOADING IN A PLANE OF SYMMETRY





$$\sigma = \frac{\mathbf{P}}{A} - \frac{\mathbf{M} \cdot y}{I}$$


y
C
σ_x
+
y
C
σ_x
=
y
C
σ_x

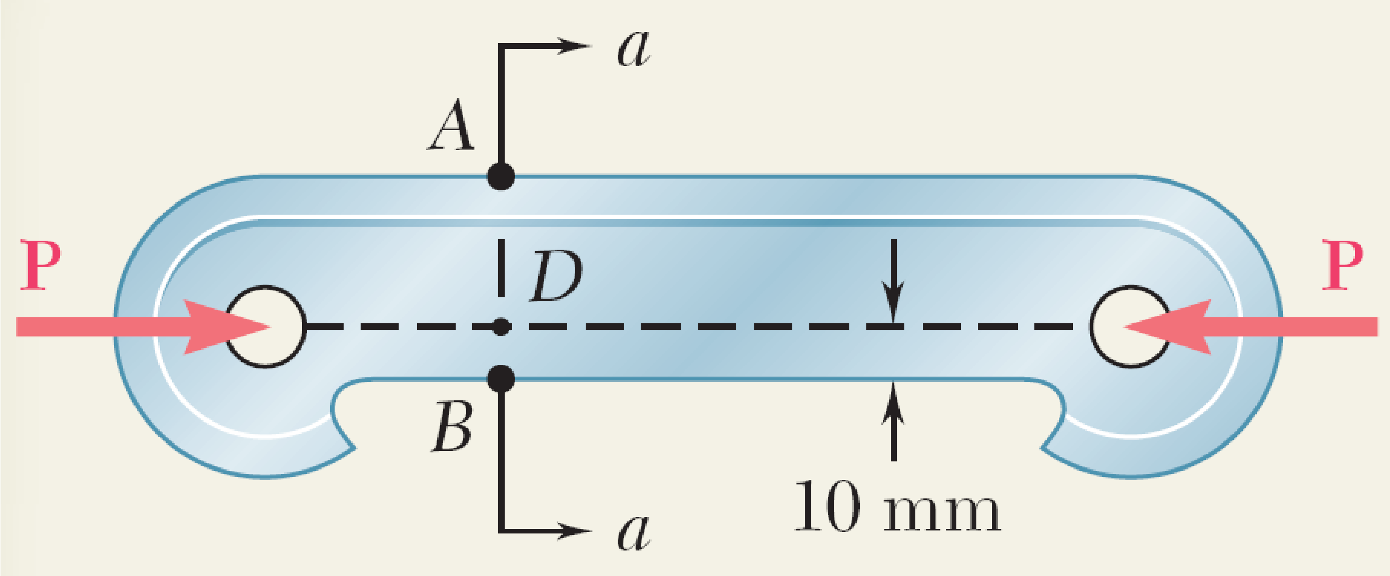
Example :

$(\sigma_{all})_t = 30$ MPa

$(\sigma_{all})_c = 120$ MPa

$E = 200$ GPa

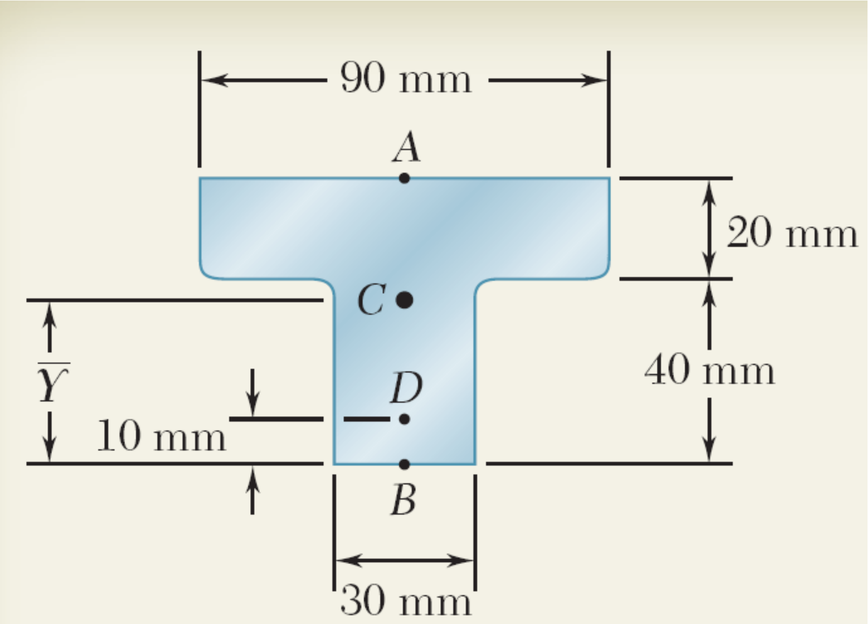
Find **P**

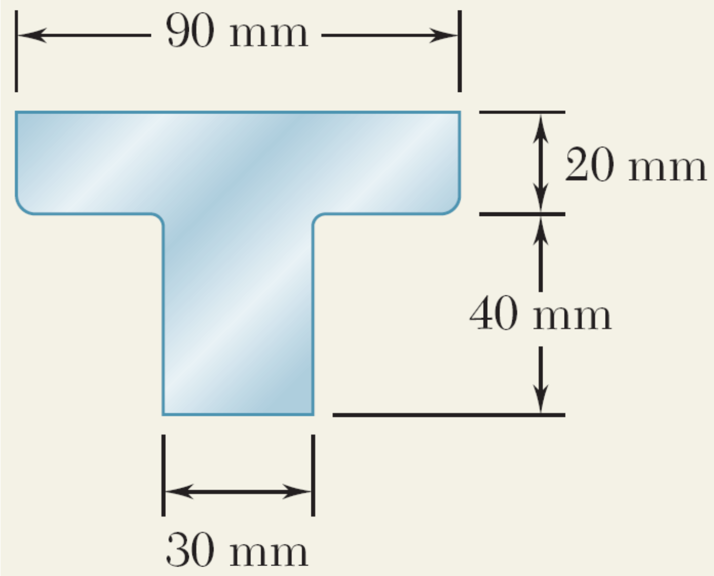


Solution :

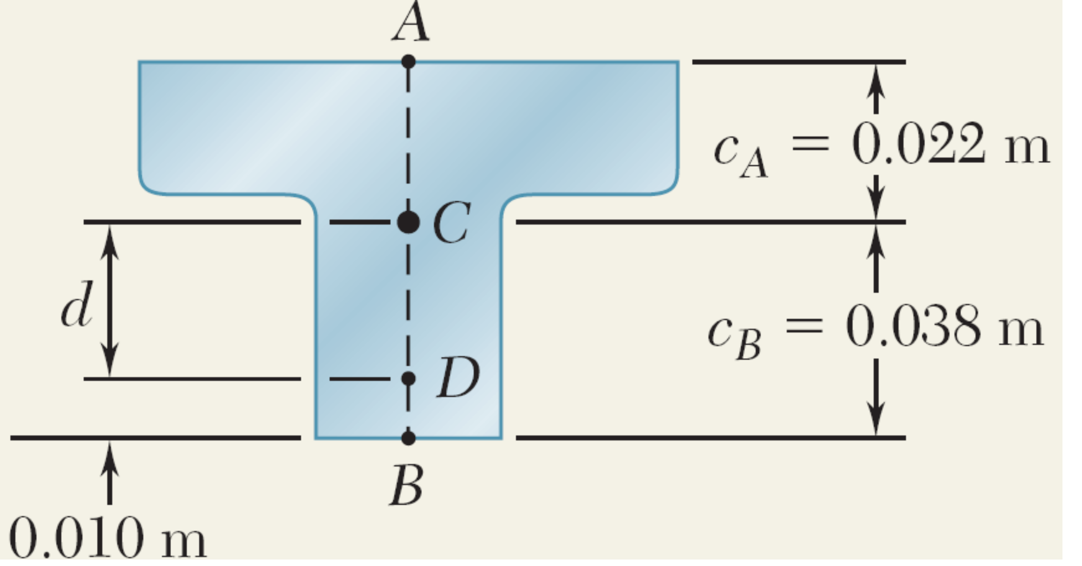
$\bar{Y} = 38$ mm

$I = 868 \times 10^{-9}\ \text{m}^4$ (Previous slides)





$$\mathbf{M} = (\bar{Y} - 0.01)\mathbf{P} = 0.028\mathbf{P}$$



$$\sigma_0 = \frac{\mathbf{P}}{A} = \frac{\mathbf{P}}{3\times10^{-3}} = 333P \quad \text{(Compression)}$$

$$\sigma_A = -\frac{\mathbf{P}}{A} + \frac{\mathbf{M}\cdot c_A}{I} = -\frac{\mathbf{P}}{3\times10^{-3}} + \frac{0.028\mathbf{P}\times0.022}{868\times10^{-9}} = 377\mathbf{P} \quad \text{(Tension)}$$

$$\sigma_B = -\frac{\mathbf{P}}{A} - \frac{\mathbf{M}\cdot c_B}{I} = -\frac{\mathbf{P}}{3\times10^{-3}} - \frac{0.028\mathbf{P}\times0.038}{868\times10^{-9}} = 1559\mathbf{P} \quad \text{(Compression)}$$

$$30\times10^6 = 377\mathbf{P} \quad \rightarrow \quad \mathbf{P} = 79.6 \text{ kN}$$

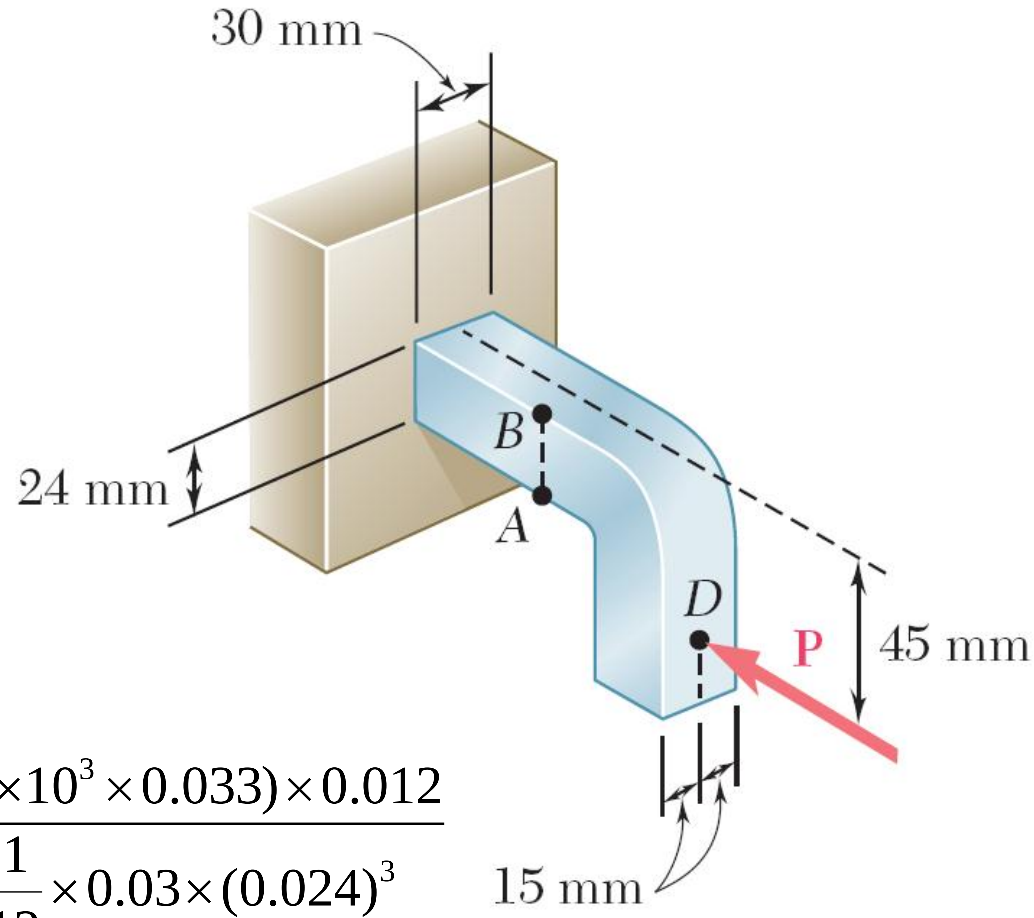
$$120\times10^6 = 1559\mathbf{P} \quad \rightarrow \quad \mathbf{P} = 77 \text{ kN}$$

Example :

$\mathbf{P} = 8\text{ kN}$

Find

$\sigma_A,\ \sigma_B$



Solution :

$$\sigma_A = -\frac{\mathbf{P}}{A} - \frac{\mathbf{M}\cdot c}{I} = -\frac{8\times 10^3}{0.03\times 0.024} - \frac{(8\times 10^3\times 0.033)\times 0.012}{\frac{1}{12}\times 0.03\times (0.024)^3}$$

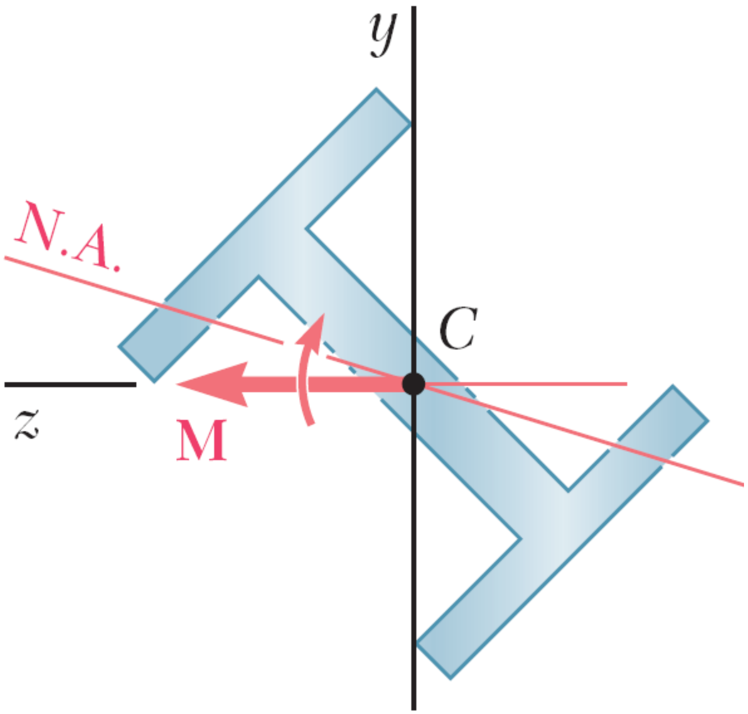
$$= -11.11\text{ MPa} - 91.67\text{ MPa} = 102.78\text{ MPa (Compression)}$$

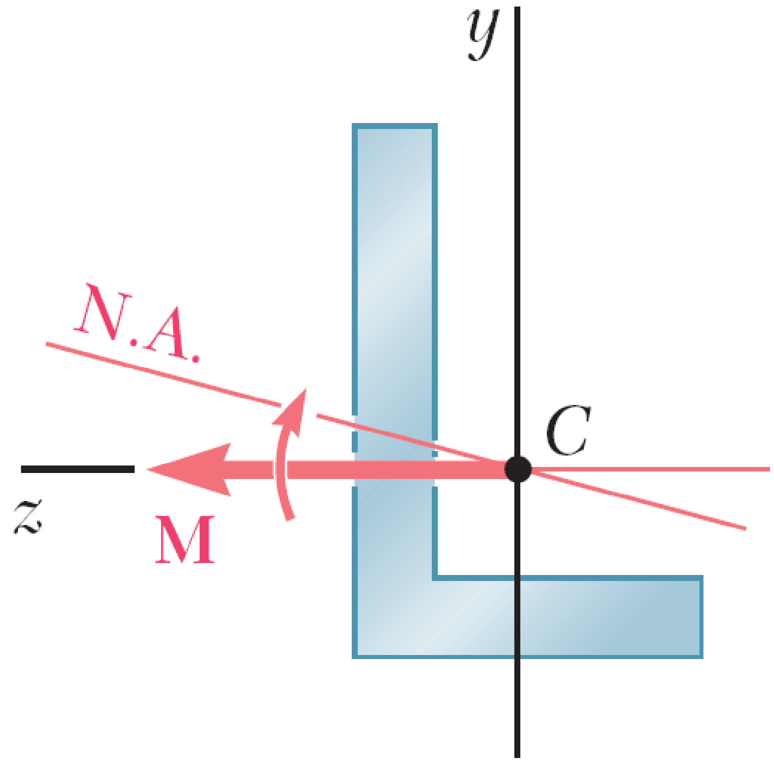
$$\sigma_B = -\frac{\mathbf{P}}{A} + \frac{\mathbf{M}\cdot c}{I} = -\frac{8\times 10^3}{0.03\times 0.024} + \frac{(8\times 10^3\times 0.033)\times 0.012}{\frac{1}{12}\times 0.03\times (0.024)^3}$$

$$= -11.11\text{ MPa} + 91.67\text{ MPa} = 80.56\text{ MPa (Tension)}$$

4.13 UNSYMMETRIC BENDING

Two types of problems are considered





1- The bending is not on the symmetric plane.

2- No plane of symmetry is existed. (not required in the course)

$$\int \sigma_x \, dA = 0$$

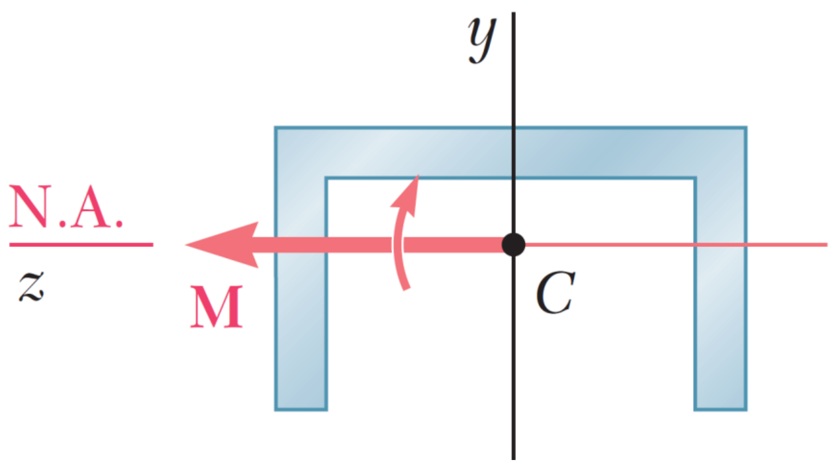
$$\int z \, \sigma_x \, dA = 0$$

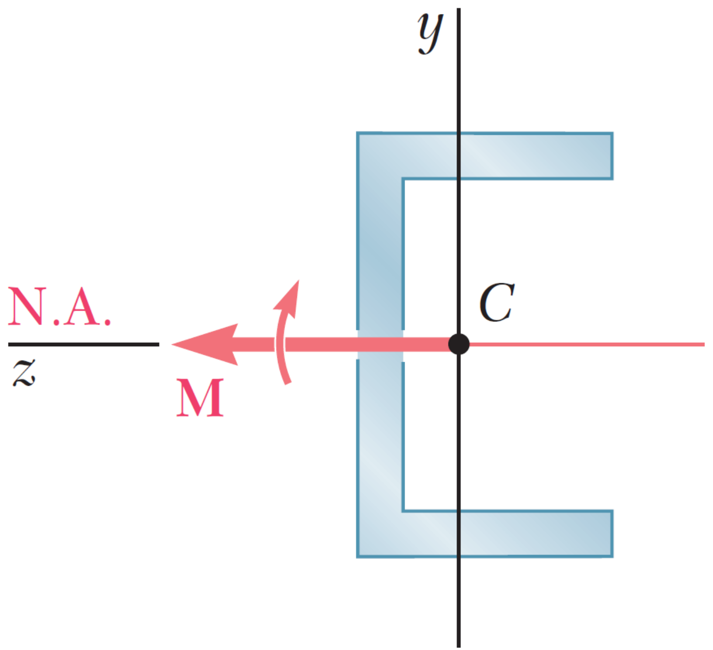
$$\int -y \, \sigma_x \, dA = M$$

from the second equation

$$\int z \left(\frac{-y}{c} \right) \sigma_{\max} \, dA = 0 \quad \text{or} \quad \int yz \, dA = 0$$

*** The N.A will coincide with **M** if and only if **M** is directed along one of the principal centroidal axis





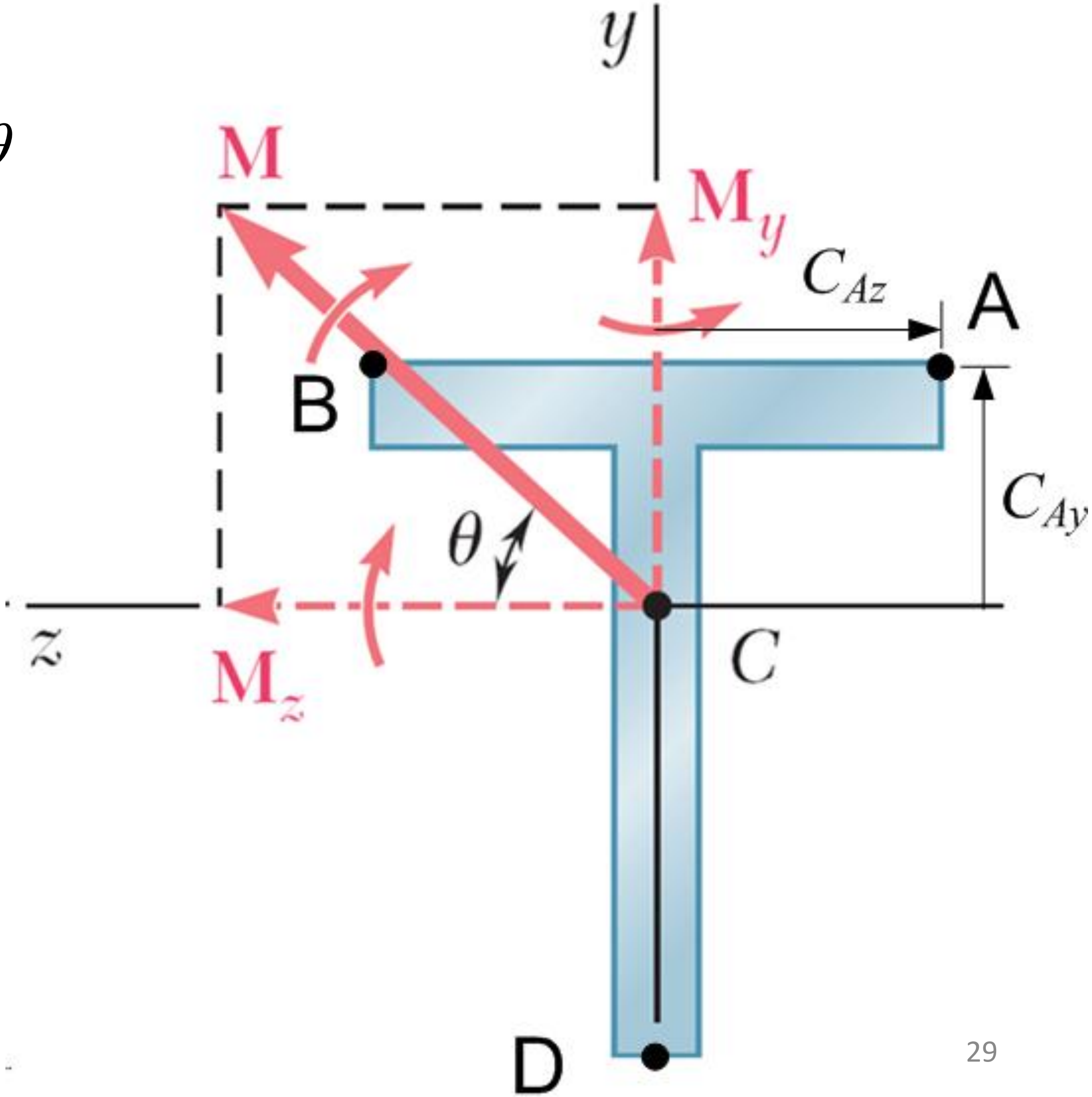
****M is not coincident with the principal centroidal axis**

$$\mathbf{M_z} = \mathbf{M}\cos\theta, \quad \mathbf{M_y} = \mathbf{M}\sin\theta$$

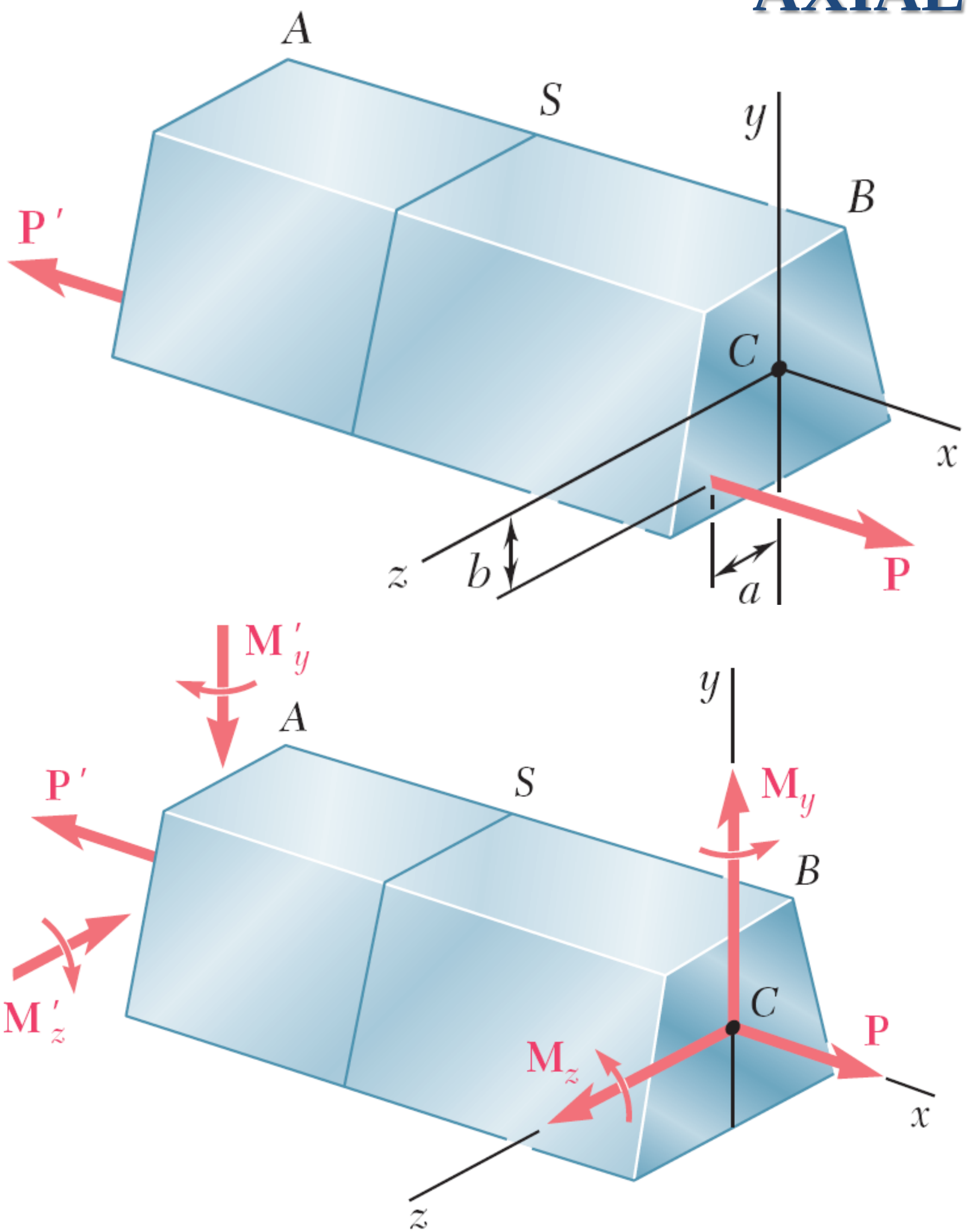
$$\sigma_A = -\frac{\mathbf{M_y} \cdot c_{Az}}{I_y} - \frac{\mathbf{M_z} \cdot c_{Ay}}{I_z}$$

$$\sigma_B = +\frac{\mathbf{M_y} \cdot c_{Bz}}{I_y} - \frac{\mathbf{M_z} \cdot c_{By}}{I_z}$$

$$\sigma_D = 0 + \frac{\mathbf{M_z} \cdot c_{Dy}}{I_z}$$



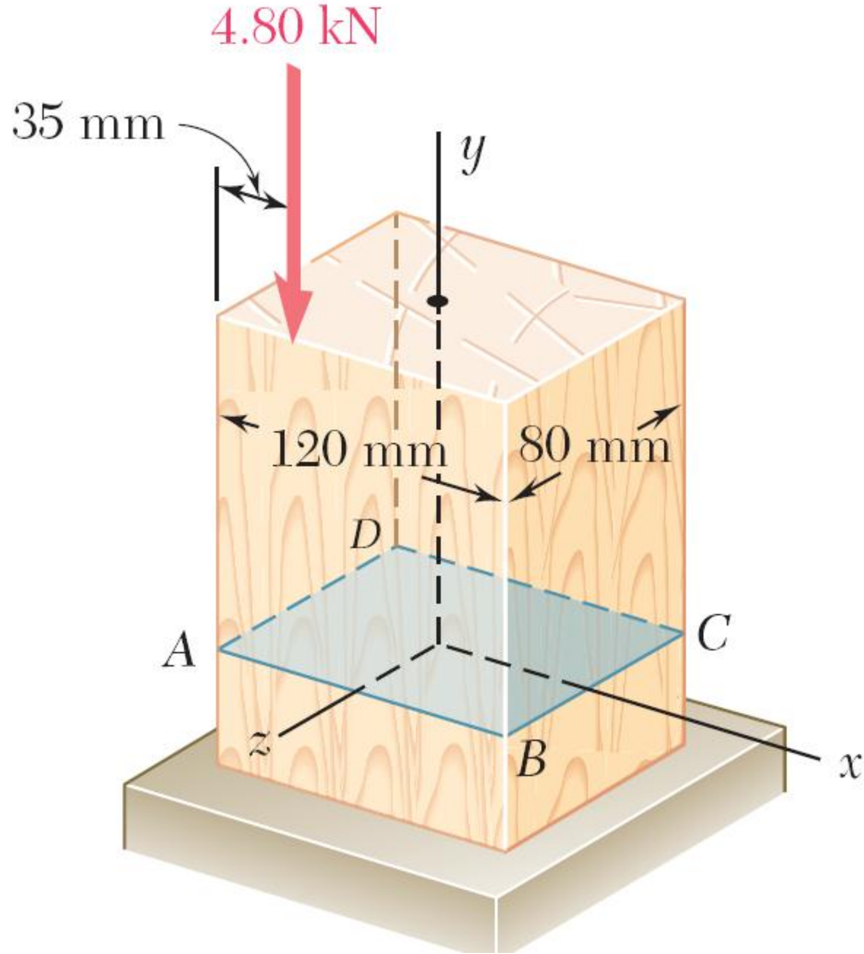
4.14 GENERAL CASE OF ECCENTRIC AXIAL LOADING



$$\sigma_x = \frac{\mathbf{P}}{A} + \frac{\mathbf{M_y} \cdot z}{I_y} - \frac{\mathbf{M_z} \cdot y}{I_z}$$

Example :

Find σ_A, σ_B, σ_C and σ_D,



Solution :

$$\mathbf{M_x} = 4.8\times10^3\times0.04 = 192 \text{ N.m}$$

$$\mathbf{M_z} = 4.8\times10^3\times0.025 = 120 \text{ N.m}$$

$$I_x = \frac{1}{12}\times0.12\times(0.08)^3 = 5.12\times10^{-6} \text{ m}^4$$

$$I_z = \frac{1}{12}\times0.08\times(0.12)^3 = 11.52\times10^{-6} \text{ m}^4$$

$$\sigma_0 = -\frac{\mathbf{P}}{A} = -0.5 \text{ MPa}$$

$$\sigma_1 = \frac{\mathbf{M_x}\cdot z}{I_x} = \frac{192\times0.04}{5.12\times10^{-6}} = 1.5 \text{ MPa}$$

$$\sigma_2 = \frac{\mathbf{M_z}\cdot x}{I_z} = \frac{120\times0.06}{11.52\times10^{-6}} = 0.625 \text{ MPa}$$

$$\sigma_A = -0.5 \text{ MPa} - 1.5 \text{ MPa} - 0.625 \text{ MPa} = -2.625 \text{ MPa}$$

$$\sigma_B = -0.5 \text{ MPa} - 1.5 \text{ MPa} + 0.625 \text{ MPa} = -1.375 \text{ MPa}$$

$$\sigma_C = -0.5 \text{ MPa} + 1.5 \text{ MPa} + 0.625 \text{ MPa} = 1.625 \text{ MPa}$$

$$\sigma_D = -0.5 \text{ MPa} + 1.5 \text{ MPa} - 0.625 \text{ MPa} = 0.375 \text{ MPa}$$

Example :

$t = 12$ mm

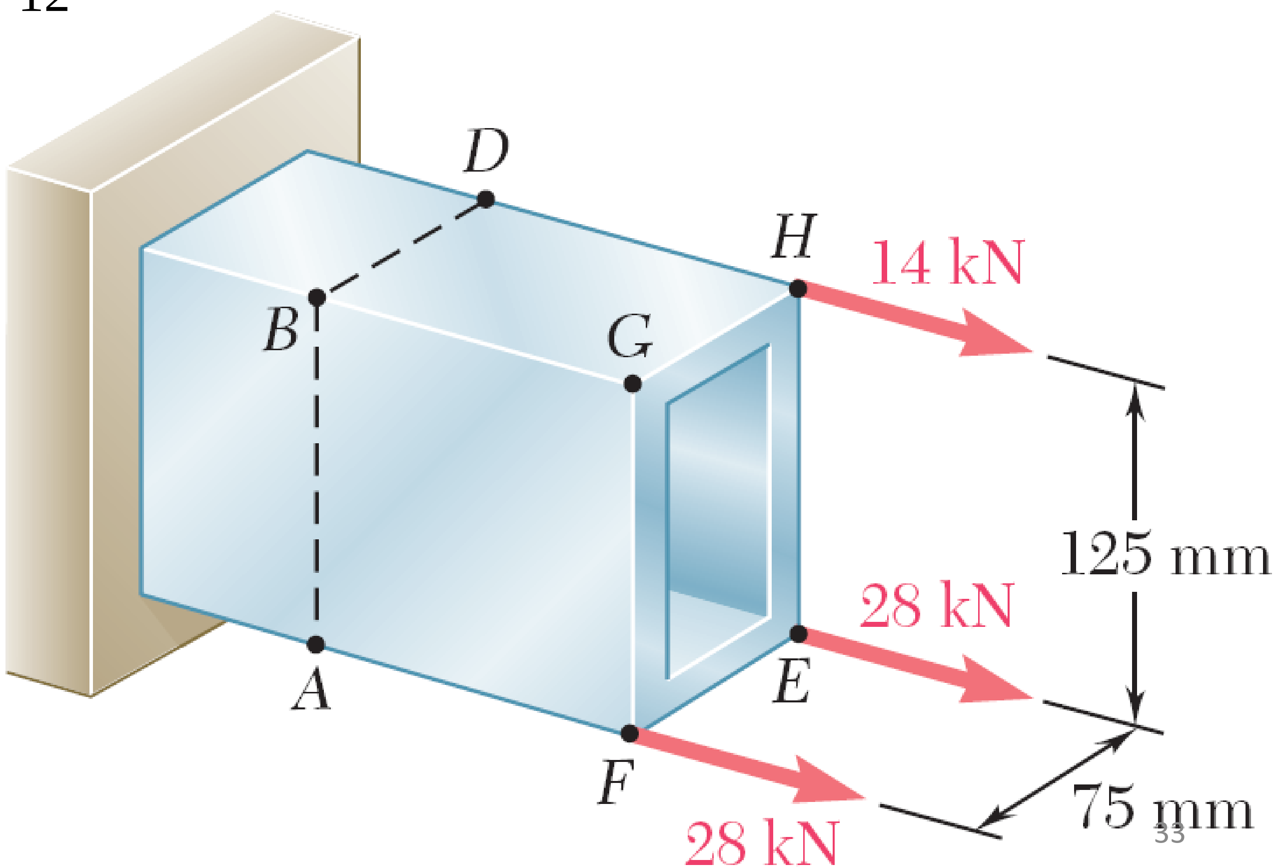
$A = 4.224 \times 10^{-3}$ m^2

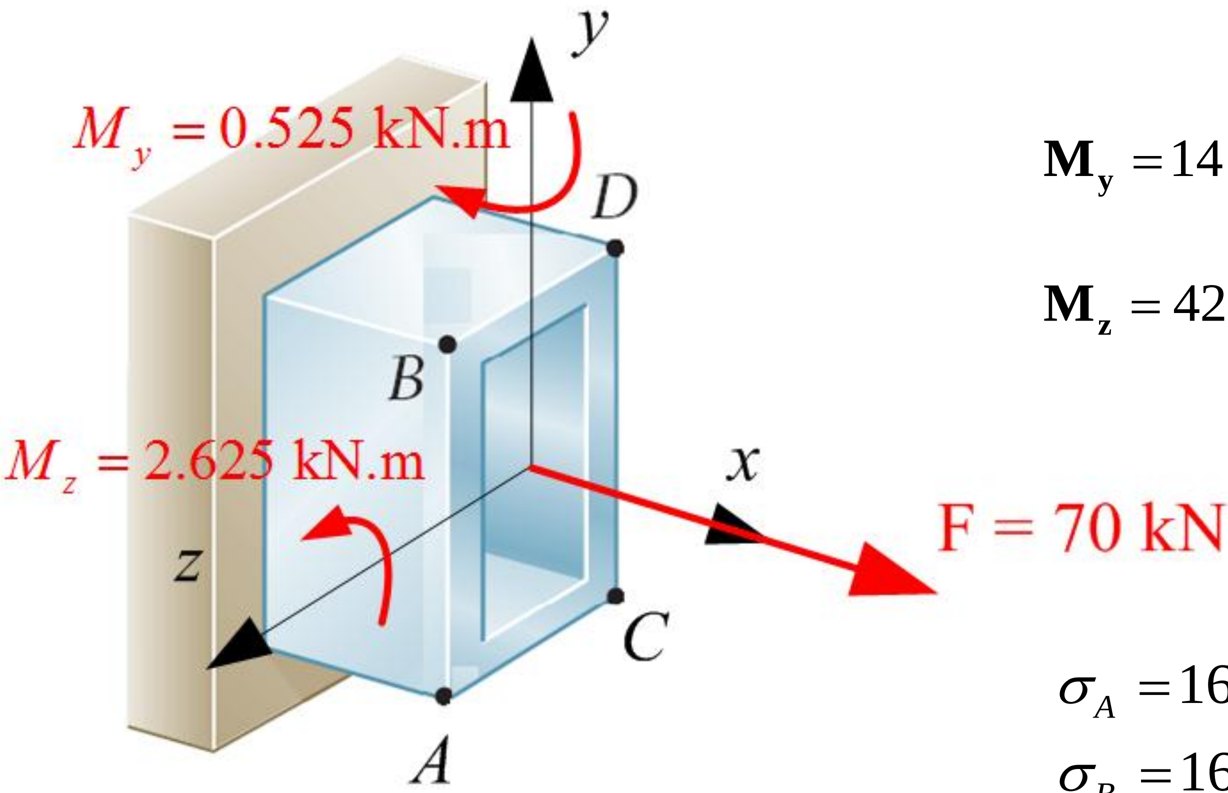
$$I_y = \frac{1}{12} \times 0.125 \times (0.075)^3 - \frac{1}{12} \times 0.101 \times (0.051)^3 = 3.278 \times 10^{-6}$$

$$I_z = \frac{1}{12} \times 0.075 \times (0.125)^3 - \frac{1}{12} \times 0.051 \times (0.101)^3 = 7.828 \times 10^{-6}$$

find

$\sigma_A, \sigma_B, \sigma_C$ and σ_D





$$\mathbf{M_y} = 14\times10^3 \times \frac{0.075}{2} = 0.525 \text{ kN.m}$$

$$\mathbf{M_z} = 42\times10^3 \times \frac{0.125}{2} = 2.625 \text{ kN.m}$$

$$\sigma_A = 16.57 - 6 + 20.96 = 31.53 \text{ MPa}$$

$$\sigma_B = 16.57 - 6 - 20.96 = -10.39 \text{ MPa}$$

$$\sigma_C = 16.57 + 6 + 20.96 = 43.53 \text{ MPa}$$

$$\sigma_D = 16.57 + 6 - 20.96 = 1.61 \text{ MPa}$$

$$\sigma_0 = \frac{\mathbf{F}}{A} = \frac{70\times10^3}{4.224\times10^{-3}} = 16.57 \text{ MPa}$$

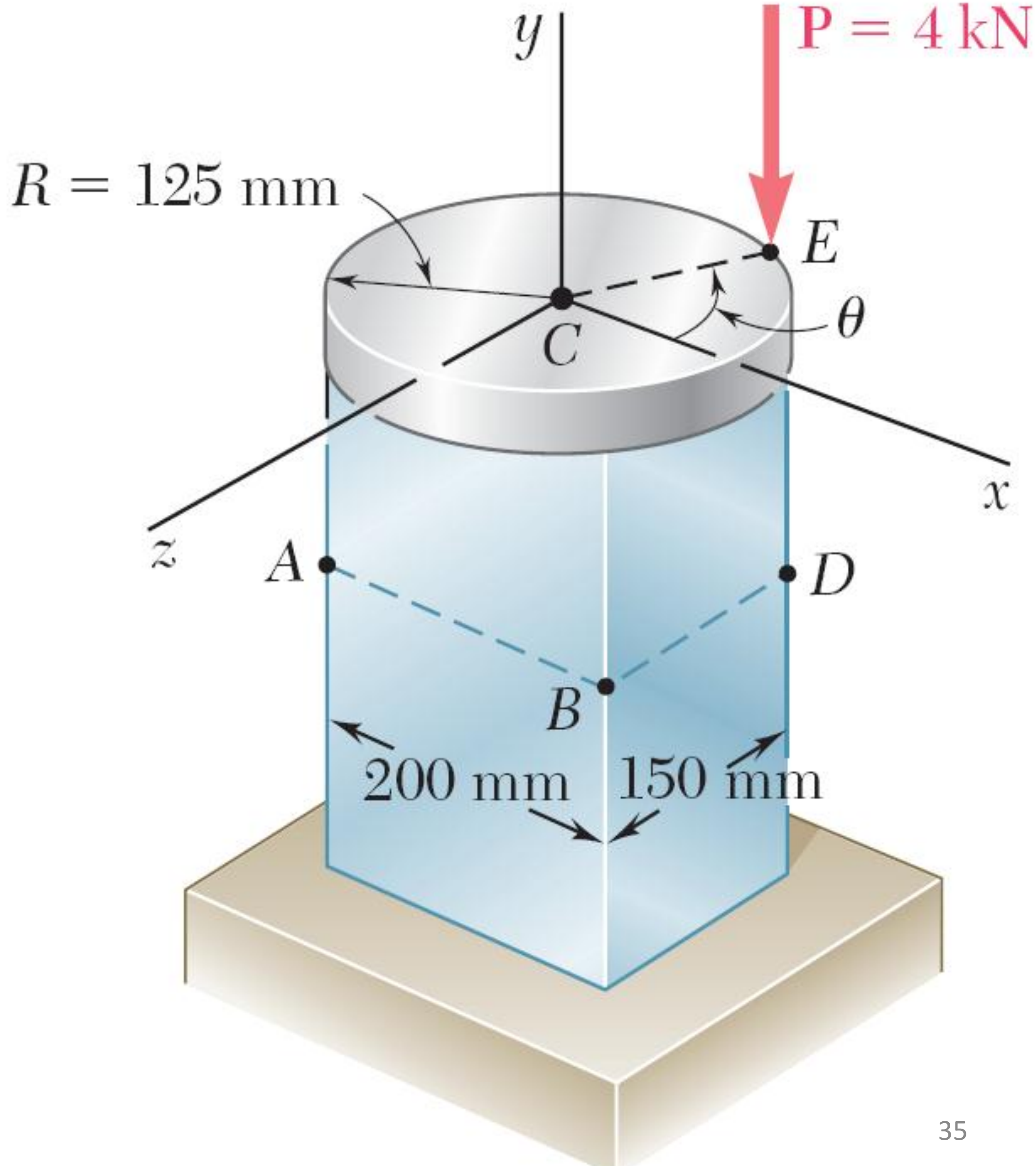
$$\sigma_1 = \frac{\mathbf{M_y}\cdot z_{\max}}{I_y} = \frac{0.525\times10^3\times0.0375}{3.278\times10^{-6}} = 6 \text{ MPa}$$

$$\sigma_2 = \frac{\mathbf{M_z}\cdot y_{\max}}{I_z} = \frac{2.625\times10^3\times0.0625}{7.828\times10^{-6}} = 20.96 \text{ MPa}$$

Problem:

θ = 30 deg.

Find normal stresses at points *A*, *B* and *D*



END OF CHAPTER FOUR

MECHANICS OF MATERIALS

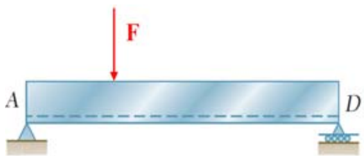
CHAPTER FIVE

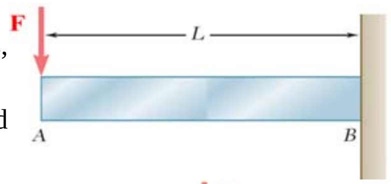
ANALYSIS AND DESIGN OF BEAMS FOR BENDING

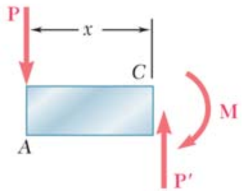
Prepared by : Dr. Mahmoud Rababah

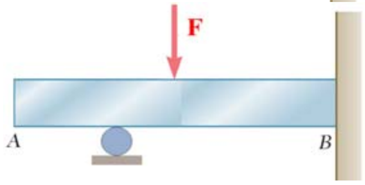
5.1 INTRODUCTION

- Members supporting perpendicular loadings (transverse) are called beams
- Beams are classified on loading basis:
  - simply supported beams
  - cantilever beams
  - overhanging beams, and etc.
- Beams are in buildings, aircraft wings, bridges, etc.
- Beams developed internal shear and moment









SIGN CONVENTION

Distributed load

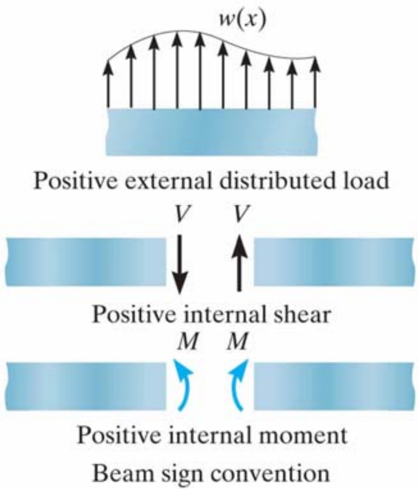
Upward is positive

Shear

If the internal shear rotates the segment cw, the shear is then positive.

Moment

If the internal moment causes compression on the top surface (holding the water), the moment is then positive

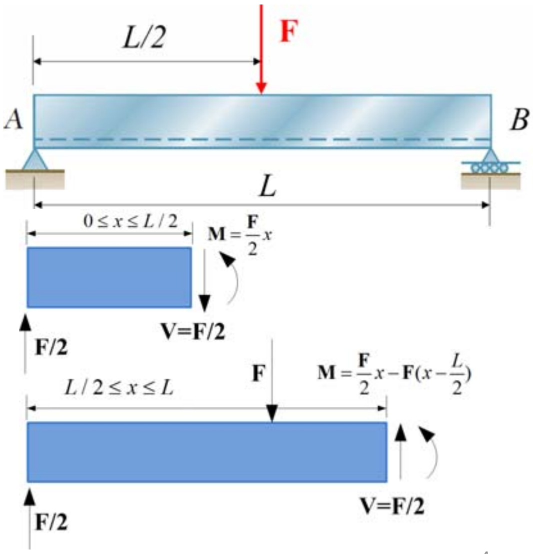


3

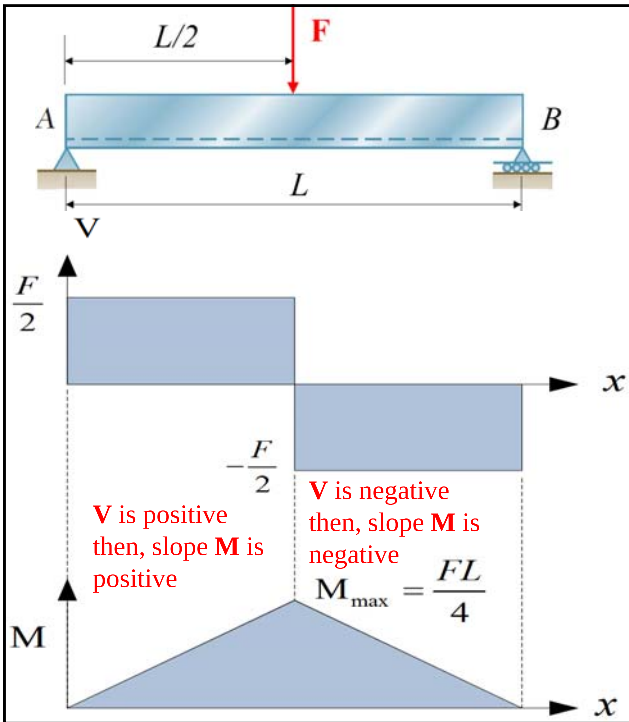
5.2 SHEAR AND BENDING MOMENT DIAGRAMS

Example:

Draw the shear and the moment diagram.



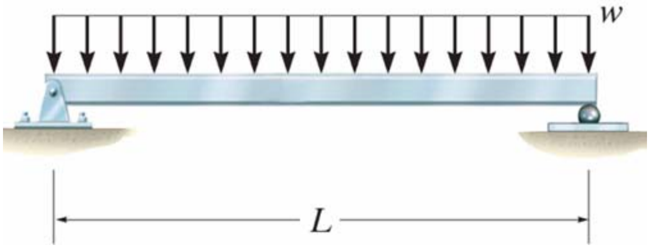
4

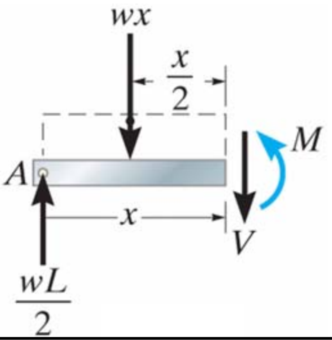


Slope **V** is constant then, **M** is linear

5

Example: Draw the shear and moment diagram.

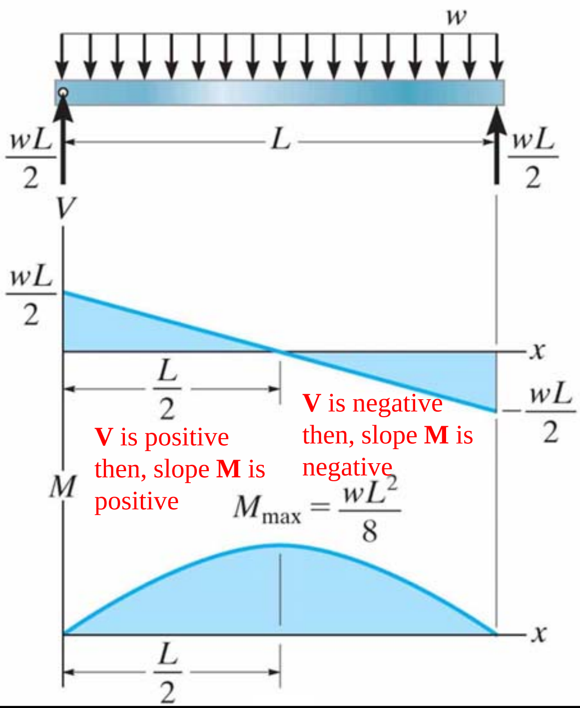




$$\mathbf{V} = w\left(\frac{L}{2} - x\right)$$

$$\mathbf{M} = \frac{wl}{2}x - wx \cdot \frac{x}{2} = \frac{w}{2}\left(Lx - x^2\right)$$

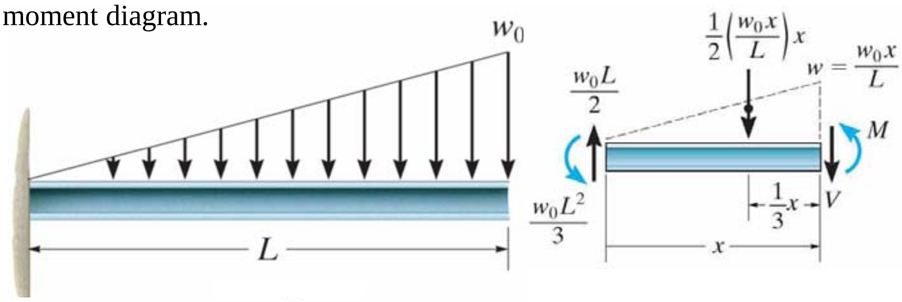
6

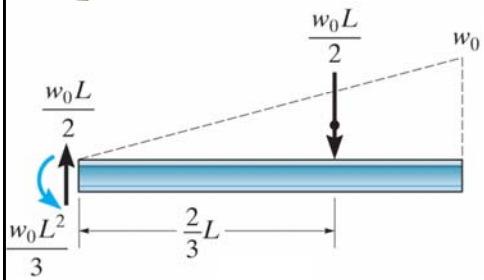


Slope V is negative; then, **M** is concave down

7

Example: Draw the shear and the moment diagram.

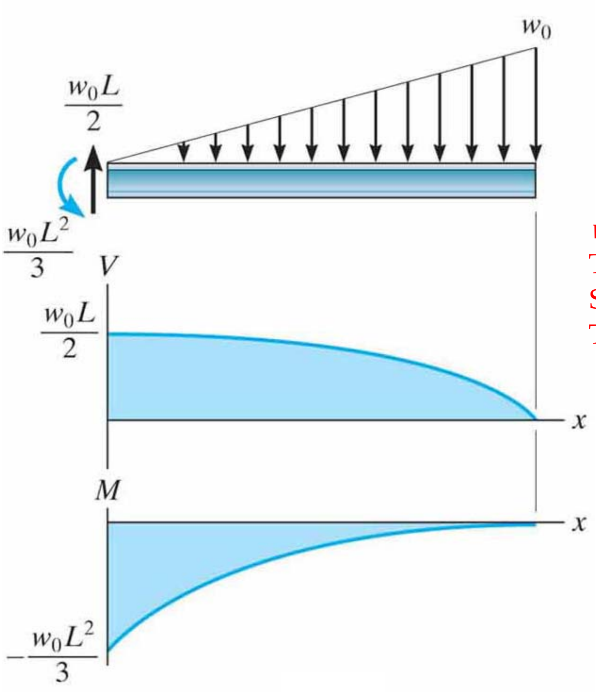




$$v = \frac{w_0 L}{2} - \frac{1}{2}\left(\frac{w_0 x}{L}\right)x$$

$$M = -\frac{w_0 L^2}{3} + \frac{w_0 L}{2}x - \frac{1}{2}\left(\frac{w_0 x}{L}\right)x\left(\frac{1}{3}x\right)$$

8



w is negative
Then, slope **V** is negative.
Slope w is negative
Then, **V** is concave down

V is positive
Then, slope **M** is positive.
Slope **V** is negative
Then, **M** is concave down

9

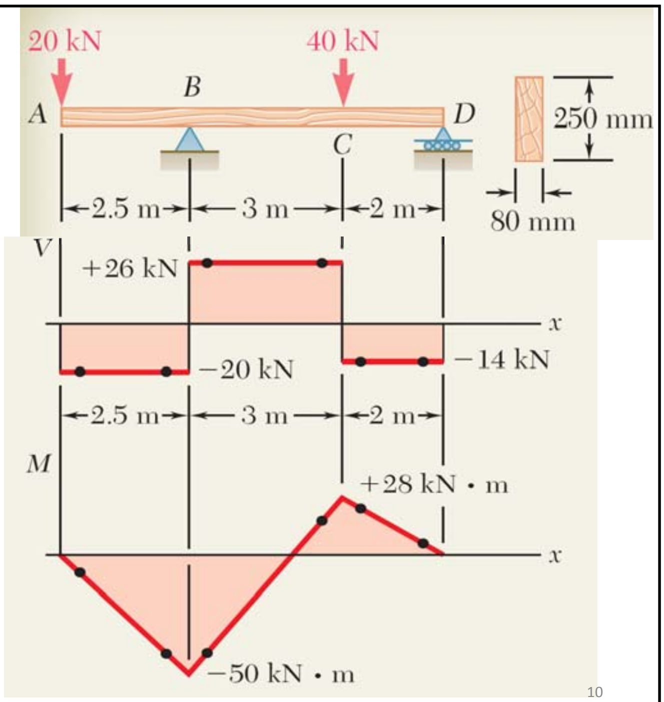
Example :

$\mathbf{R_B} = 46$ kN

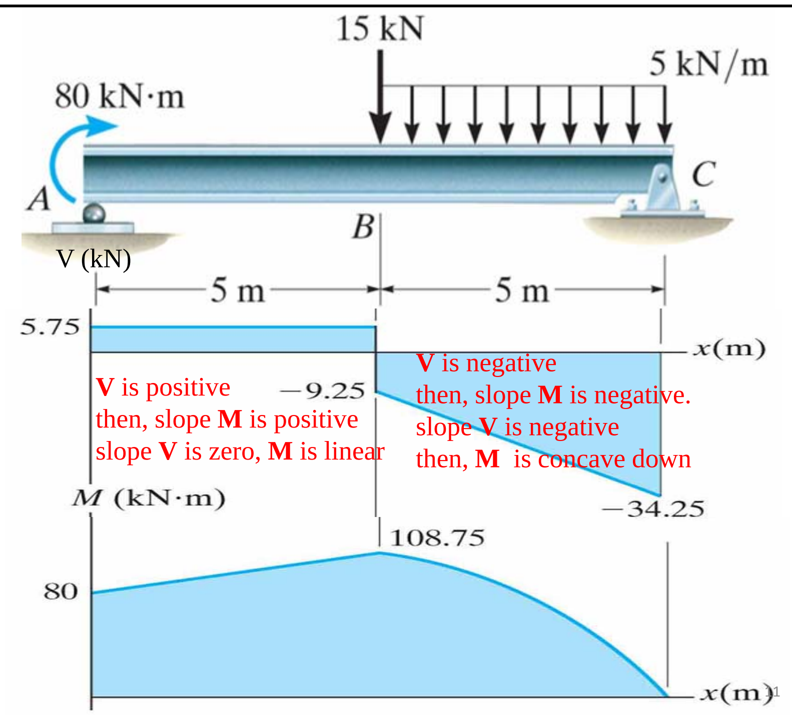
$\mathbf{R_D} = 14$ kN

$\mathbf{V_{max}} = 26$ kN

$\mathbf{M_{max}} = 50$ kN

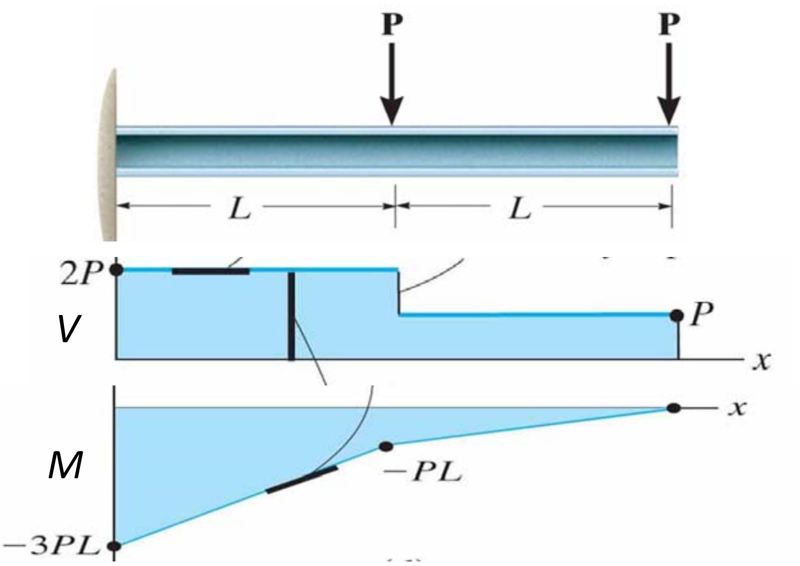


10



11

Example: Draw the shear and the moment diagram.



12

5.3 RELATIONS AMONG LOAD, SHEAR AND BENDING MOMENT

$$\frac{d\mathbf{V}}{dx} = w(x)$$

$$\frac{d\mathbf{M}}{dx} = \mathbf{V}$$

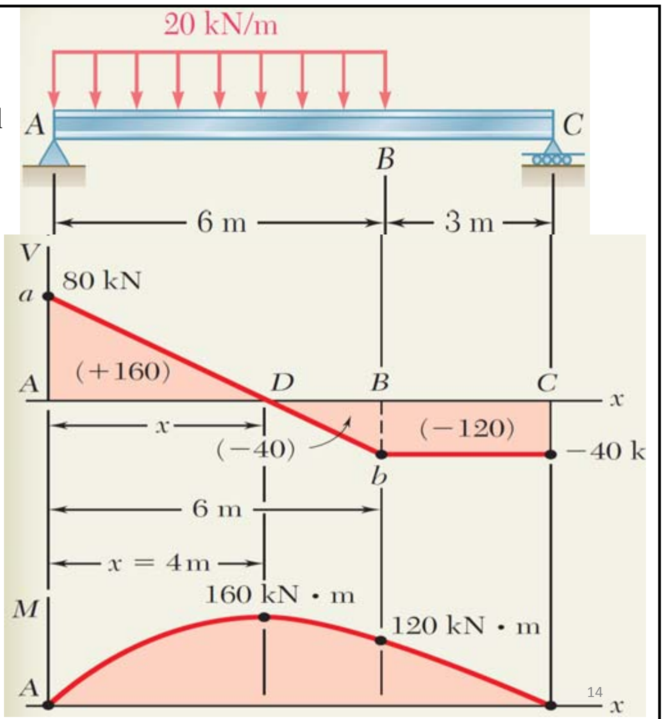
$$\Delta\mathbf{V} = \int w(x)\, dx$$

$$\Delta\mathbf{M} = \int \mathbf{V}(x)\, dx$$

13

Example:

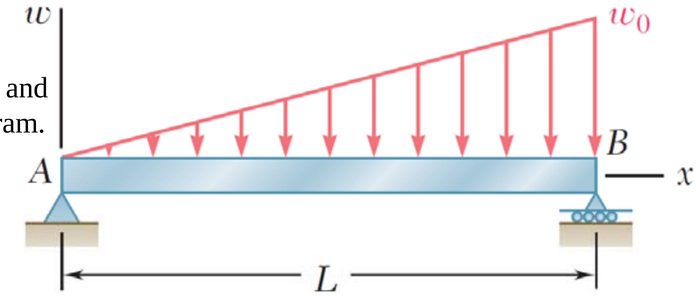
Draw the shear and the moment diagram.



14

Example:

Draw the shear and the moment diagram.



$$w = -\frac{x}{L} w_0$$

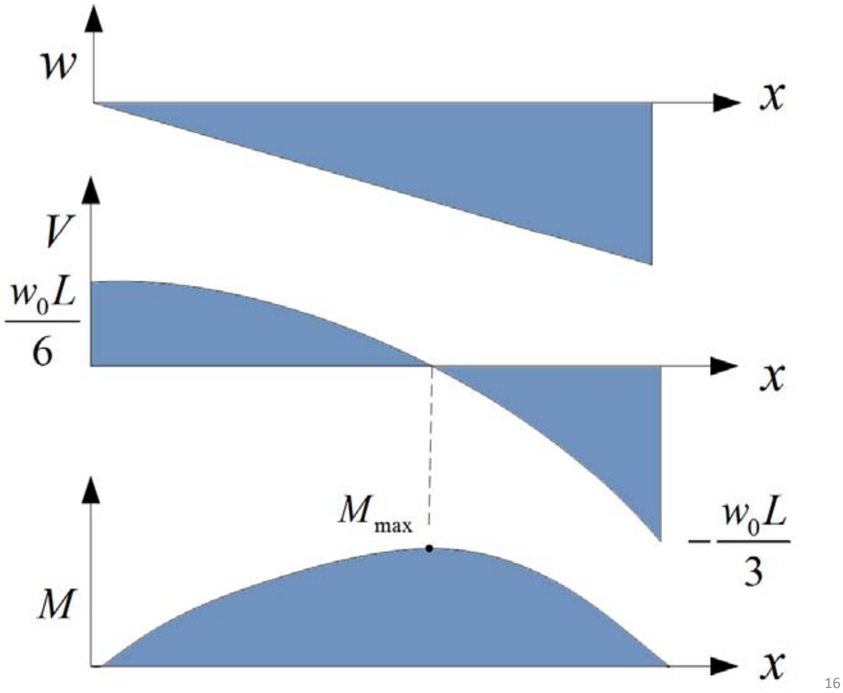
$$\mathbf{V} = \int w(x)\,dx = -\frac{x^2}{2L} w_0 + c_1$$

$$\mathbf{V}\Big|_{x=0} = \mathbf{R_A} \quad \rightarrow \quad c_1 = \frac{w_0 L}{6}$$

$$\mathbf{M} = \int \mathbf{V}(x)\,dx = -\frac{x^3}{6L} w_0 + \frac{w_0 L}{6} x + c_2$$

$$\mathbf{M}\Big|_{x=0} = 0 \quad \rightarrow \quad c_2 = 0$$

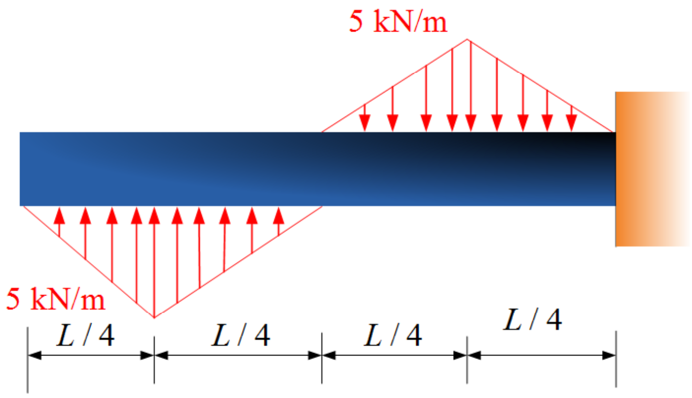
15



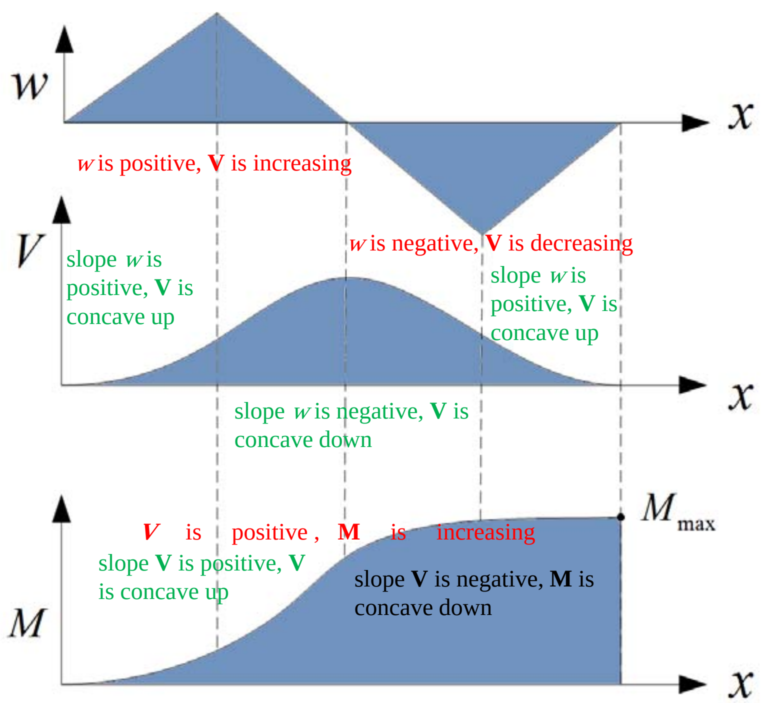
16

Example:

Draw the shear and the moment diagram.



17



18

END OF CHAPTER FIVE

19

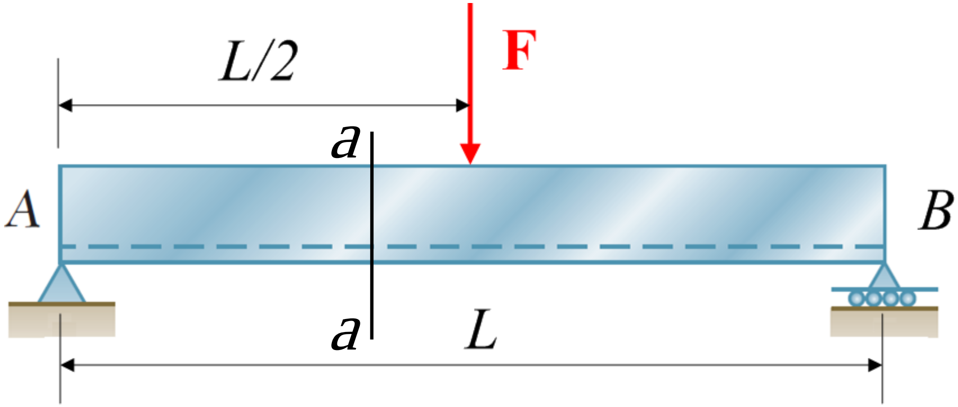
MECHANICS OF MATERIALS

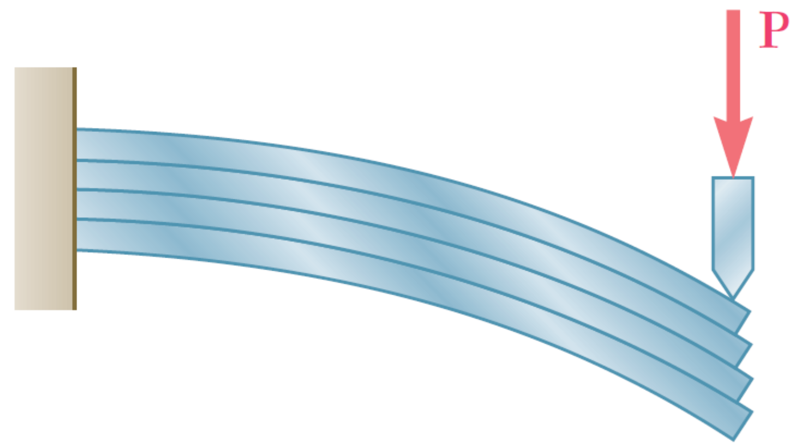
CHAPTER SIX

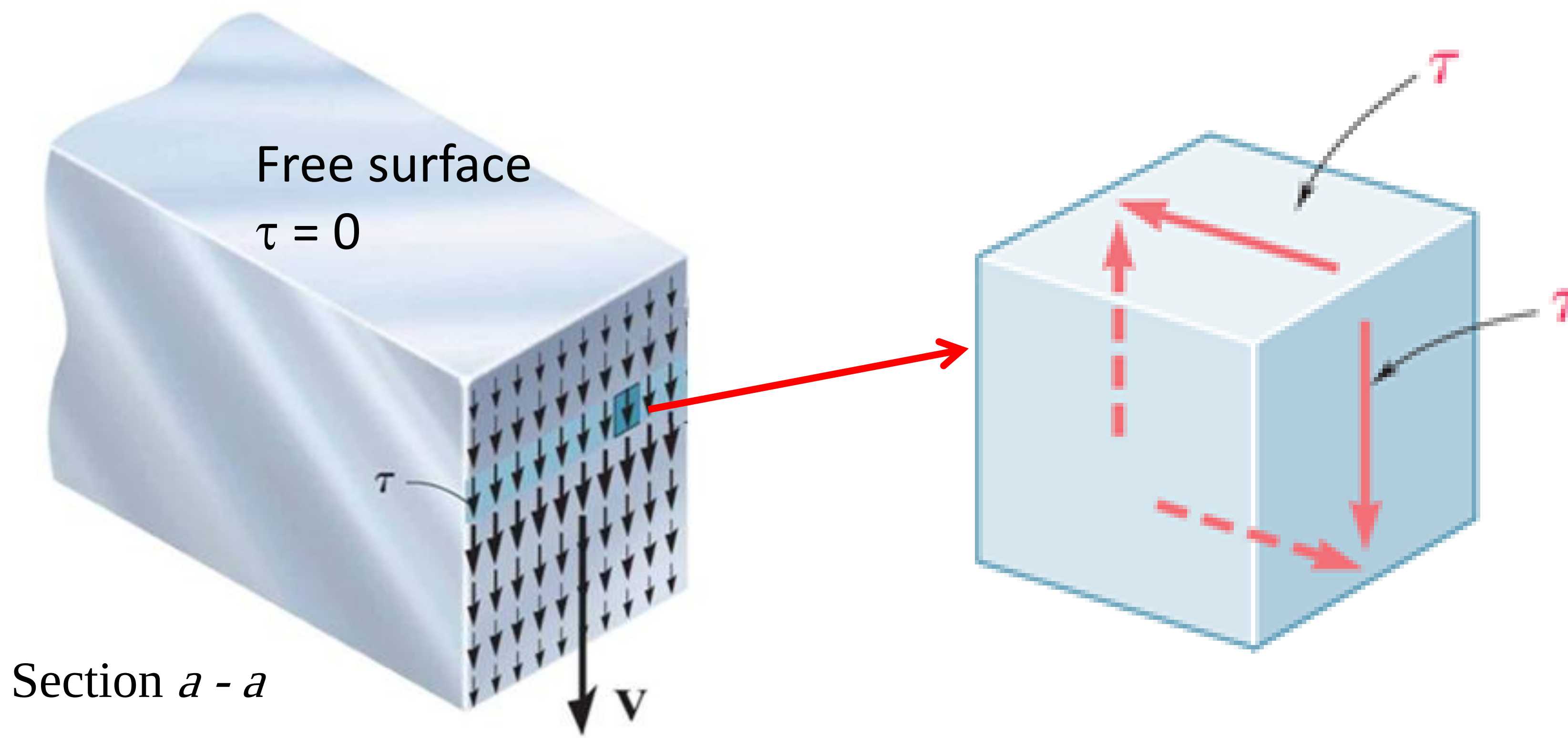
SHEARING STRESSES IN BEAMS AND THIN-WALLED MEMBERS

Prepared by : Dr. Mahmoud Rababah

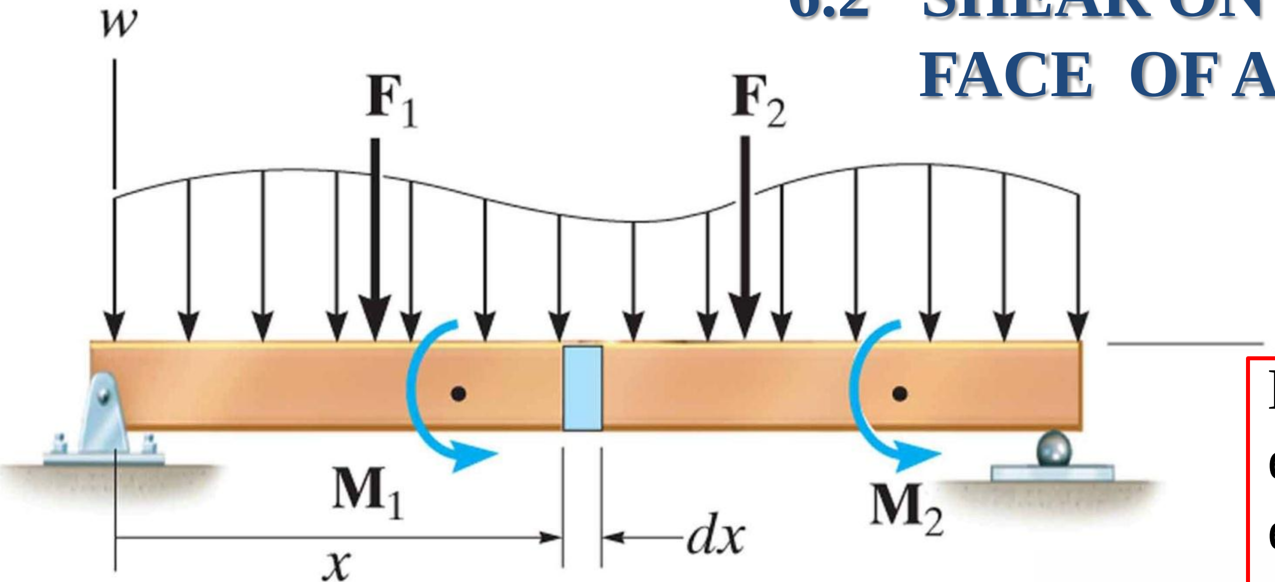
6.1 INTRODUCTION



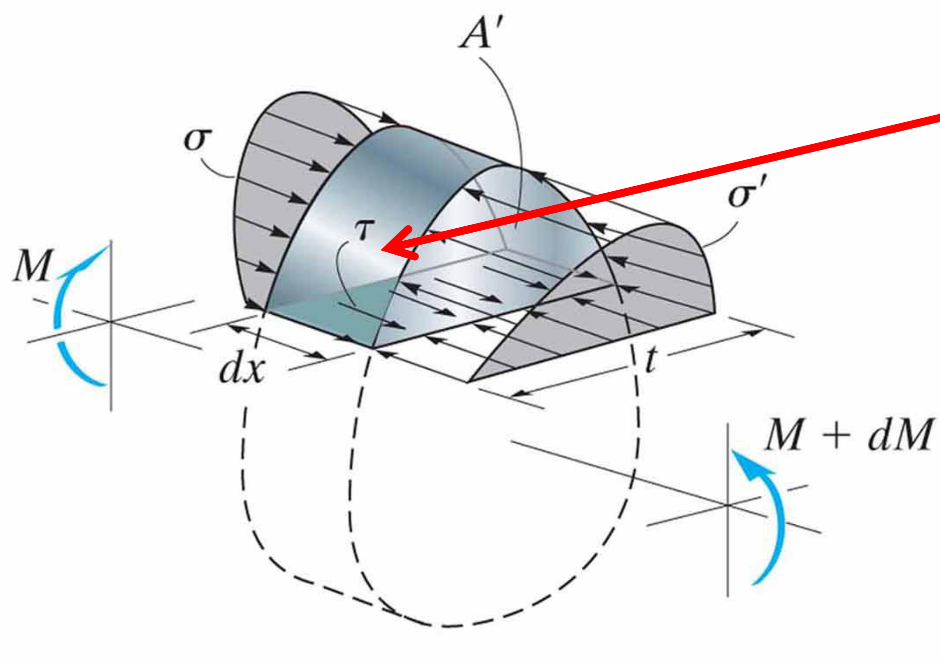




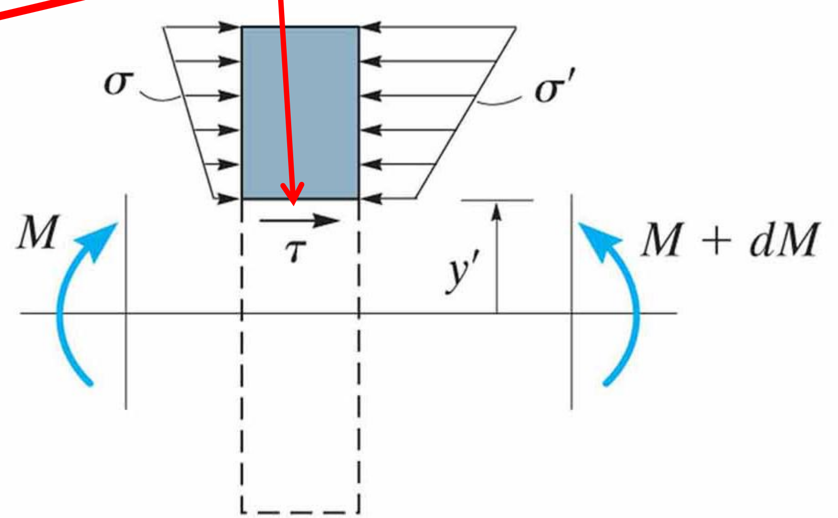
6.2 SHEAR ON THE HORIZONTAL FACE OF A BEAM ELEMENT



Longitudinal shear must exist to satisfy the equilibrium along x-axis



Three-dimensional view



Profile view

SHEAR FORMULA

$$\overset{+}{\leftarrow}\sum \mathbf{F_x} = 0$$

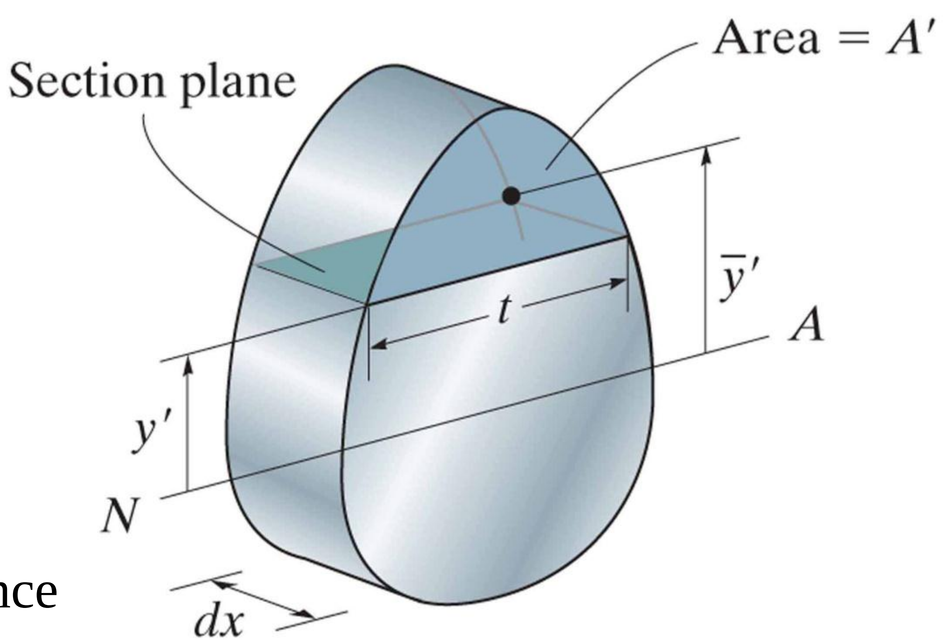
$$\int_{A'} \sigma' dA' - \int_{A'} \sigma dA' - \tau(tdx) = 0$$

$$\int_{A'} \left(\frac{\mathbf{M} + d\mathbf{M}}{I}\right) y dA' - \int_{A'} \left(\frac{\mathbf{M}}{I}\right) y dA' - \tau(tdx) = 0$$

$$\left(\frac{d\mathbf{M}}{I}\right) \int_{A'} y dA' = \tau(tdx)$$

$$\tau = \frac{1}{It}\left(\frac{d\mathbf{M}}{dx}\right) \int_{A'} y dA'$$

$$\tau = \frac{\mathbf{V}Q}{It}$$



V the internal shear force at the considered section.

Q the first moment of area = $\bar{y}' A'$

I the second moment of inertia.

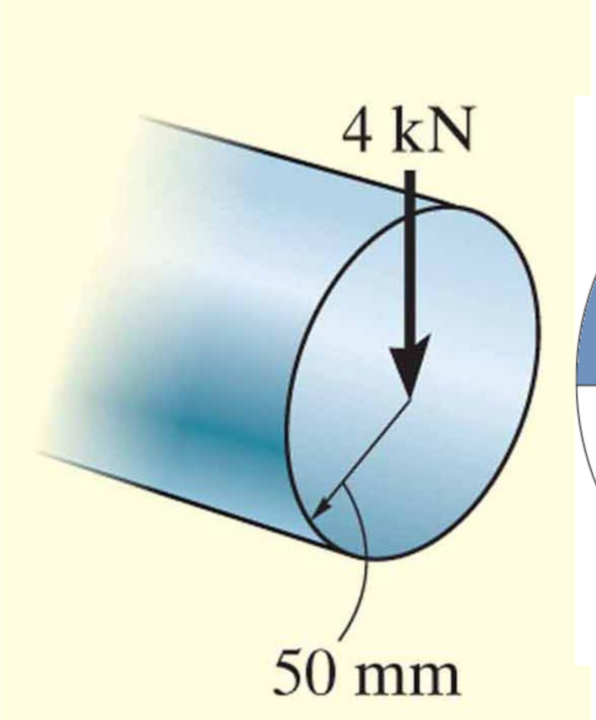
t the thickness at the considered distance from the neutral axis.

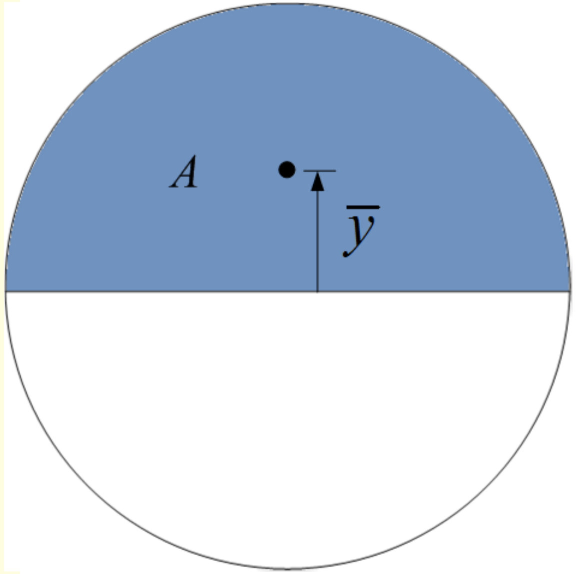
Example :

$$V = 4 \text{ kN}$$

$$c = 50 \text{ mm}$$

$$I = \frac{\pi}{4} c^4 = 4.909 \times 10^{-6} \text{ m}^4$$

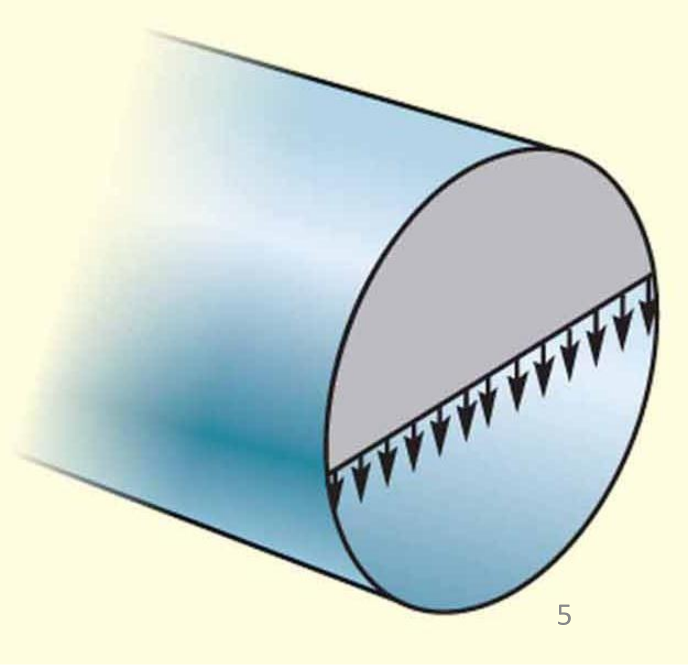




$$Q = \bar{y} \cdot A = \left(\frac{4c}{3\pi}\right)\left(\frac{\pi c^2}{2}\right) = 83.33 \times 10^{-6} \text{ m}^3$$

$$\tau = \frac{\mathbf{V}Q}{It} = 679 \text{ kPa}$$

$t = 2c$



Example:

Find the shear stress distribution over the cross-section.

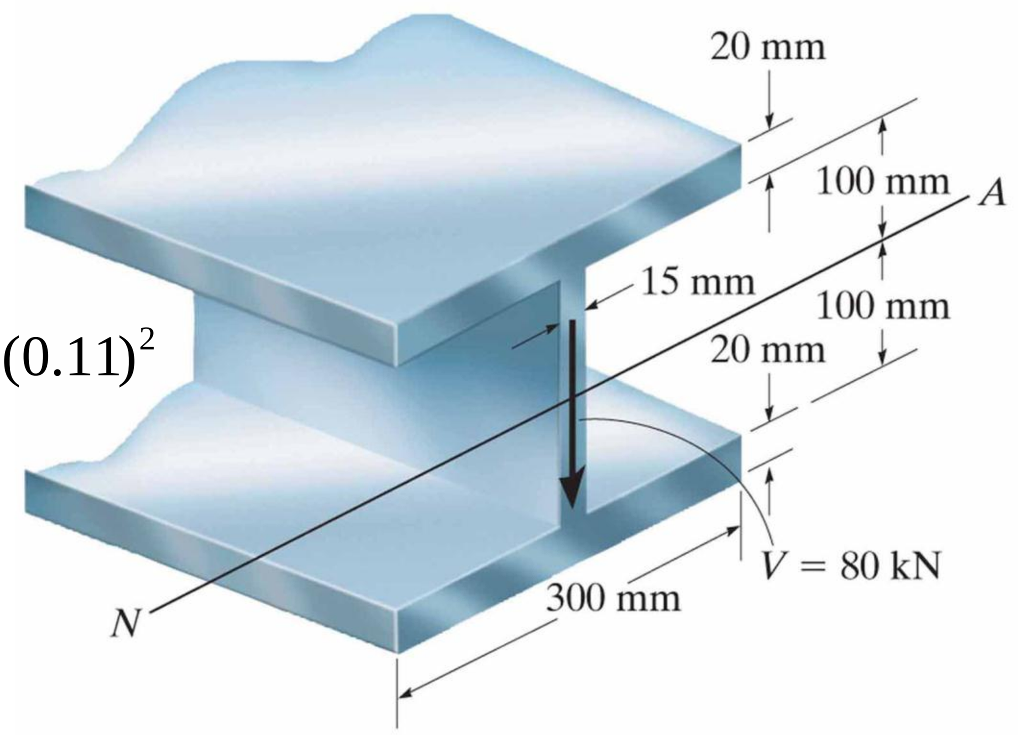
Solution :

$$\mathbf{V} = 80 \text{ kN}$$

$$I_1 = \frac{1}{12} \times 0.015 \times (0.2)^3$$

$$I_2 = \frac{1}{12} \times 0.3 \times (0.02)^3 + 0.3 \times 0.02 \times (0.11)^2$$

$$I = I_1 + 2I_2 = 155.6 \times 10^{-6} \text{ m}^4$$



$t_A = 0.3,$

$Q_A = 0$

$t_{B'} = 0.3\ \text{m}$

$Q_{B'} = 0.3 \times 0.02 \times 0.11 = 660 \times 10^{-6}\ \text{m}^3$

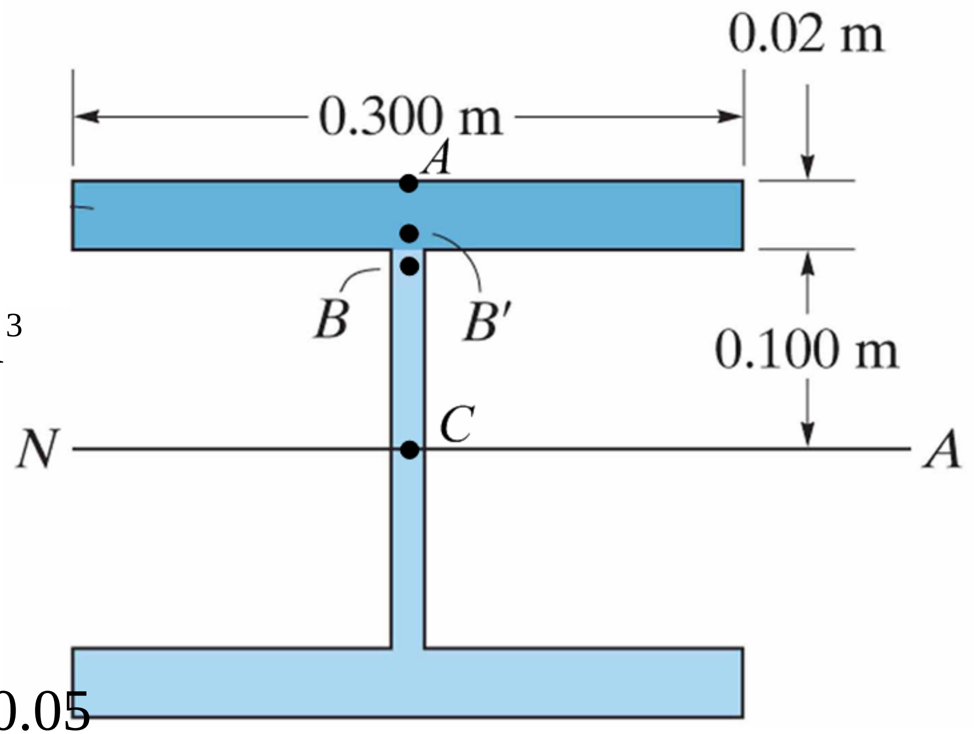
$t_B = 0.015\ \text{m}$

$Q_B = Q_{B'} = 660 \times 10^{-6}\ \text{m}^3$

$t_C = 0.015\ \text{m}$

$Q_C = 0.3 \times 0.02 \times 0.11 + 0.015 \times 0.1 \times 0.05$

$= 735 \times 10^{-6}\ \text{m}^3$



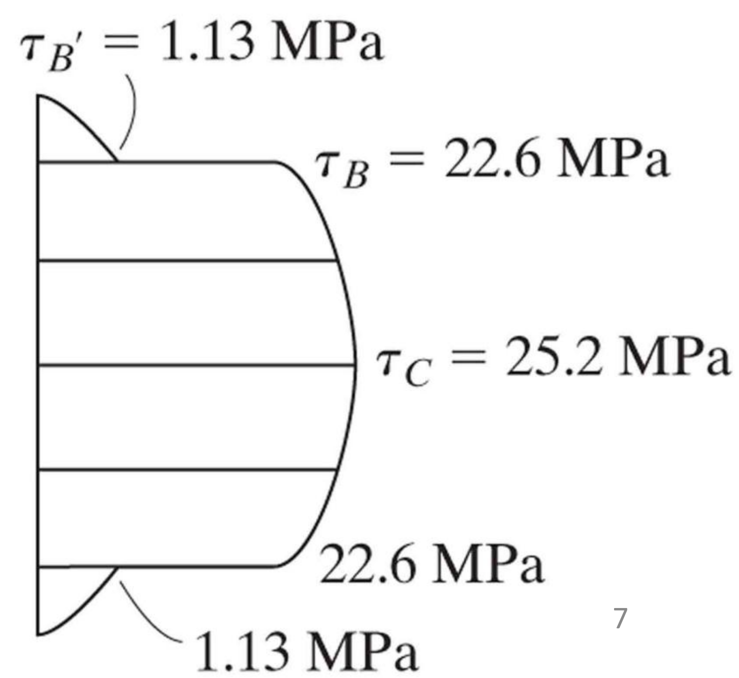
apply the shear formula, you get

$\tau_A = 0$

$\tau_{B'} = 1.13\ \text{MPa}$

$\tau_B = 22.6\ \text{MPa}$

$\tau_C = 25.2\ \text{MPa}$



- The shear distribution is not linear because τ is function of Q and t. Even if t is constant Q will be

$$Q = \bar{y} \times w \times h$$

Quadratic variation

Linear variation

Linear variation

Example :

$\mathbf{V} = 100\ \text{kN}$

Find the shear stress at point *A*.

Solution :

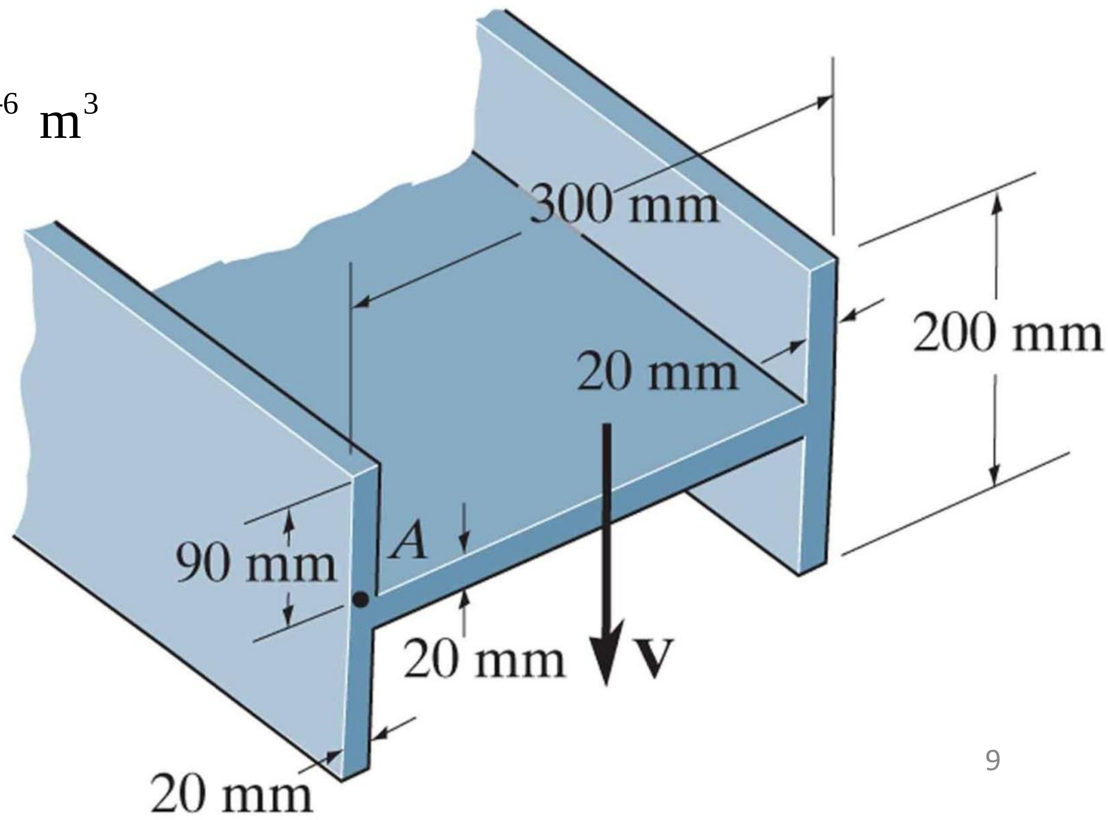
$$I = \frac{1}{12}\times 0.26\times(0.02)^3 + 2\left[\frac{1}{12}\times 0.02\times(0.2)^3\right]$$

$$= 26.84\times 10^{-6}\ \text{m}^4$$

$$Q = 0.09\times 0.02\times 0.055 = 99\times 10^{-6}\ \text{m}^3$$

$$t = 0.02\ \text{m}$$

$$\tau = \frac{\mathbf{V}Q}{It} = 18.44\ \text{MPa}$$



Note: you can consider

$Q = 2\times 0.09\times 0.02\times 0.055$

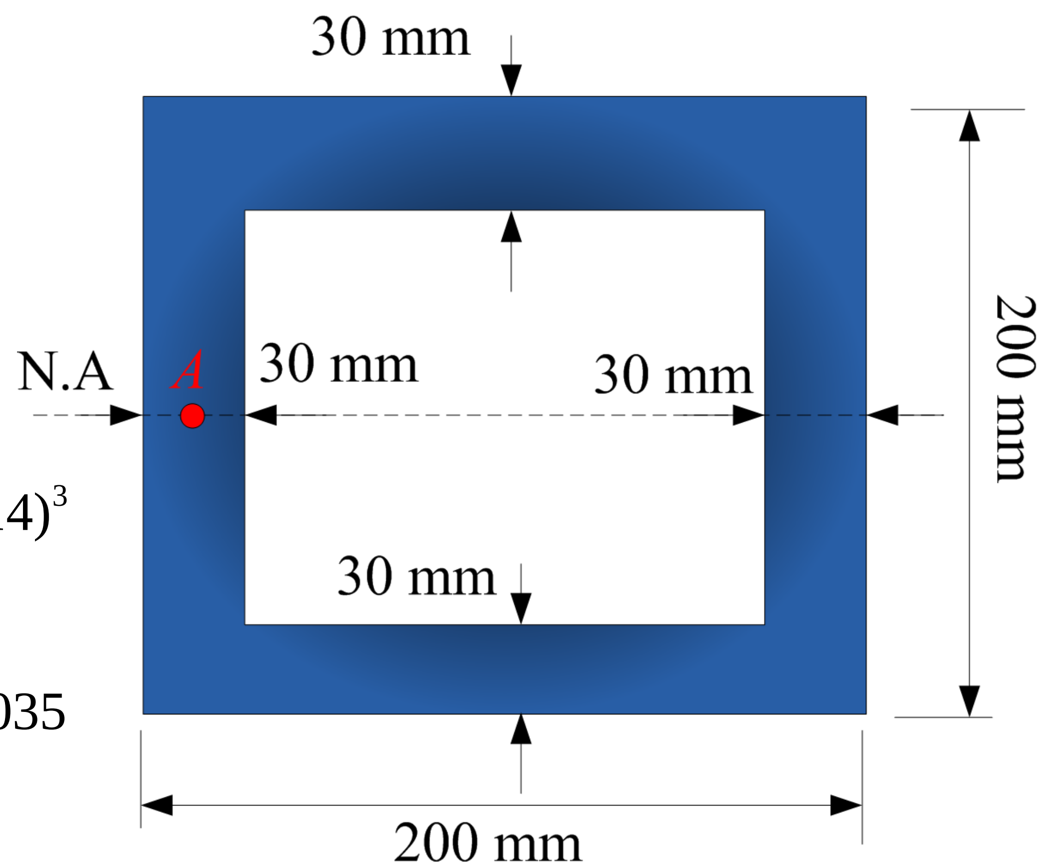
but then,

$t = 2\times 0.02$

Example :

$\mathbf{V} = 50\text{ kN}$

Find τ at point A



Solution :

$$I = \frac{1}{12} \times 0.2 \times (0.2)^3 - \frac{1}{12} \times 0.14 \times (0.14)^3$$

$$= 101.32 \times 10^{-6}\ \text{m}^4$$

$$Q = 0.2 \times 0.1 \times 0.05 - 0.14 \times 0.07 \times 0.035$$

$$= 657 \times 10^{-6}\ \text{m}^3$$

$$t = 0.06\text{ m}$$

$$\tau_A = \frac{\mathbf{V}Q}{It} = 5.4\text{ MPa}$$

6.4 SHEARING STRESSES τ_{xy} IN COMMON TYPES OF BEAMS

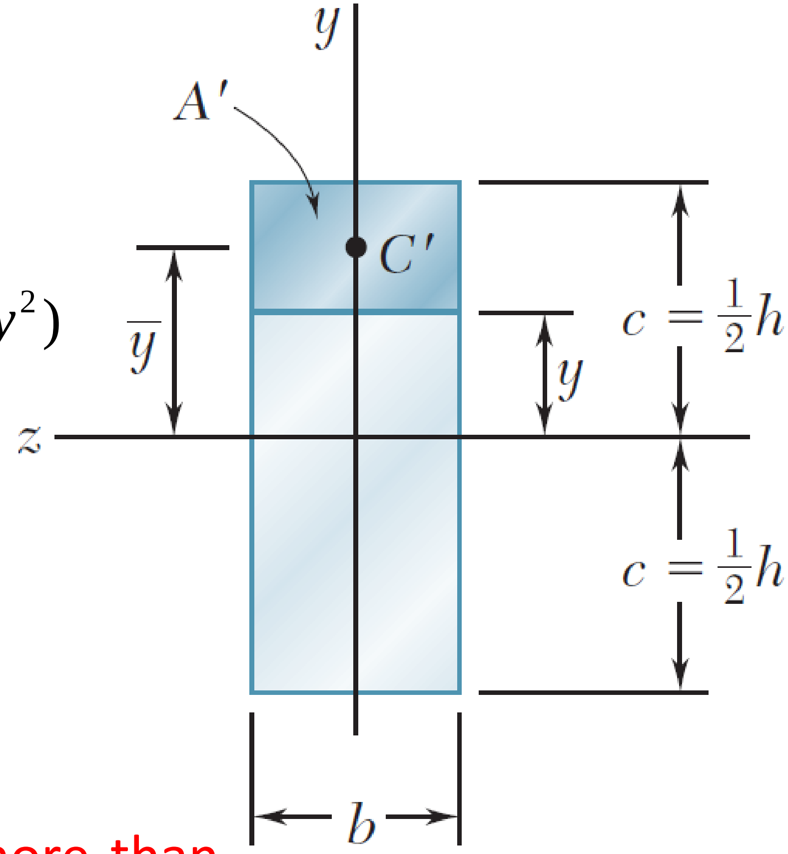
For rectangular beams

$$\tau_{xy} = \frac{\mathbf{V}Q}{It}$$

$$Q = A \cdot \bar{y} = b(c-y)\cdot\left(\frac{c+y}{2}\right) = \frac{1}{2}b\cdot(c^2 - y^2)$$

$$\tau_{xy} = \frac{3}{2}\frac{\mathbf{V}}{A}\left(1 - \frac{y^2}{c^2}\right)$$

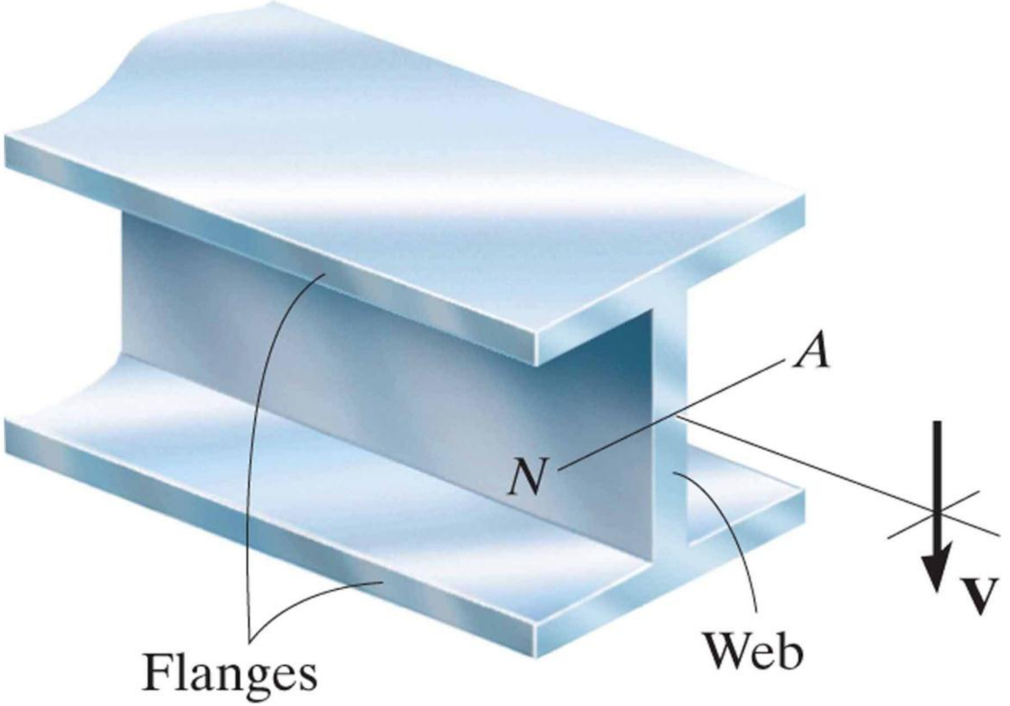
$$\tau_{max} = \frac{3}{2}\frac{\mathbf{V}}{A}$$



Note: the maximum shear is 50% more than the average shear calculated before.

- For wide flange beams the maximum shear stress can be approximated as :

$$\tau_{max} = \frac{\mathbf{V}}{A_{web}}$$



For the previous example, apply

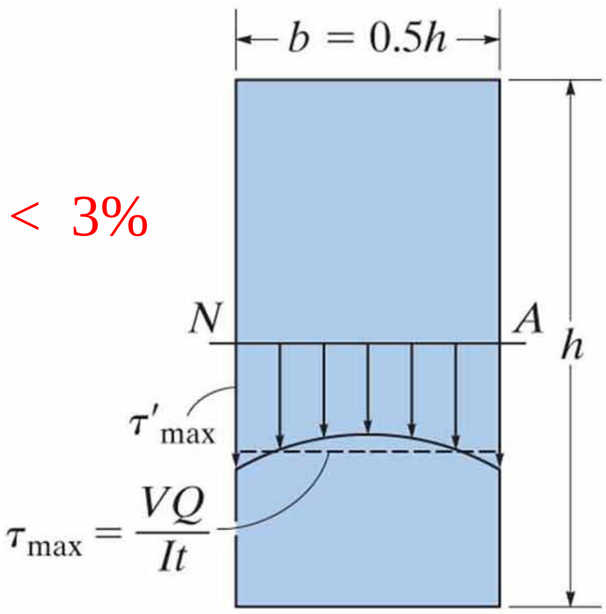
$$\tau_{max} = \frac{\mathbf{V}}{A_{web}} = \frac{80\times10^{3}}{0.2\times0.015} = 26.67 \text{ MPa}$$

(close to the one obtained from the shear formula)

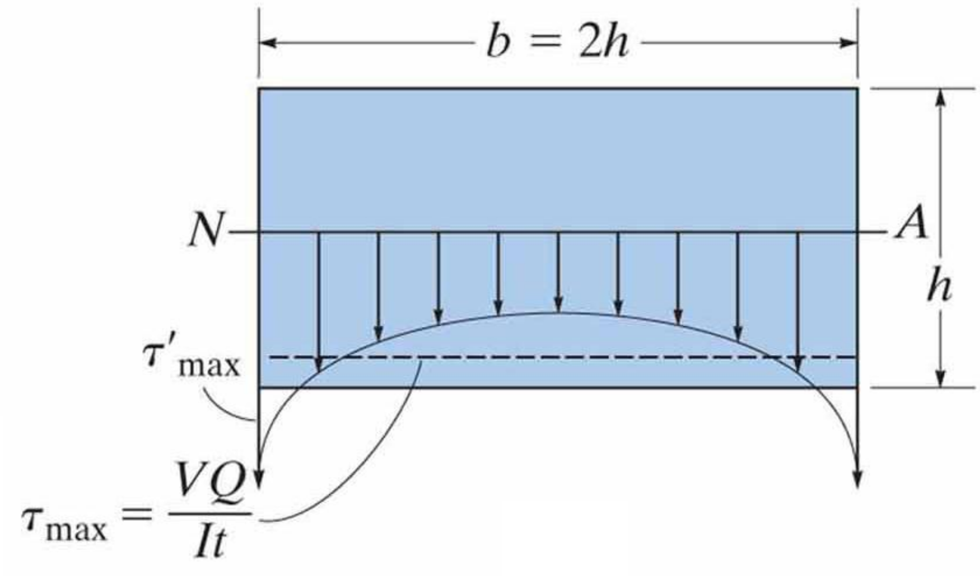
SHEAR FORMULA LIMITATIONS

I. The difference between the shear stress obtained from the shear formula and the actual one increases for wider beams

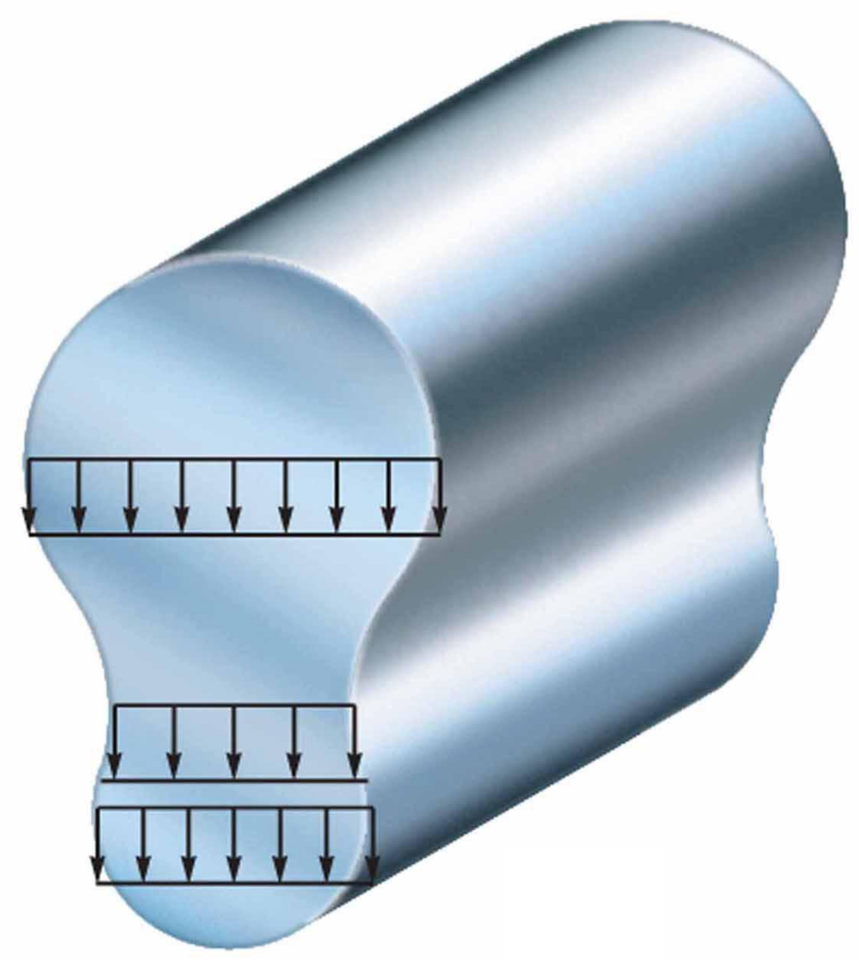
error < 3%



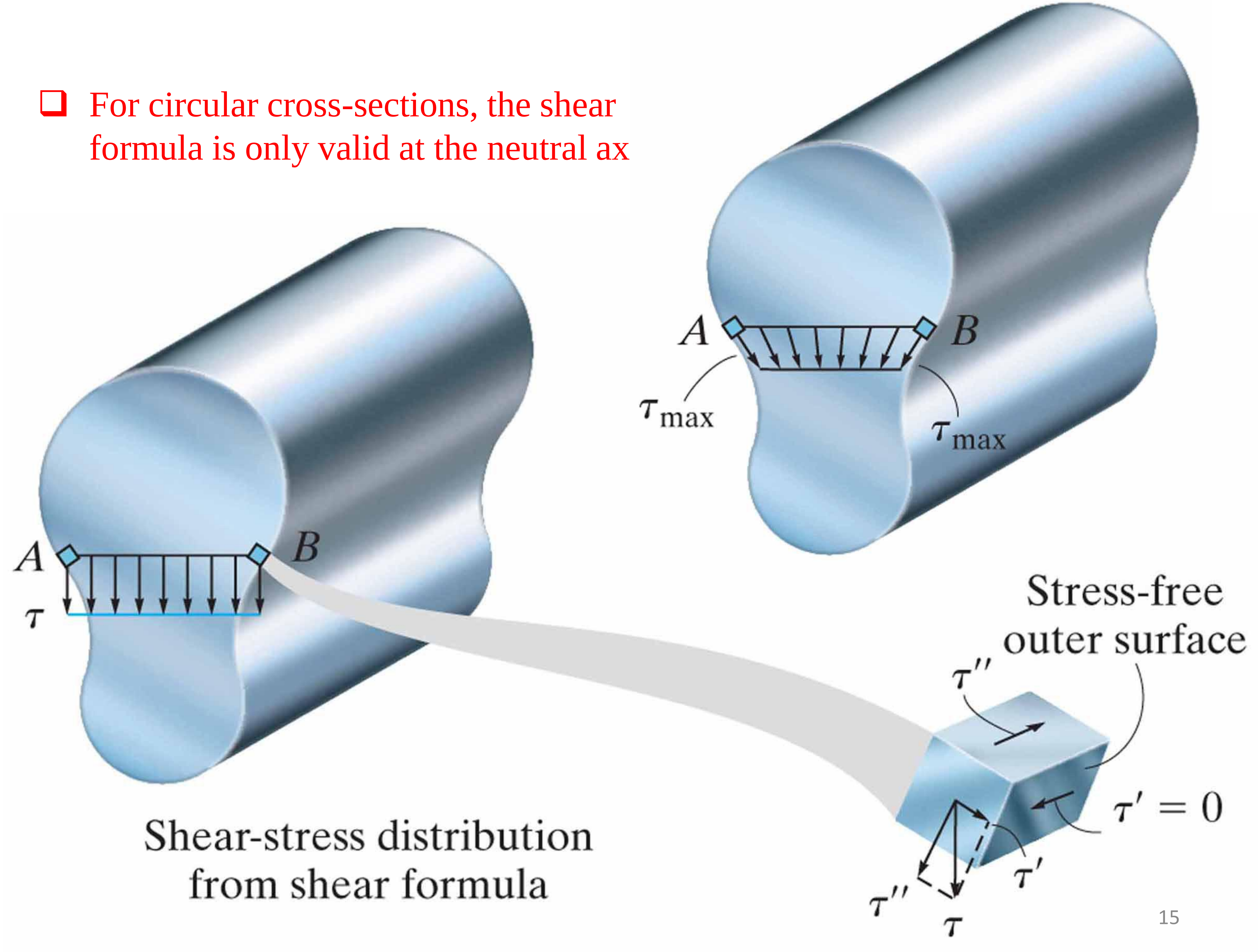
error > 39%



II. shear formula is only valid at sections intersecting the boundary of the members at angle 90^{o}



- For circular cross-sections, the shear formula is only valid at the neutral ax

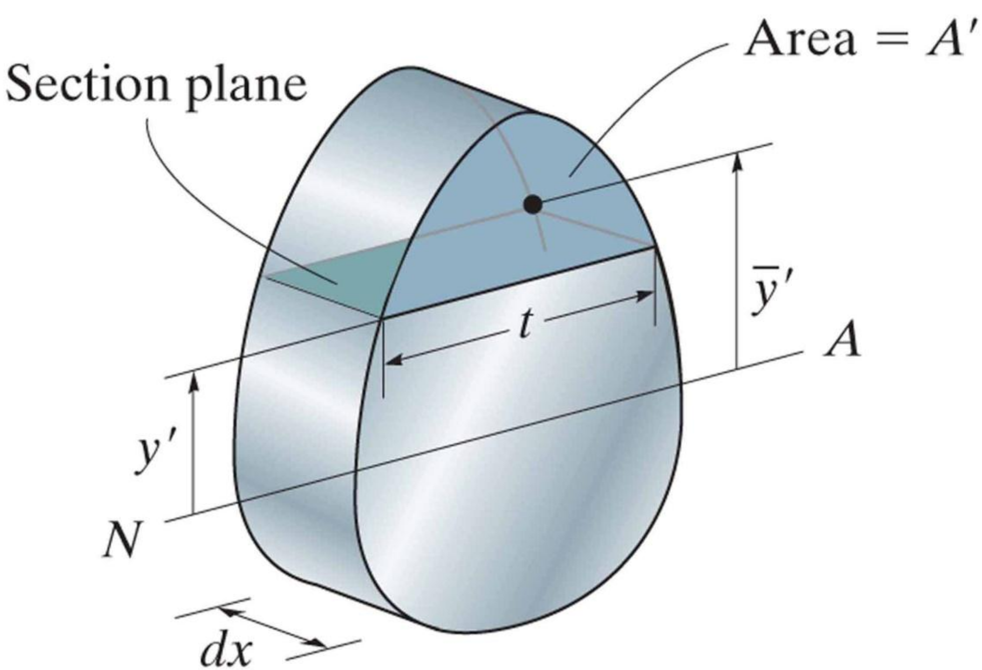


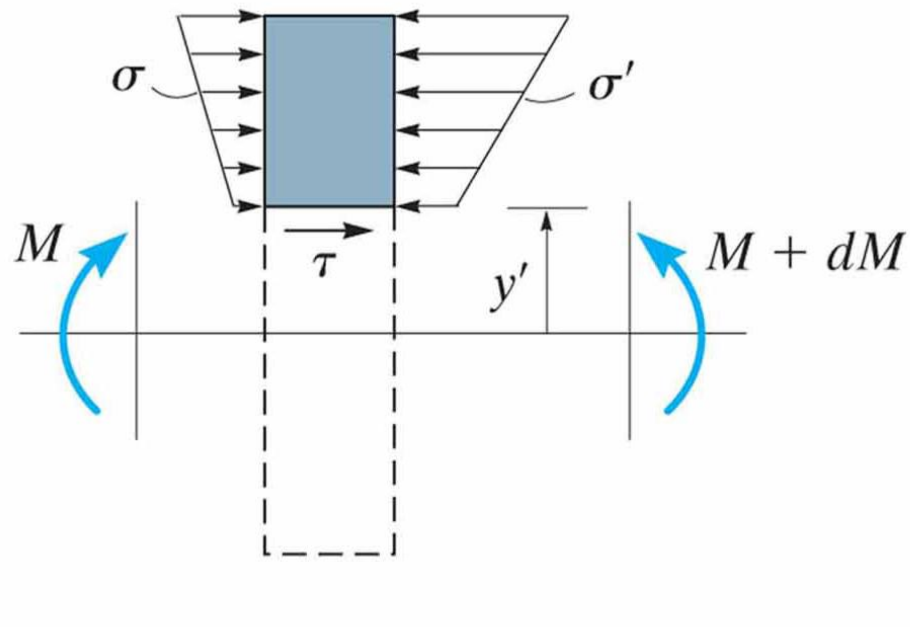
SHEAR FLOW

Let **the horizontal force** for the shown element

$dH = \tau t dx,$ then

$$\frac{dH}{dx} = \tau t = \frac{VQ}{I} = q \quad \text{(Shear flow)}$$





Profile view

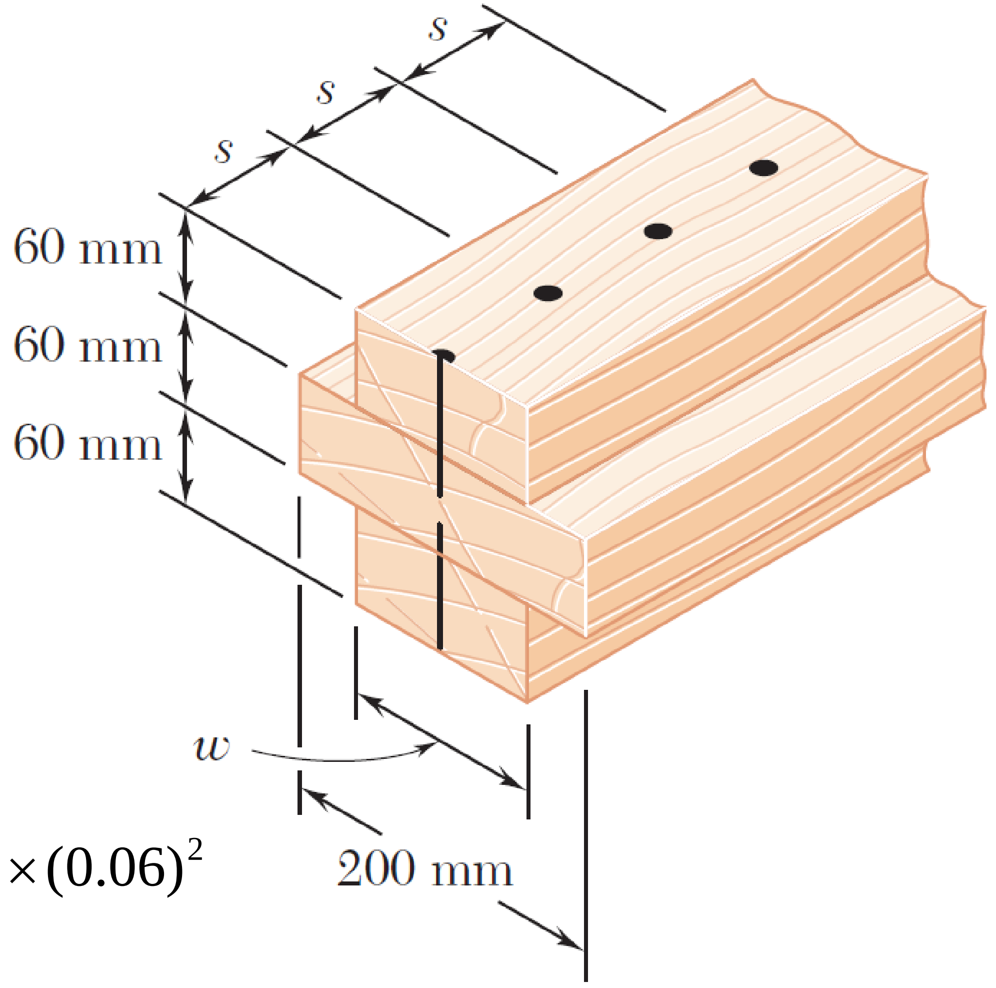
Example :

$s = 75$ mm

$w = 120$ mm

$H_{\max}^{nail} = 400$ N

Find $V_{\max}$



Solution :

$$I_1 = \frac{1}{12} \times 0.2 \times (0.06)^3$$

$$I_2 = \frac{1}{12} \times 0.12 \times (0.06)^3 + 0.12 \times 0.06 \times (0.06)^2$$

$$I = I_1 + 2I_2 = 31.68 \times 10^{-6} \text{ m}^4$$

$$Q = 0.12 \times 0.06 \times 0.06 = 432 \times 10^{-6} \text{ m}^3$$

$$q = \frac{H_{\max}^{nail}}{s} = \frac{V_{\max} Q}{I}$$

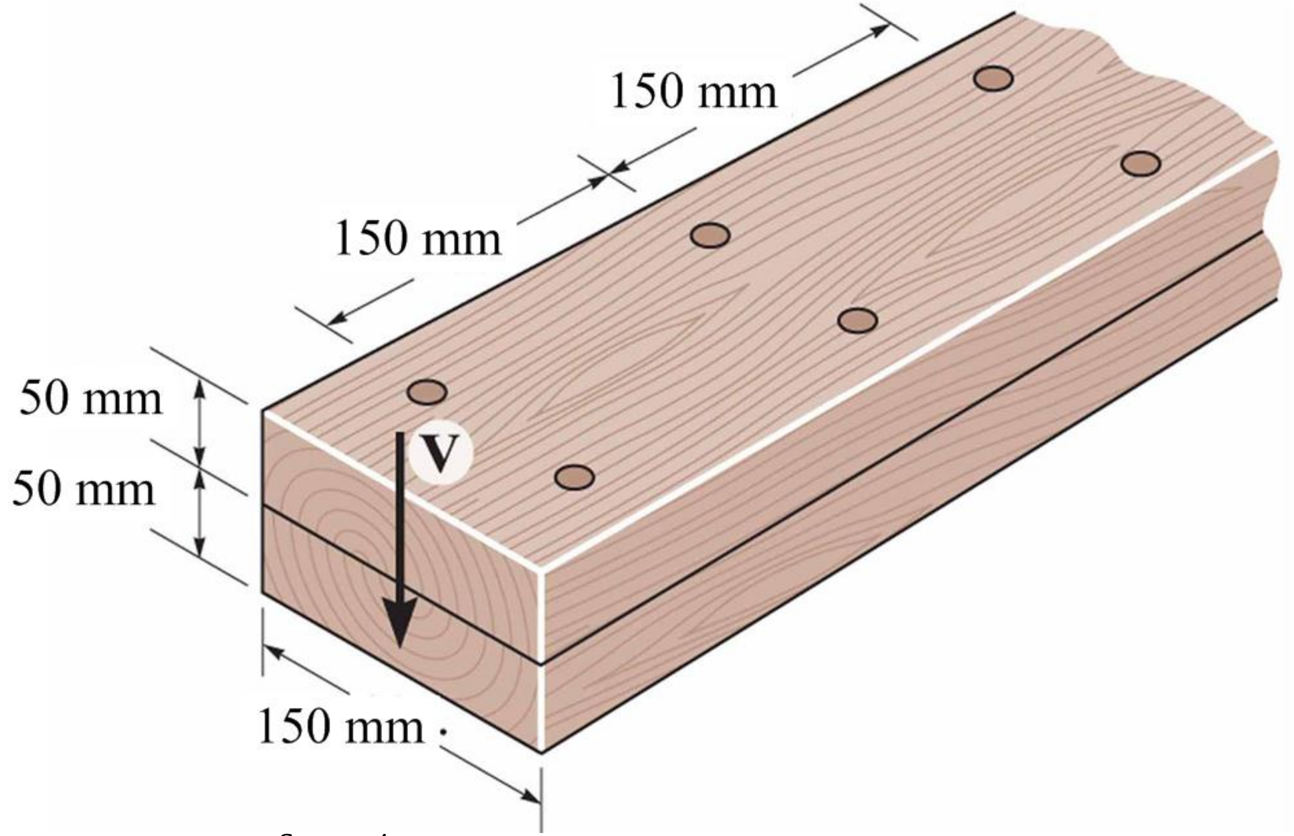
$$V_{\max} = 391 \text{ N}$$

Example :

$H_{\max}^{nail} = 200\text{ N}$

$s = 0.15\text{ m}$

Find $\mathbf{V}_{\mathbf{max}}$



Solution :

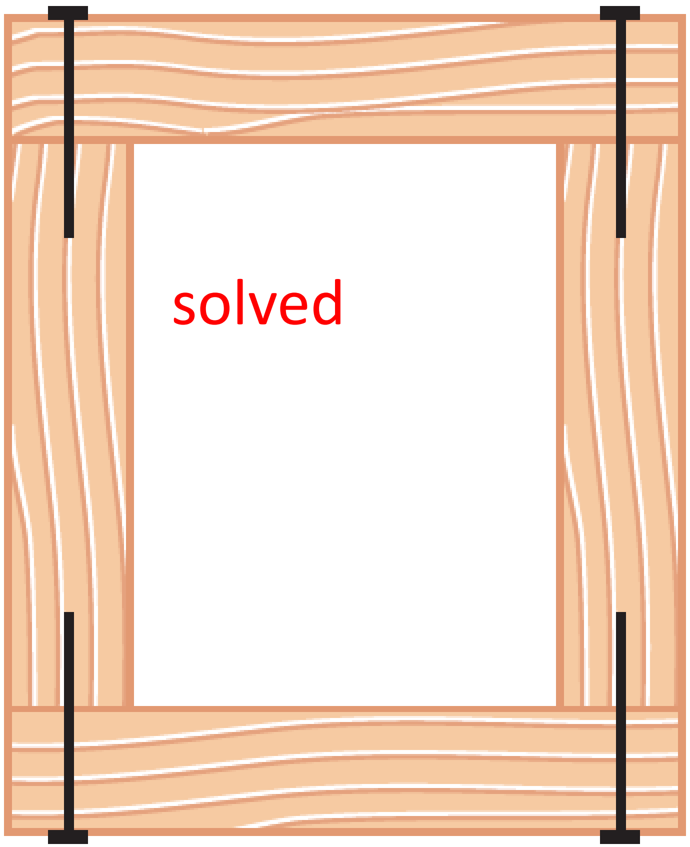
$$I = \frac{1}{12} \times 0.15 \times (0.1)^3 = 12.5 \times 10^{-6}\text{ m}^4$$

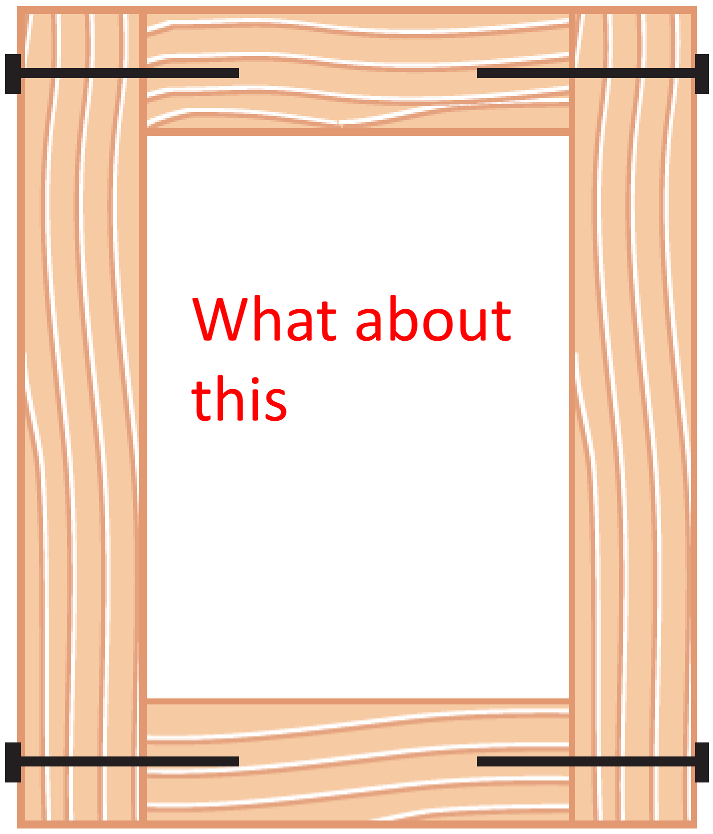
$$Q = 0.15 \times 0.05 \times 0.025 = 187.5 \times 10^{-6}\text{ m}^3$$

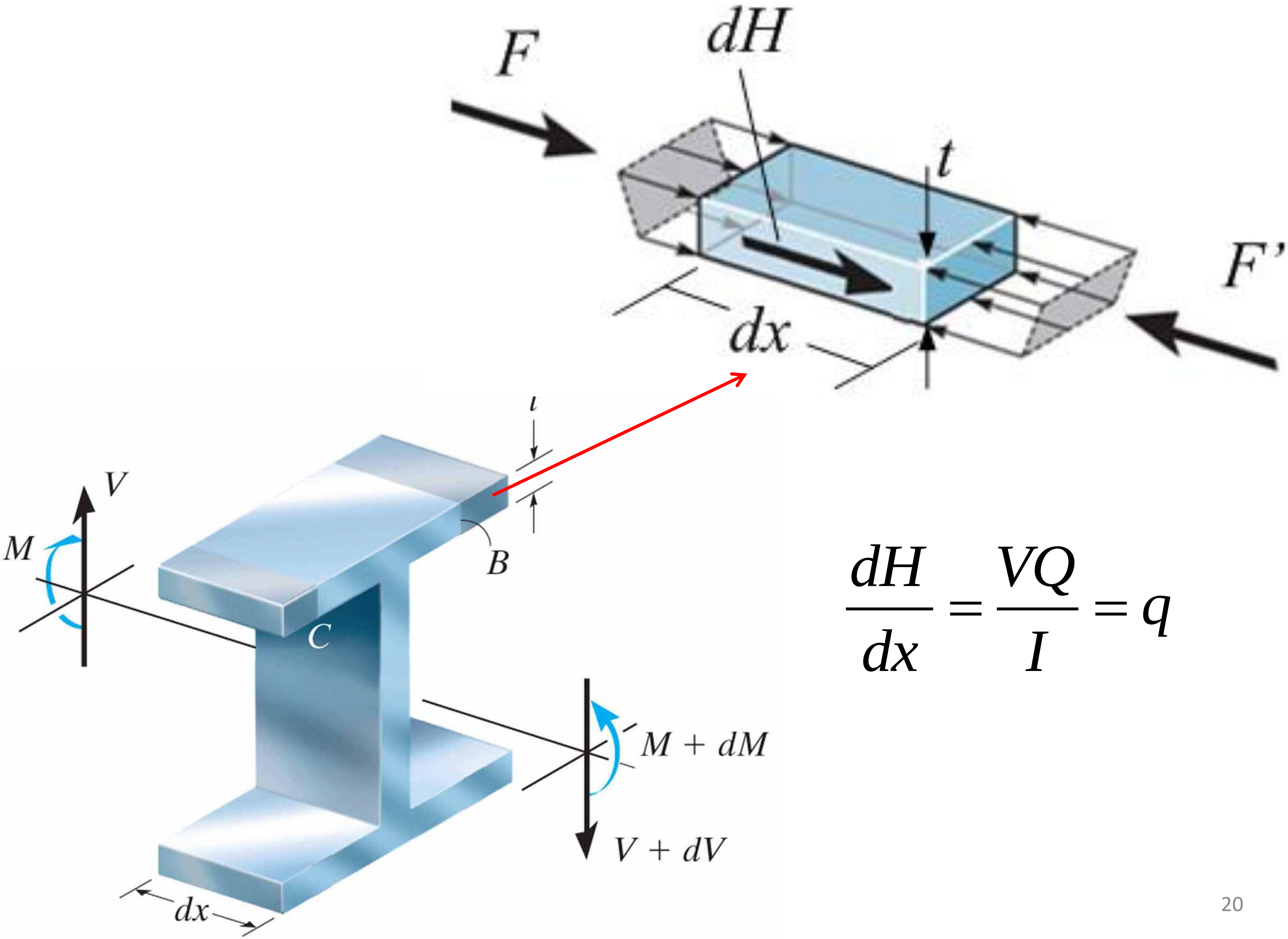
$$q = \frac{2 \times H_{\max}^{nail}}{s} = \frac{\mathbf{V}_{\mathbf{max}} Q}{I}$$

$$\mathbf{V}_{\mathbf{max}} = 177.78\text{ N}$$

6.6 LONGITUDINAL SHEAR ON A BEAM ELEMENT OF ARBITRARY SHAPE







$$\frac{dH}{dx} = \frac{VQ}{I} = q$$

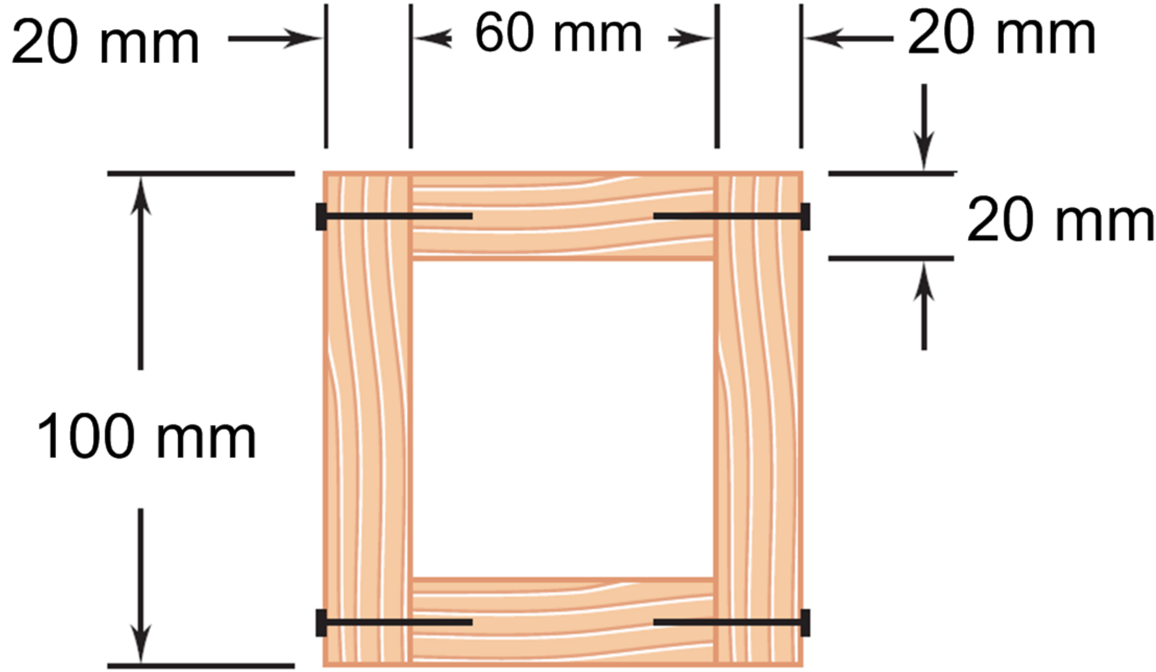
Example :

$s = 50$ mm

$\mathbf{V} = 1$ kN

Find

H^{nail}



Solution :

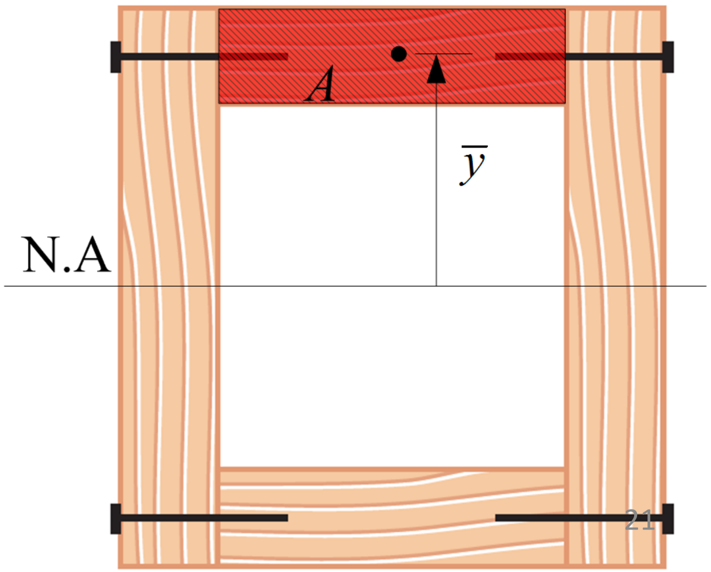
$$Q = \bar{y} \cdot A = 0.04 \times 0.06 \times 0.02 = 48 \times 10^{-6} \text{ m}^3$$

$$I = \frac{1}{12} \times 0.1 \times (0.1)^3 - \frac{1}{12} \times 0.06 \times (0.06)^3$$

$$= 7.25 \times 10^{-6} \text{ m}^4$$

$$\frac{H^{nail}}{s} = \frac{1}{2} \times \frac{\mathbf{V}Q}{I}$$

$$H^{nail} = 165.5 \text{ N}$$



Example :

$\mathbf{V} = 8 \text{ kN}$

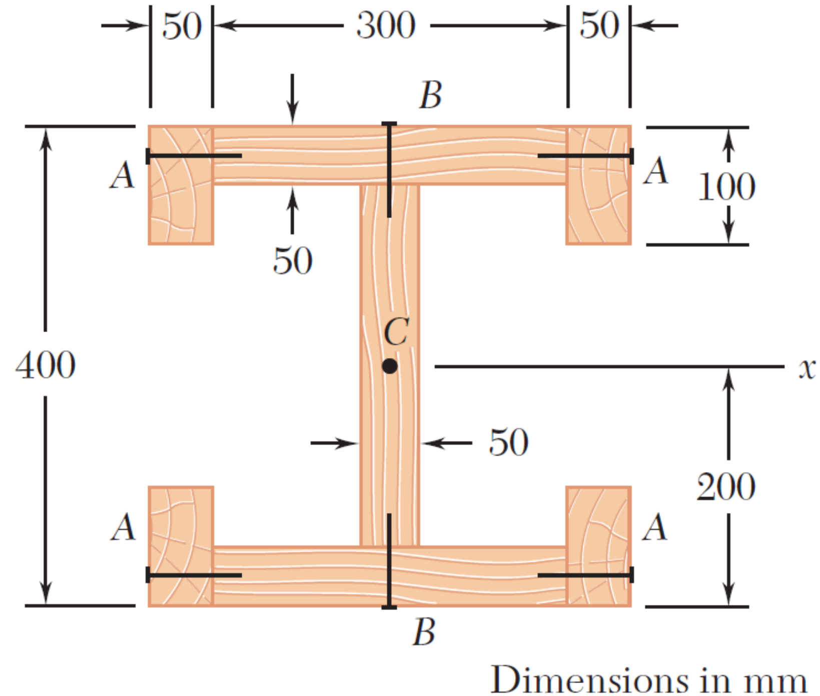
$s_A = 60 \text{ mm}$

$s_B = 25 \text{ mm}$

$I_x = 1.504 \times 10^{-3} \text{ m}^4$

Find

H_A, H_B



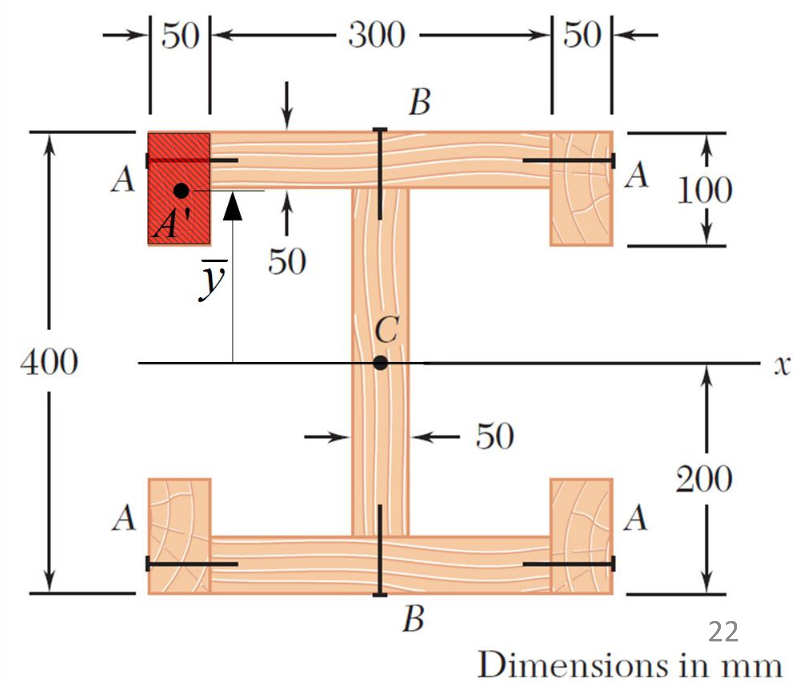
Solution :

$Q_A = \bar{y} \cdot A'$

$= 0.05 \times 0.1 \times 0.15 = 750 \times 10^{-6} \text{ m}^3$

$$\frac{H_A}{s_A} = \frac{\mathbf{V} Q_A}{I}$$

$H_A = 239.4 \text{ N}$



For pin B

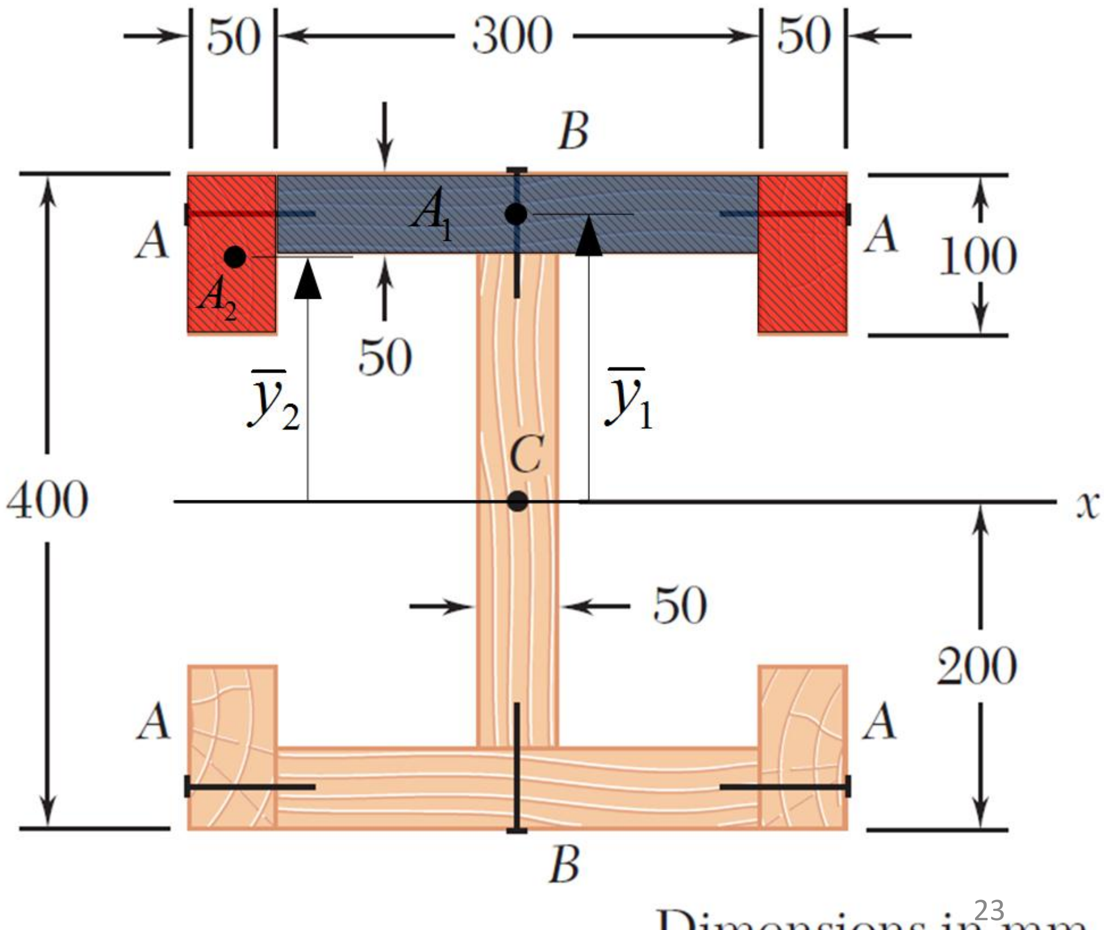
$$Q_B = \bar{y}_1 A_1 + 2\bar{y}_2 A_2$$

$$= 0.175 \times 0.3 \times 0.05 + 2 \times 0.15 \times 0.05 \times 0.1$$

$$= 4.125 \times 10^{-3} \text{ m}^3$$

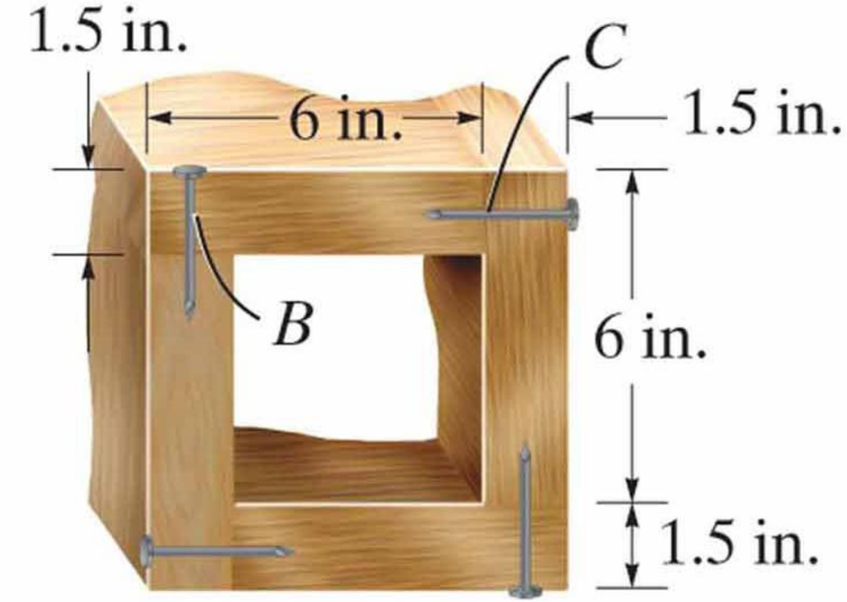
$$\frac{H_B}{s_B} = \frac{\mathbf{V}Q_B}{I}$$

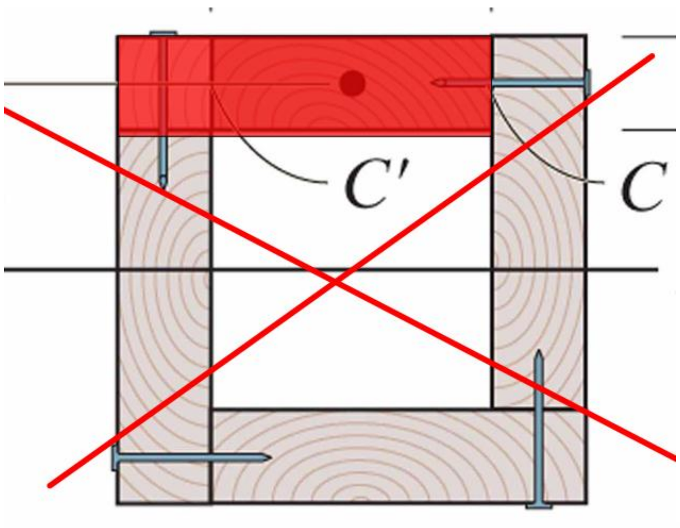
$$H_B = 548.5 \text{ N}$$

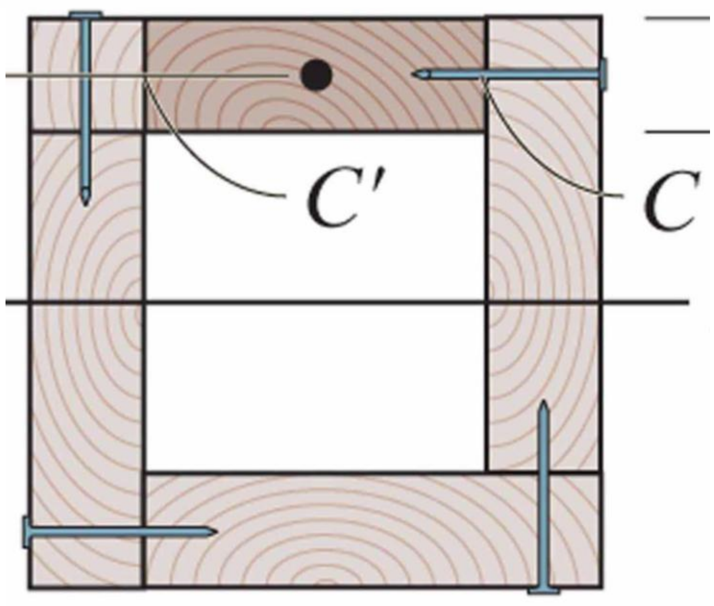


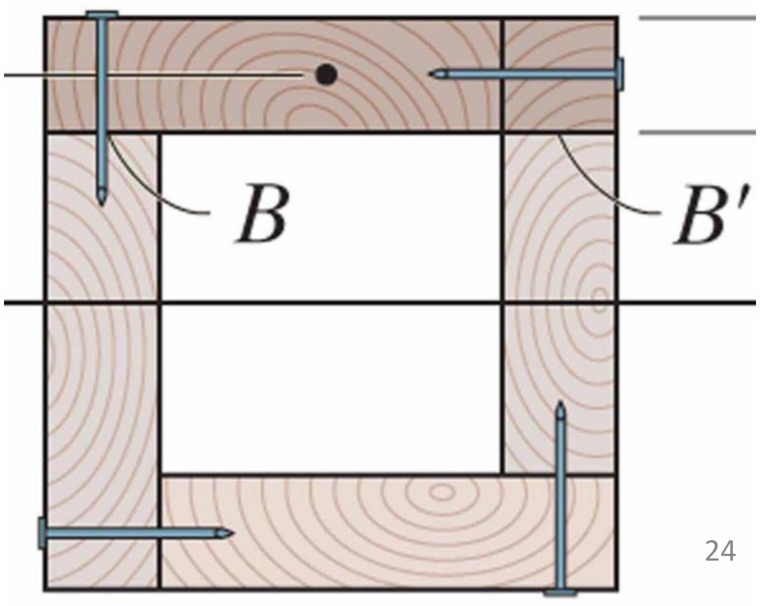
Further discussion

The segment considered for calculating the shear flow should be symmetric.

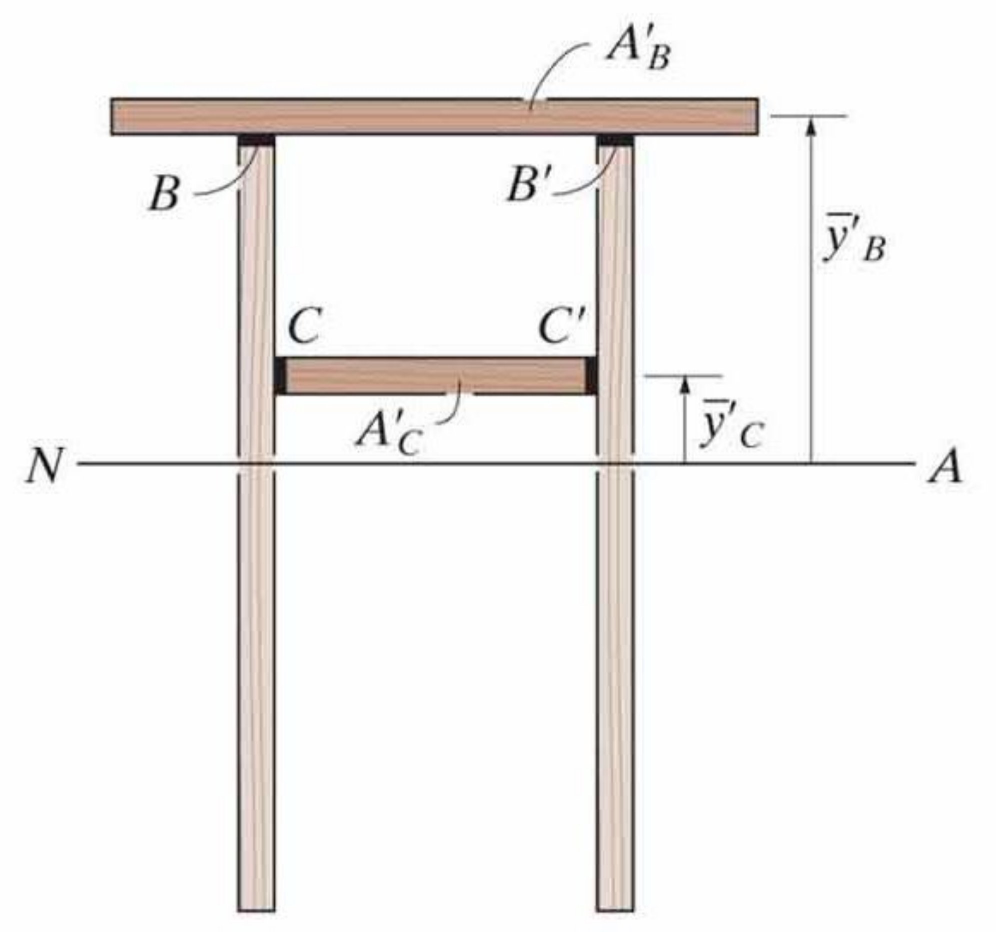




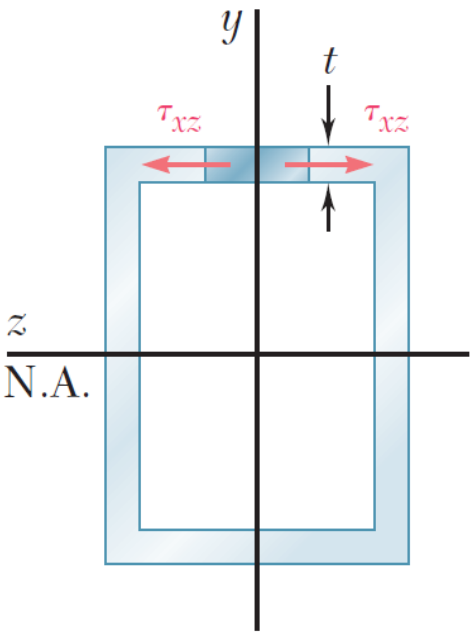


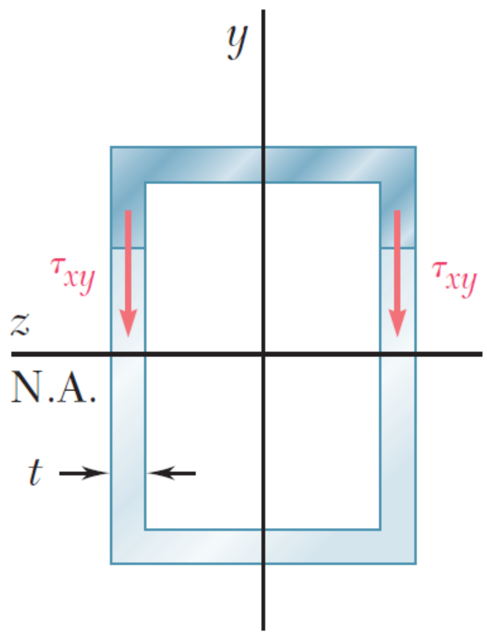


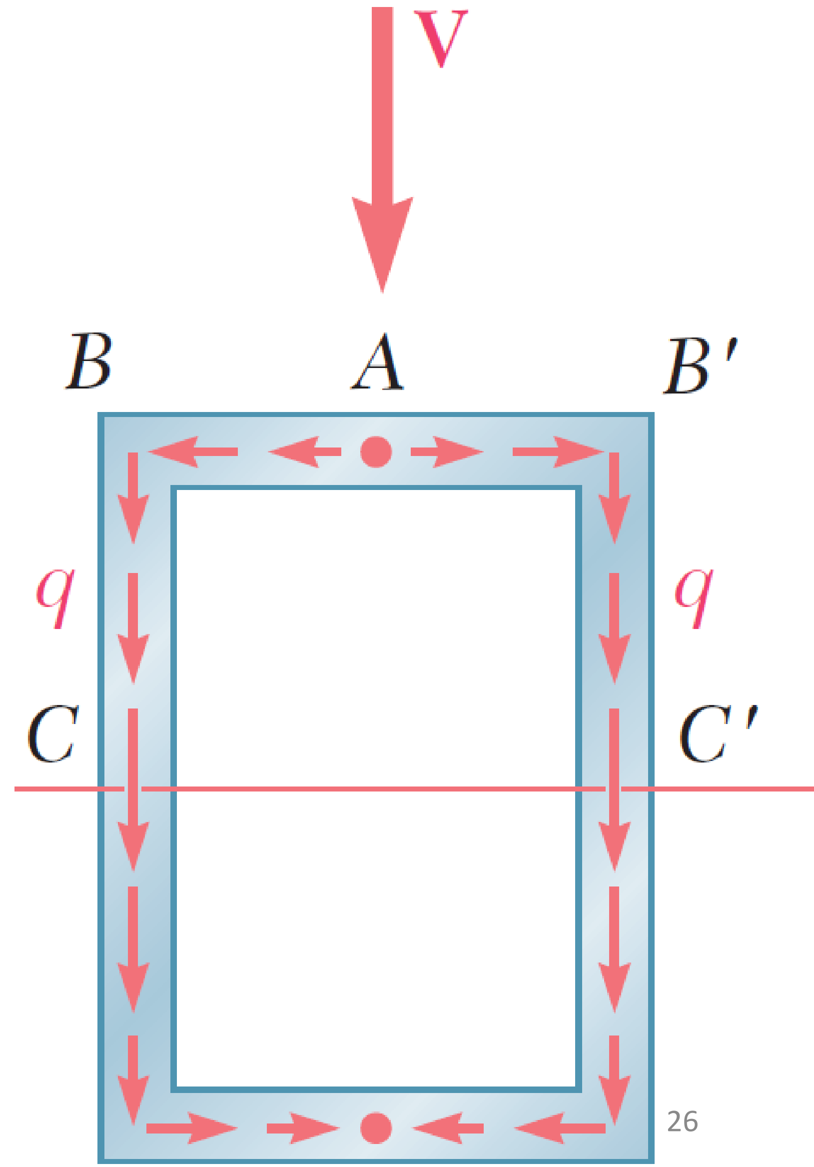
How to calculate the shear at joints C and C’



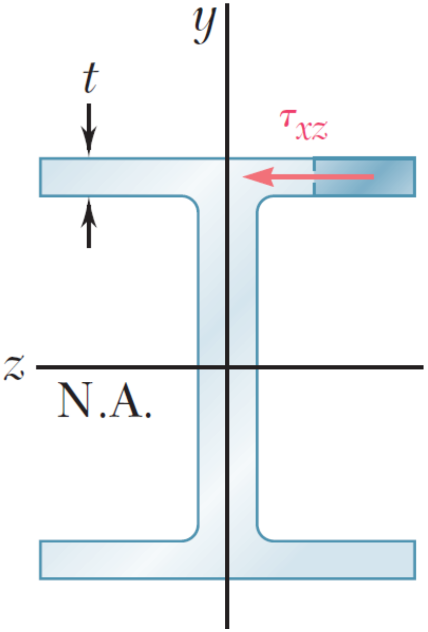
6.7 SHEARING STRESSES IN THIN-WALLED MEMBERS

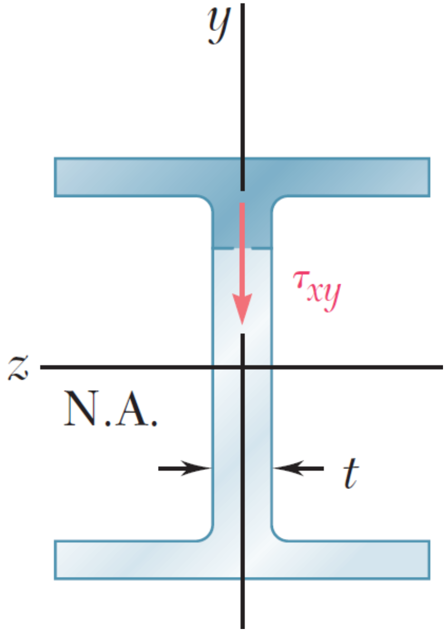


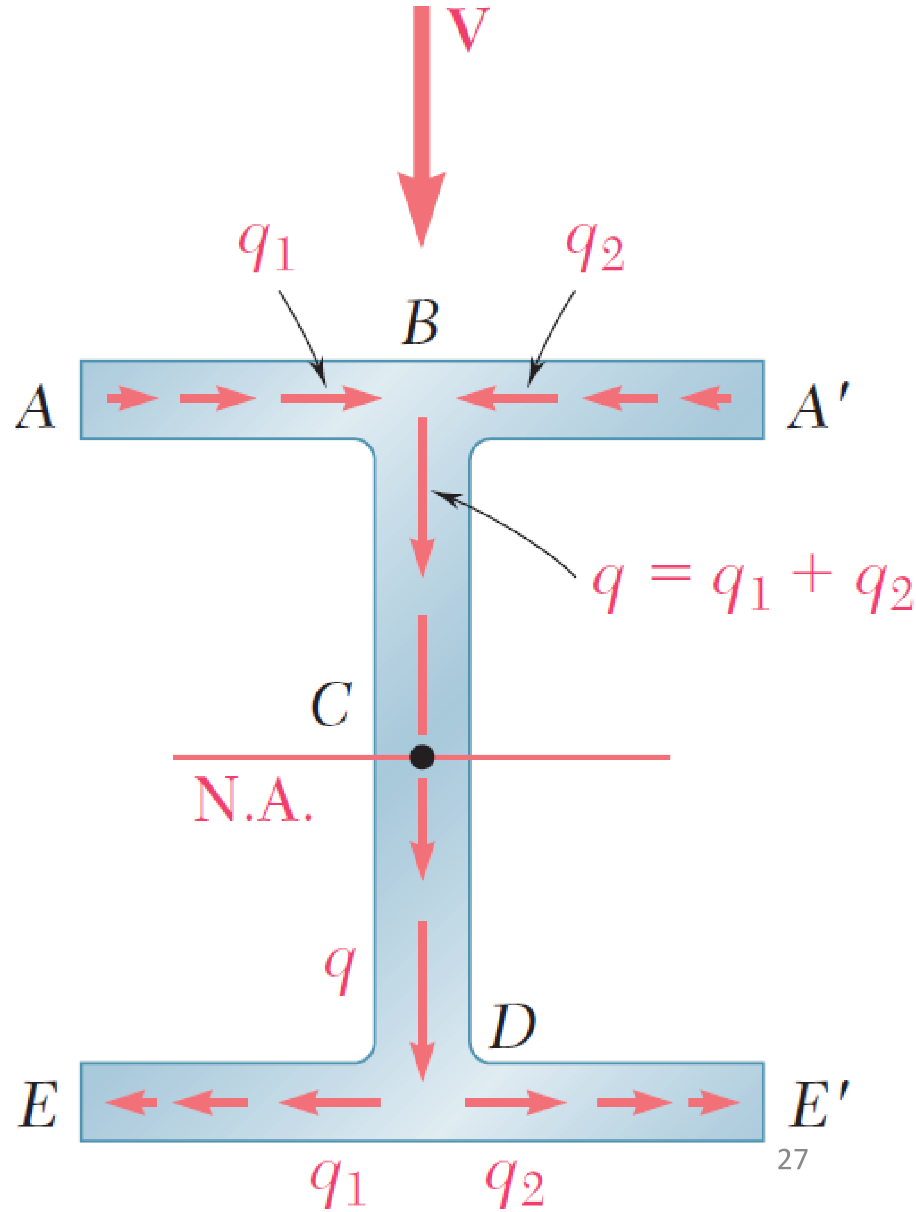




The stress variation along the thickness is small for thin walled members


y
t
τ_{xz}
z
N.A.


y
τ_{xy}
z
N.A.
t


V
q_1
q_2
B
A
A′
$q = q_1 + q_2$
C
N.A.
q
D
E
E′
q_1
q_2

END OF CHAPTER SIX

MECHANICS OF MATERIALS

CHAPTER SEVEN

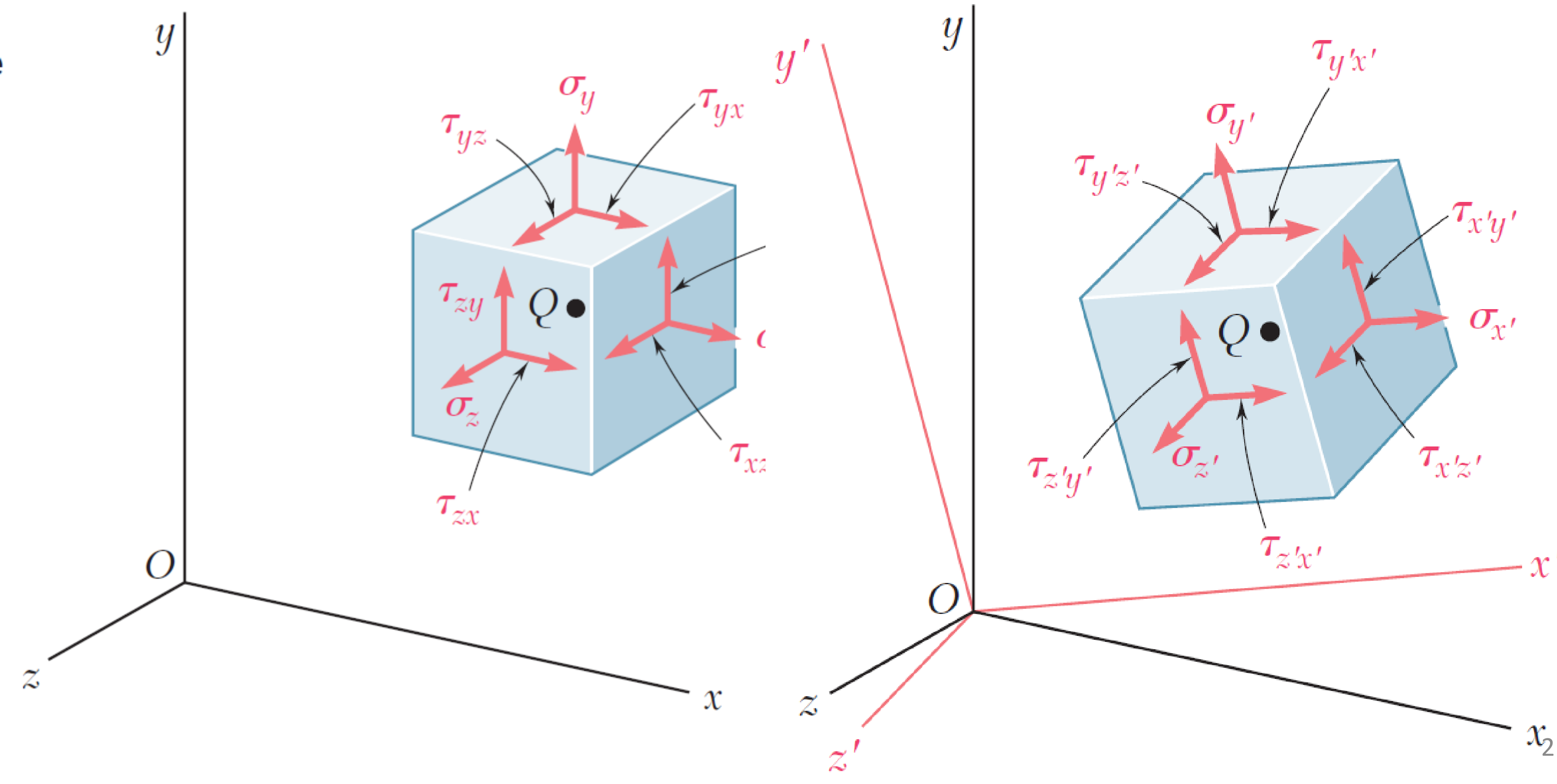
TRANSFORMATIONS OF STRESS AND STRAIN

Prepared by : Dr. Mahmoud Rababah

7.1 INTRODUCTION

- Failure can occur in any angle.
- It is irrelevant to the assumed coordinates.

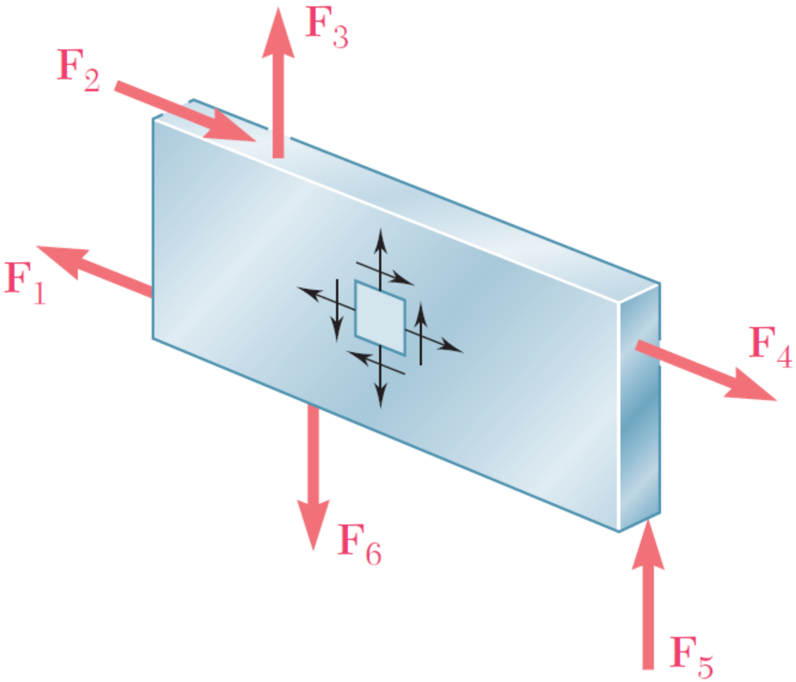
General loading condition is:

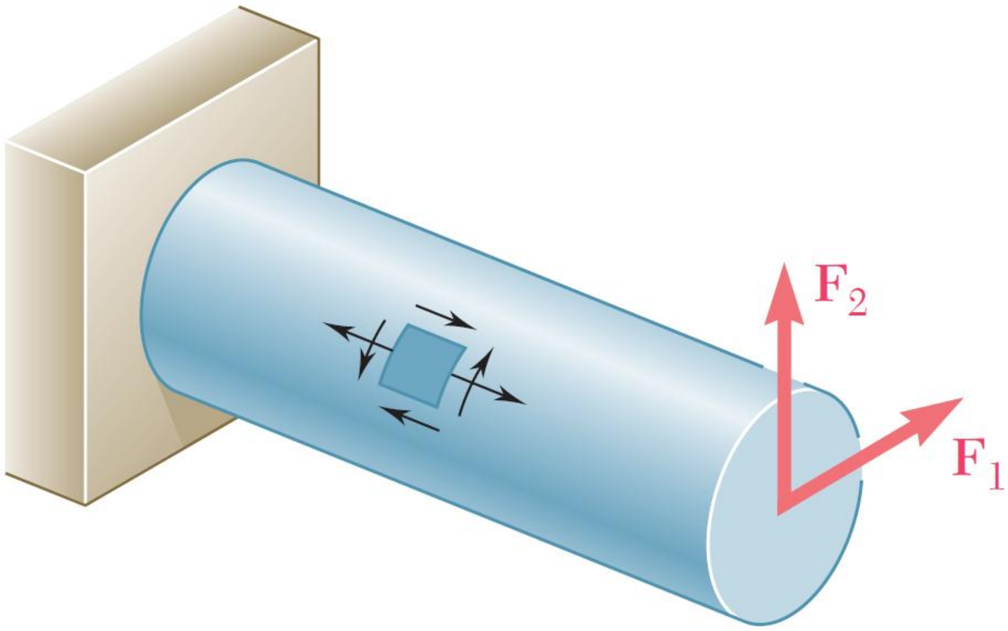


2

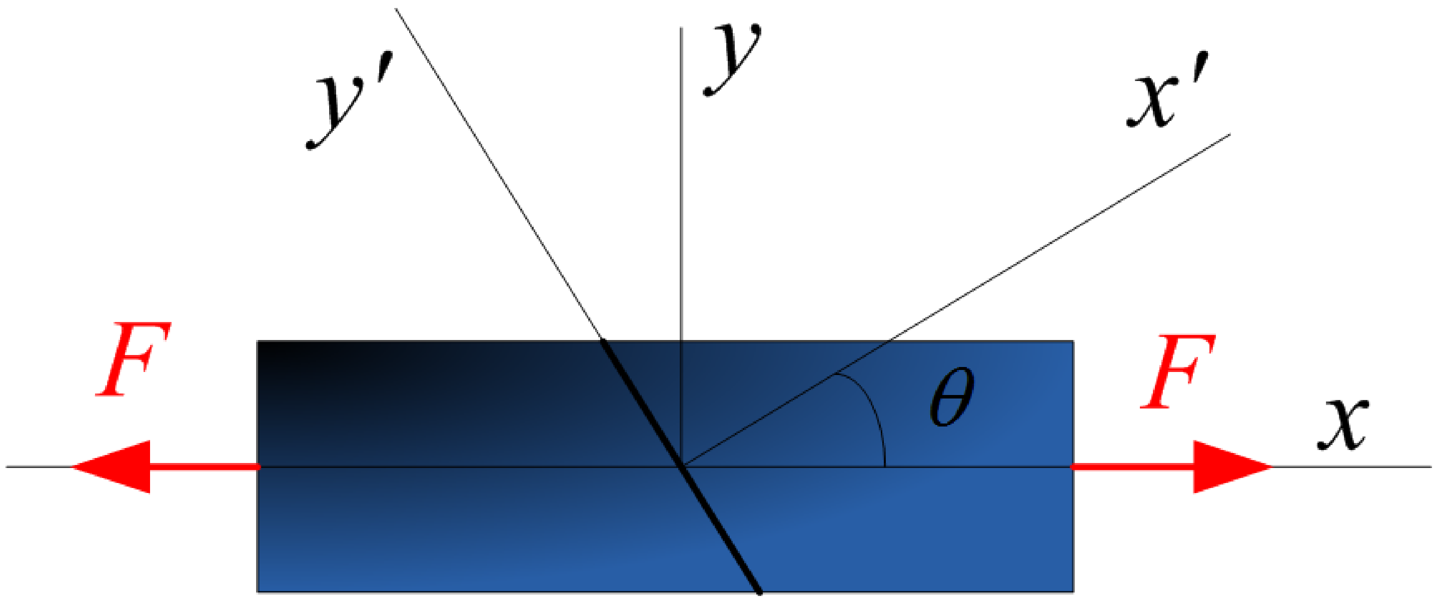
A special case of stress is the plane stress where only σ_x, σ_y and τ_{xy} components are existed.

Examples of plane stress are:





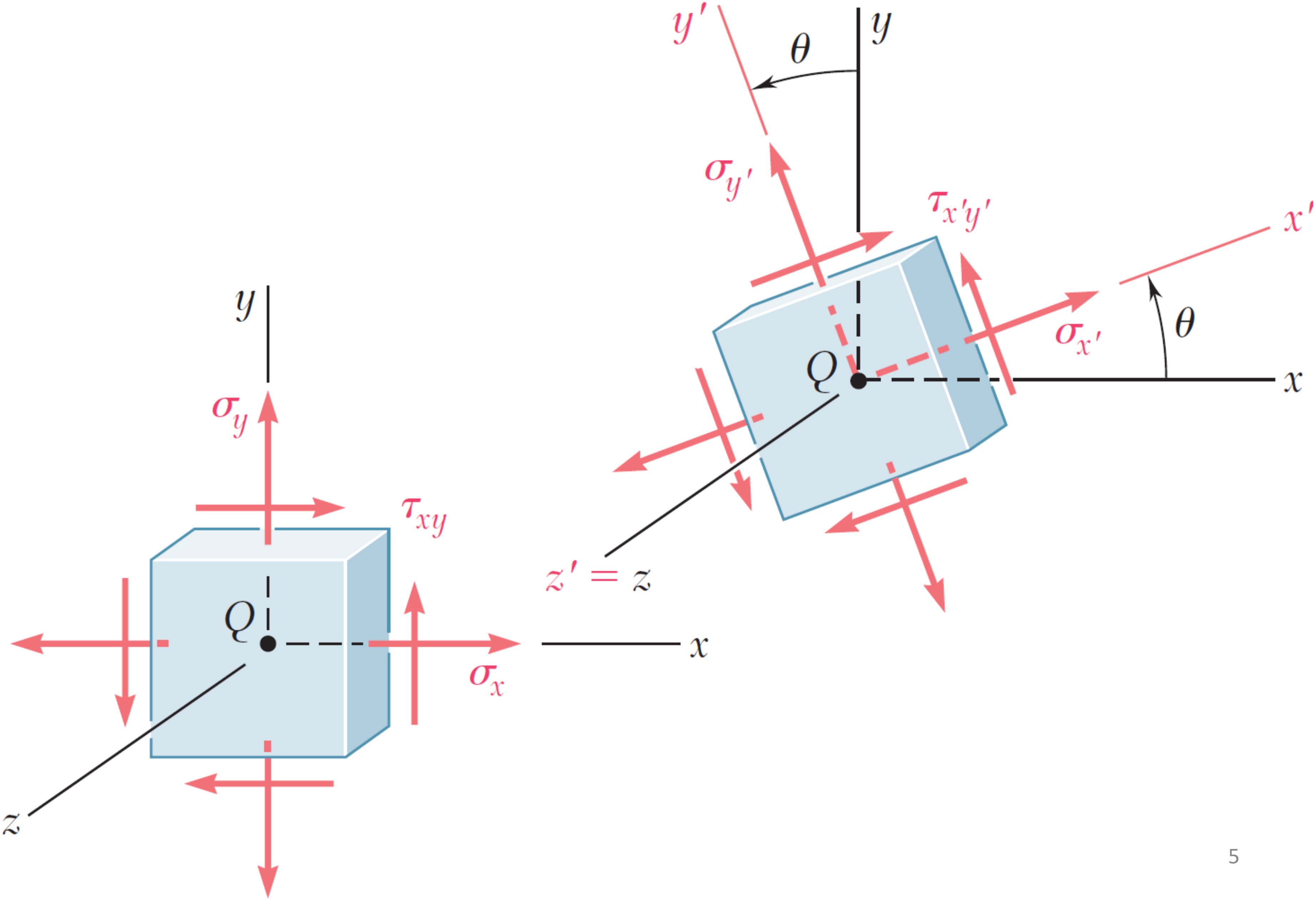
Recall section 1.11

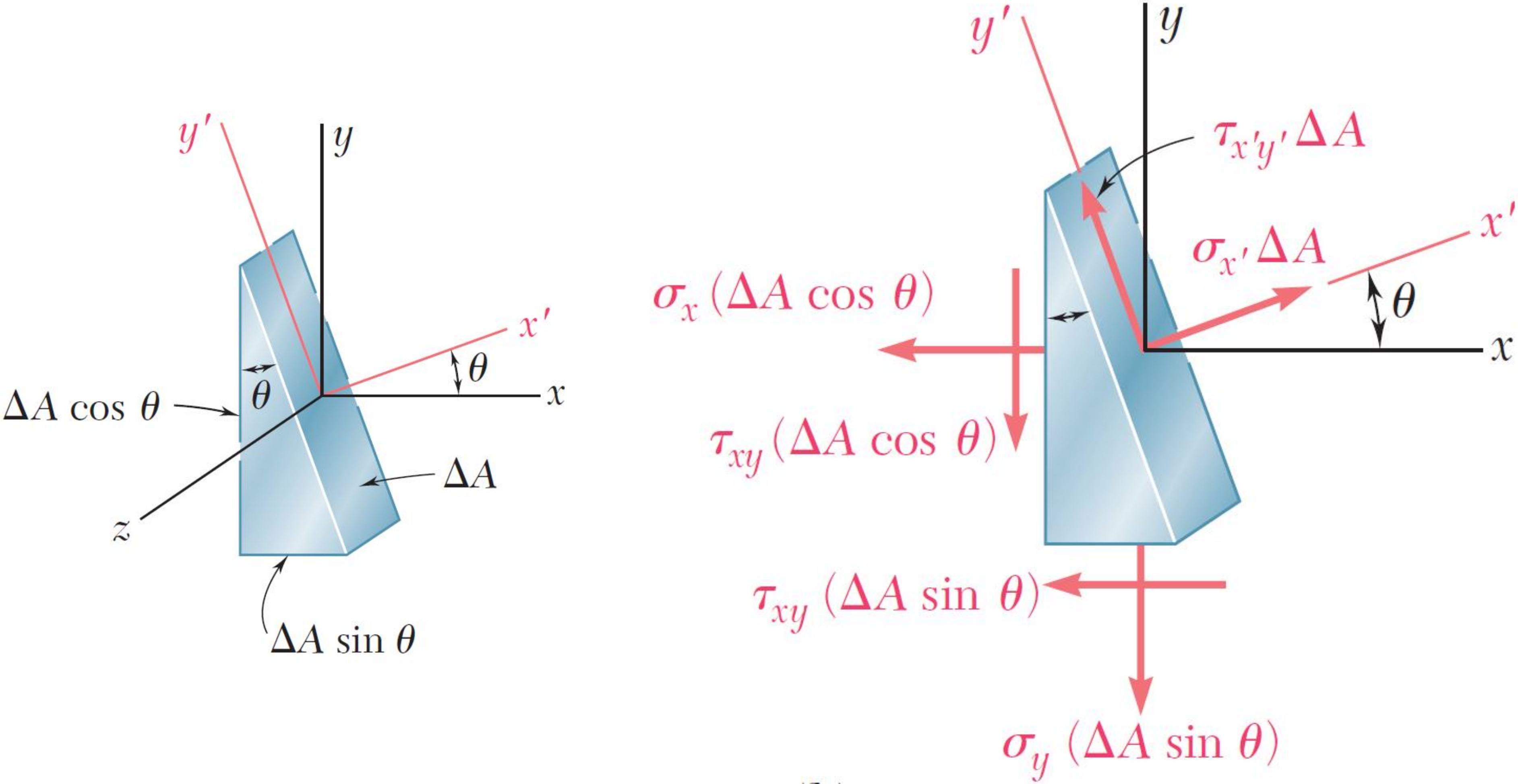


$$\sigma_x = \frac{\mathbf{F}}{A} \qquad \tau_{xy} = 0$$

$$\sigma_{x'} = \frac{F}{A}\cos^2\theta \qquad \tau_{x'y'} = -\frac{F}{A}\sin\theta\cos\theta$$

7.2 TRANSFORMATION OF PLANE STRESS





$$\sum F_{x'} = 0: \ \sigma_{x'}\Delta A - \sigma_x(\Delta A\cos\theta)\cos\theta - \tau_{xy}(\Delta A\cos\theta)\sin\theta$$
$$-\sigma_y(\Delta A\sin\theta)\sin\theta - \tau_{xy}(\Delta A\sin\theta)\cos\theta = 0$$

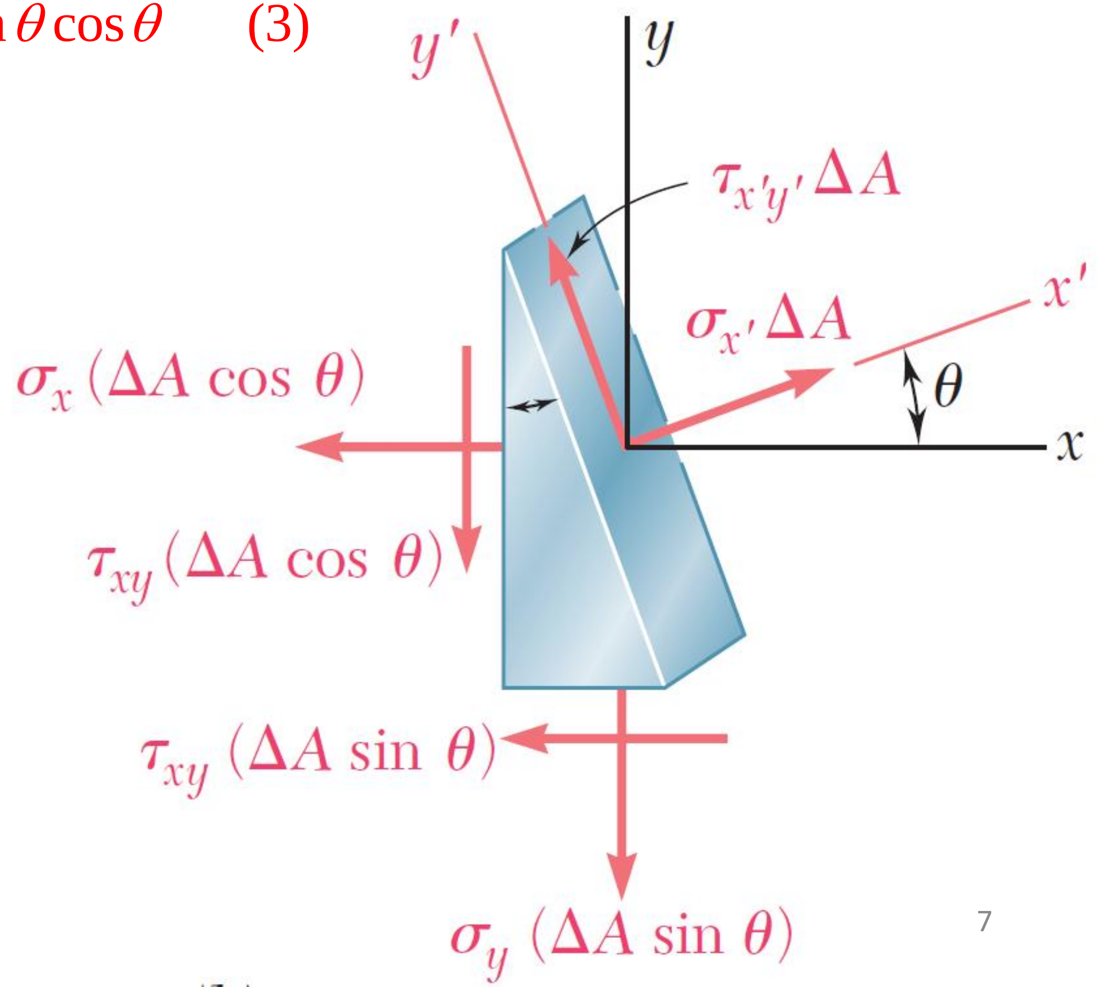
$$\sigma_{x'} = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \qquad (1)$$

$$\sum F_{y'} = 0: \ \tau_{x'y'}\Delta A + \sigma_x(\Delta A\cos\theta)\sin\theta - \tau_{xy}(\Delta A\cos\theta)\cos\theta$$
$$-\sigma_y(\Delta A\sin\theta)\cos\theta + \tau_{xy}(\Delta A\sin\theta)\sin\theta = 0$$

$$\tau_{x'y'} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \quad (2)$$

in same manner, $\sigma_{y'}$ is obtained as

$$\sigma_{y'} = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta \qquad (3)$$



TRANSFORMATION EQUATIONS SUMMARY

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta \qquad (1)$$

$$\sigma_{y'} = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \sin\theta\cos\theta \qquad (2)$$

$$\tau_{x'y'} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \qquad (3)$$

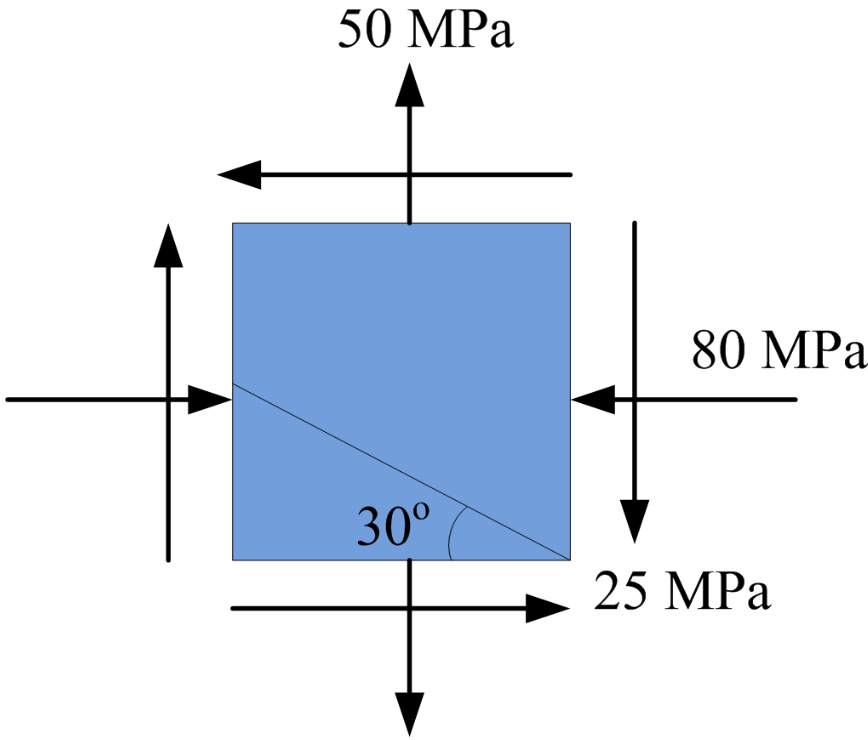
The equations can also be rewritten as:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (1)$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \qquad (2)$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (3)$$

Example: Find the stress on a surface making an angle 30° as shown in the figure aside.



Solution :

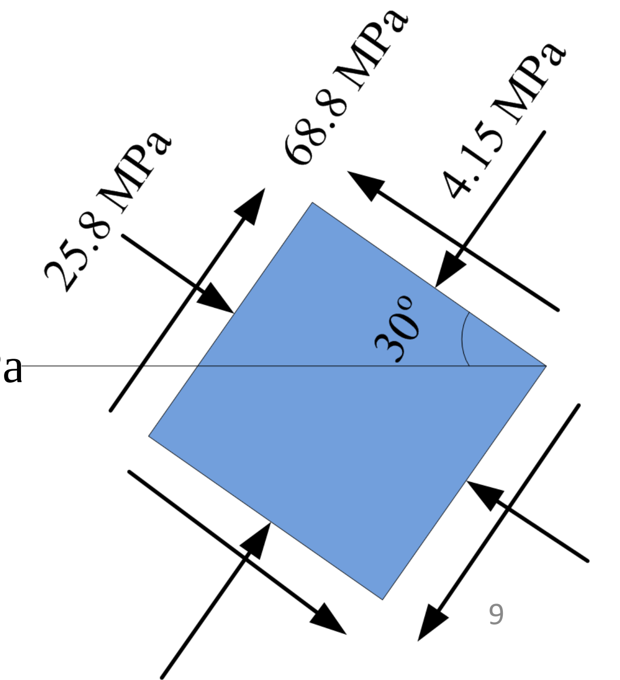
To coincide σ_x with the required surface,

a rotation of 60^o ccw is required, then :

$$\sigma_{x'} = -80\cos^2 60 + 50\sin^2 60 - 50\sin 60\cos 60 = -4.15 \text{ MPa}$$

$$\sigma_{y'} = -80\sin^2 60 + 50\cos^2 60 + 50\sin 60\cos 60 = -25.8 \text{ MPa}$$

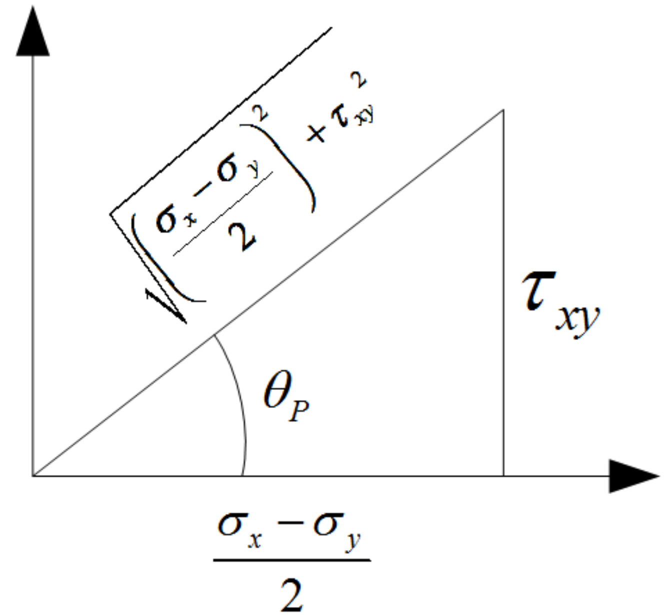
$$\tau_{x'y'} = (80+50)\sin 60\cos 60 - 25(\cos^2 60 - \sin^2 60) = 68.79 \text{ MPa}$$



7.3 PRINCIPAL STRESSES (MAXIMUM SHEARING STRESS)

- The maximum stresses can be obtained by finding the angle where the stress is maximum. i.e.

$$\frac{d\sigma_{x'}}{d\theta} = 0 \quad \rightarrow \quad \tan 2\theta_P = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$



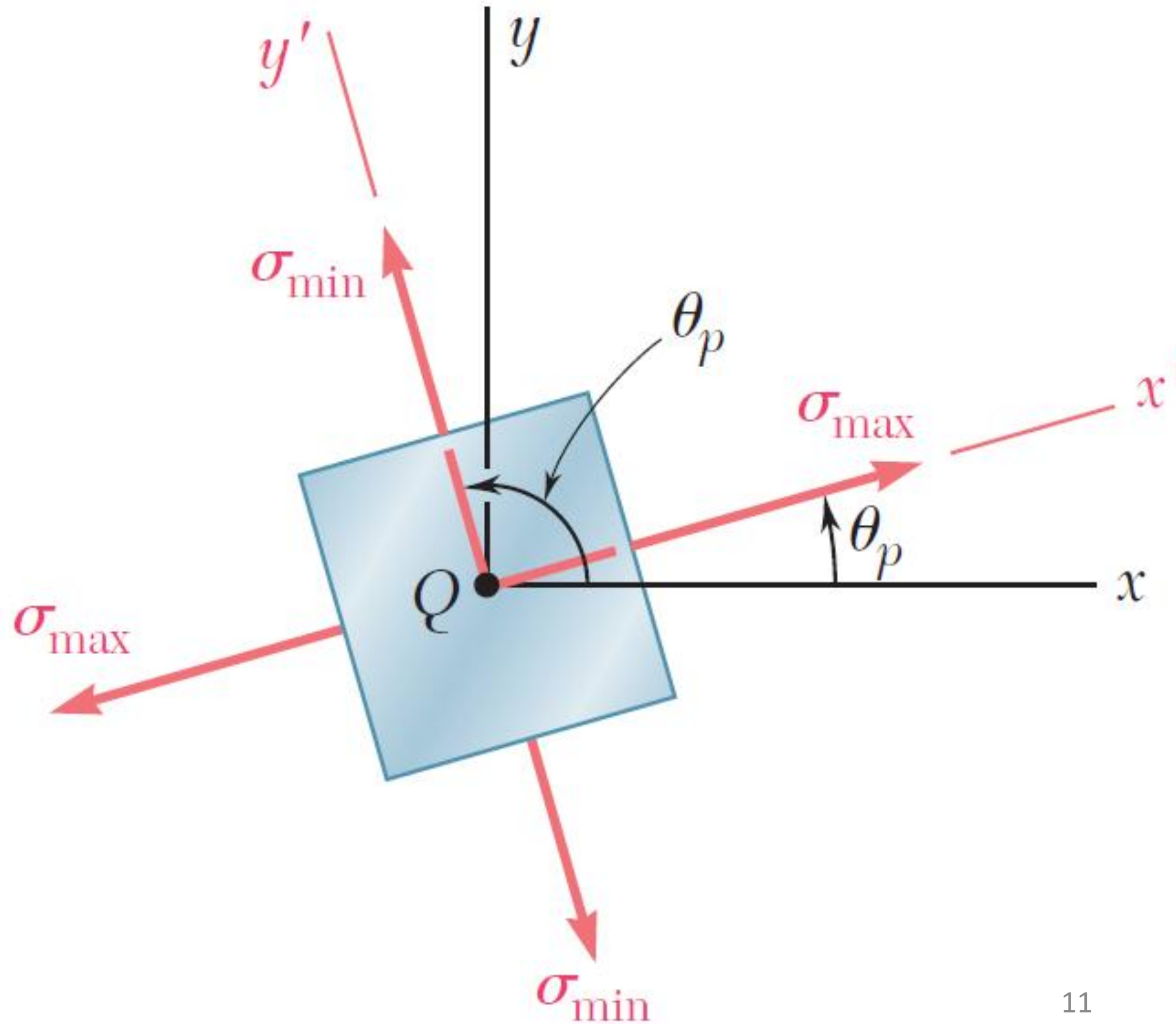
$$\sigma_{\min,\max} = \sigma_{x'}\Big|_{\theta=\theta_P} \quad \rightarrow \quad \sigma_{\min,\max} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\,2}}$$

$$\tau\Big|_{\theta=\theta_P} = 0$$

The stresses are called principal stresses and denoted as σ_1 and σ_2, where $\sigma_1 > \sigma_2$

- When the stress is principal (maximum and minimum), shear stresses does not exist

$$\tau\Big|_{\theta=\theta_P}=0$$



The maximum shear stresses can be obtained by finding the angle where the shear stress is maximum. i.e.

$$\frac{d\tau_{x'y'}}{d\theta} = 0 \quad \rightarrow \quad \tan 2\theta_S = \frac{-(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

note that $\tan 2\theta_S$ is the negative reciprocal of $\tan 2\theta_P$

$i.e.$ $2\theta_S$ and $2\theta_P$ are 90^o apart.

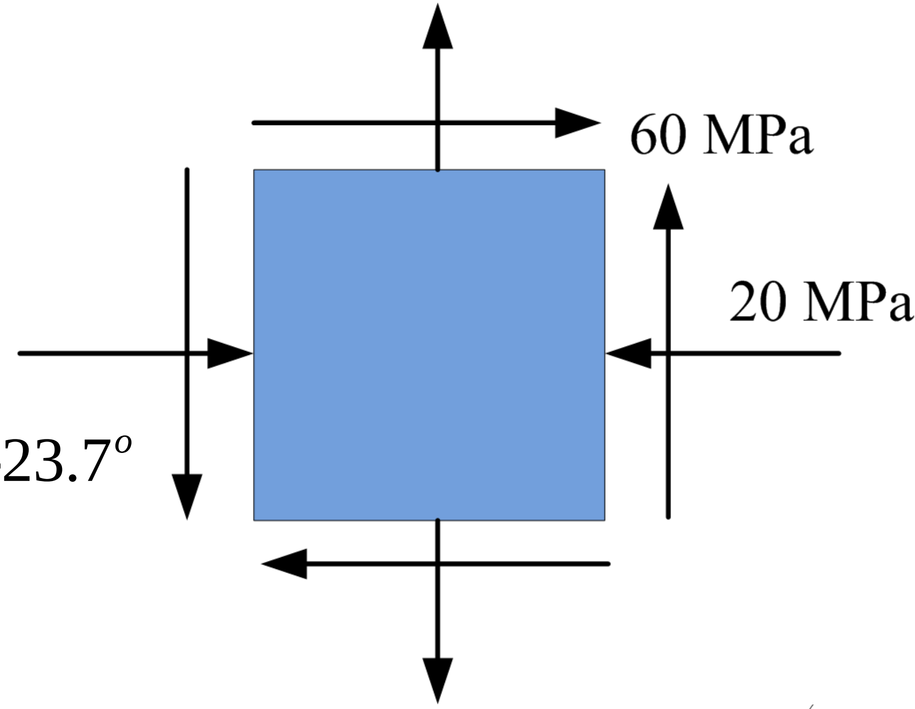
$i.e.$ the maximum shear stress is located 45^o from the principal planes.

$$\tau_{\max} = \tau_{x'y'}\Big|_{\theta=\theta_S} \quad \rightarrow \quad \tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

$$\sigma\Big|_{\theta=\theta_S} = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$

When the shear stress is maximum, normal stresses are still exist

Example: Find the principal stresses and the maximum shear stress and their planes.



Solution :

$$\tan 2\theta_P = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \rightarrow \quad \theta_{P_{1,2}} = 66.3^o, -23.7^o$$

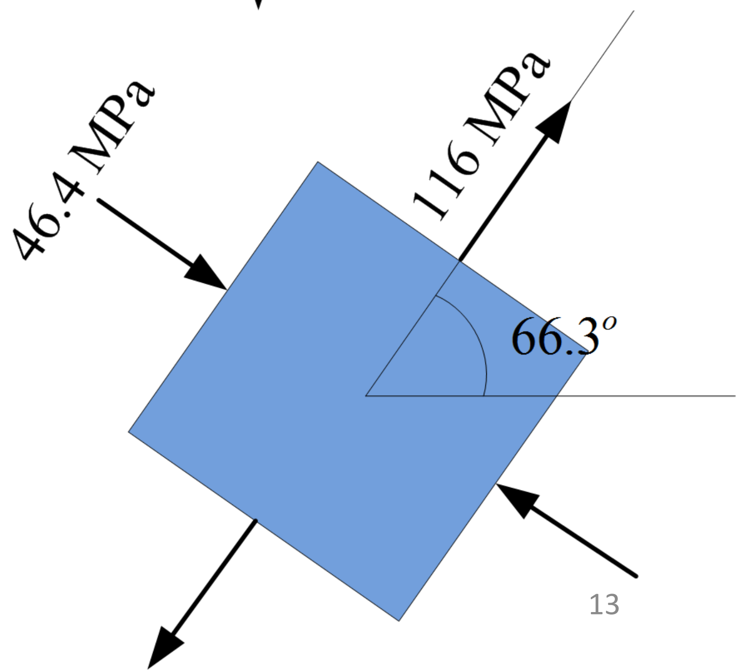
$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$\sigma_1 = 116$ MPa, $\sigma_2 = -46.4$ MPa

to know which angle correspoinds to σ_1

apply $\sigma_{x'} \big|_{\theta=\theta_{P1}}$.

Conclude that θ_{P1} corresponds σ_1



the maximum shear stress is

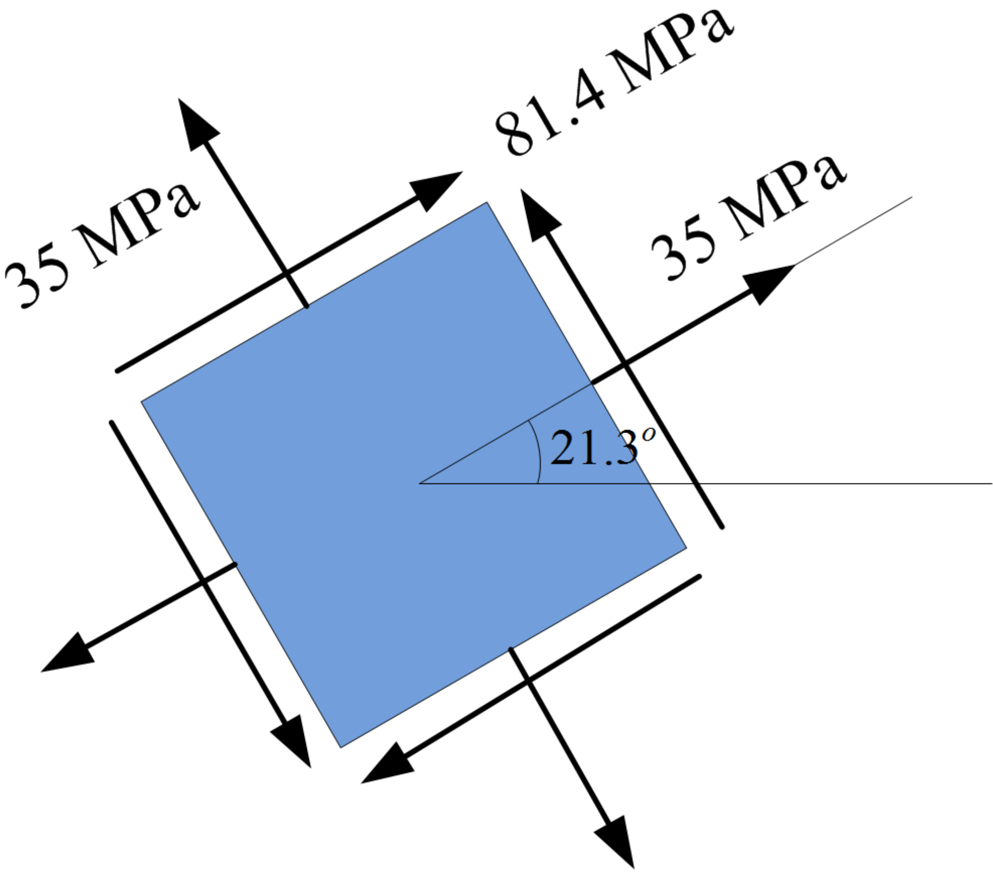
$$\tau_{1,2} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} = \pm 81.4 \text{ MPa}$$

$$\tan 2\theta_S = \frac{-(\sigma_x - \sigma_y)}{2\tau_{xy}}$$

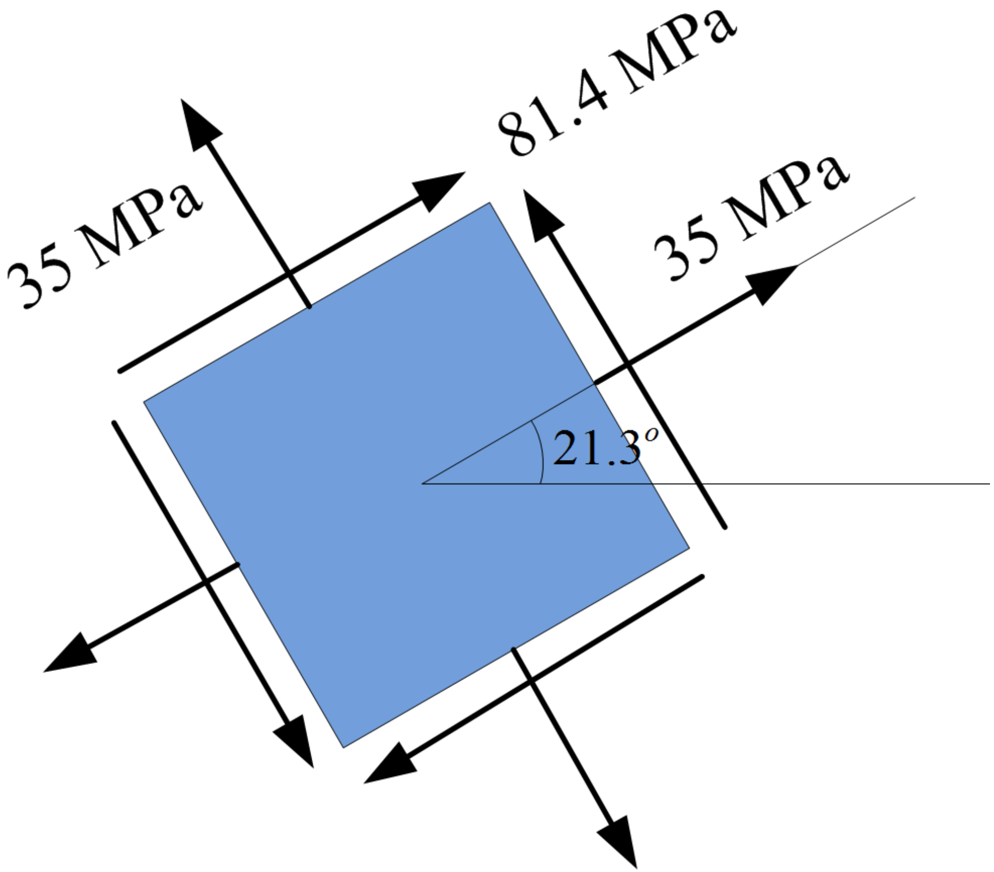
$$\theta_{S_1} = 21.3^o, \quad \theta_{S_2} = 111.3^o$$

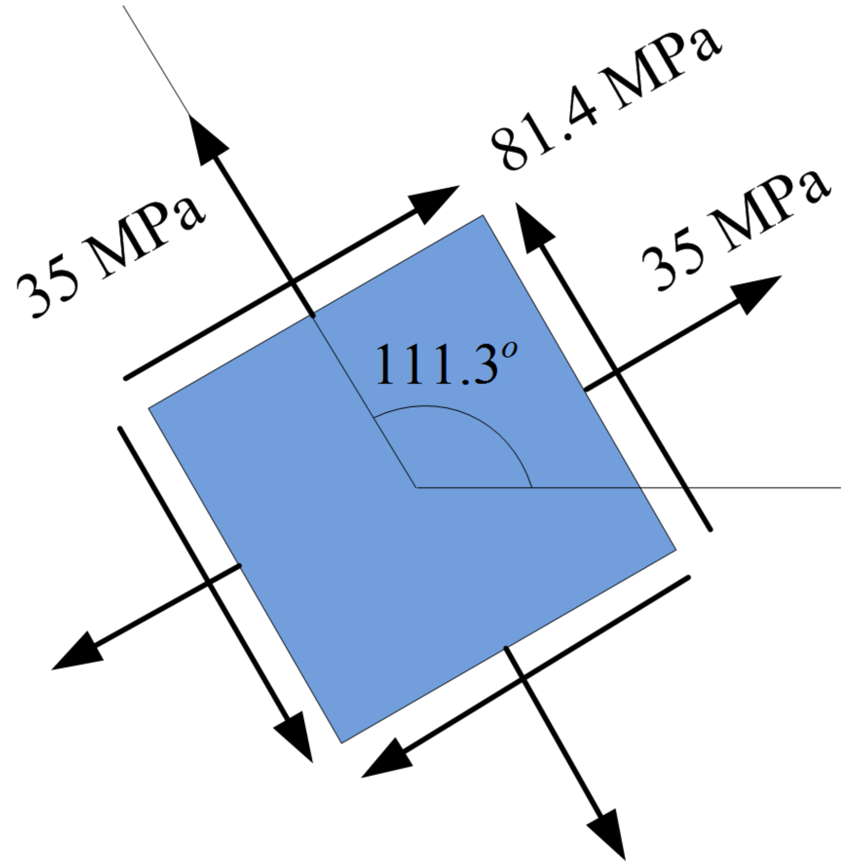
$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = 35 \text{ MPa}$$

check at $\tau_{x'y'} \big|_{\theta=\theta_{S1}} = +81.4 \text{ MPa}$

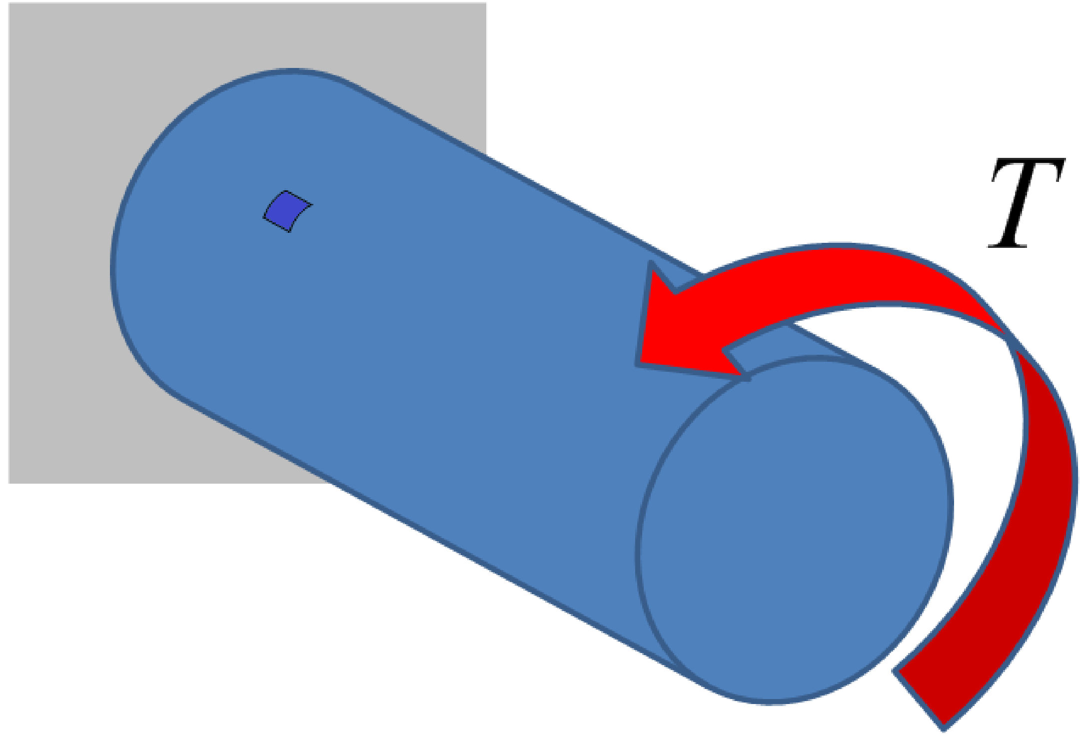


FURTHER DISCUSSION





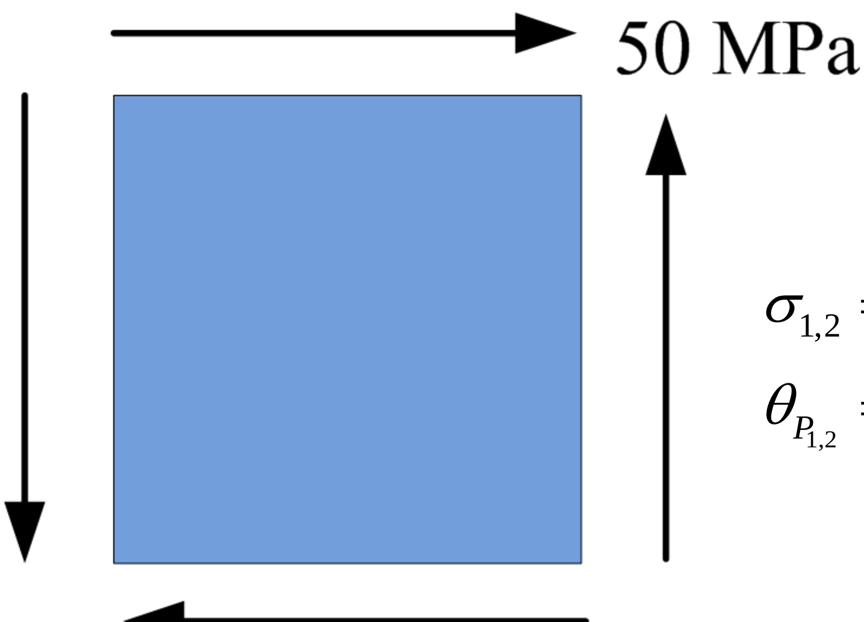
Example: Find the principal stresses and the maximum shear stress for the rod shown.



Solution :

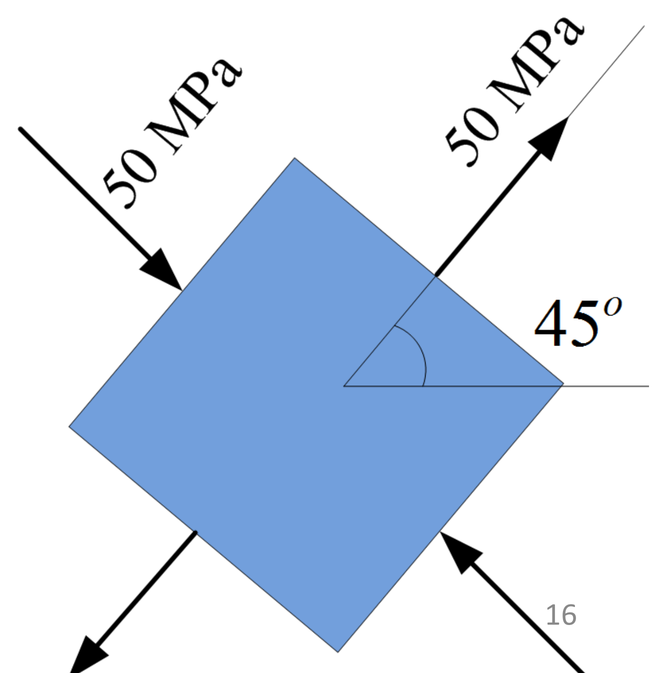
apply $\tau = \dfrac{\mathbf{T}\cdot c}{J}$, to get τ

assume $\tau = 50$ MPa

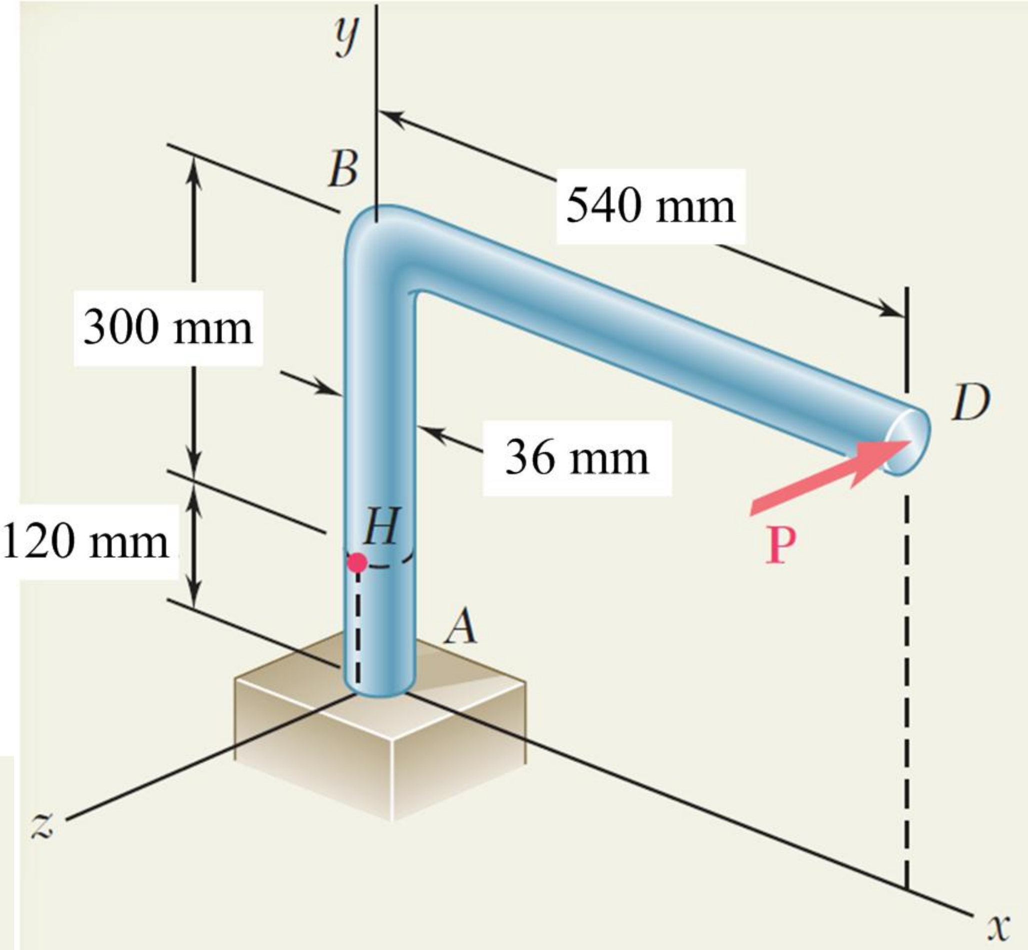


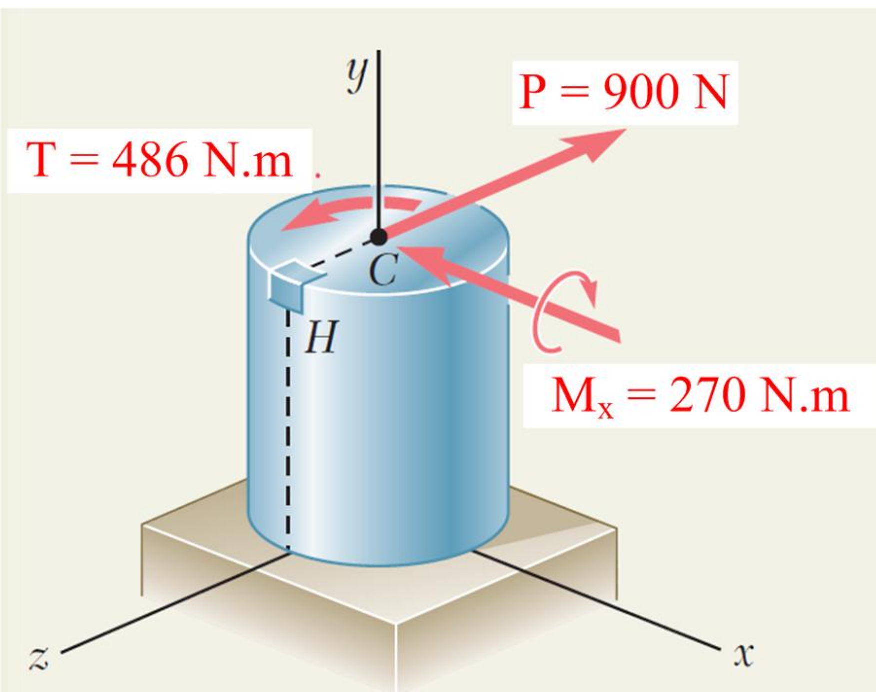
$$\sigma_{1,2} = \pm 50 \text{ MPa}$$

$$\theta_{P_{1,2}} = \pm 45^{o}$$



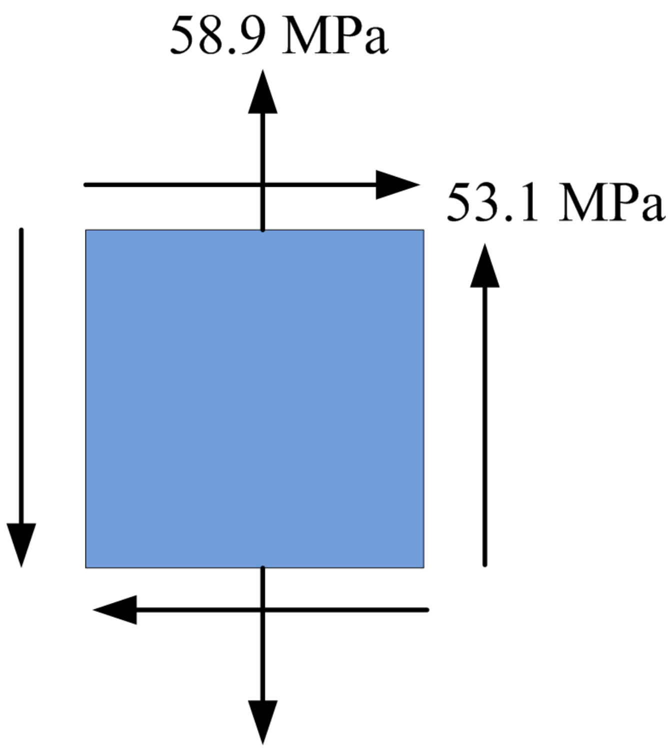
Example: Find the maximum normal stress for the member shown aside





$$\sigma_y = \frac{\mathbf{M_x} \cdot c}{I} = \frac{270 \times 0.018}{\frac{\pi}{64}(0.036)^4} = 58.9 \text{ MPa}$$

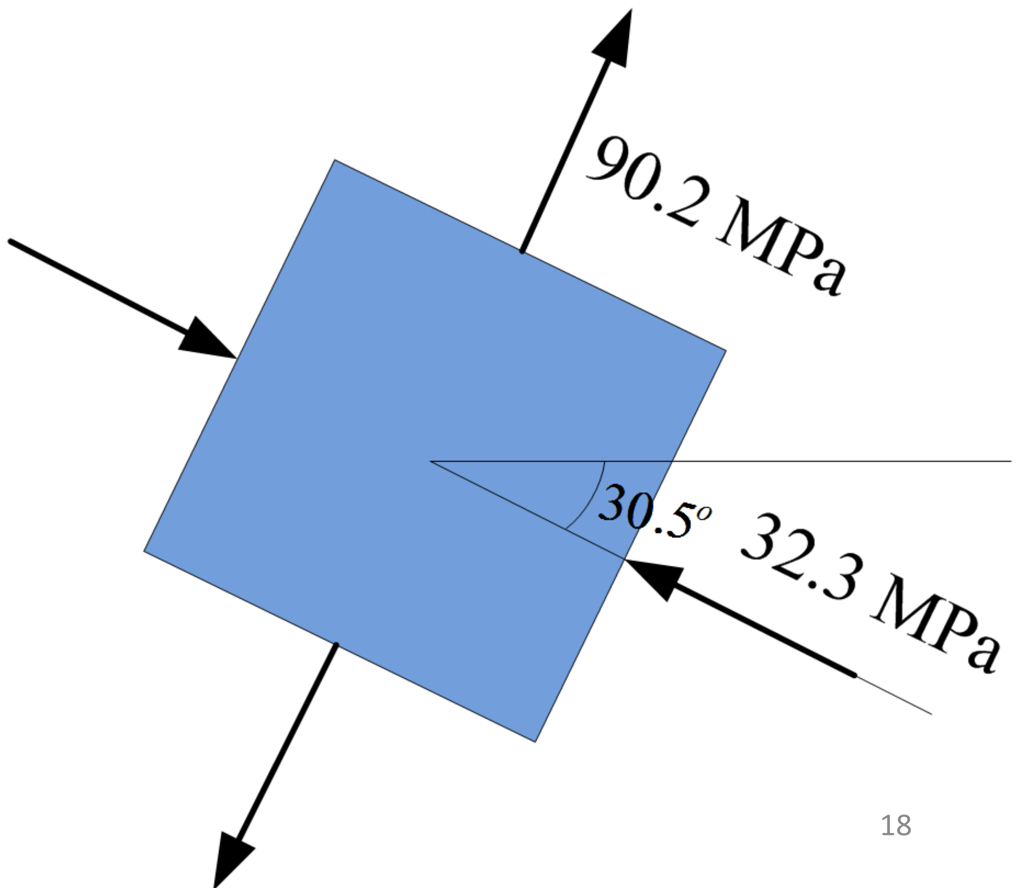
$$\tau_{xy} = \frac{\mathbf{T} \cdot c}{J} = \frac{486 \times 0.018}{\frac{\pi}{32}(0.036)^4} = 53.1 \text{ MPa}$$



$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

$$\sigma_1 = 90.2 \text{ MPa}, \quad \sigma_2 = -32.3 \text{ MPa}$$

$$\tan 2\theta_P = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \rightarrow \quad \theta_{P_{1,2}} = -30.5^o, 59.5^o$$



7.4 MOHR'S CIRCLE FOR PLANE STRESS

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (1)$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (2)$$

square both equations and sum them, we will get

$$\left[\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right]^2 + \left[\tau_{x'y'}\right]^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2 \qquad (*)$$

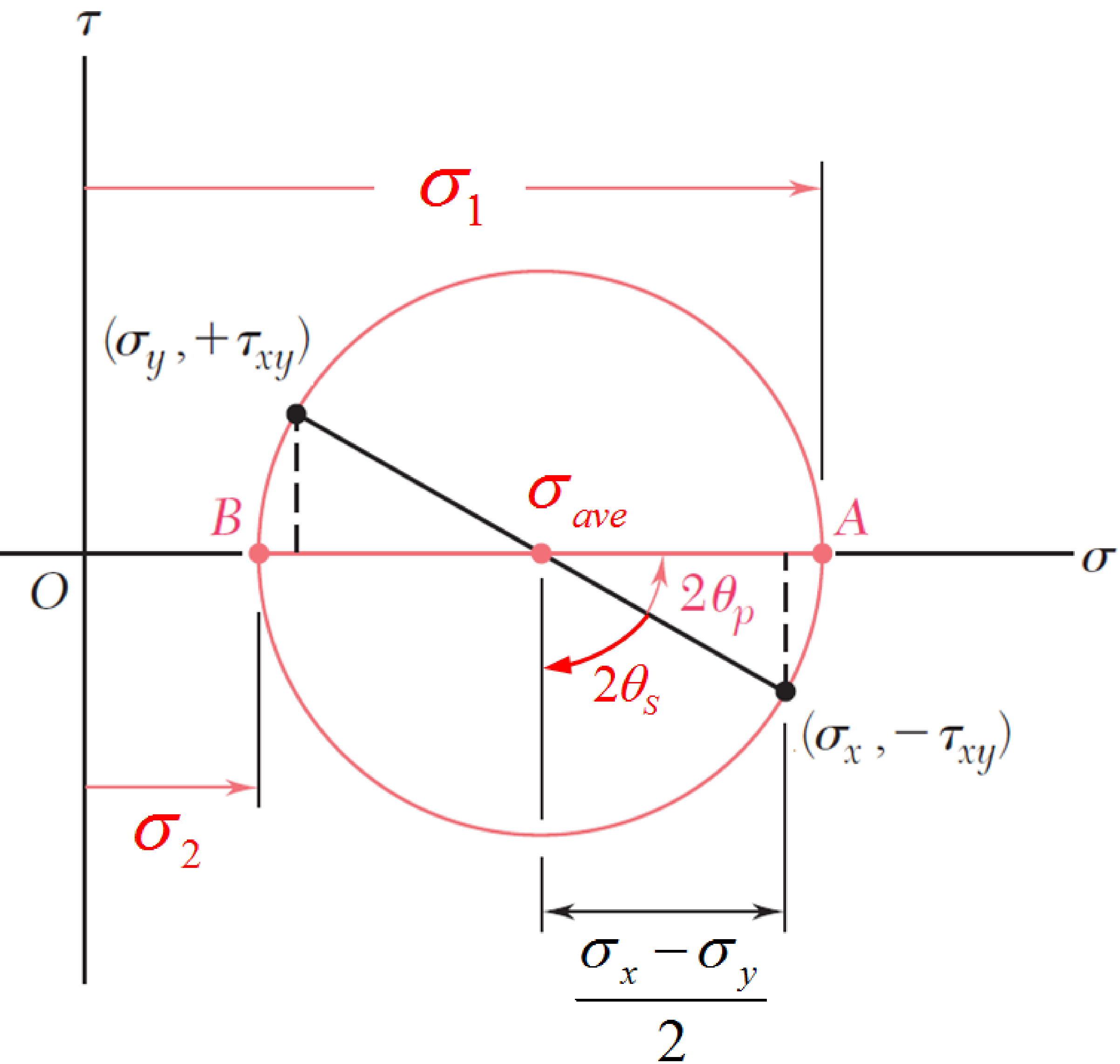
let

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}$$

then, * will be rewritten as

$$\left(\sigma_{x'} - \sigma_{ave}\right)^2 + \left(\tau_{x'y'}\right)^2 = R^2$$ (Equation of a circle)



$$\text{center} = (\sigma_{ave}, 0)$$

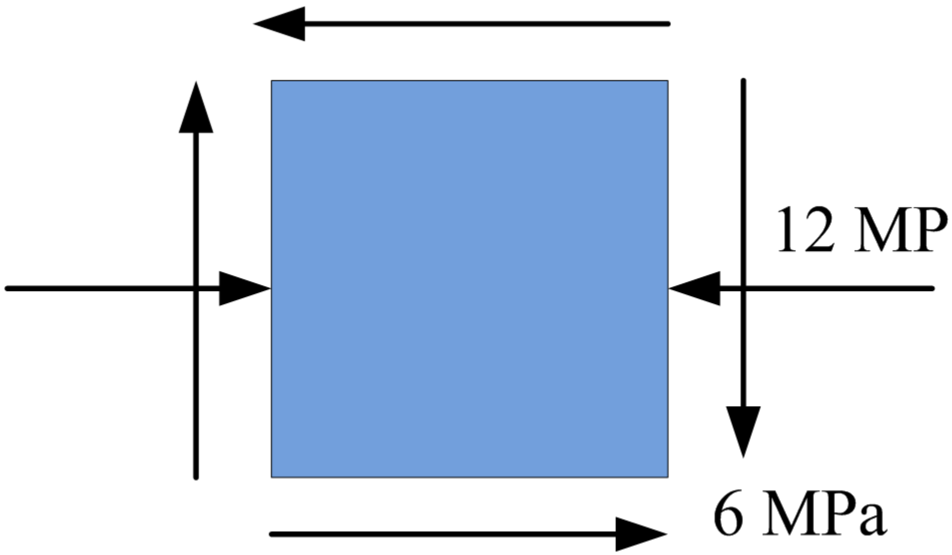
points on the circle

$$(\sigma_x, -\tau_{xy})$$

$$(\sigma_y, \tau_{xy})$$

- At the stress orientation represented by the black line; if you rotate the element ccw by θ_P you will get the principal stresses.
- If you rotate cw by θ_S you will get the maximum shear

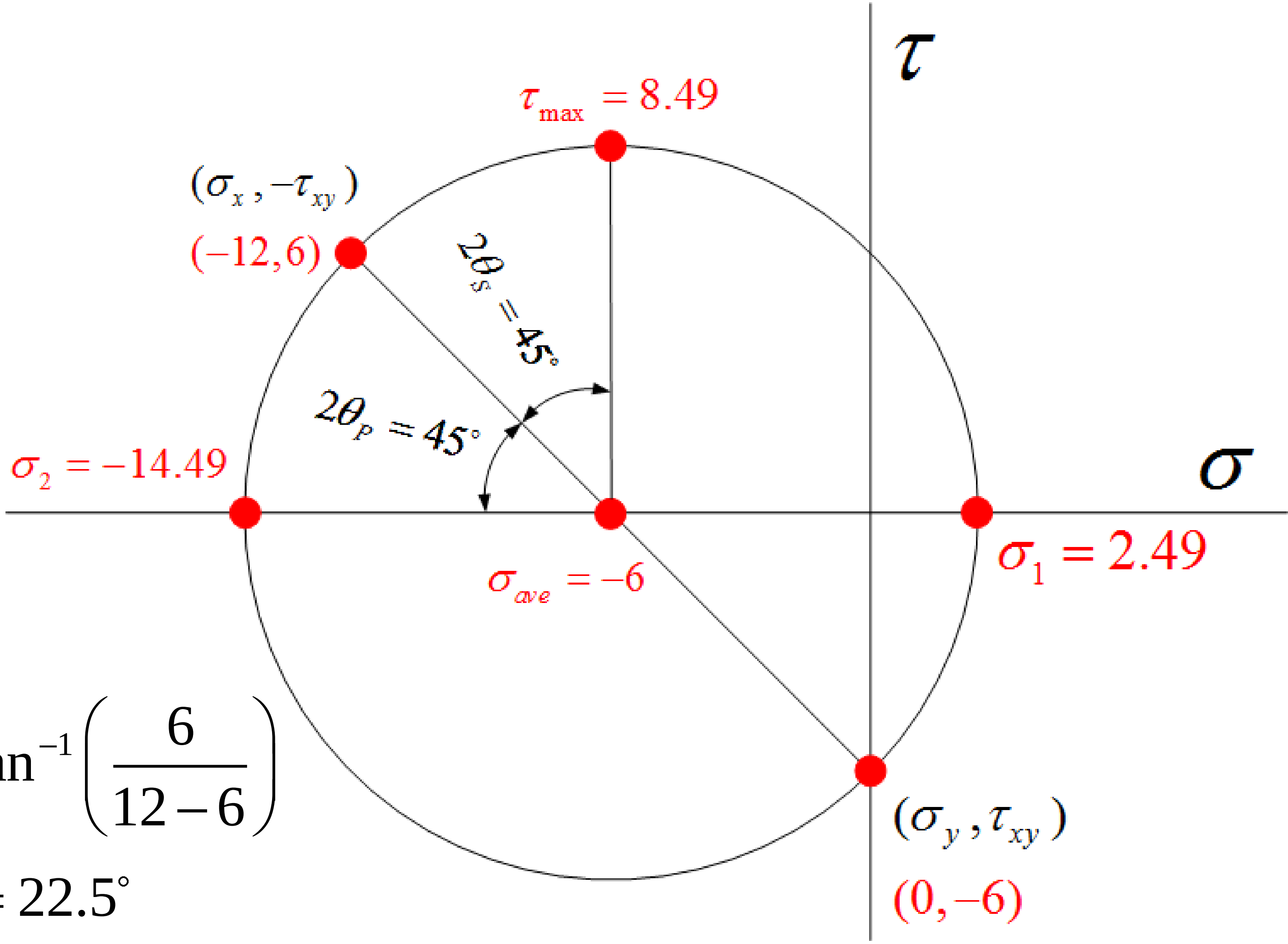
Example: Draw the mohr's circle for the shown element.



Solution :

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = -6 \text{ MPa}$$

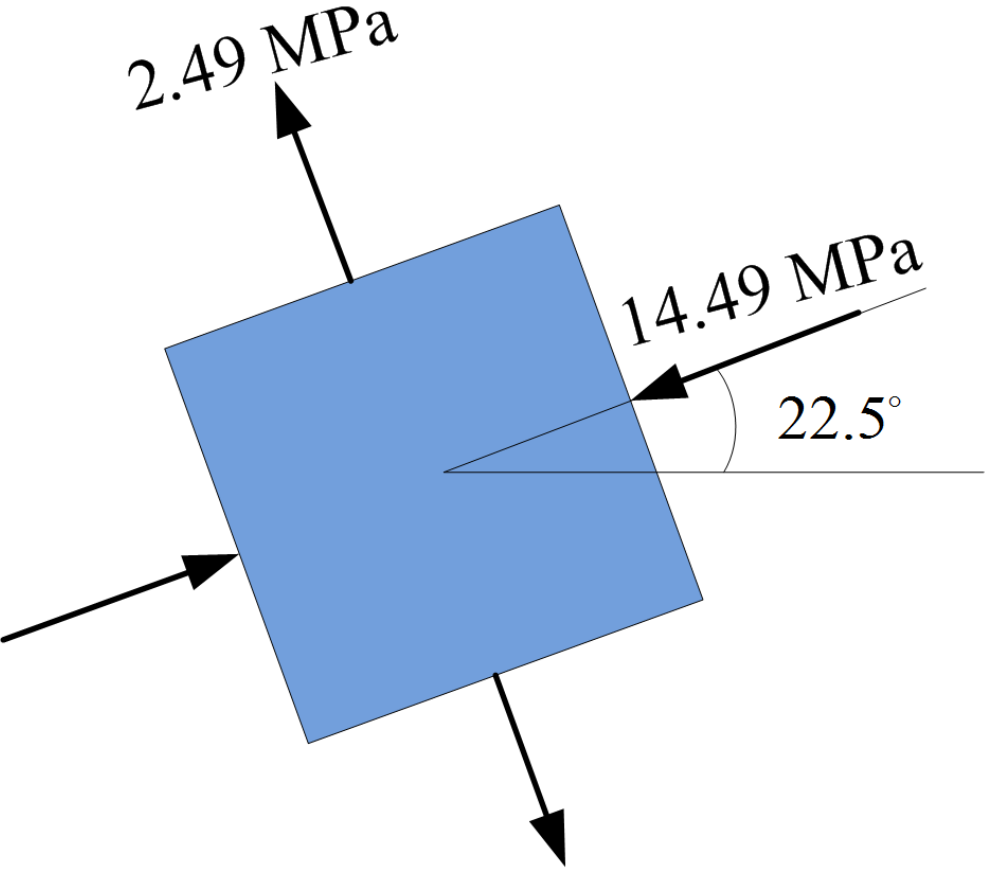
$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2} = 8.49 \text{ MPa}$$



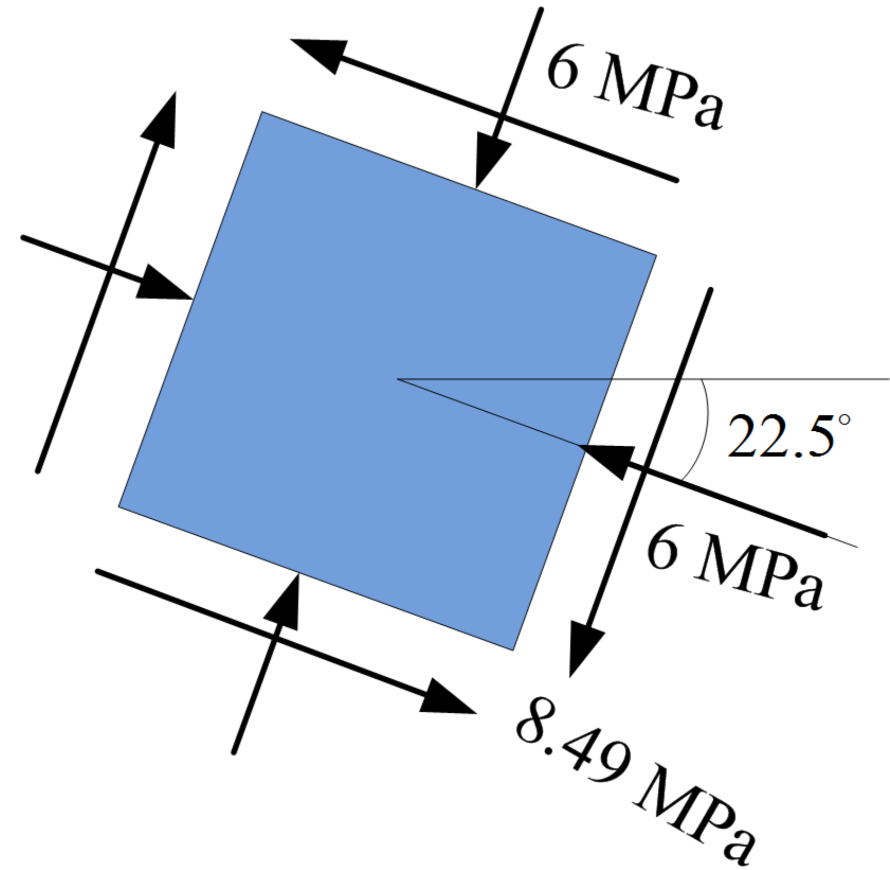
$$2\theta_P = \tan^{-1}\left(\frac{6}{12-6}\right)$$

$$\rightarrow \theta_P = 22.5^\circ$$

$$\rightarrow \theta_S = 22.5^\circ$$

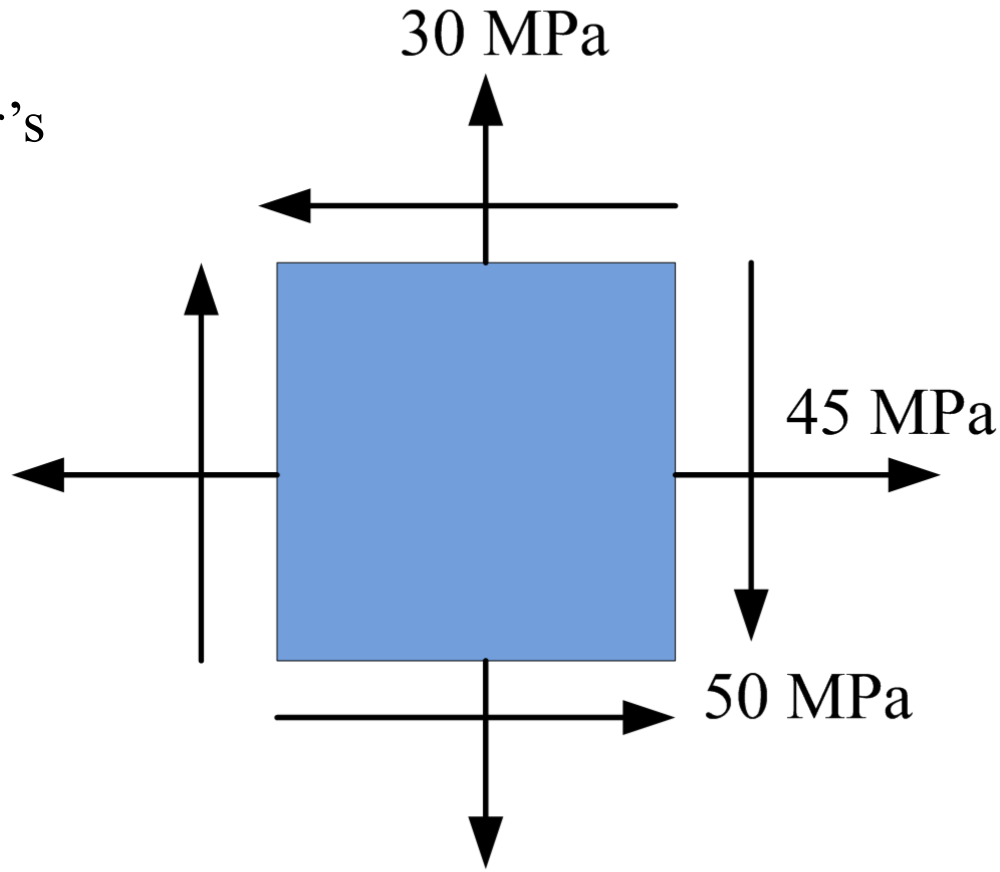


- if you rotate the element ccw by 22.5 deg. you will get the principal stresses.



- If you rotate cw by 22.5 deg. you will get the maximum shear stress.

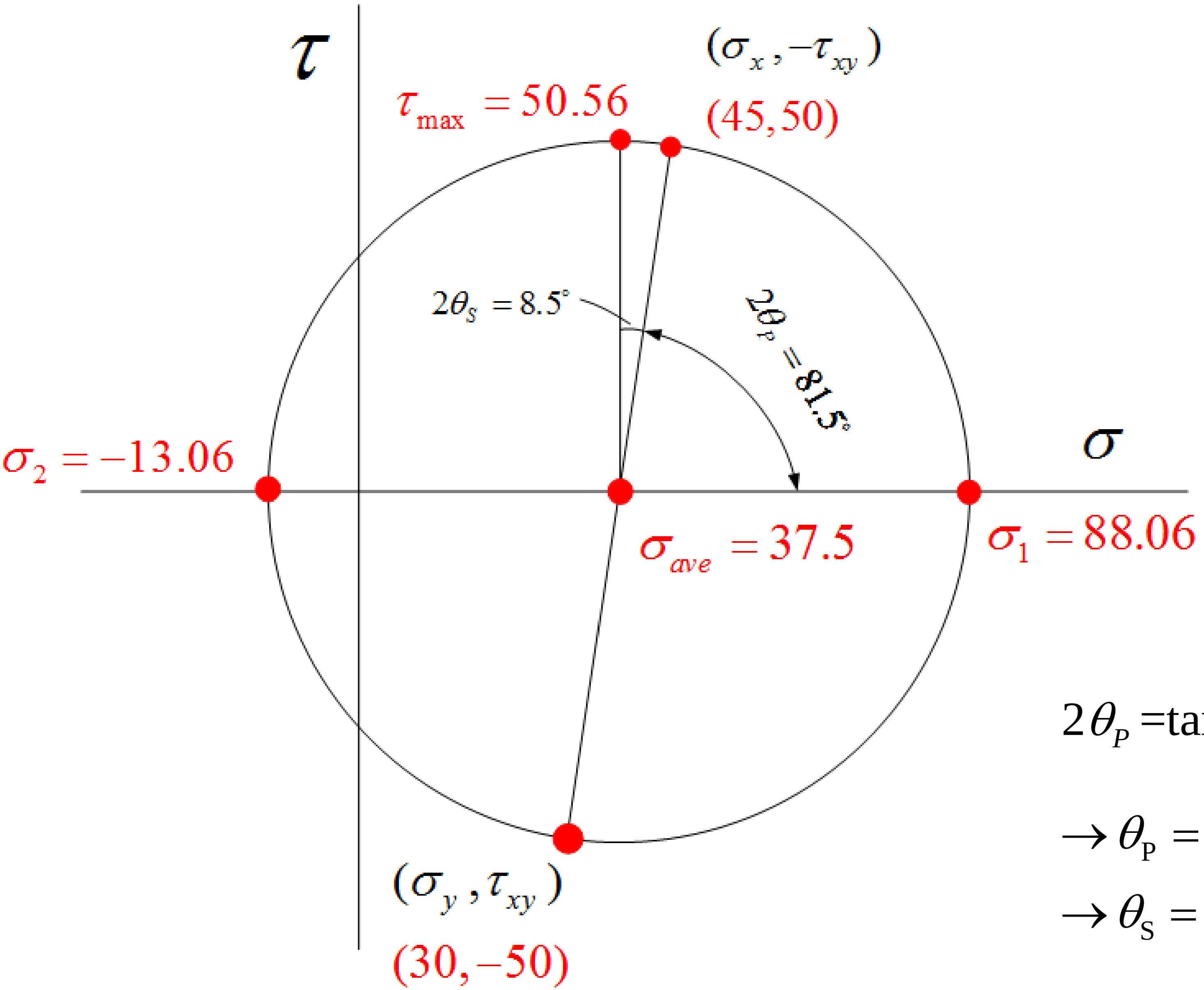
Example: Draw the mohr's circle for the shown element.



Solution :

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = 37.5 \text{ MPa}$$

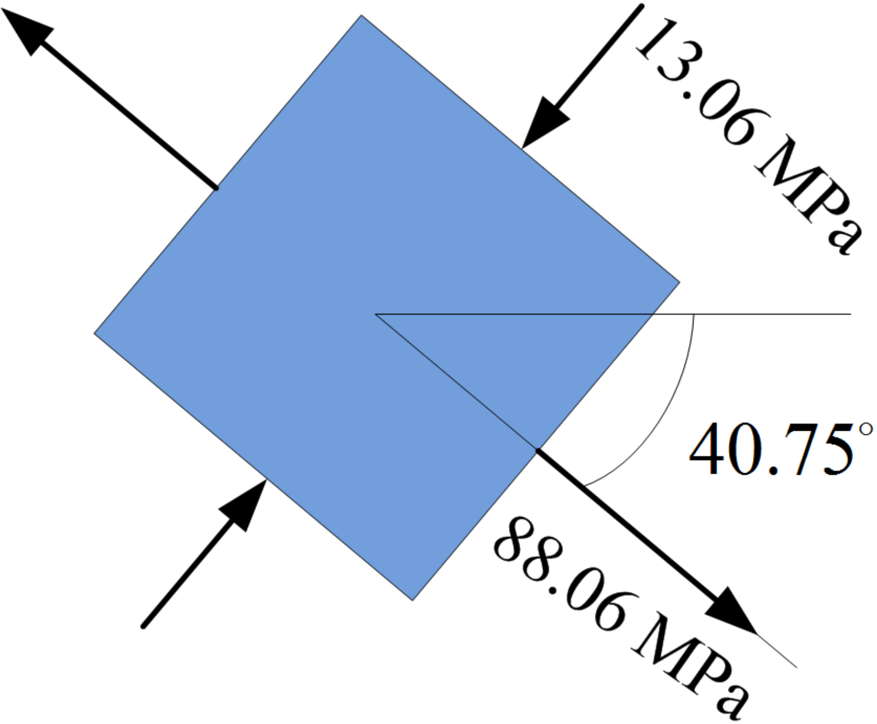
$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2} = 50.56 \text{ MPa}$$

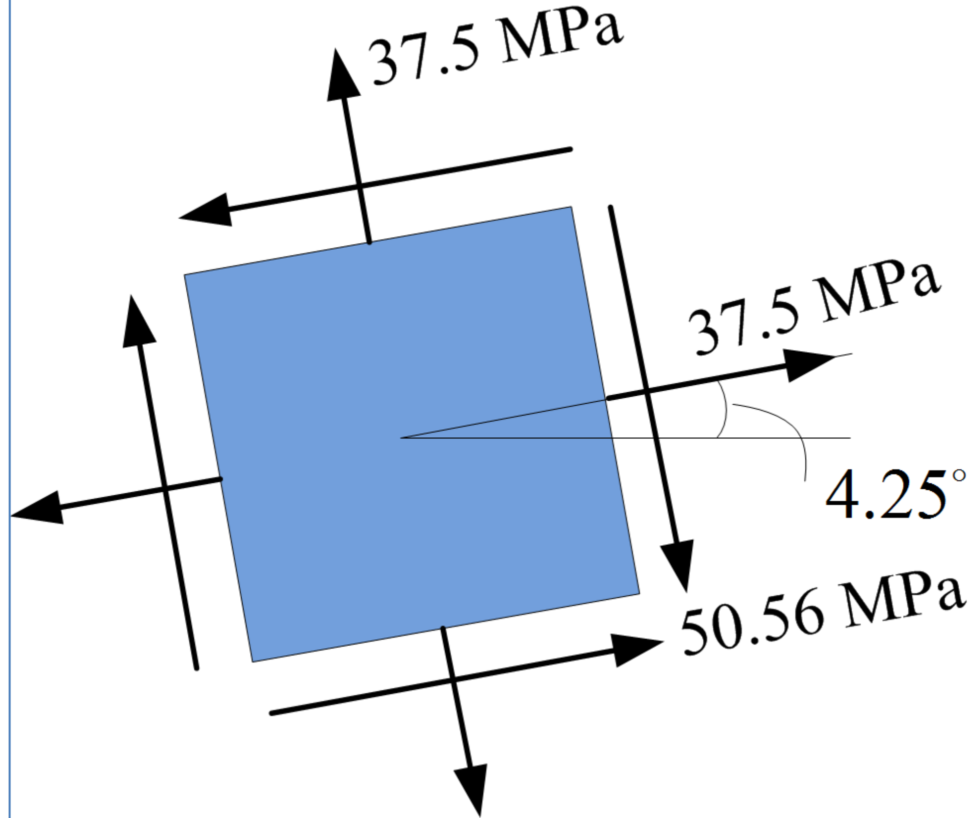


$$2\theta_P = \tan^{-1}\left(\frac{50}{45-37.5}\right)$$

$$\rightarrow \theta_P = 40.75^\circ$$

$$\rightarrow \theta_S = 4.25^\circ$$

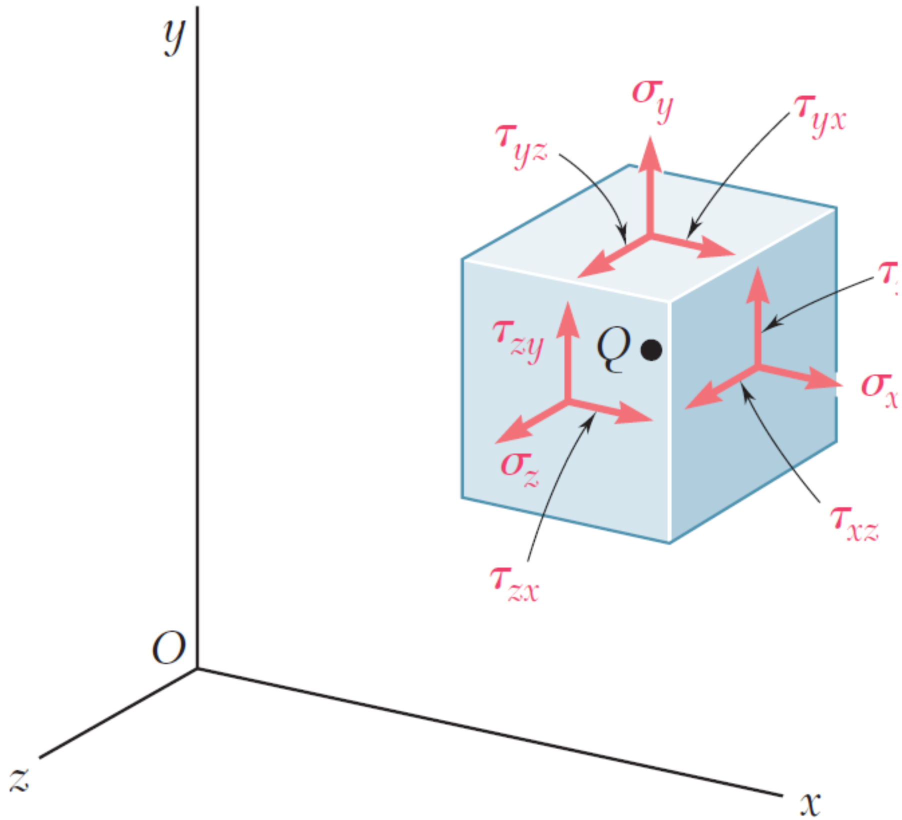


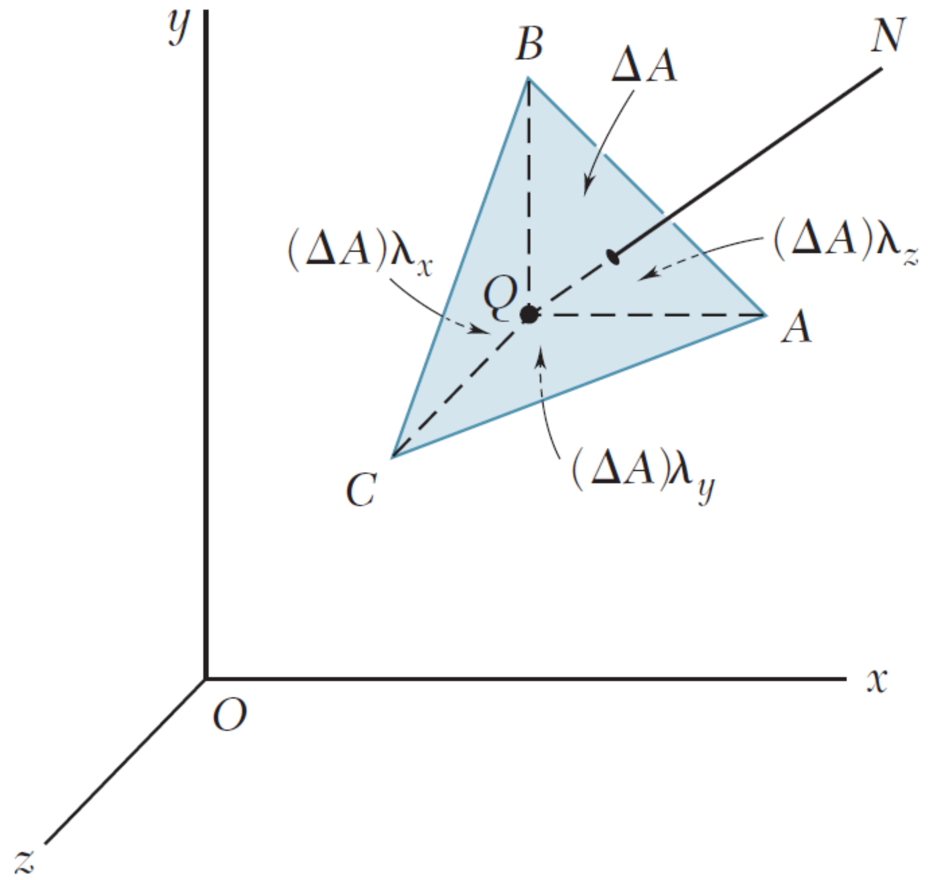


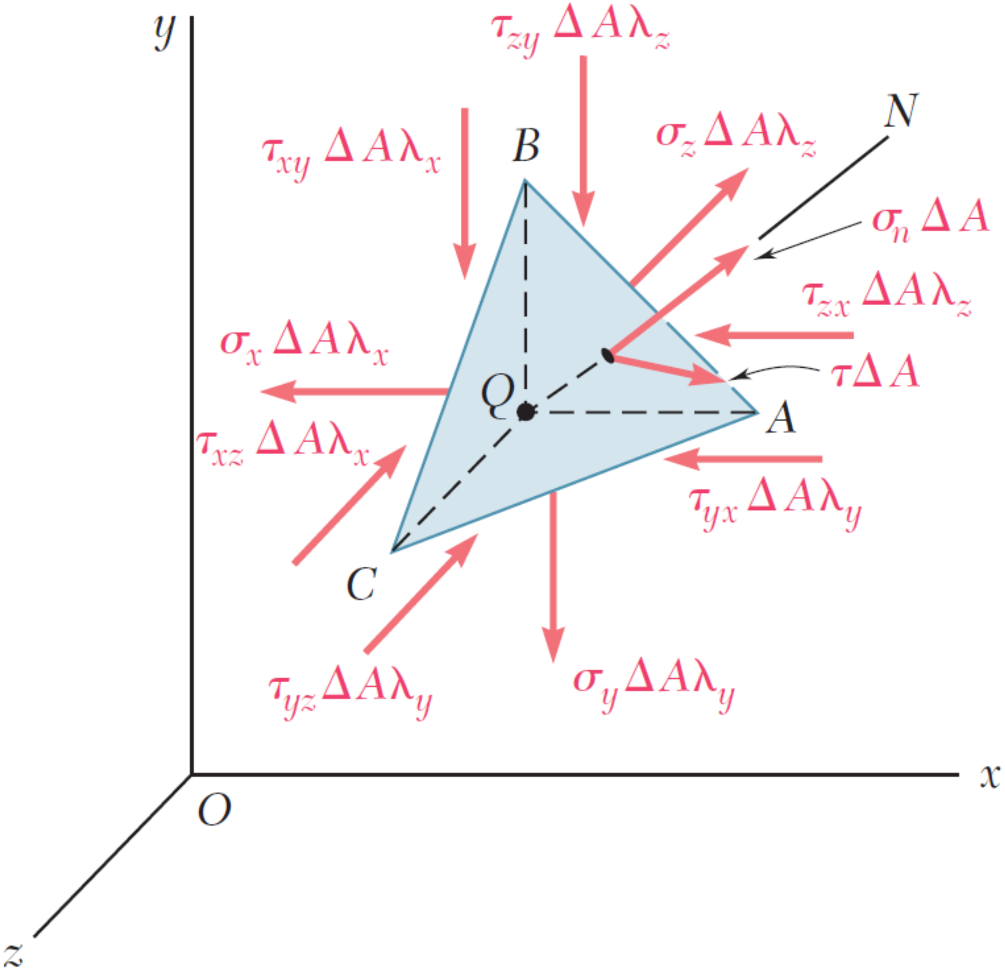
- if you rotate the element cw by 40.75 deg. you will get the principal stresses.
- If you rotate ccw by 4.25 deg. you will get the maximum shear stress.

7.5 GENERAL STATE OF STRESS

$$\mathbf{N} = \lambda_x \mathbf{i} + \lambda_y \mathbf{j} + \lambda_z \mathbf{k}$$





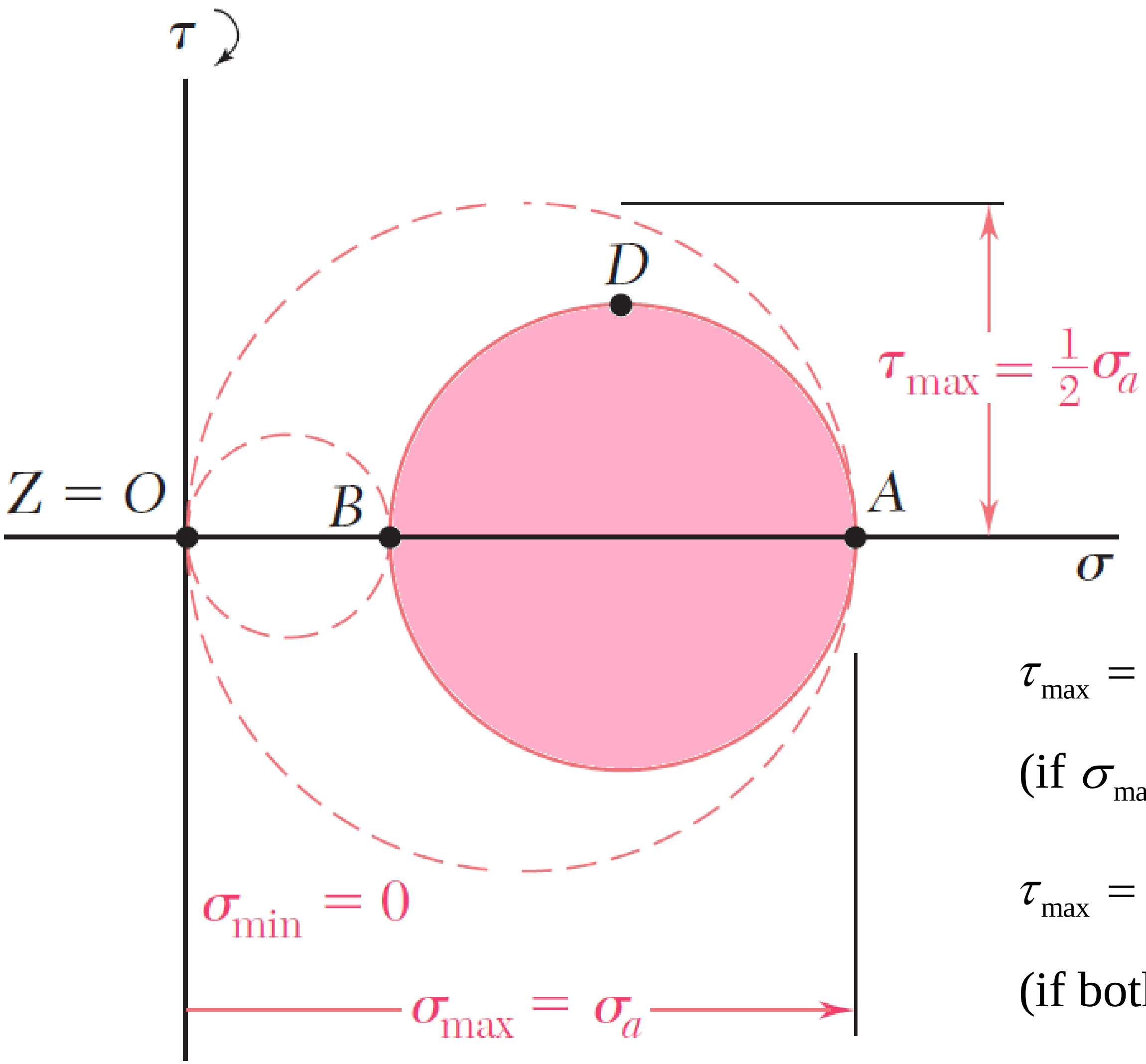


$$\sum F_n = 0$$

$$\sigma_n = \sigma_x\lambda_x^2 + \sigma_y\lambda_y^2 + \sigma_z\lambda_z^2 + \tau_{xy}\lambda_x\lambda_y + \tau_{xz}\lambda_x\lambda_z + \tau_{yz}\lambda_y\lambda_z$$

$$\sigma_n = \sigma_1\lambda_1^2 + \sigma_2\lambda_2^2 + \sigma_3\lambda_3^2$$

7.6 APPLICATION OF MOHR'S CIRCLE TO THE THREE DIMENSIONAL ANALYSIS OF STRESS (MAXIMUM SHEAR STRESS)

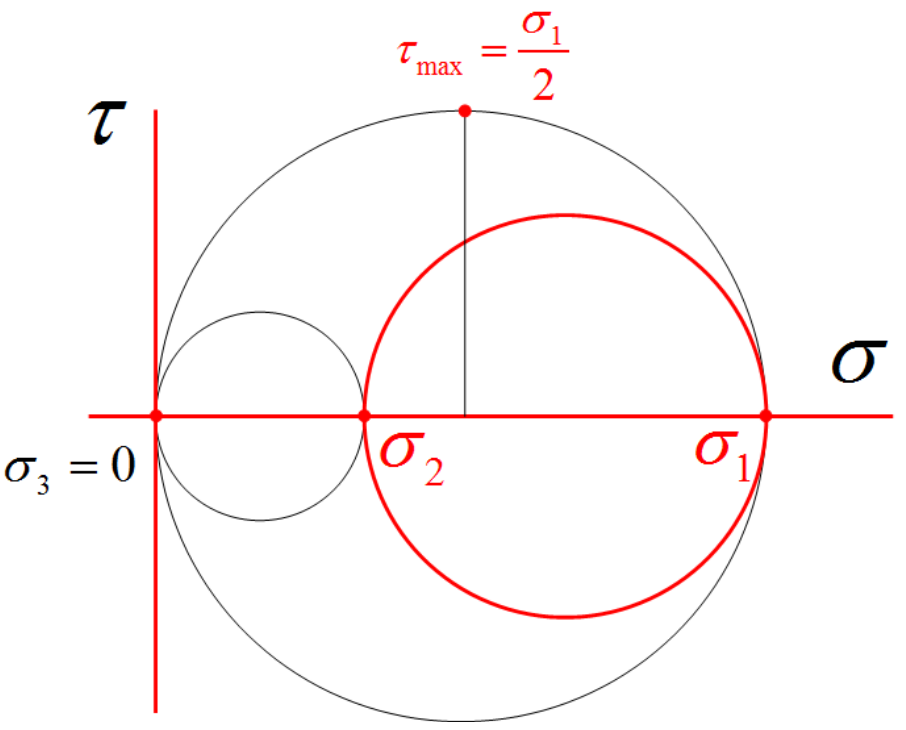


$$\tau_{max} = \frac{|\sigma_{max} - \sigma_{min}|}{2}$$

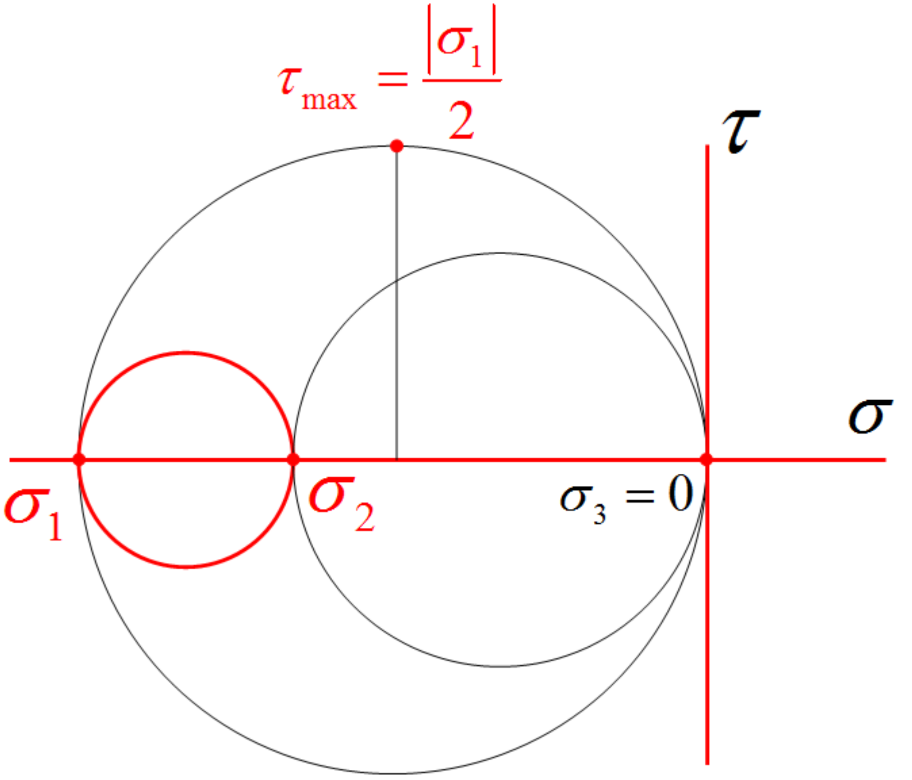
(if σ_{max} and σ_{min} having opposite signs)

$$\tau_{max} = \frac{|\sigma_{max}|}{2}$$

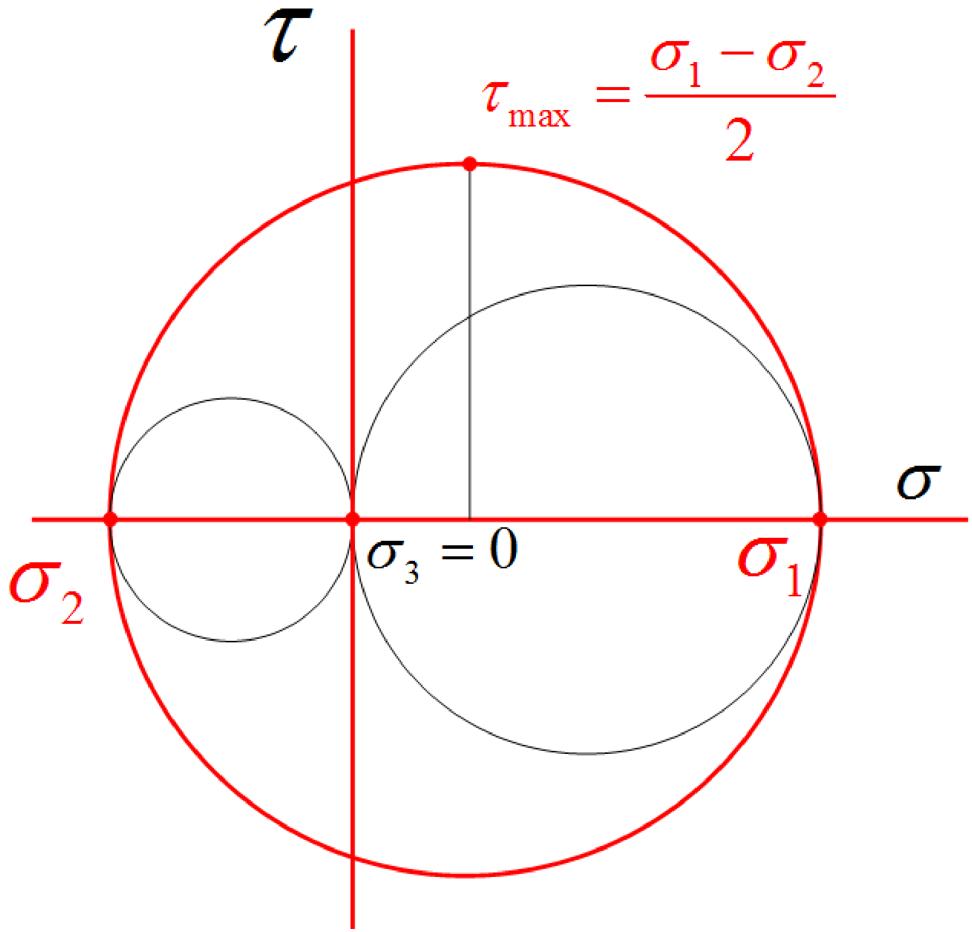
(if both σ_{max} and σ_{min} having same signs)



$$\sigma_1 > \sigma_2 > 0$$



$$\sigma_1 < \sigma_2 < 0$$



$$\sigma_1 > 0 > \sigma_2$$

7.7 YIELD CRITERIA FOR DUCTILE MATERIALS UNDER PLANE STRESS

Failure of a ductile material

1- Maximum-Shearing-Stress Criterion.

Failure occurs by slippage of the material along oblique surfaces and is due primarily to shearing stresses (form a cone shape of angle 45°).

$\tau_{\max} < \tau_y$ (No failure occurs)

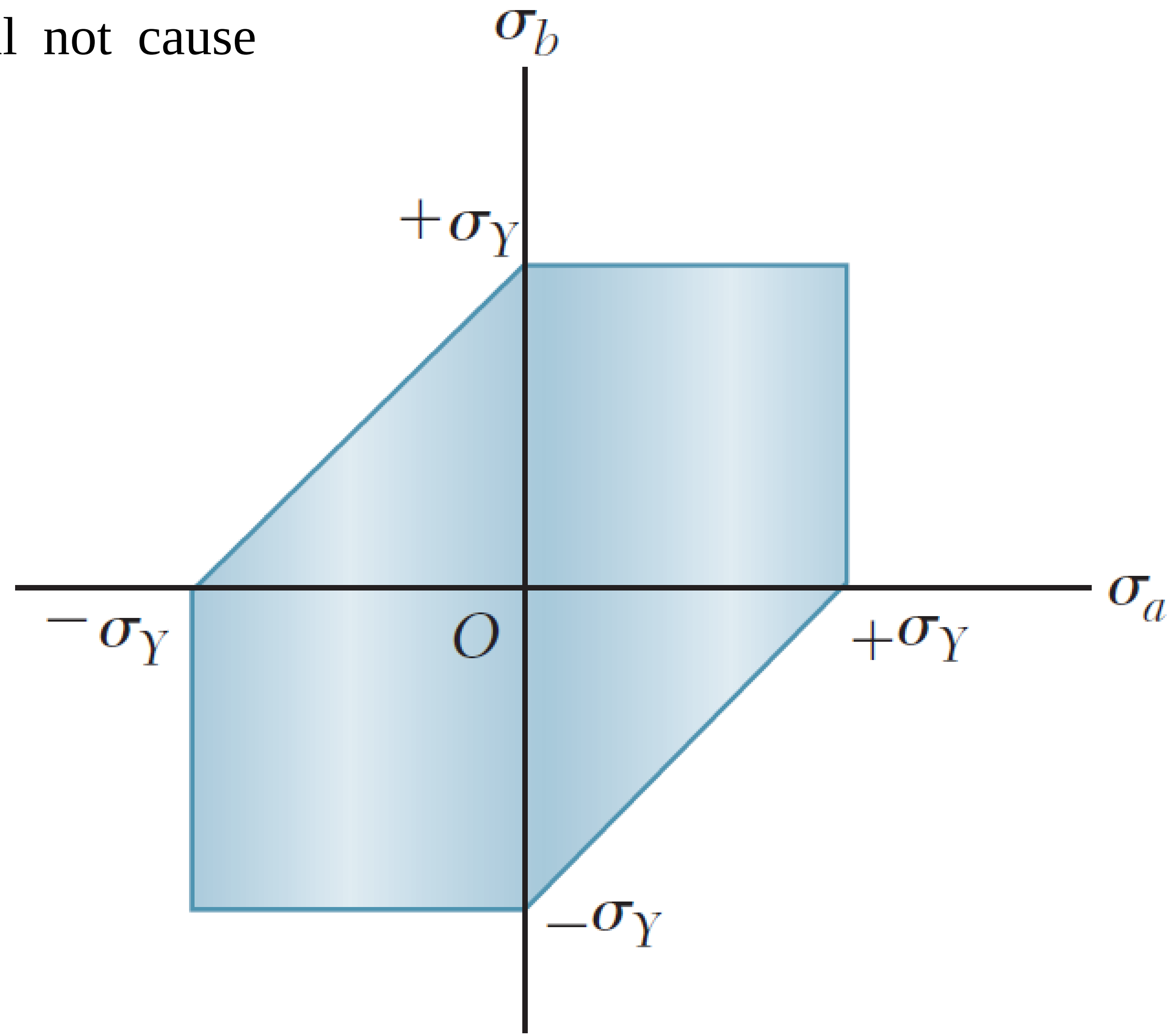
in uniaxial loading $\tau_{\max} = \dfrac{\sigma_{\max}}{2}$. Thus to avoid failure

$|\sigma_a| < \sigma_y \qquad |\sigma_b| < \sigma_y$

in plane stress loading

$|\sigma_a - \sigma_b| < \sigma_y$ (σ_a and σ_b are having opposite signs)

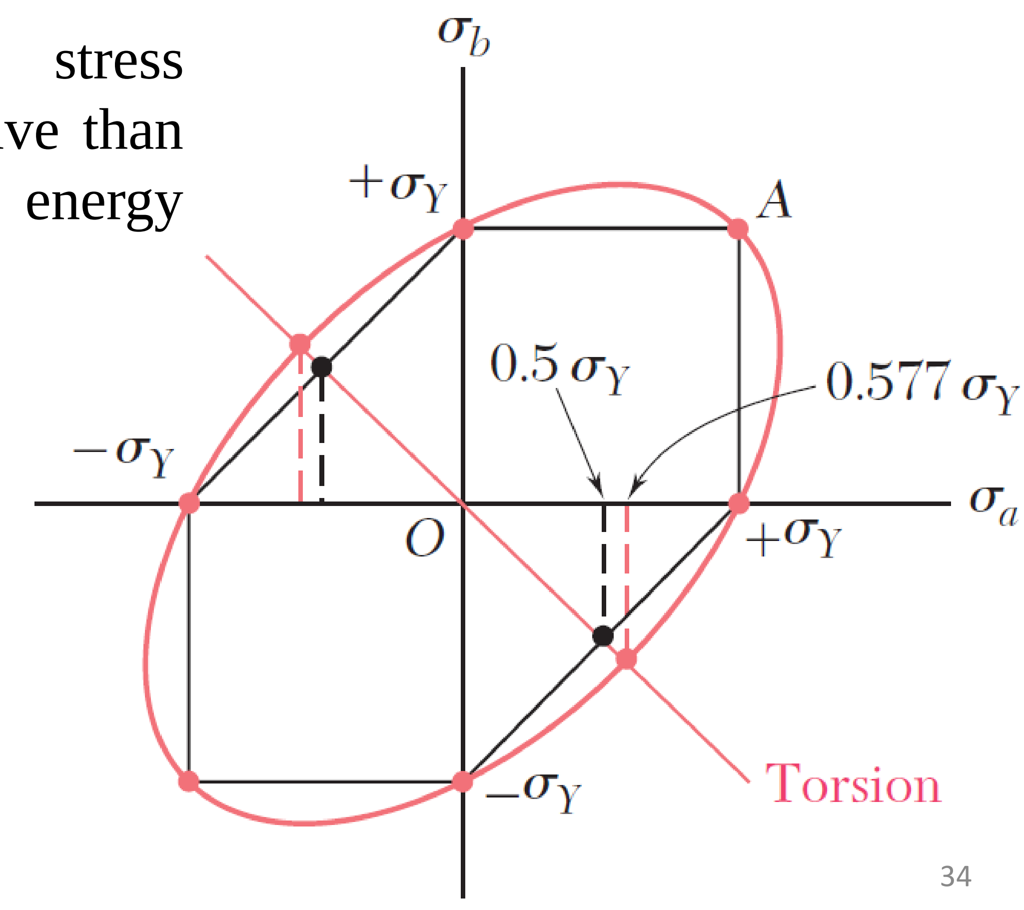
- Stress state located in the blue region will not cause failure.



2- Maximum-Distortion-Energy Criterion.(*von Mises criterion*)

$$\sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2 < \sigma_y$$

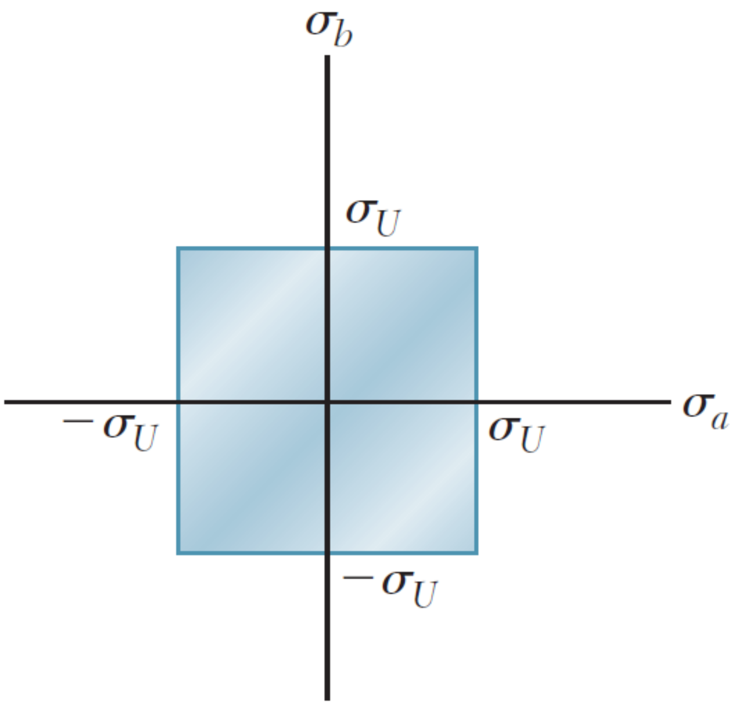
the maximum-shearing stress criterion is more conservative than the maximum distortion energy criterion.



7.8 FRACTURE CRITERIA FOR BRITTLE MATERIALS UNDER PLANE STRESS

1- Maximum-Normal-Stress Criterion.

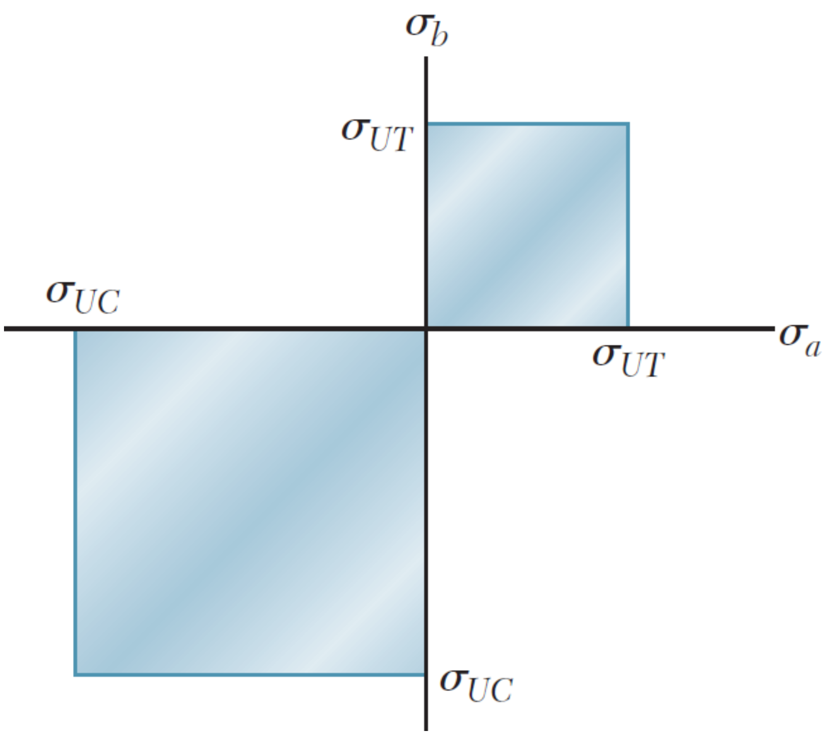
$$|\sigma_a| < \sigma_U \qquad |\sigma_b| < \sigma_U$$



Microscopic cracks or cavities, which tend to weaken the material in tension, will not affect its resistance to compressive failure. Thus

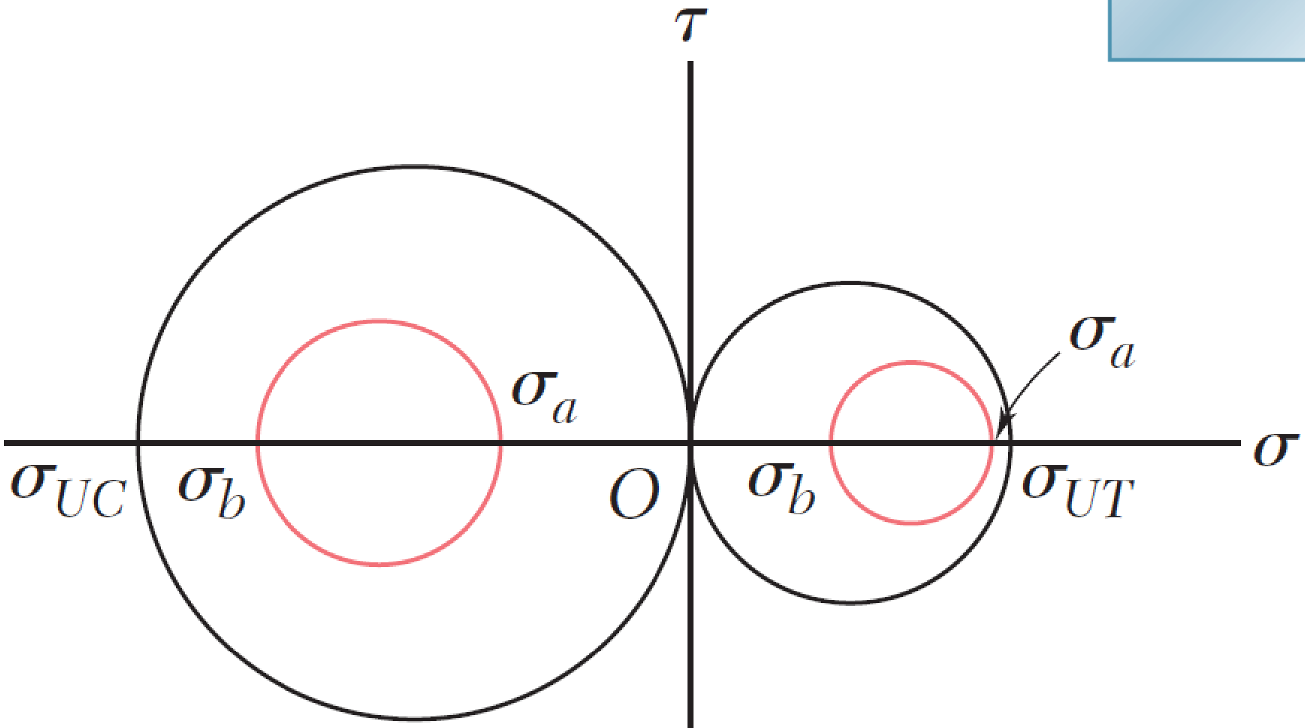
$$\sigma_{UC} > \sigma_{UT}$$

2- Mohr's Criterion.

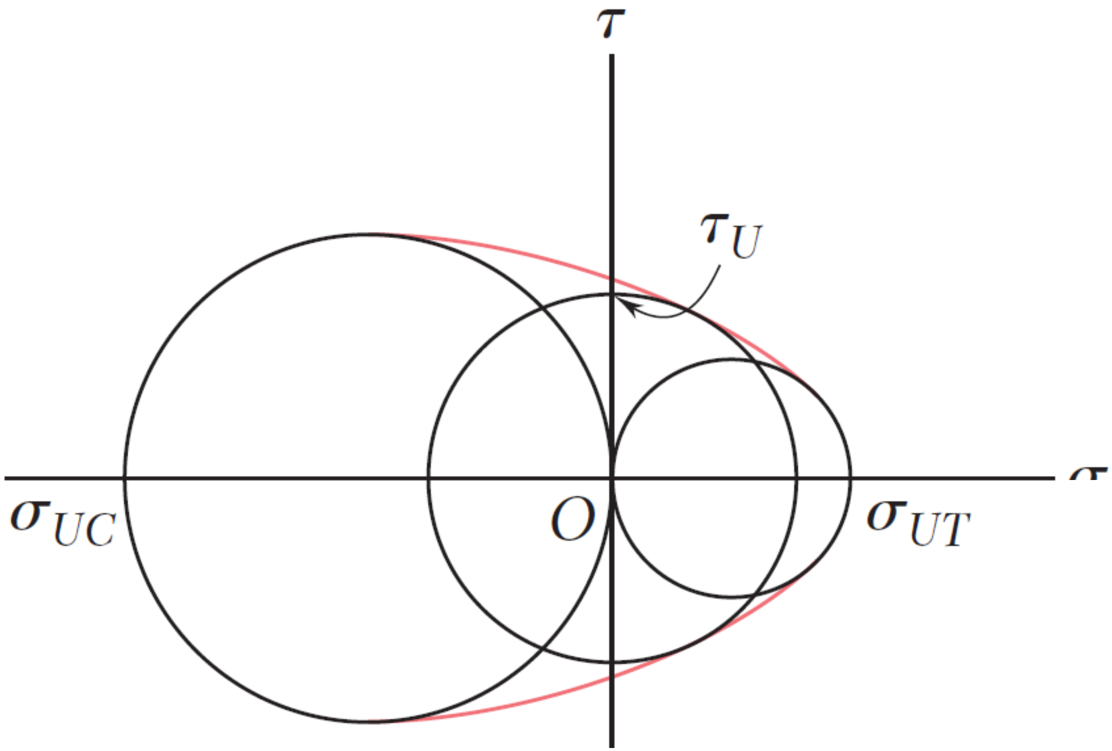


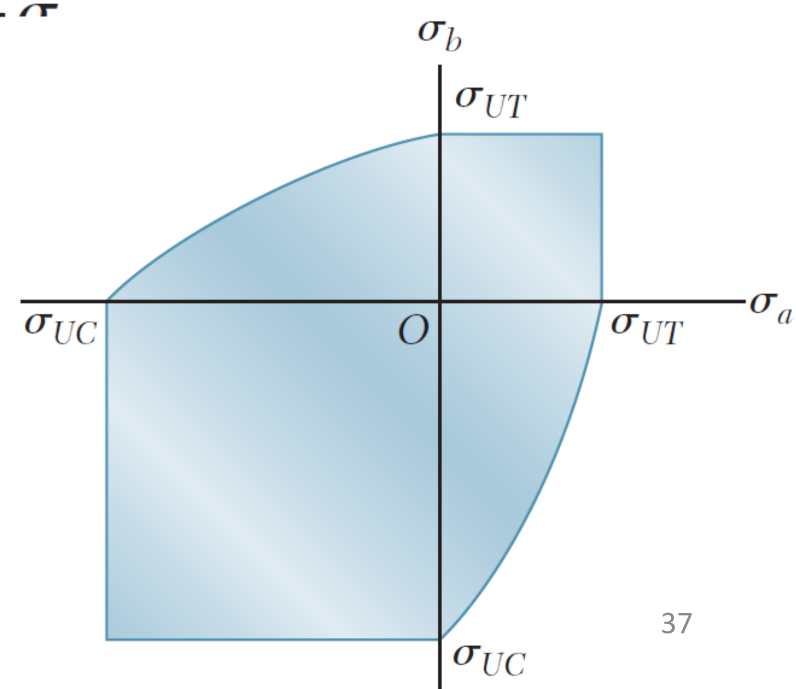
$\sigma_a < \sigma_U \qquad \sigma_b < \sigma_U \qquad \sigma_a + \text{ and } \sigma_b +$

$|\sigma_a| < \sigma_U \qquad |\sigma_b| < \sigma_U \qquad \sigma_a - \text{ and } \sigma_b -$



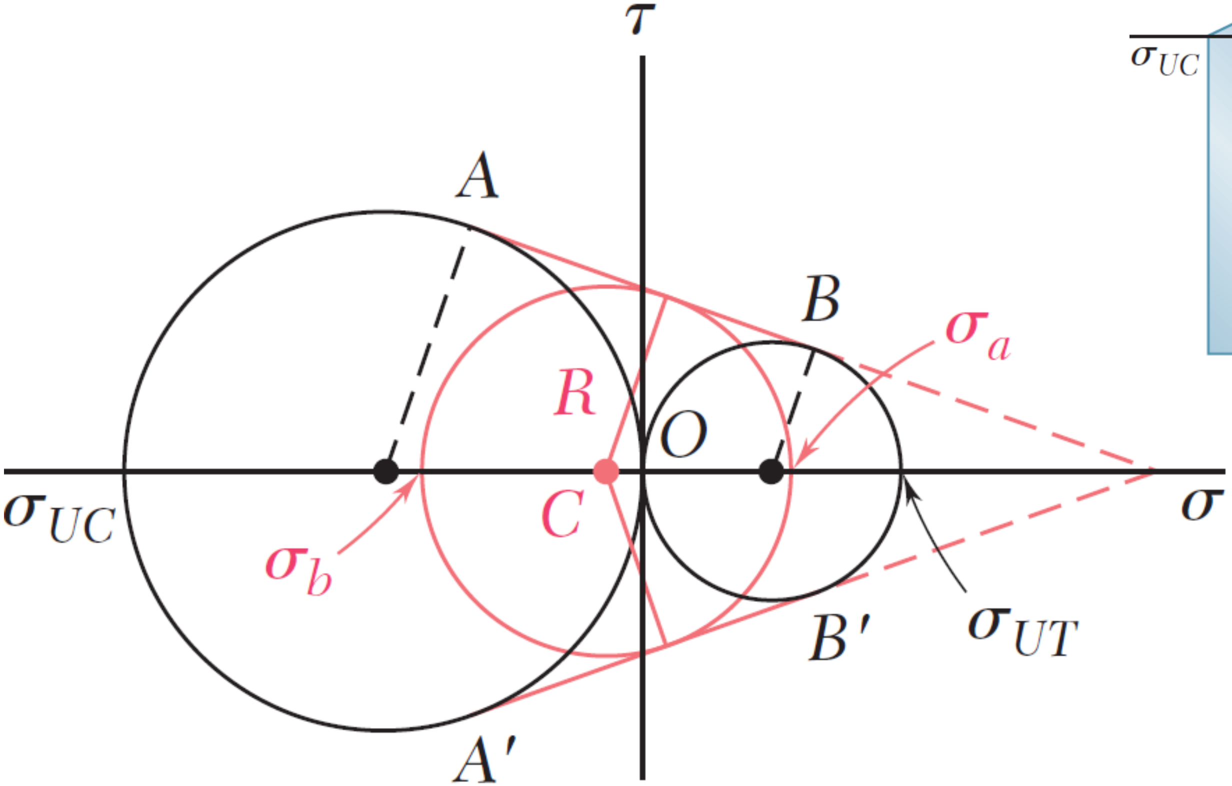
- Many tests can be conducted to draw circles between the two circles to draw the envelope (red curve)

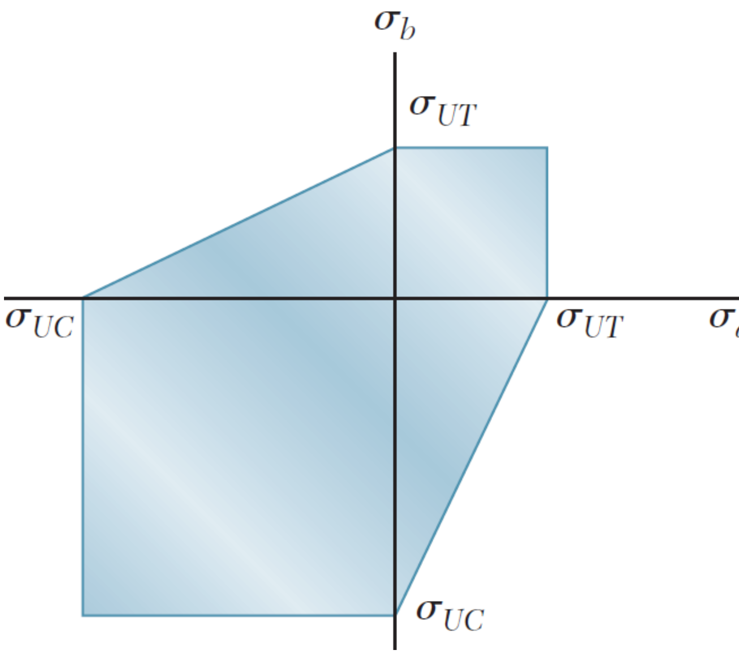




3- simplified mohr's circle criterion

or to save conducting many tests assume the envelope as a straight tangent lines between the circles (it will then be more conservative).



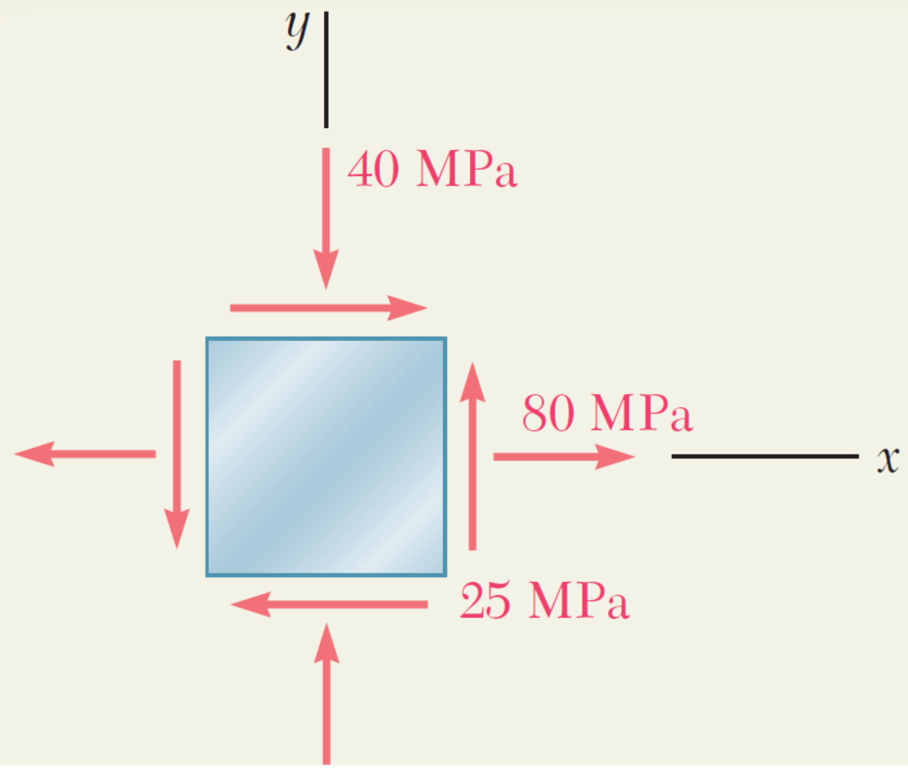


Example: Find the factor of safety usin

(a) Maximum shearing stress criterion.

(b) von mises criterion.

given $\sigma_Y = 250$ MPa



Solution :

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

$\sigma_1 = 85$ MPa, $\sigma_2 = -45$ MPa

(a) Maximum-Shearing-Stress Criterion

$$\tau_Y = \frac{1}{2}\sigma_Y = 125 \text{ MPa}$$

$$\tau_{\max} = \frac{85-(-45)}{2} = 65 \text{ MPa}$$

$$F.S = \frac{120 \text{ MPa}}{65 \text{ MPa}} = 1.92$$

(b) Maximum-Distortion-Energy Criterion

$$\sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2 = \left(\frac{\sigma_Y}{F.S}\right)^2$$

$$F.S = 2.19$$

7.9 STRESSES IN THIN-WALLED PRESSURE VESSELS

if $\frac{r}{t} \geq 10$ (thin-walled vessel)

r → radius

t → thickness

- Two types of vessels are mainly considered.

  1- The cylindrical vessels.

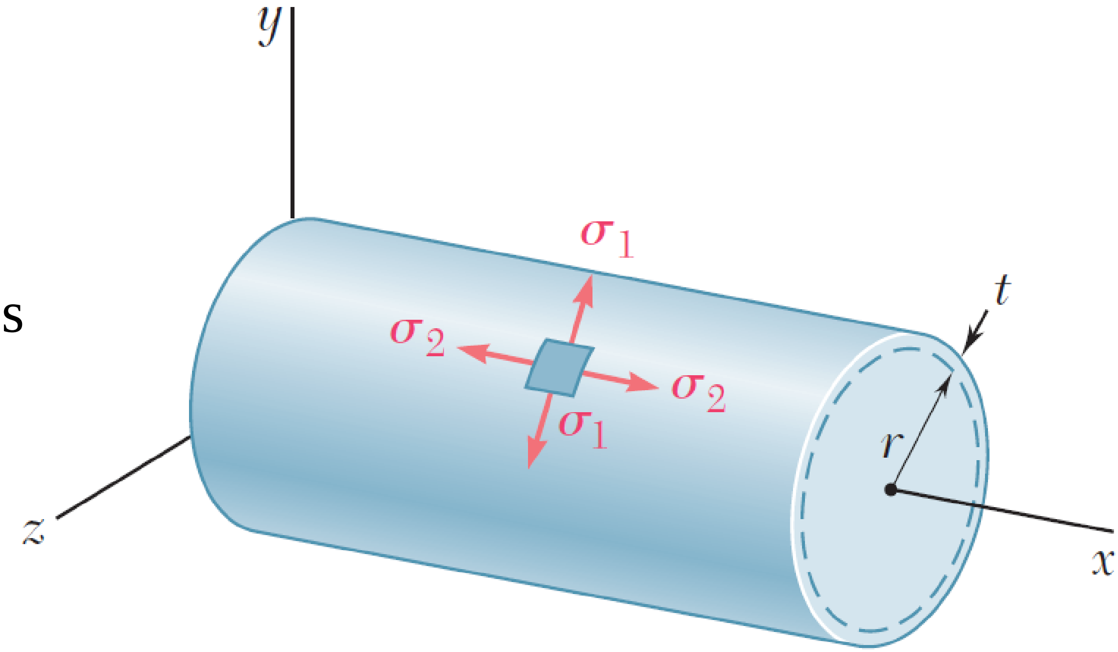
  2- The spherical vessels.

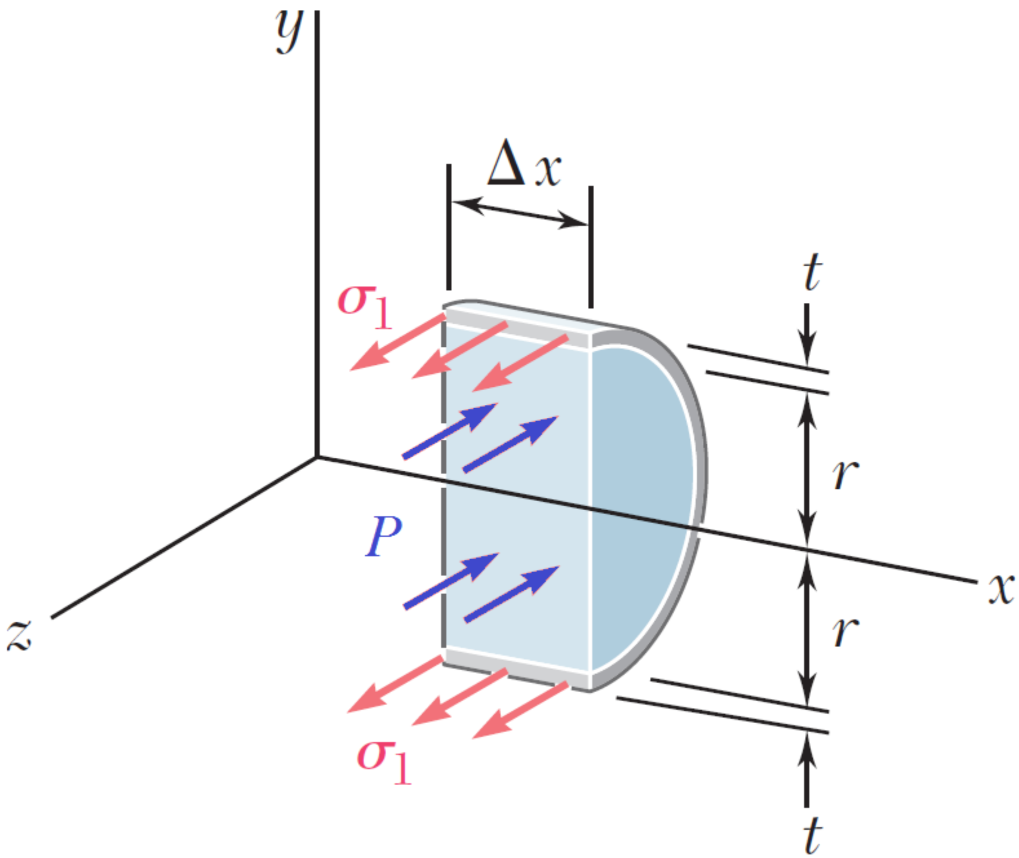
- Since the thickness is small, the stress distribution throughout the thickness will not vary significantly. (state of plane stress)
- The pressure inside the vessels is measured as gauge pressure (*i.e.* The difference between the inside and the outside pressures).

Cylindrical vessels

σ_1 is called hoop stress

σ_2 is called longitudinal stress

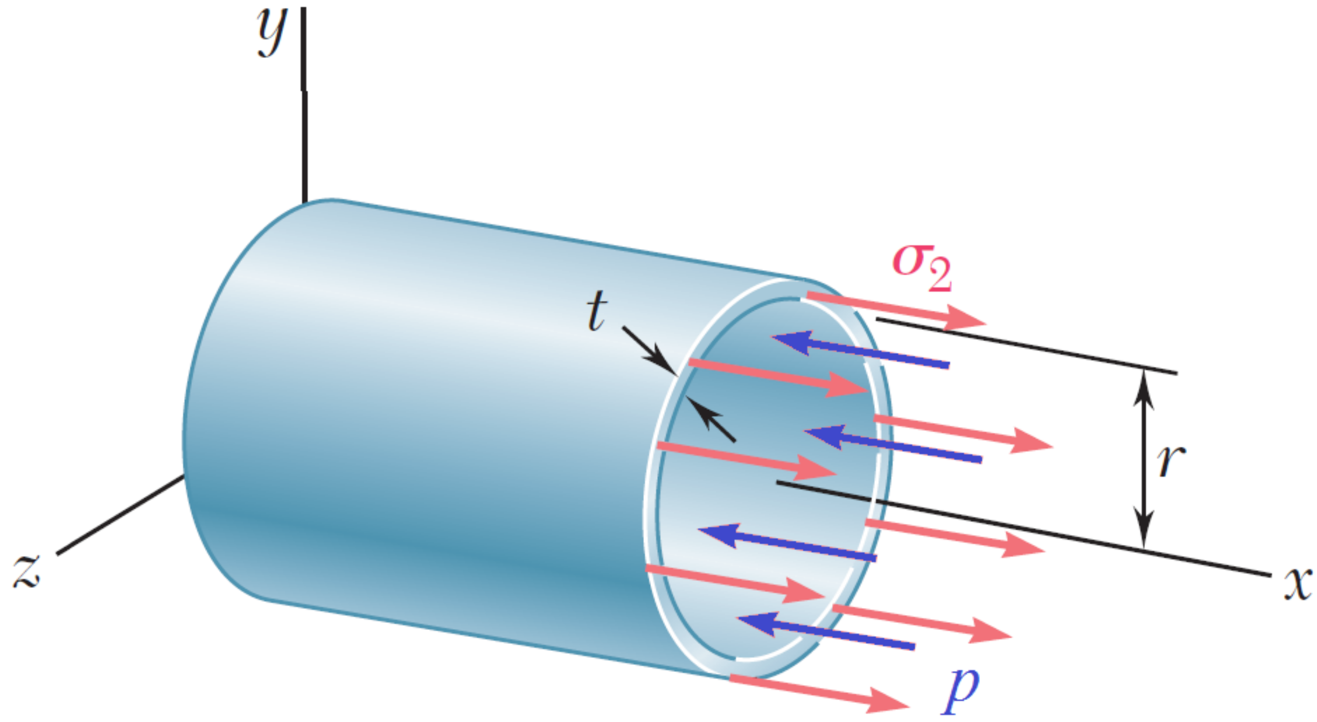




$$\sum F = 0$$

$$2\sigma_1 t \Delta x = p * 2r * \Delta x$$

$$\sigma_1 = \frac{pr}{t}$$

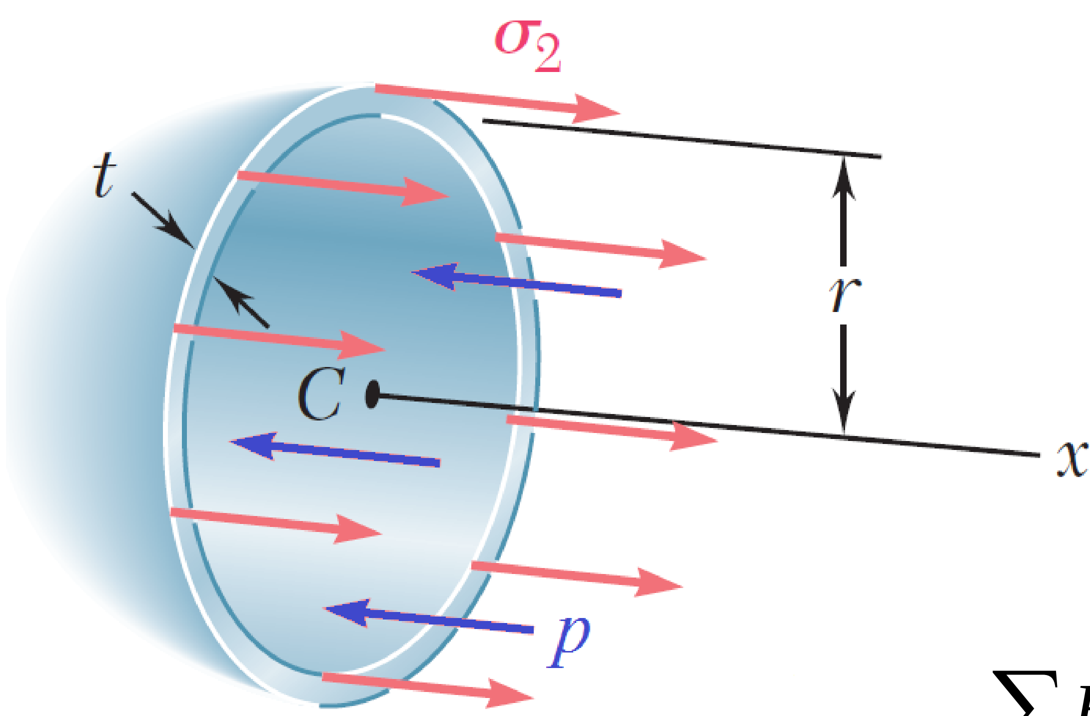


$$\sum F = 0$$

$$\sigma_2 (2\pi r t) = p * \pi r^2$$

$$\sigma_2 = \frac{pr}{2t}$$

Spherical vessels



$$\sum F = 0$$

$$\sigma_2(2\pi rt) = p * \pi r^2$$

$$\sigma_1 = \sigma_2 = \frac{pr}{2t}$$

Example :

$\sigma_{all} = 140$ MPa for both cylindrical and spherical vessels.

$r = 1.2$ m. and $t = 12$ mm. Find p_{all}

Solution :

for cylinder

$$\sigma_{all} = \frac{pr}{t} \quad \rightarrow \quad p_{all} = 2.8 \text{ MPa}$$

for spherical

$$\sigma_{all} = \frac{pr}{2t} \quad \rightarrow \quad p_{all} = 5.6 \text{ MPa}$$

Example :

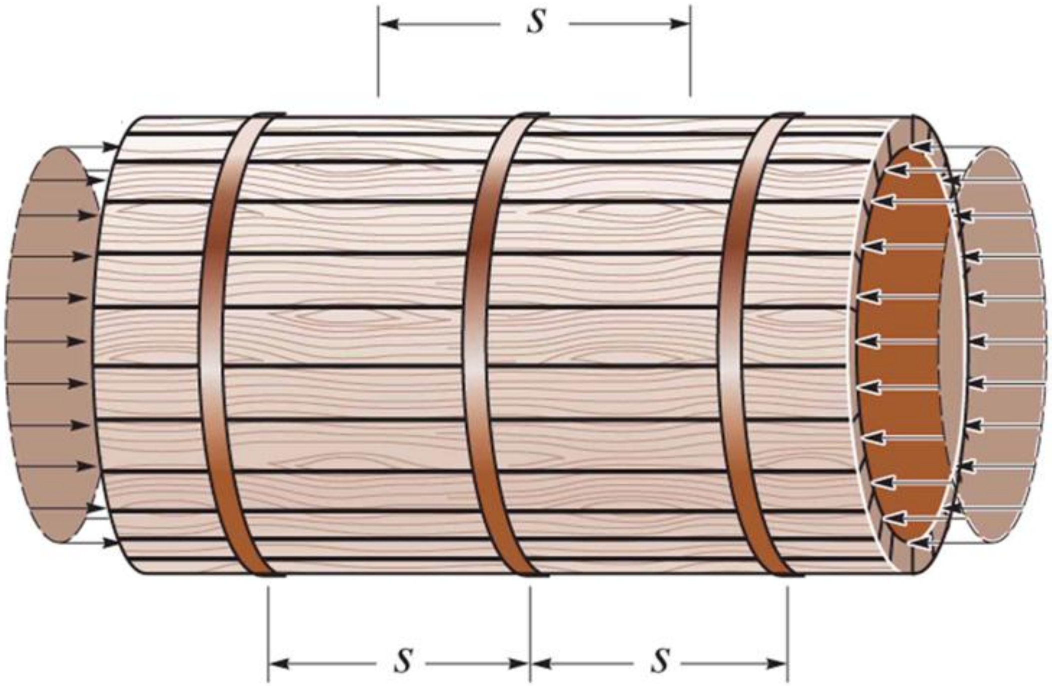
$d = 0.9$ mm

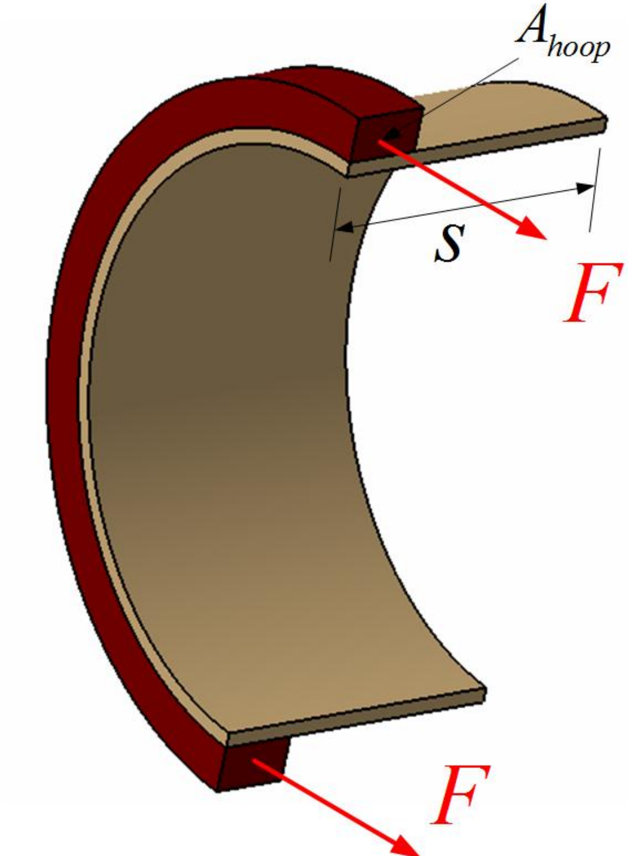
$A_{hoop} = 125\ \text{mm}^2$ (steel ring)

$\sigma_{all} = 84$ MPa

$p = 28$ kPa

Find s





Solution :

$$\sum F = 0$$

$$p \times 2r \times s - 2F = 0$$

$$F = prs = 12600s$$

$$\sigma_{all} = \frac{F}{A} = 84 \text{ MPa}$$

$$s = 833.33 \text{ mm}$$

Example :

$r = 0.6$ m

$t = 6$ mm

$p = -70$ kPa

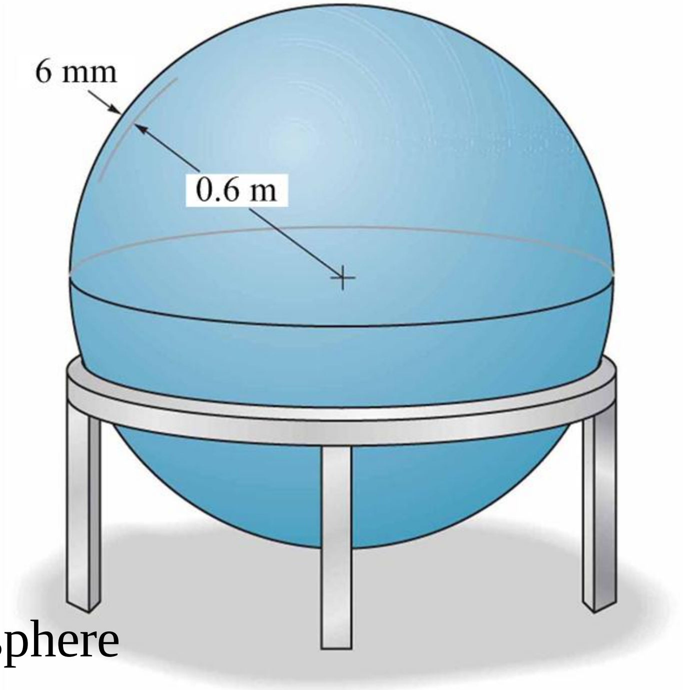
$\mu_s = 0.5$

Find:

(1) Torque T to initiate the rotation of the upper hemisphere

(2) The vertical force required to seperate the hemispheres.

(3) the horizontal force required to slide the hemispheres.



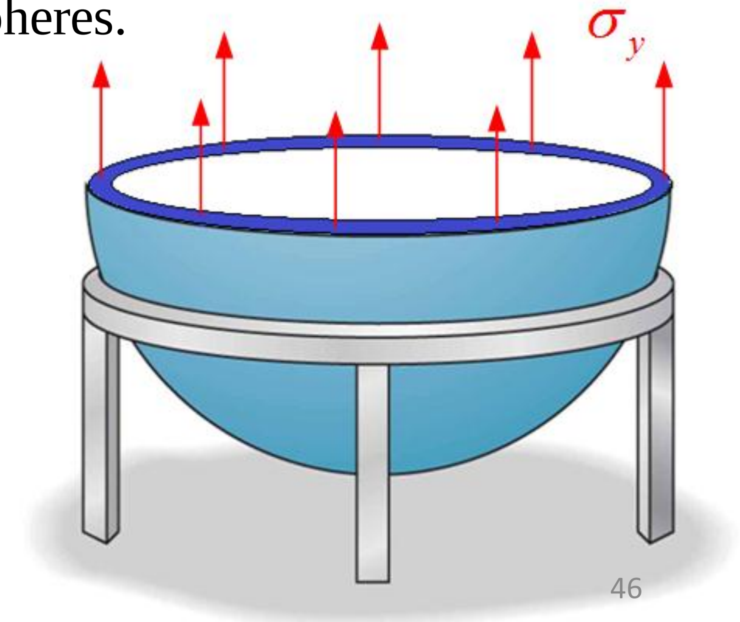
Solution :

$$\sum F = 0$$

$$F_y = p \times \pi r^2 = 79.12 \text{ kN} \quad \text{(vertical force)}$$

$$F_x = \mu_s F_y = 39.56 \text{ kN} \quad \text{(sliding force)}$$

$$T = F_x \times r = 23.74 \text{ kN.m}$$

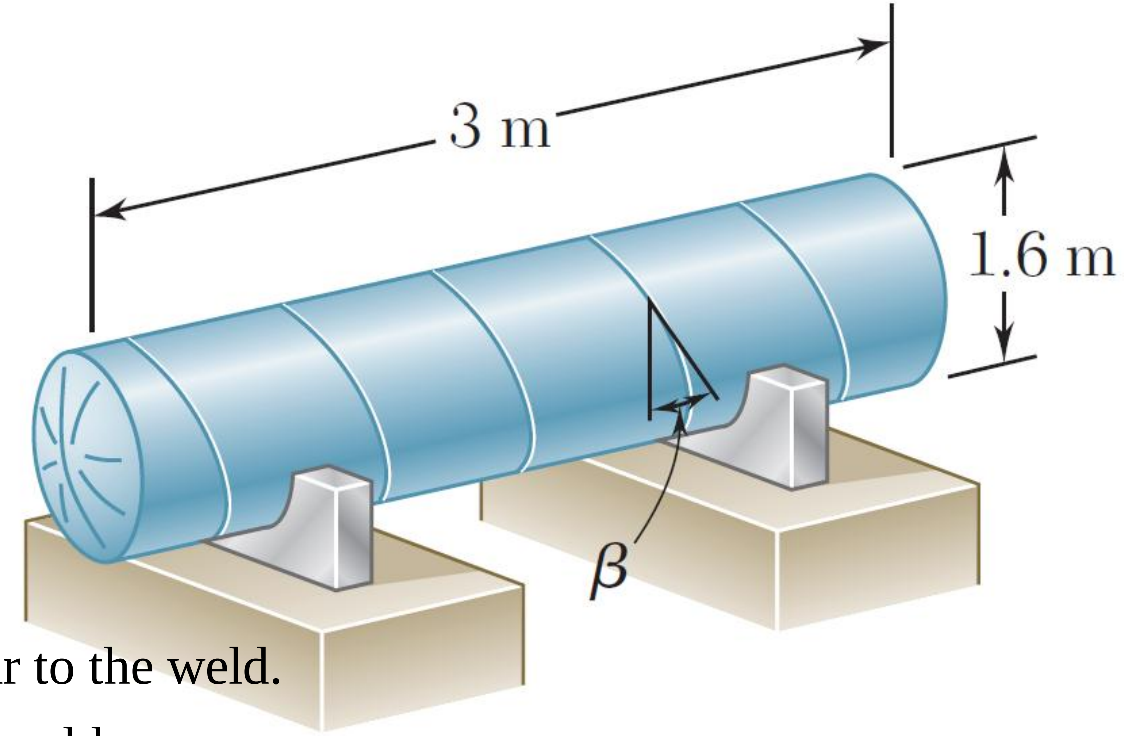


Example :

$p = 600$ kPa

$t = 8$ mm

$\beta = 20^o$

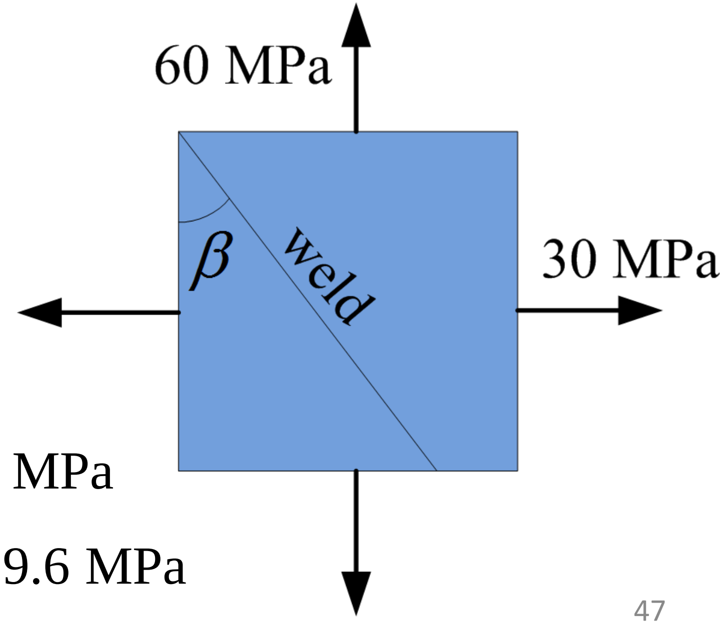


Find

(1) The normal stress perpendicular to the weld.

(2) The shear stress parallel to the weld.

***S*olution :**



$$\sigma_1 = \frac{pr}{t} = 60 \text{ MPa}$$

$$\sigma_2 = \frac{pr}{2t} = 30 \text{ MPa}$$

$$\sigma_w = \sigma_2 \cos^2\beta + \sigma_1 \sin^2\beta + 2\tau_{xy} \sin\beta\cos\beta = 33.5 \text{ MPa}$$

$$\tau_w = -(\sigma_2 - \sigma_1)\sin\beta\cos\beta + \tau_{xy}(\cos^2\beta - \sin^2\beta) = 9.6 \text{ MPa}$$

Example :

$r_{in} = 450$ mm

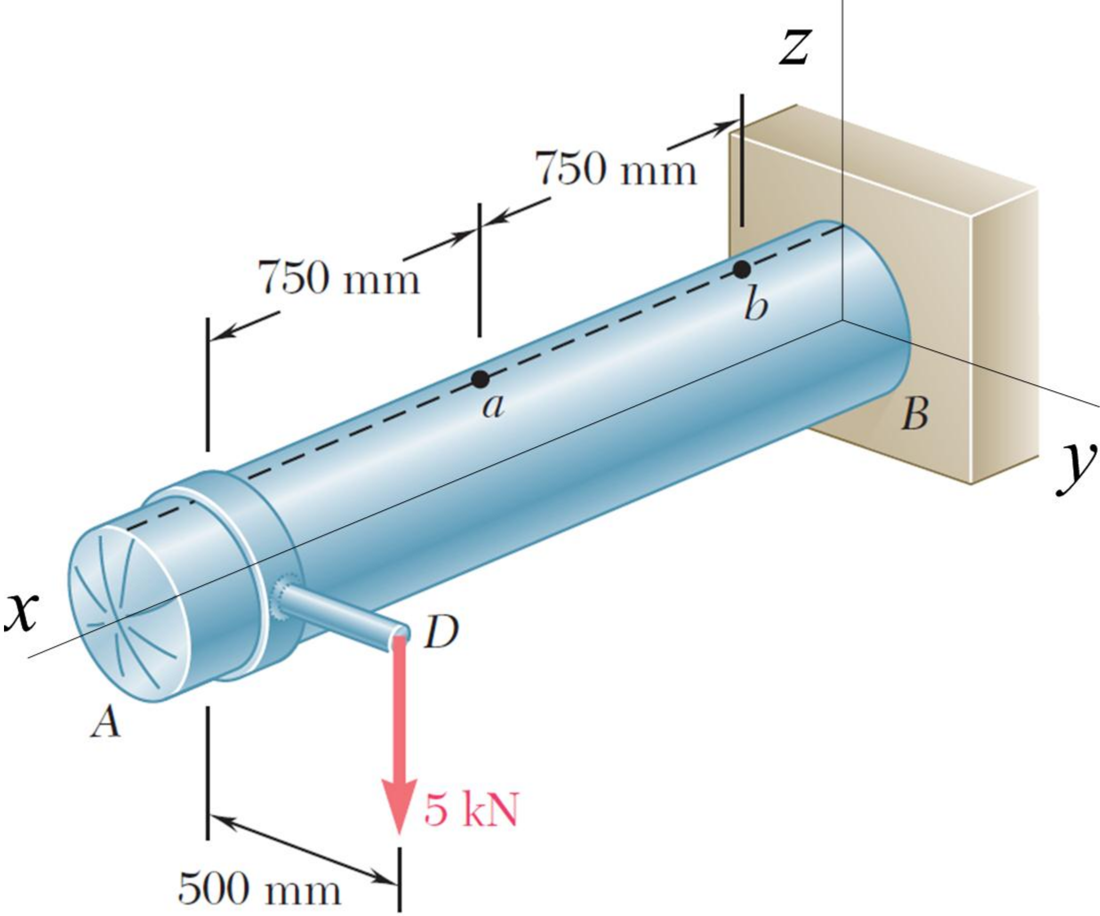
$t = 6$ mm

$p = 1.2$ MPa

For point a find

$\sigma_{\max}$

$\tau_{\max}$ (in-plane)



Solution :

$$\sigma_x = \frac{pr}{2t} + \frac{\mathbf{M_a} \cdot c}{I}$$

$$= \frac{1.2 \times 10^6 \times 0.456}{2 \times 6 \times 10^{-3}} + \frac{(5 \times 10^3 \times 0.75) \times (0.456)}{\frac{\pi}{4}\left((0.456)^4 - (0.45)^4\right)}$$

$$= 45.6 \text{ MPa} + 0.976 \text{ MPa} = 46.58 \text{ MPa}$$

$$\sigma_y = \frac{pr}{t} = 91.2 \text{ MPa}$$

$$\tau_{xy} = \frac{\mathbf{T} \cdot c}{J} = \frac{(5\times10^3 \times 0.5)\times(0.456)}{\frac{\pi}{2}\left((0.456)^4 - (0.45)^4\right)} = 0.325 \text{ MPa}$$

since τ_{xy} is very small compared to σ_x and σ_y

it is good approximation to make

(verify by apply the principal stress equation)

$$\sigma_1 = \sigma_y = 91.2 \text{ MPa}$$

$$\sigma_2 = \sigma_x = 46.58 \text{ MPa}$$

$$\tau_{\max}(\text{in-plane}) = \frac{91.2 - 46.58}{2} = 22.3 \text{ MPa}$$

$$\tau_{\max} = \frac{91.2}{2} = 45.6 \text{ MPa}$$

END OF CHAPTER SEVEN

MECHANICS OF MATERIALS

CHAPTER EIGHT

PRINCIPAL STRESSES UNDER A GIVEN LOADING

Prepared by : Dr. Mahmoud Rababah

8.4 STRESSES UNDER COMBINED LOADINGS

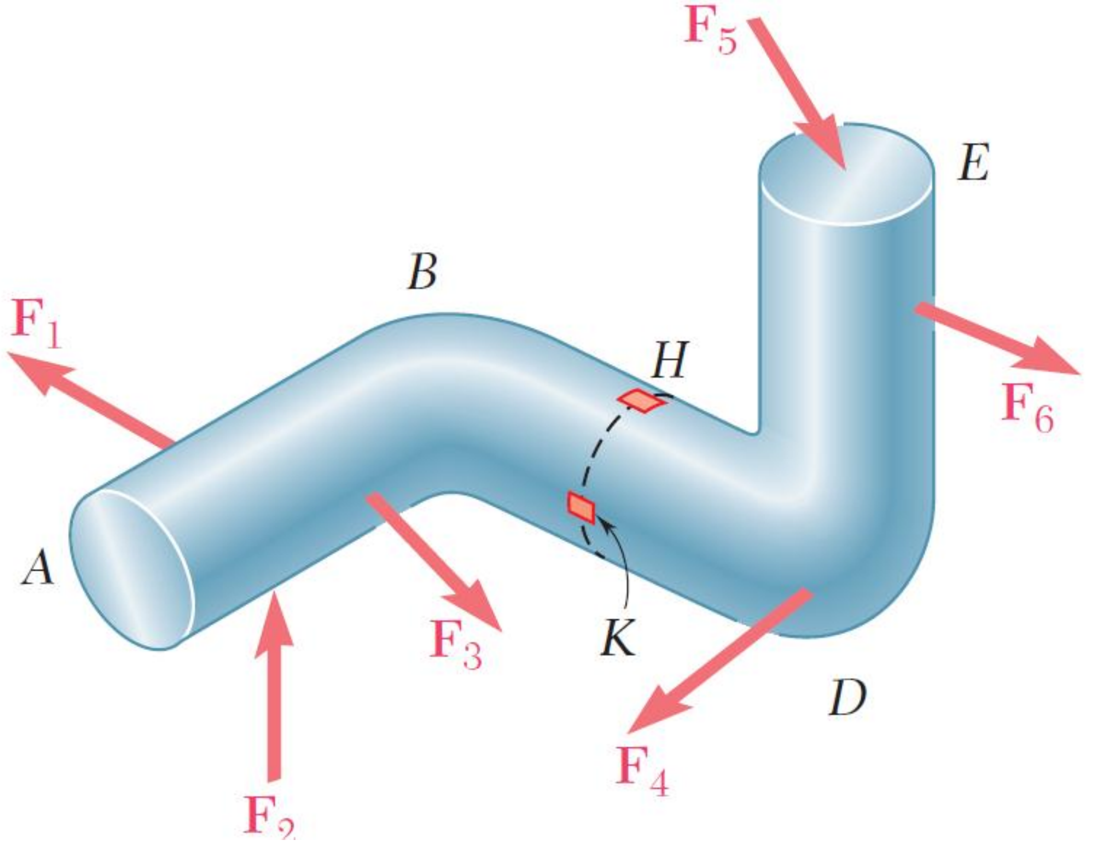
1- We first determine the internal forces affecting the element.

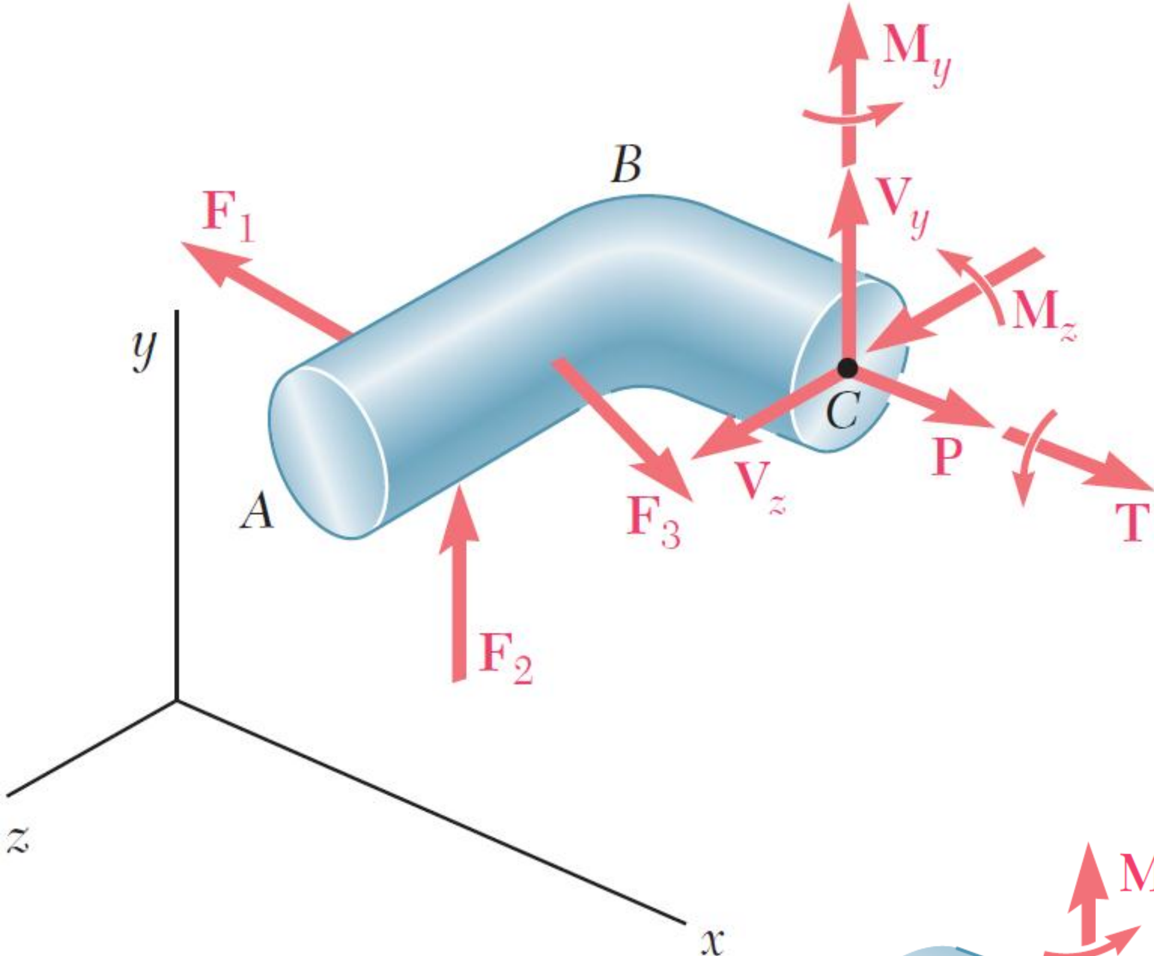
2- since the element is on the surface; the problem is plane stress problem.

3- find the state of stress on the element.

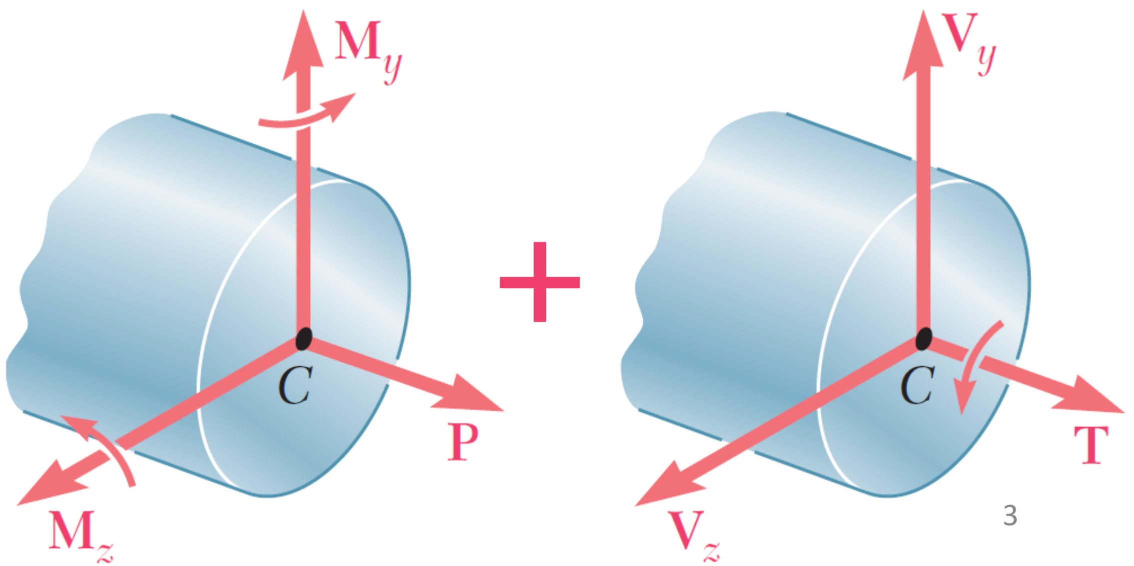
4- Obtain the principal stresses and the maximum shear stress.

5- Compare with the allowable values.





- In general, the internal forces result in:
  - Two shear forces.
  - Two bending moments.
  - One torsion moment.
  - One axial loading.



Two shear forces are :

$\mathbf{V_y}$ and $\mathbf{V_z}$ result in

$$\tau_{xy1} = \frac{\mathbf{V_y} Q}{It}, \quad \tau_{xz1} = \frac{\mathbf{V_z} Q}{It}$$

Two moments are :

$\mathbf{M_y}$ and $\mathbf{M_z}$ result in

$$\sigma_{x1} = \frac{-\mathbf{M_y} \cdot z}{I_y}, \quad \sigma_{x2} = \frac{-\mathbf{M_z} \cdot y}{I_z}$$

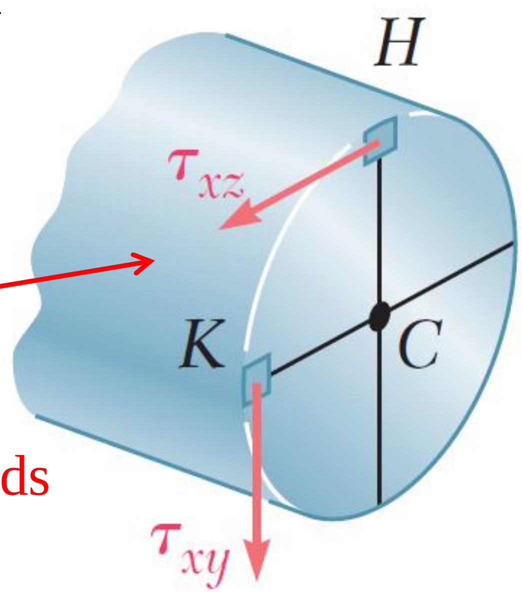
The axial force P results in

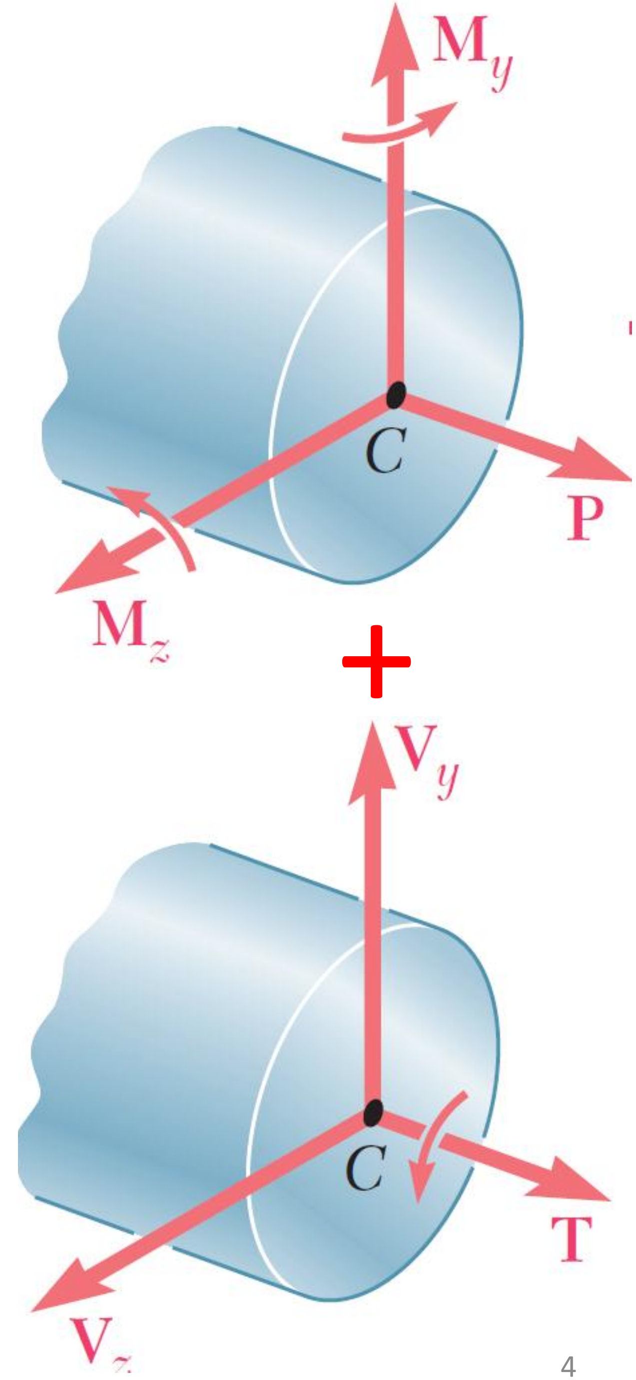
$$\sigma_{x3} = \frac{\mathbf{P}}{A}$$

The torsion **T** results in

$$\tau = \frac{\mathbf{T} \cdot c}{J}$$

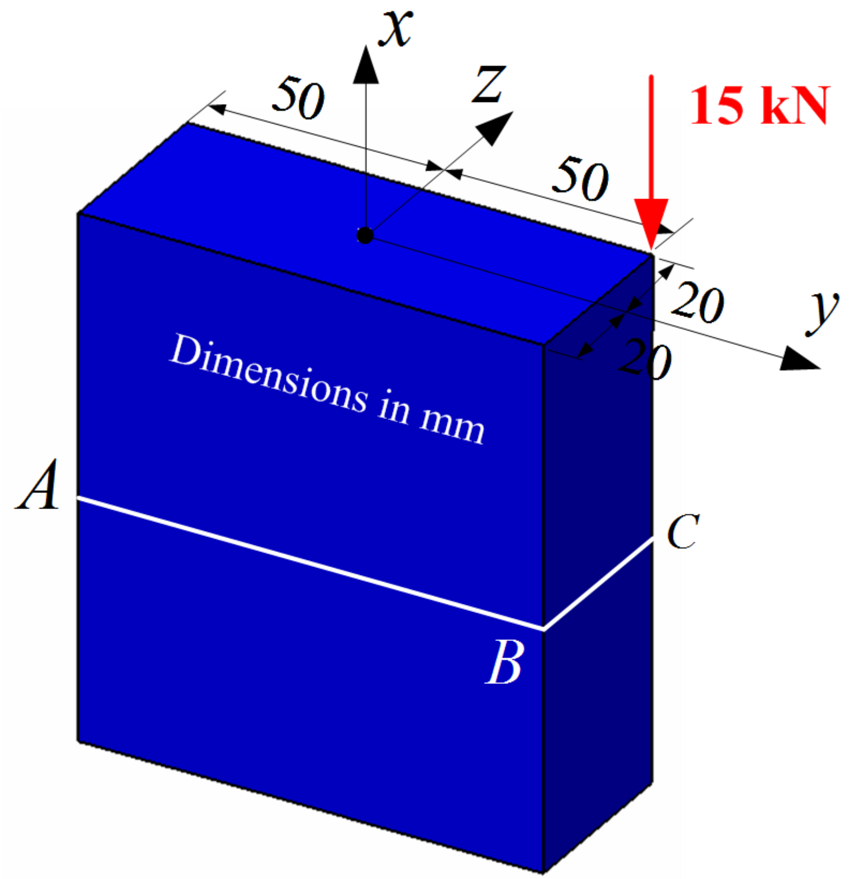
(it can be τ_{xy2} or τ_{xz2} depends on the element location)





Example :

Find the normal stresses at points *A*, *B* and *C*



Solution :

$$\sigma_{axial} = \frac{-15\times10^3}{0.1\times0.04} = 3.75\text{ MPa} \ \ \text{(compression)}$$

$$\sigma_{bend1} = \frac{(15\times10^3\times0.02)\times0.02}{\frac{1}{12}\times0.1\times(0.04)^3} = 11.25\text{ MPa}$$

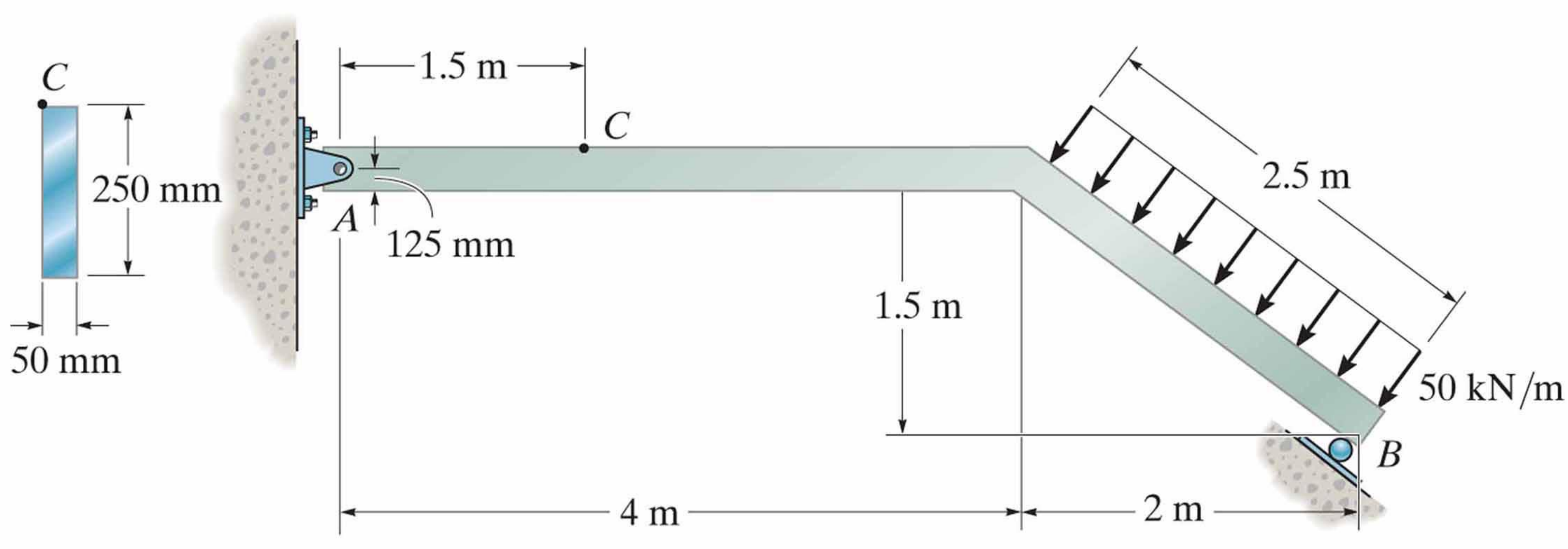
$$\sigma_{bend2} = \frac{(15\times10^3\times0.05)\times0.05}{\frac{1}{12}\times0.04\times(0.1)^3} = 11.25\text{ MPa}$$

$$\sigma_A = \sigma_{axial} + \sigma_{bend1} + \sigma_{bend2} = 18.75\text{ MPa}$$

$$\sigma_B = \sigma_{axial} + \sigma_{bend1} - \sigma_{bend2} = -3.75\text{ MPa}$$

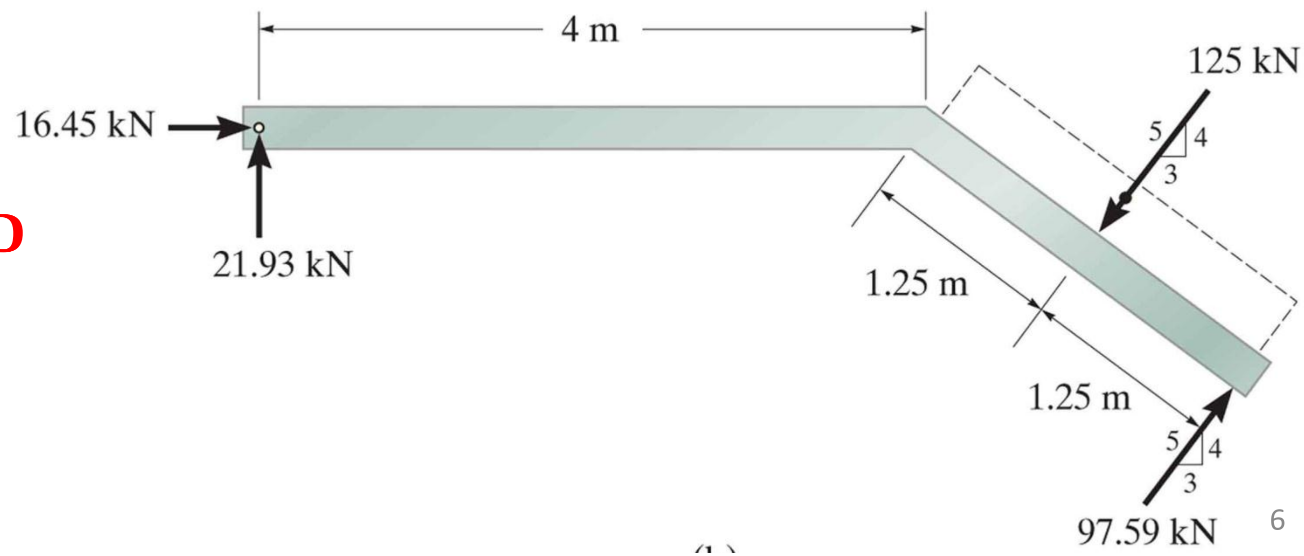
$$\sigma_C = \sigma_{axial} - \sigma_{bend1} - \sigma_{bend2} = -26.25\text{ MPa}$$

Example : Find normal and shear stresses at point *C*



Solution:

1- Find the FBD

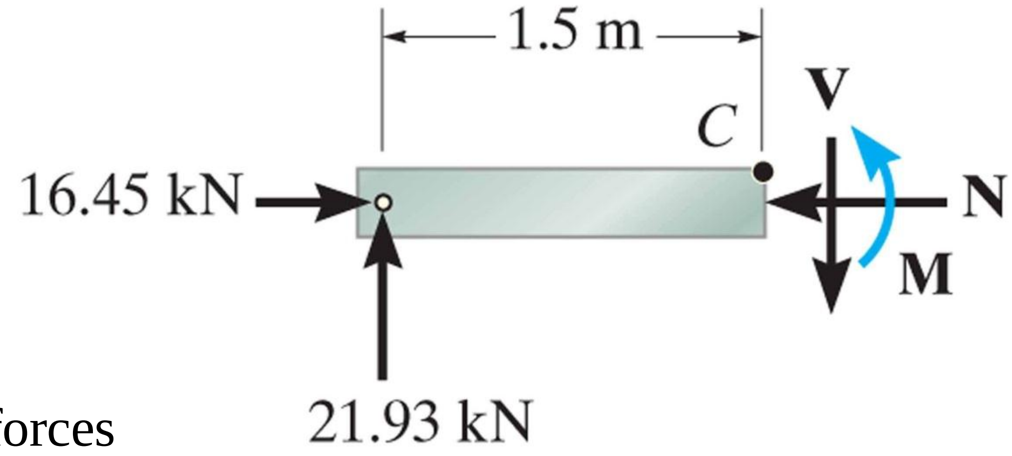


2- Take section at point *C*

$$\mathbf{V} = 21.93 \text{ kN}$$

$$\mathbf{M} = 21.93 \times 1.5 = 32.89 \text{ kN.m}$$

$$\mathbf{N} = 16.45 \text{ kN}$$



Find the stresses resulted from the forces

$$\sigma_{axial} = \frac{\mathbf{N}}{A} = 1.32 \text{ MPa} \ \ \text{(compression)}$$

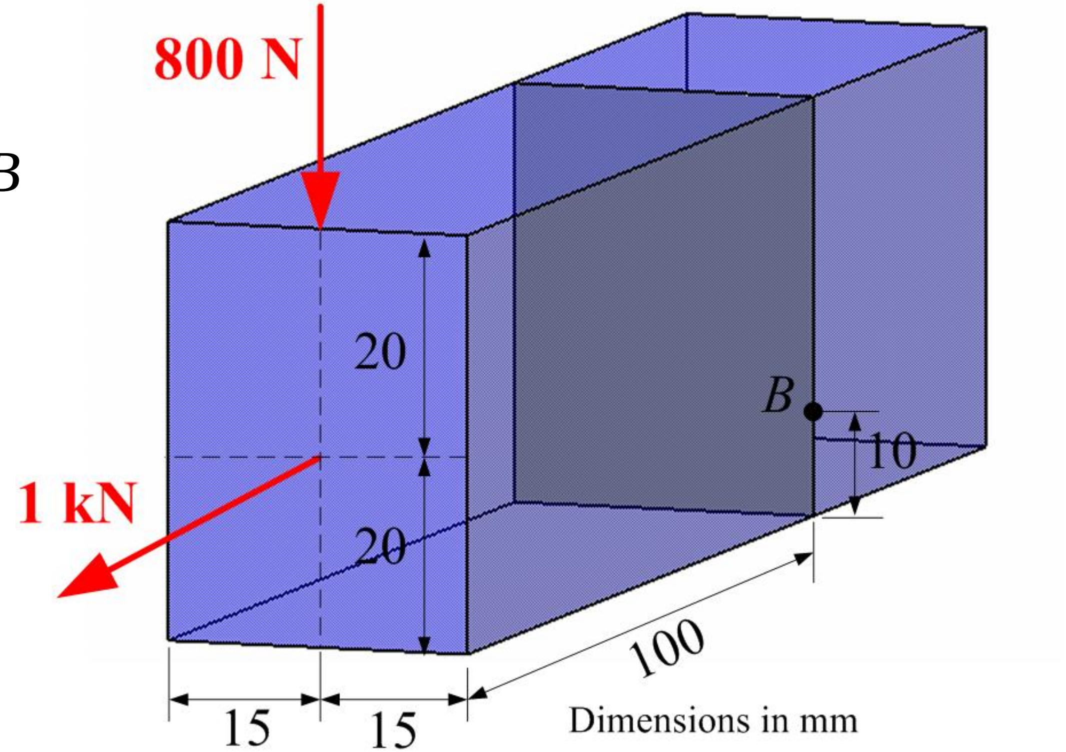
$$\sigma_{bend} = \frac{\mathbf{M} \cdot c}{I} = \frac{32.89 \times 10^3 \times 0.125}{\frac{1}{12} \times 0.05 \times (0.25)^3} = 63.16 \text{ MPa}$$

$$\sigma_C = \sigma_{axial} + \sigma_{bend} = 64.5 \text{ MPa}$$

$$\tau_C = \frac{\mathbf{V}Q}{It} = 0$$

Example :

Find the state of stress at point *B*

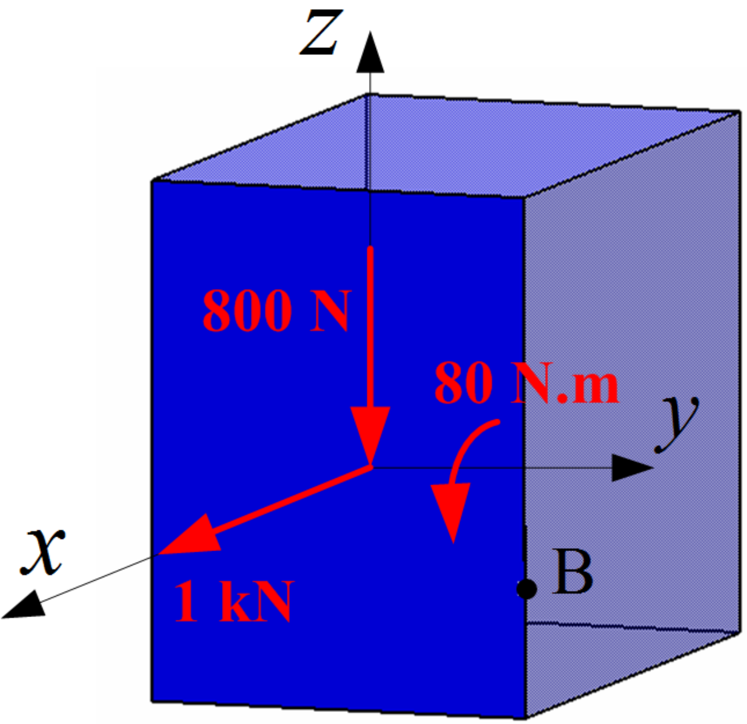


Solution :

$$\mathbf{F_x} = 1 \text{ kN}$$

$$\mathbf{F_z} = -800 \text{ N}$$

$$\mathbf{M_y} = 80 \text{ N.m}$$



$$\sigma_B = \frac{\mathbf{F_x}}{A} - \frac{\mathbf{M_y} \cdot z}{I_y}$$

$$= \frac{1\times10^3}{0.03\times0.04} - \frac{80\times0.01}{\frac{1}{12}\times0.03\times(0.04)^3} = -4.167 \text{ MPa}$$

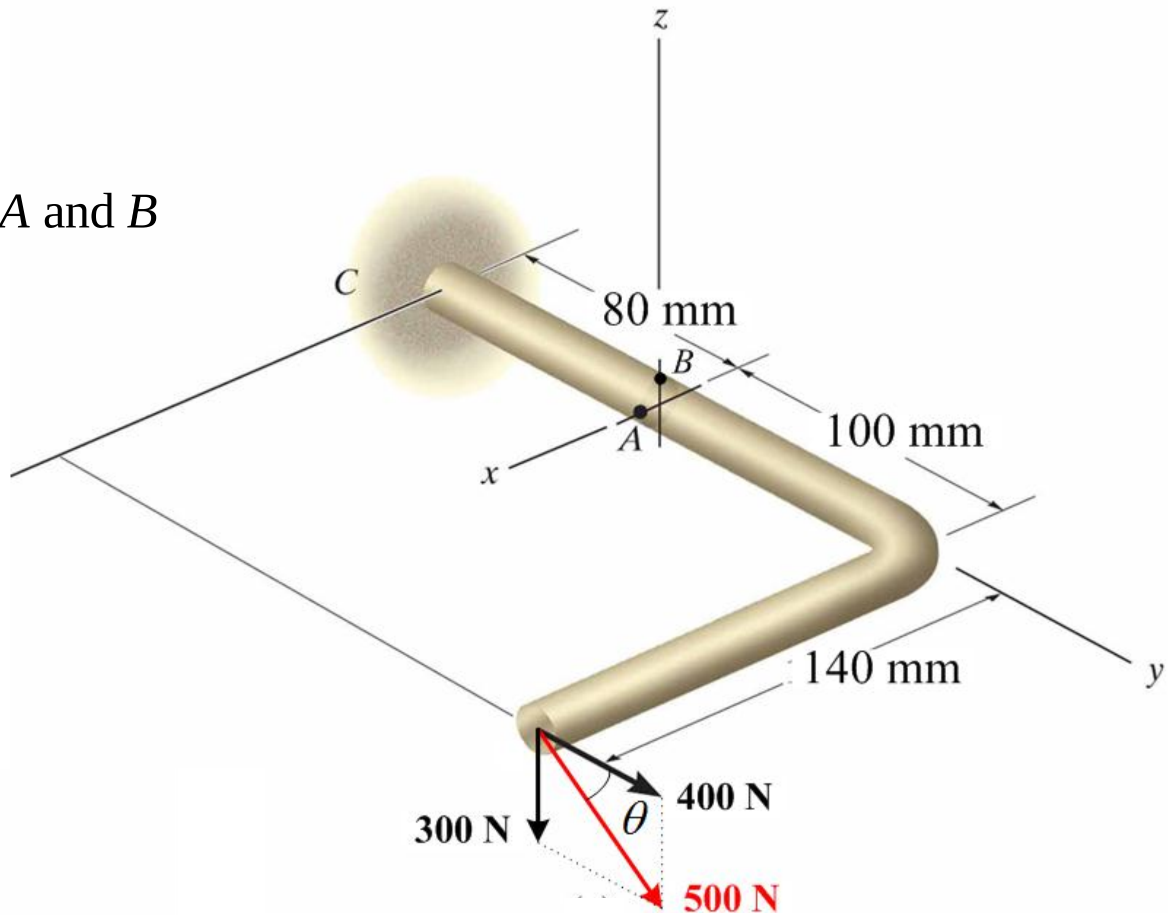
$$\tau_B = \frac{\mathbf{F_z} Q}{I_y t} = \frac{800\times[0.01\times0.03\times0.015]}{\frac{1}{12}\times0.03\times(0.04)^3\times0.03} = 750 \text{ kPa}$$

$$\tau_{xz}\Big|_B = -750 \text{ kPa}$$

Example :

$\theta = 36.87^o, \ d = 15 \text{ mm}$

Find the state of stress at points *A* and *B*



Solution :

take section at points *A* and *B*

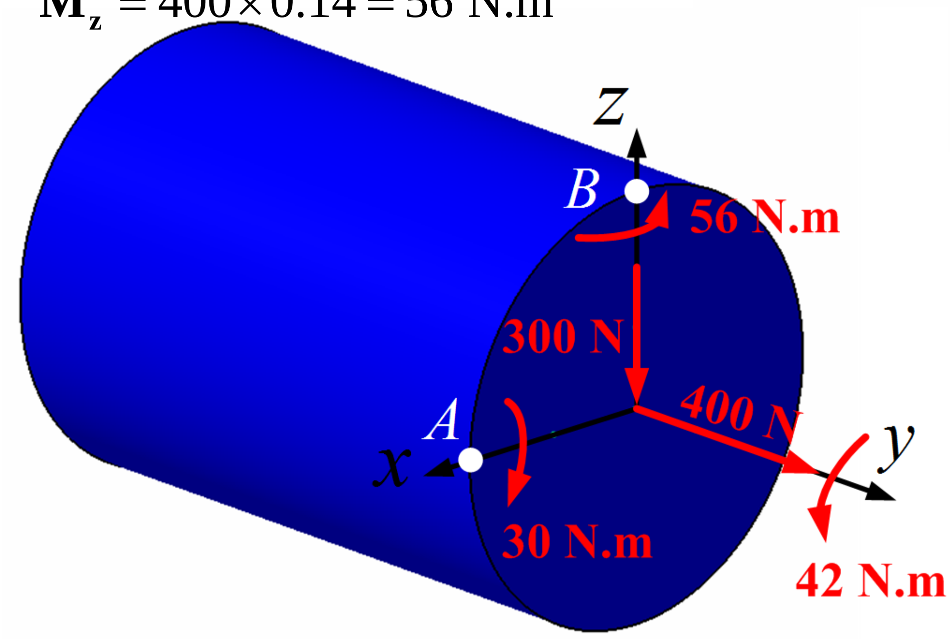
and calculate the internal forces

$\mathbf{F_x} = 0, \ \mathbf{F_y} = 400 \text{ N}, \ \mathbf{F_z} = -300 \text{ N}$

$\mathbf{M_x} = -300 \times 0.1 = -30 \text{ N.m}$

$\mathbf{T_y} = 300 \times 0.14 = 42 \text{ N.m}$

$\mathbf{M_z} = 400 \times 0.14 = 56 \text{ N.m}$



$$\sigma_A = \frac{\mathbf{F_y}}{A} + \frac{\left|\mathbf{M_z}\right| r}{I} = \frac{400}{\frac{\pi}{4}(0.015)^2} + \frac{56 \times 7.5 \times 10^{-3}}{\frac{\pi}{64}(0.015)^2}$$

$$= 2.26 \text{ MPa} + 169.01 \text{ MPa} = 171.27 \text{ MPa}$$

$$\sigma_B = \frac{\mathbf{F_y}}{A} + \frac{\left|\mathbf{M_x}\right| r}{I} = \frac{400}{\frac{\pi}{4}(0.015)^2} + \frac{30 \times 7.5 \times 10^{-3}}{\frac{\pi}{64}(0.015)^4}$$

$$= 2.26 \text{ MPa} + 90.54 \text{ MPa} = 92.8 \text{ MPa}$$

$$Q = \bar{A} \cdot \bar{y} = \frac{\pi}{8}(0.015)^2 \times \frac{4 \times 7.5 \times 10^{-3}}{3\pi} = 281 \times 10^{-9} \text{ m}^3$$

$$\tau_A = \frac{F_z Q}{It} + \frac{\mathbf{T_y} \cdot r}{J} = \frac{300 \times 281 \times 10^{-9}}{\frac{\pi}{64}(0.015)^4 \times 0.015} + \frac{42 \times 7.5 \times 10^{-3}}{\frac{\pi}{32}(0.015)^4}$$

$$= 2.26 \text{ MPa} + 63.38 \text{ MPa} = 65.64 \text{ MPa}$$

$$\tau_B = \frac{\mathbf{T_y} \cdot r}{J} = \frac{42 \times 7.5 \times 10^{-3}}{\frac{\pi}{32}(d)^4} = 63.38 \text{ MPa}$$

Example :

cylinder radius ($r = 20$ mm)

Find

(a) the normal and shearing stresses at points H and K

(b) the principal axes and principal stresses at K

(c) the maximum shearing stress at K.

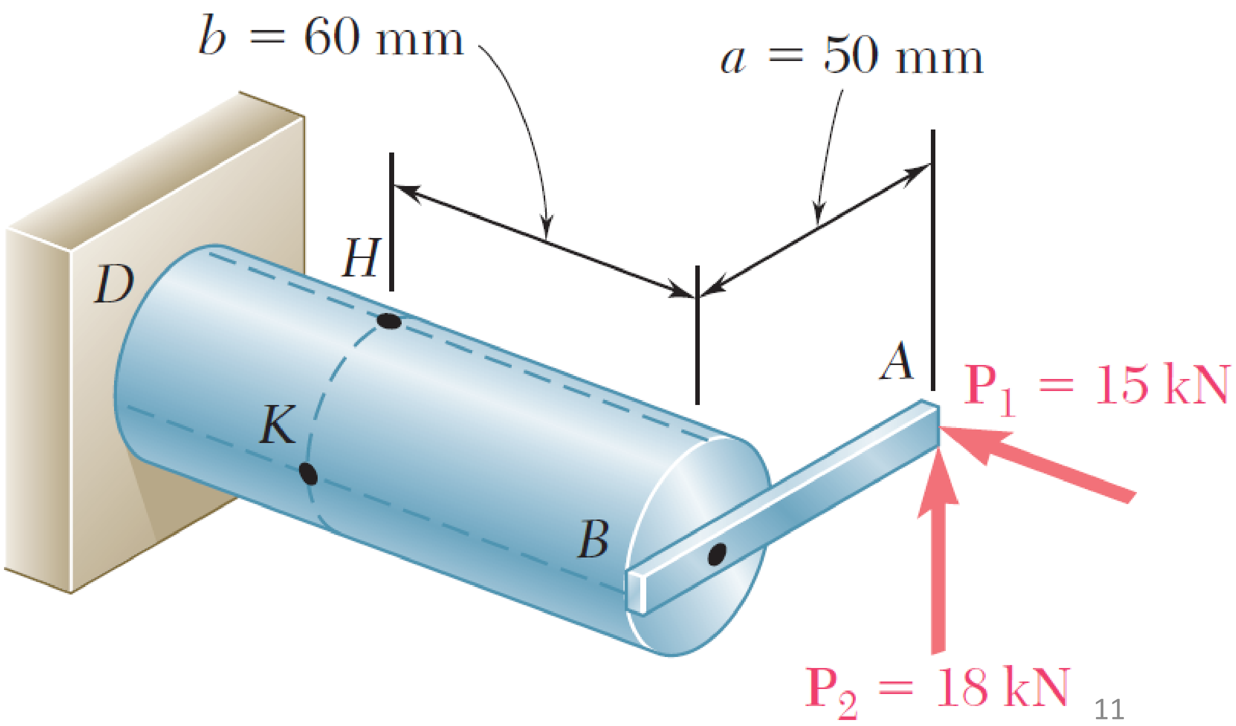
Solution :

$F_x = -15$ kN, $F_y = 18$ kN

$T_x = 18\times10^3\times0.05 = 900$ N.m

$M_y = 15\times10^3\times0.05 = 750$ N.n

$M_z = 18\times10^3\times0.06 = 1080$ N.m



Point H

$$\sigma_x = -\frac{|\mathbf{F_x}|}{A} - \frac{|\mathbf{M_z}|r}{I} = \frac{-15\times 10^3}{\pi(0.02)^2} - \frac{1080\times 0.02}{\frac{\pi}{4}(0.02)^4}$$

$$= -11.9 \text{ MPa } -171.9 \text{ MPa} = -183.8 \text{ MPa}$$

$$\tau_{xz} = \frac{\mathbf{T_x}\cdot r}{J} = \frac{900\times 0.02}{\frac{\pi}{2}(0.02)^4} = +71.6 \text{ MPa}$$

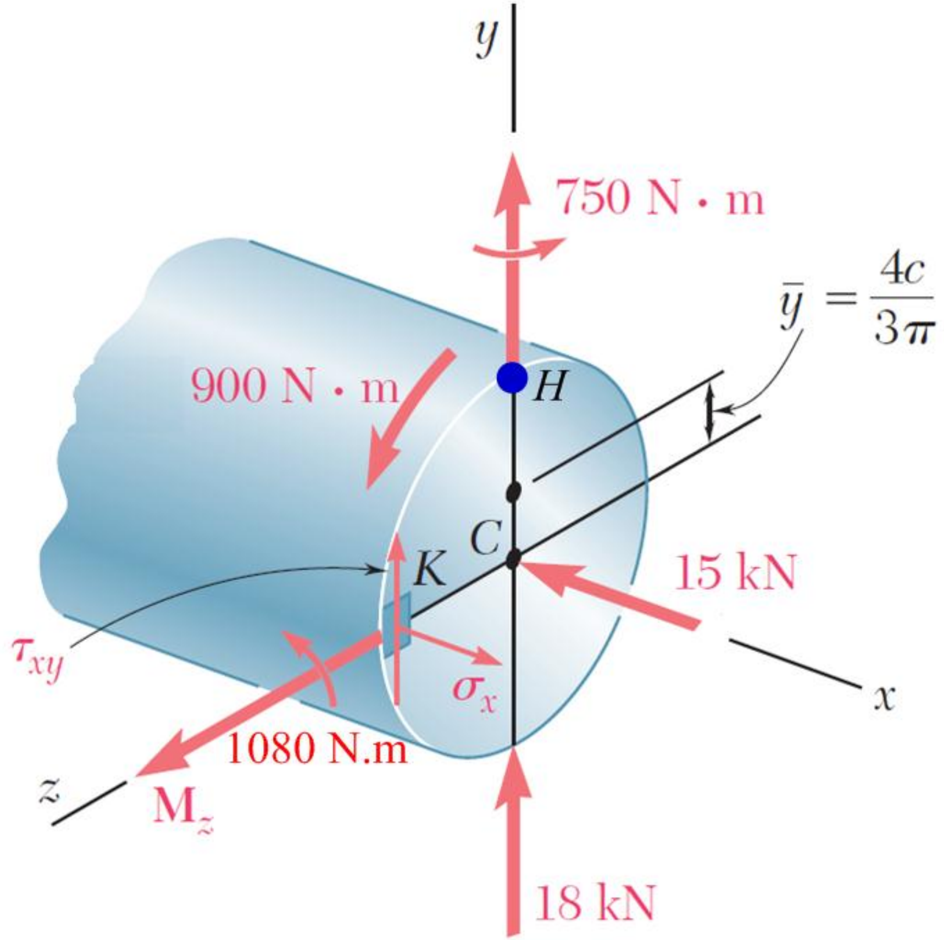
Point K

$$\sigma_x = -\frac{|\mathbf{F_x}|}{A} + \frac{|\mathbf{M_y}|r}{I} = \frac{-15\times 10^3}{\pi(0.02)^2} + \frac{750\times 0.02}{\frac{\pi}{4}(0.02)^4}$$

$$= -11.9 \text{ MPa} + 119.3 \text{ MPa} = 107.4 \text{ MPa}$$

$$\tau_{xy} = \frac{F_y Q}{It} - \frac{\mathbf{T_x}\cdot r}{J} = \frac{18\times 10^3 \times 5.33\times 10^{-6}}{\frac{\pi}{4}(0.02)^4 \times 0.02} - \frac{900\times 0.02}{\frac{\pi}{2}(0.02)^4}$$

$$= 19.1 \text{ MPa} - 71.6 \text{ MPa} = -52.5 \text{ MPa}$$



$\sigma_x = 107.4 \text{ MPa}, \quad \tau_{xy} = -52.5$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

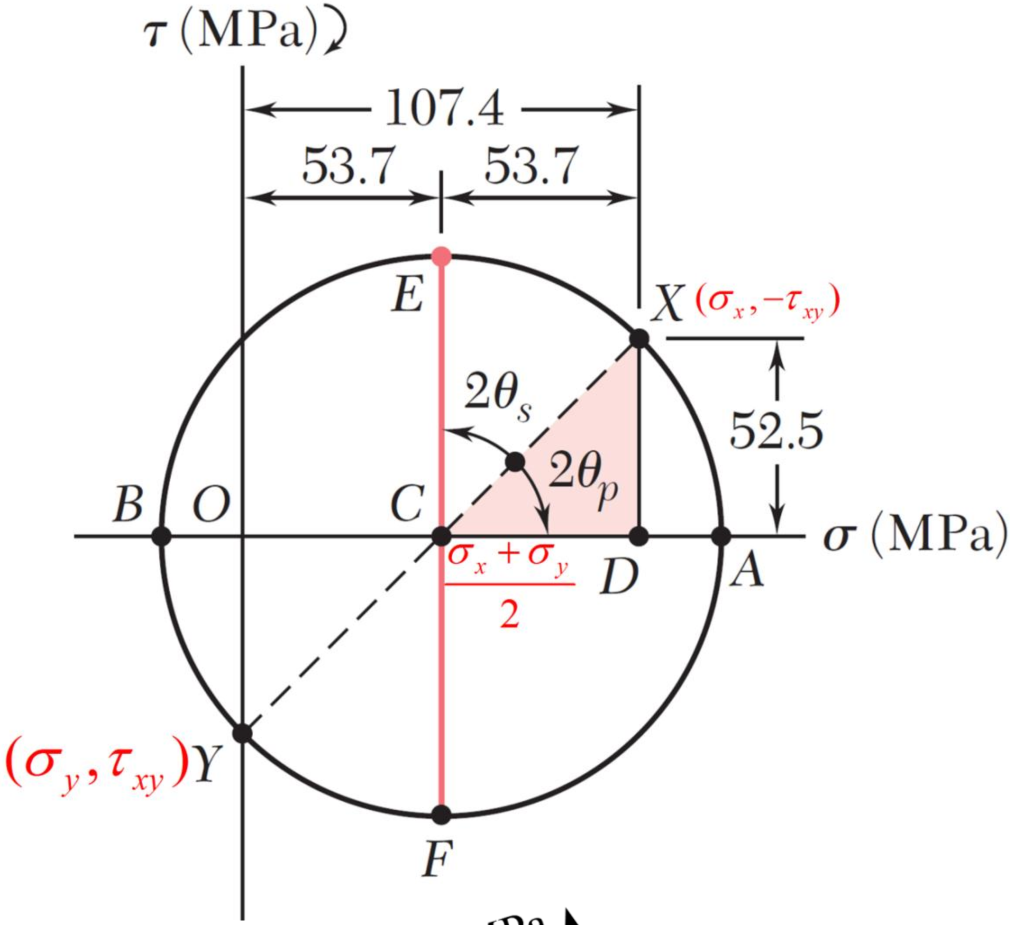
$\sigma_1 = 128.8 \text{ MPa}, \quad \sigma_2 = -21.4 \text{ MPa}$

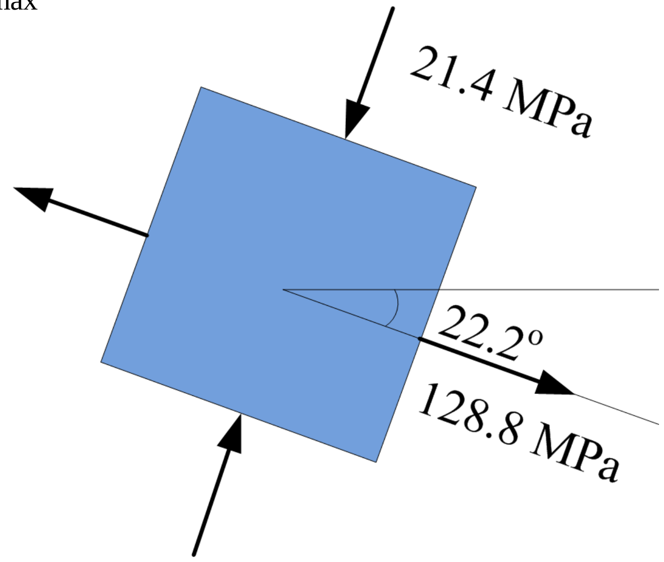
$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}} = 75.1 \text{ MPa}$$

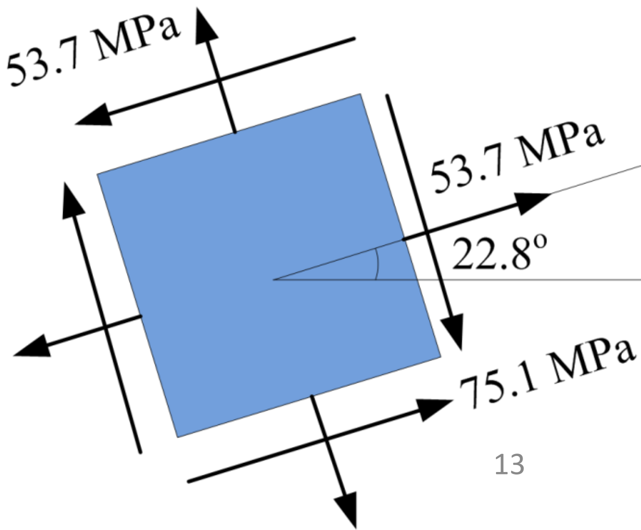
$$\tan 2\theta_P = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \rightarrow \quad \theta_{P1} = 22.2^0 \ \text{(cw)}$$

thus $\theta_{S1} = 22.8^0$ (ccw)

$\tau_{max} = R = 75.1 \text{ MPa}$







Example : Find

1. the state of stress at points E and H.
2. the principal stresses and their planes at point H.
3. the maximum shearing stress at point H.

Solution :

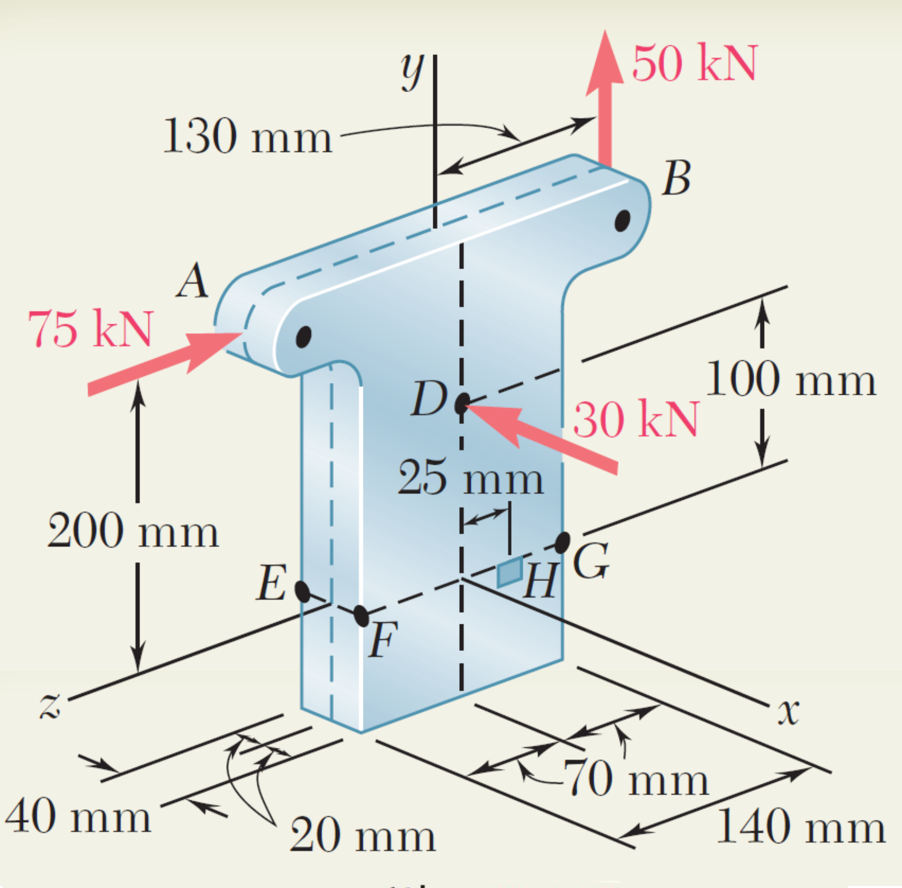
$$\mathbf{F_x} = -30 \text{ kN}$$

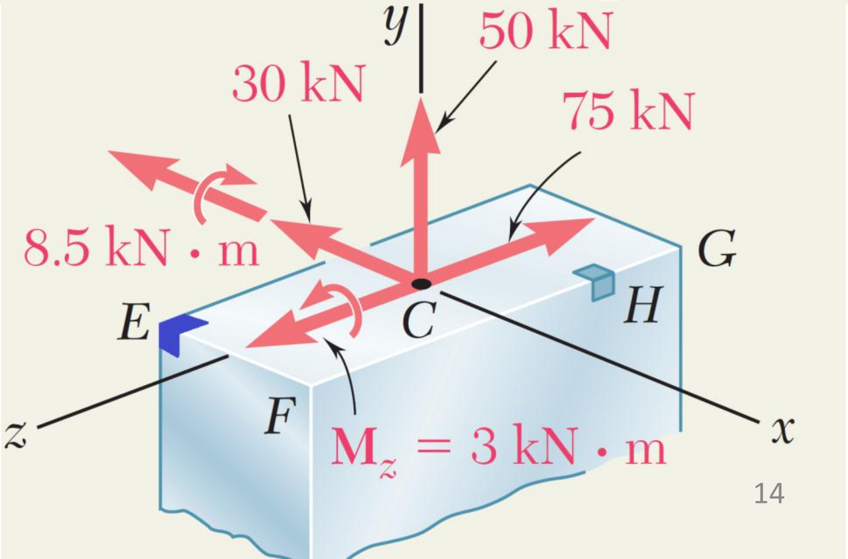
$$\mathbf{F_y} = 50 \text{ kN}$$

$$\mathbf{F_z} = -75 \text{ kN}$$

$$\mathbf{M_x} = 50\times10^3\times0.13 - 75\times10^3\times0.2$$
$$= -8.5 \text{ kN.m}$$

$$\mathbf{M_z} = 30\times10^3\times0.1 = 3 \text{ kN.m}$$





Point E

$$\sigma_y = \frac{|\mathbf{F_y}|}{A} + \frac{|\mathbf{M_x}| z_E}{I_x} - \frac{|\mathbf{M_z}| x_E}{I_z} = \frac{50\times10^3}{0.14\times0.04} + \frac{8.5\times10^3\times0.07}{\frac{1}{12}\times0.04\times(0.14)^3} - \frac{3\times10^3\times0.02}{\frac{1}{12}\times0.14\times(0.04)^3}$$

$$\tau_E = 0$$

Point H

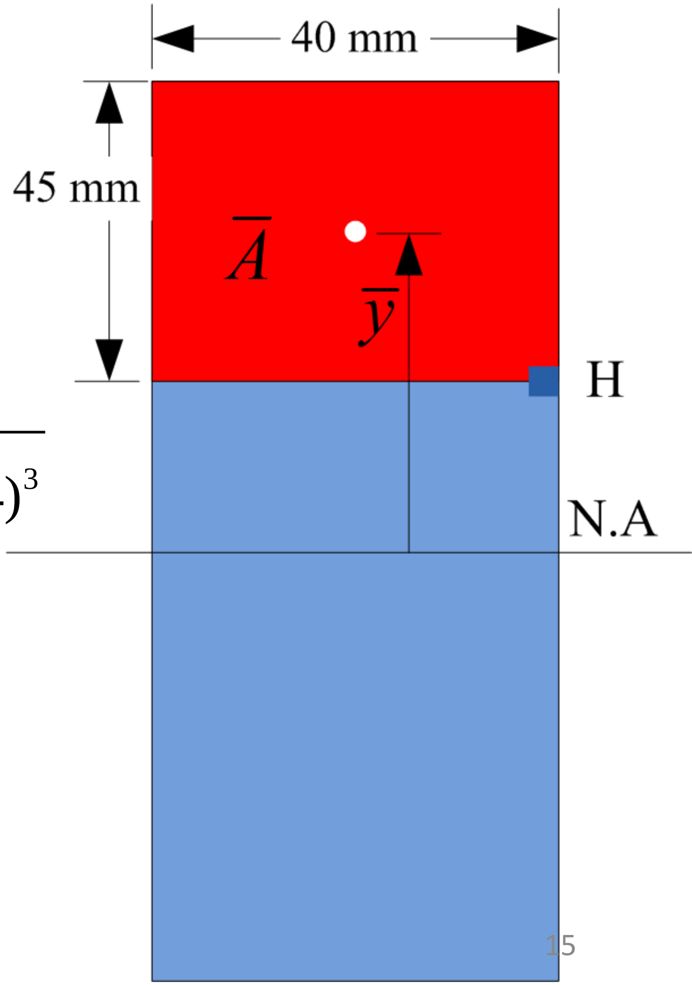
$$\sigma_y = \frac{|\mathbf{F_y}|}{A} - \frac{|\mathbf{M_x}| z_H}{I_x} + \frac{|\mathbf{M_z}| x_H}{I_z}$$

$$= \frac{50\times10^3}{0.14\times0.04} - \frac{8.5\times10^3\times0.025}{\frac{1}{12}\times0.04\times(0.14)^3} + \frac{3\times10^3\times0.02}{\frac{1}{12}\times0.14\times(0.04)^3}$$

$$= 66 \text{ MPa}$$

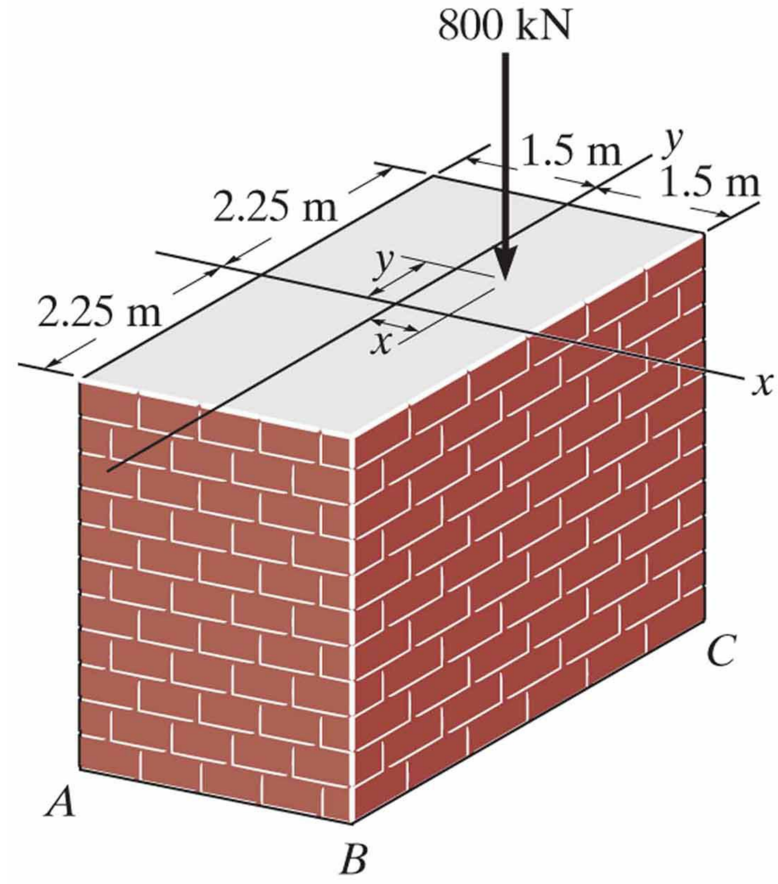
$$Q = \bar{A}\cdot\bar{y} = 0.04\times0.045\times0.0474 = 85.5\times10^{-6}$$

$$\tau_{yz} = -\frac{|F_z| Q}{I_x t} = \frac{75\times10^3\times85.5\times10^{-6}}{\frac{1}{12}\times0.04(0.14)^3\times0.04} = 17.52 \text{ MPa}$$



Example:

Find the equation *y = F(x)*, such that no tension will occur for the column.



Solution :

Tension will occur first at point *A*, thus

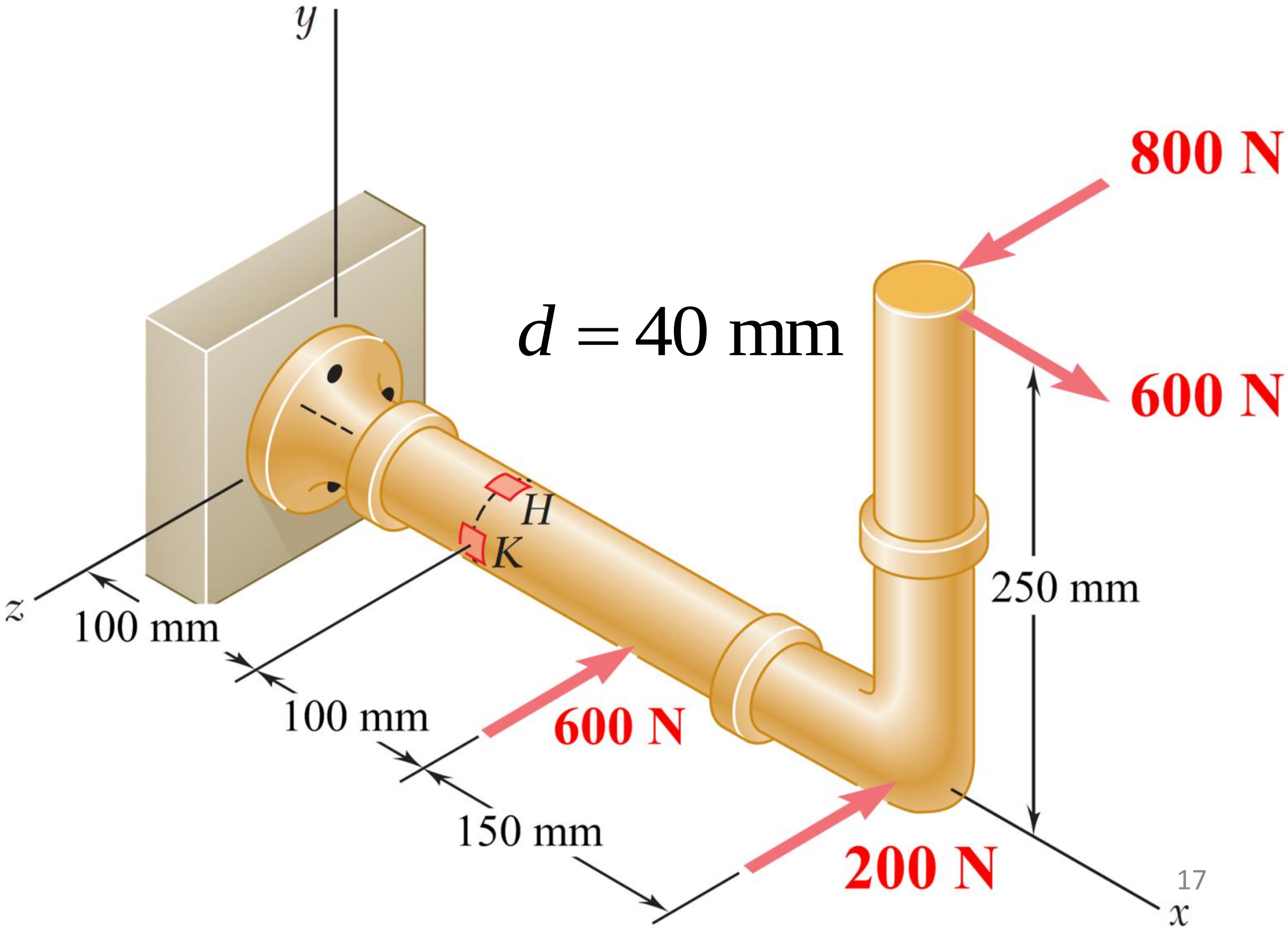
$$\sigma_A = \frac{-800\times10^3}{4.5\times3} + \frac{(800\times10^3\times y)\times2.25}{\frac{1}{12}\times3\times(4.5)^3} + \frac{(800\times10^3\times x)\times1.5}{\frac{1}{12}\times4.5\times(3)^3} = 0$$

gives

$$y = 0.75 - 1.5x$$

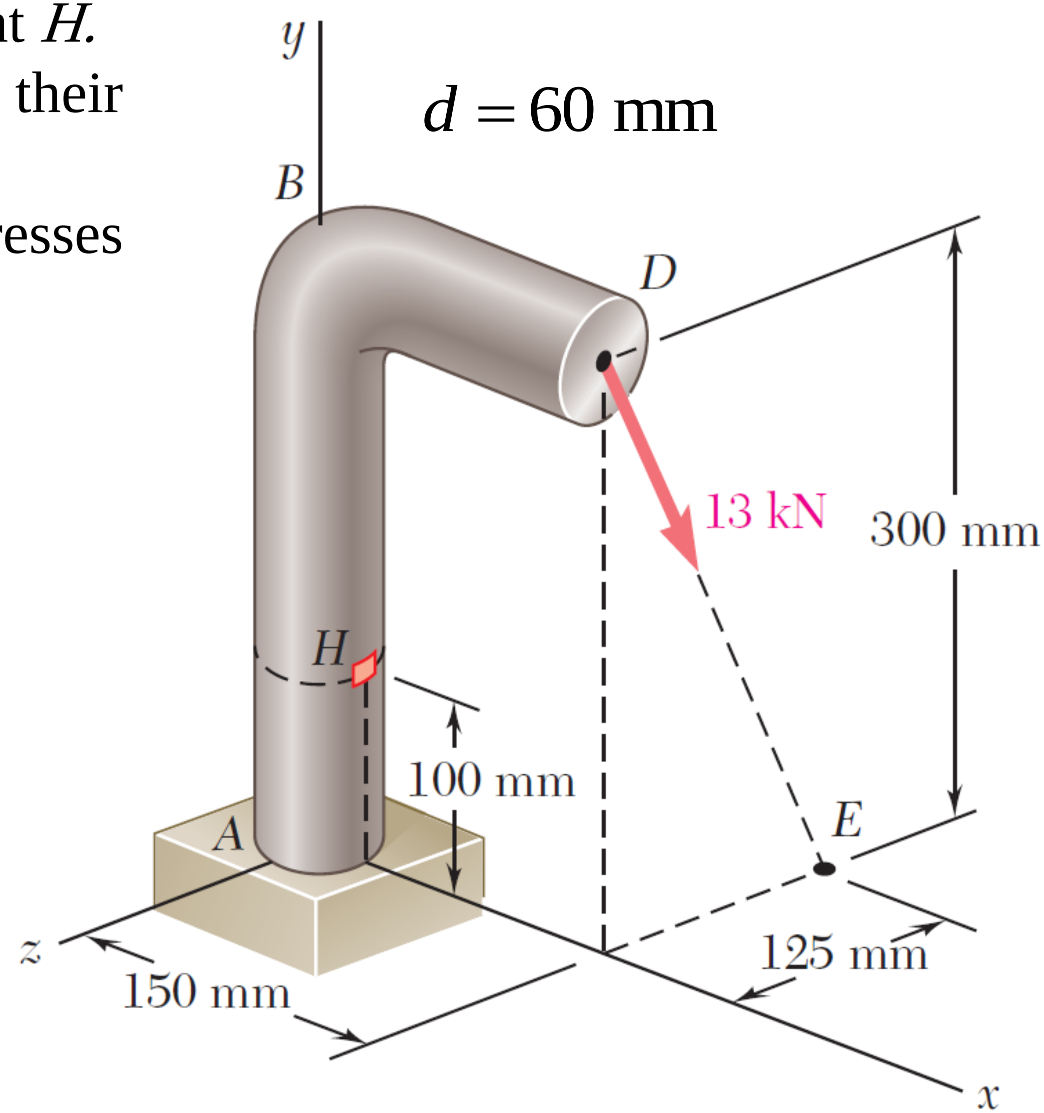
Problem: Find

1. the state of stresses at points *H* and *K.*
2. the principal stresses and their orientations.
3. the maximum shear stresses and their orientations.
4. mohr's circles for both *H* and *K.*



Problem: Find

1. the state of stresses at point *H.*
2. the principal stresses and their orientations.
3. the maximum shear stresses and their orientations.
4. draw mohr's circle.

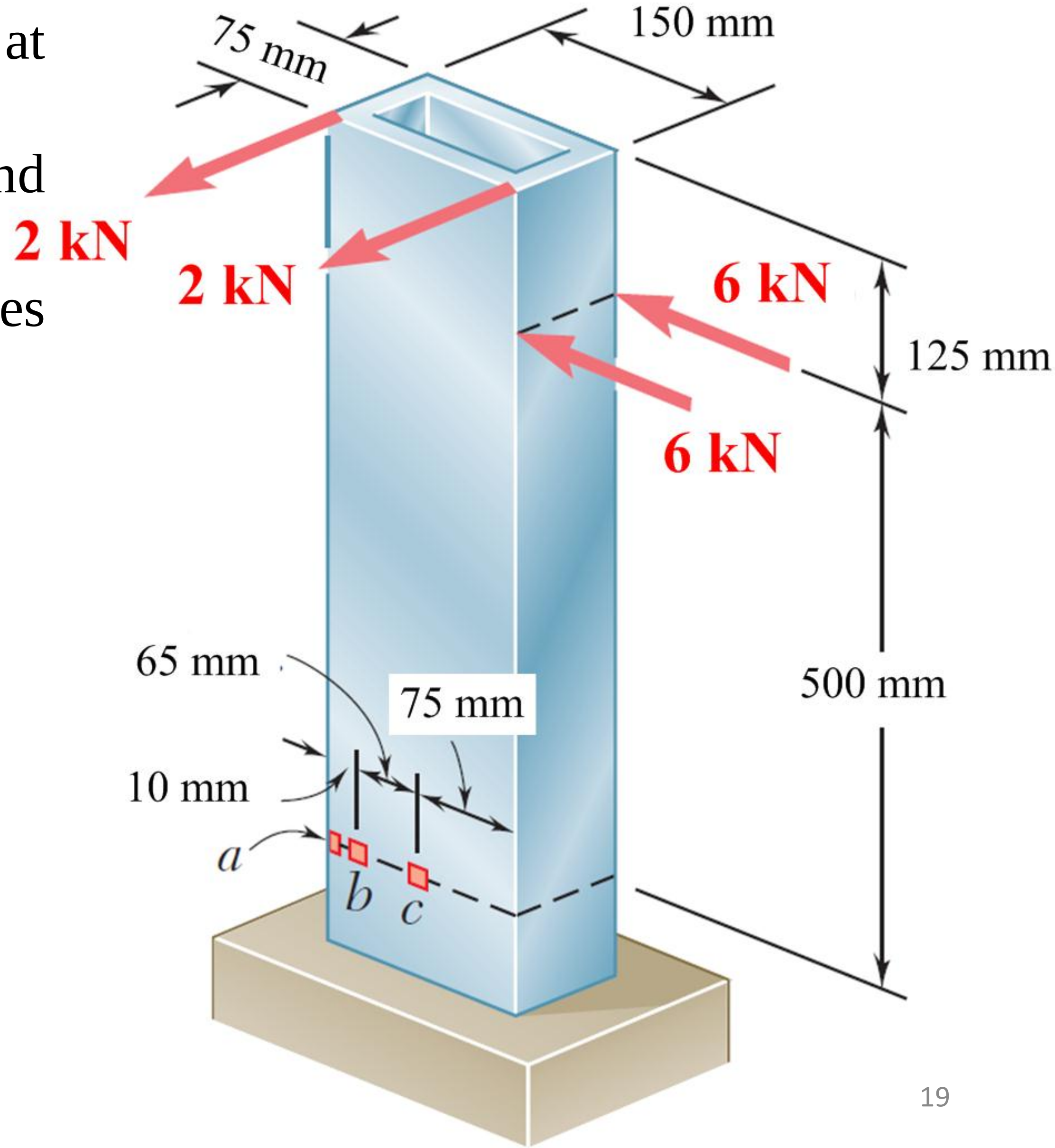


Problem: Find

1. the state of stresses at points *a*, *b* and *c*.
2. the principal stresses and their orientations.
3. the maximum shear stresses and their orientations.
4. draw mohr's circles

all walls' thicknesses

$t = 10$ mm



END CHAPTER 8

MECHANICS OF MATERIALS

CHAPTER NINE

DEFLECTION OF BEAMS

9.1 INTRODUCTION

$$\varepsilon = \frac{\Delta s' - \Delta x}{\Delta x} = \frac{(\rho - y)\Delta\theta - \rho\Delta\theta}{\rho\Delta\theta} = -\frac{y}{\rho}$$

$$\rho = -\frac{y}{\varepsilon}$$

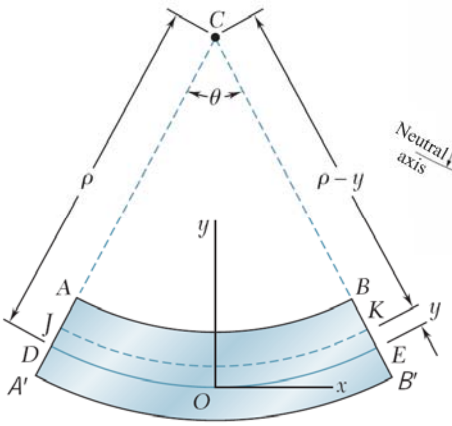
but

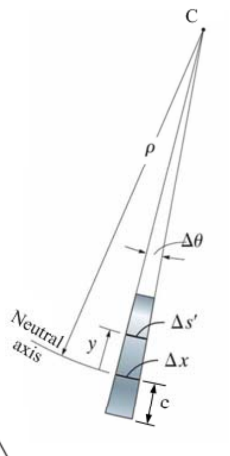
$$\varepsilon = \frac{\sigma}{E} = \frac{1}{E}\left(-\frac{\mathbf{M}\cdot y}{I}\right)$$

thus

$$\frac{1}{\rho} = \frac{\mathbf{M}}{EI}$$

Radius of curvature





9.3, 9.4 EQUATION OF THE ELASTIC CURVE

From math, we have

$$\frac{1}{\rho}=\frac{\left(\dfrac{d^2y}{dx^2}\right)}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$$

However, in most cases, the slope is very small leading to negligible term $\left(\dfrac{dy}{dx}\right)^2$ yielding

$$\frac{1}{\rho}=\frac{d^2y}{dx^2}=\frac{\mathbf{M}}{EI}$$

3

SUMMARY

$$\mathbf{M}(x)=EI\frac{d^2y}{dx^2}$$

$$\mathbf{V}(x)=\frac{d}{dx}\left(EI\frac{d^2y}{dx^2}\right)$$

$$w(x)=\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right)$$

if E and I are constants, then

$y=y(x)$ (elastic curve)

$\theta(x)=\dfrac{dy}{dx}$ (slope)

$$\mathbf{M}(x)=EI\frac{d^2y}{dx^2}$$

$$\mathbf{V}(x)=EI\frac{d^3y}{dx^3}$$

$$w(x)=EI\frac{d^4y}{dx^4}$$

4

SIGN CONVENTION (REVIEW)

Distributed load

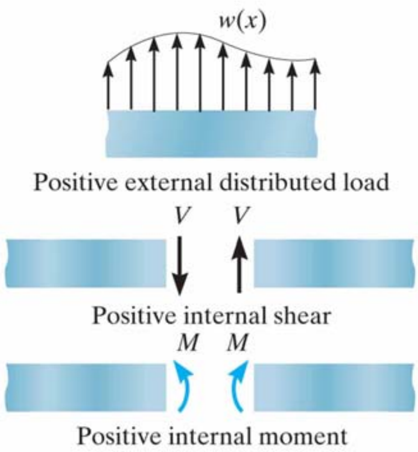
Upward is positive

Shear

If the internal shear rotates the segment cw, the shear is then positive.

Moment

If the internal moment causes compression on the top surface (holding the water), the moment is then positive



Beam sign convention

5

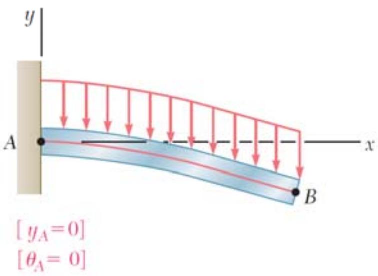
$$\mathbf{V}(x) = \int_0^x w(x)\, dx + C_1$$

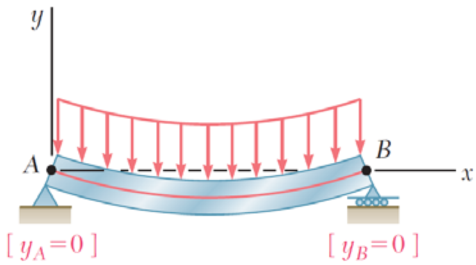
$$\mathbf{M}(x) = \int_0^x \mathbf{V}(x)\, dx + C_2$$

$$EI\theta(x) = \int_0^x \mathbf{M}(x)\, dx + C_3$$

$$y(x) = \int_0^x \theta(x)\, dx + C_4$$

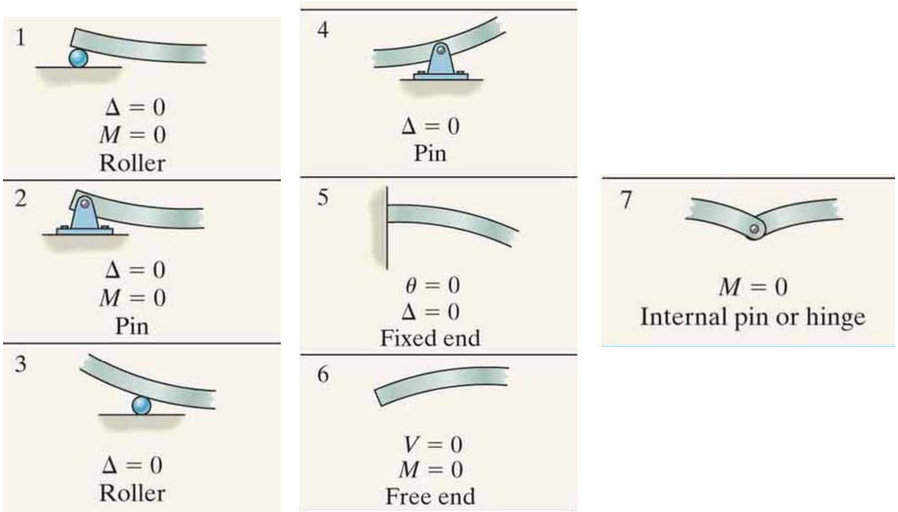
C_1, C_2, C_3 and C_4 are obtained from the boundary conditions





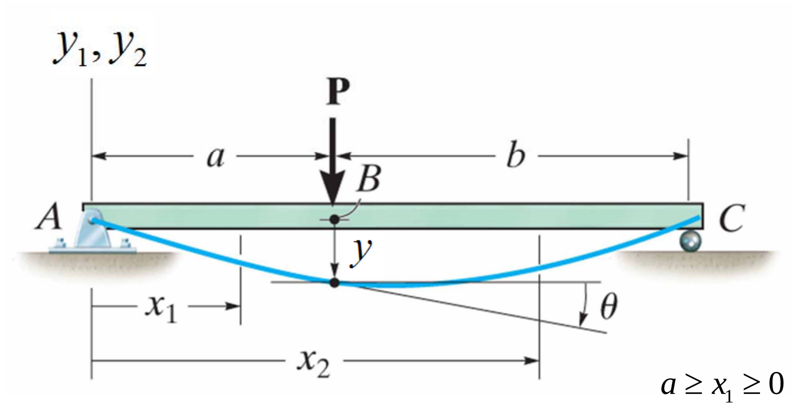
6

BOUNDARY CONDITIONS 1



7

BOUNDARY CONDITIONS 2



$a \geq x_1 \geq 0$

$a + b \geq x_2 \geq a$

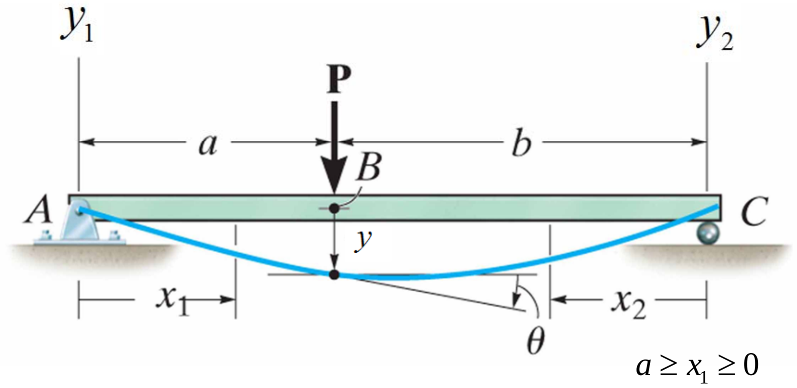
at point B

$y_1(a) = y_2(a)$

$\theta_1(a) = \theta_2(a)$

8

BOUNDARY CONDITIONS 3

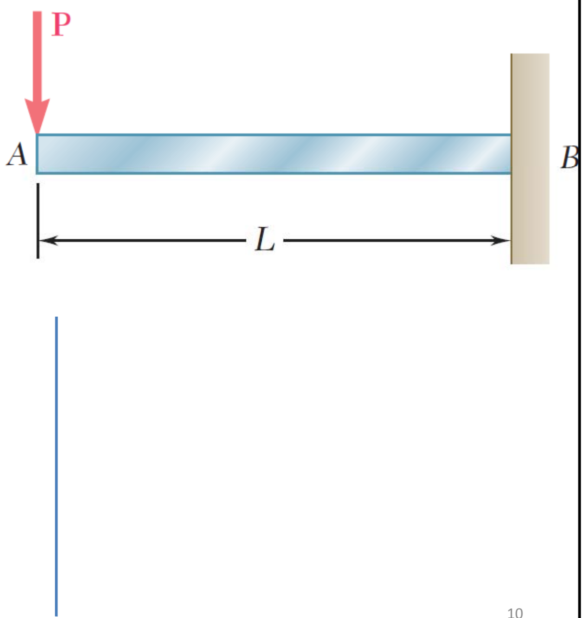


$$a \geq x_1 \geq 0$$
$$b \geq x_2 \geq 0$$
at point B
$$y_1(a) = y_2(b)$$
$$\theta_1(a) = -\theta_2(b)$$

9

Example:

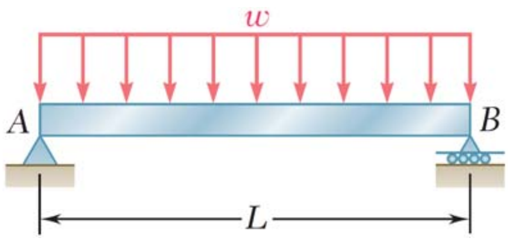
Find the elastic curve for the cantilever shown.



10

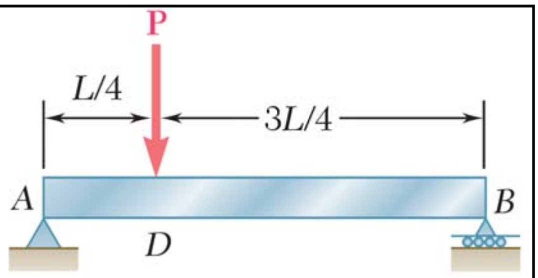
Example:

Find the elastic curve for the simply supporter shown.



Example:

Find the elastic curve for the simply supporter shown.



Solution :

$$\mathbf{M}=\begin{cases}\dfrac{3\mathbf{P}}{4}x & L/4\geq x\geq 0\\ \dfrac{3\mathbf{P}}{4}x-\mathbf{P}\left(x-\dfrac{L}{4}\right) & L\geq x\geq L/4\end{cases}$$

$$EI\theta(x)=\begin{cases}\dfrac{3\mathbf{P}}{8}x^2+C_1 & L/4\geq x\geq 0\\ \dfrac{3\mathbf{P}}{8}x^2-\mathbf{P}\left(\dfrac{x^2}{2}-\dfrac{L}{4}x\right)+C_2 & L\geq x\geq L/4\end{cases}$$

$$EIy(x)=\begin{cases}\dfrac{3\mathbf{P}}{24}x^3+C_1x+C_3 & L/4\geq x\geq 0\\ \dfrac{3\mathbf{P}}{24}x^3-\mathbf{P}\left(\dfrac{x^3}{6}-\dfrac{L}{8}x^2\right)+C_2x+C_4 & L\geq x\geq L/4\end{cases}$$

$y\big|_{x=0} = 0 \quad \rightarrow \; C_3 = 0$

$y\big|_{x=L} = 0 \qquad (1)$

$y\big|_{x=L/4} = y\big|_{x=L/4} \qquad (2)$

$\theta\big|_{x=L/4} = \theta\big|_{x=L/4} \qquad (3)$

solving the equations simultaneously, you get

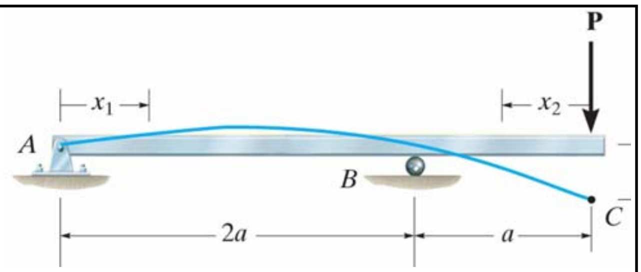
$$C_1 = -\frac{7\mathbf{P}L^2}{128},\; C_2 = -\frac{11\mathbf{P}L^2}{128} \;\text{ and } C_4 = \frac{\mathbf{P}L^3}{384}$$

$$y(x) = \frac{1}{EI}\begin{cases} \dfrac{3\mathbf{P}}{24}x^3 - \dfrac{7\mathbf{P}L^2}{128}x & L/4 \ge x \ge 0 \\ \dfrac{3\mathbf{P}}{24}x^3 - \mathbf{P}\left(\dfrac{x^3}{6} - \dfrac{L}{8}x^2\right) - \dfrac{11\mathbf{P}L^2}{128}x + \dfrac{\mathbf{P}L^3}{384} & L \ge x \ge L/4 \end{cases}$$

13

Example:

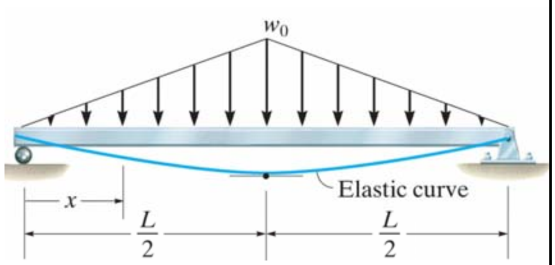
Find the elastic curve for the beam shown.



14

Example:

Find the elastic curve for the beam shown.



9.5 STATICALLY INDETERMINATE BEAMS

unknowns

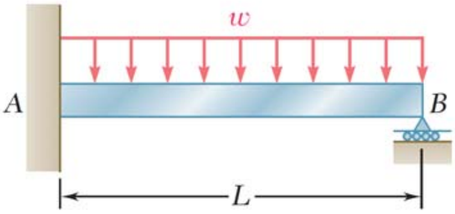
$\mathbf{A_x}, \mathbf{A_y}, \mathbf{M_A}$ and B

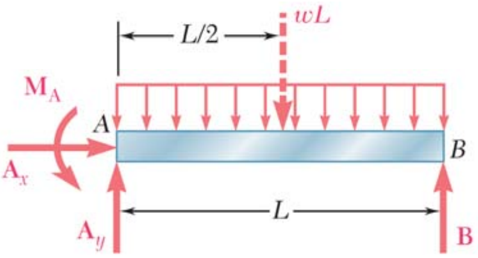
$\sum \mathbf{F_x} = 0, \sum \mathbf{F_y} = 0$ and $\sum \mathbf{M_A} = 0$

three equations and **four** unknowns

(statically indeterminate problem)

this problem can be solved using the elastic curve.





17

$$\mathbf{M} = -\frac{1}{2}wx^2 + \mathbf{A_y}x - \mathbf{M_A}$$

$$EI\theta(x) = -\frac{1}{6}wx^3 + \frac{\mathbf{A_y}}{2}x^2 - \mathbf{M_A}x + C_1$$

$$EIy(x) = -\frac{1}{24}wx^4 + \frac{\mathbf{A_y}}{6}x^3 - \frac{\mathbf{M_A}}{2}x^2 + C_1x + C_2$$

Summary

unknowns

$\mathbf{A_x}, \mathbf{A_y}, \mathbf{M_A}, B, C_1$ and C_2

equations

$\sum \mathbf{F_x} = 0, \sum \mathbf{F_y} = 0, \sum \mathbf{M_A} = 0$

$y|_{x=0} = 0, \quad y|_{x=L} = 0, \quad \theta|_{x=0} = 0$

18

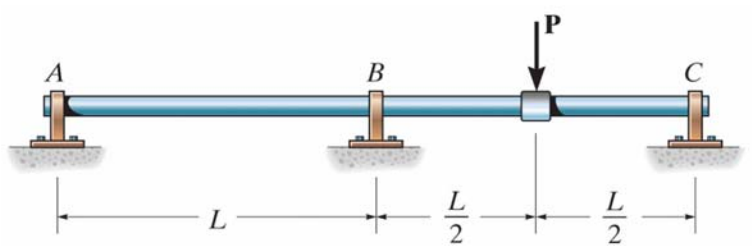
Problem : Find the reactions on the beam.

unknowns

A_y, B_y and C_y

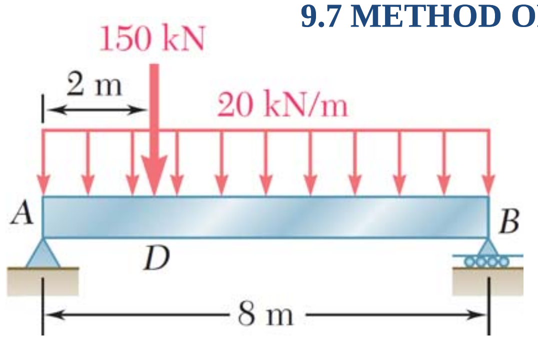
$\sum F_y = 0$ and $\sum M_A = 0$

two equations and **three** unknowns.

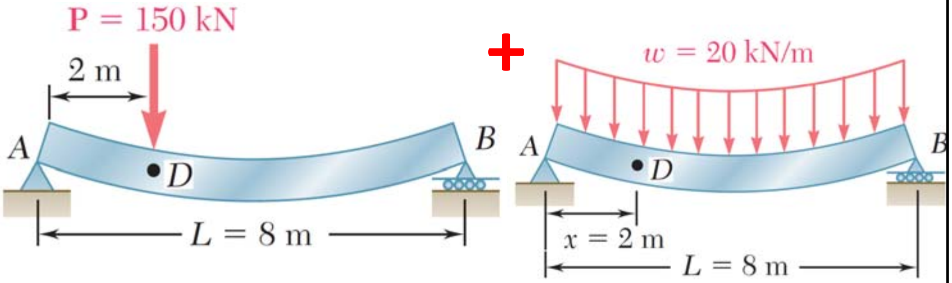


19

9.7 METHOD OF SUPERPOSITION

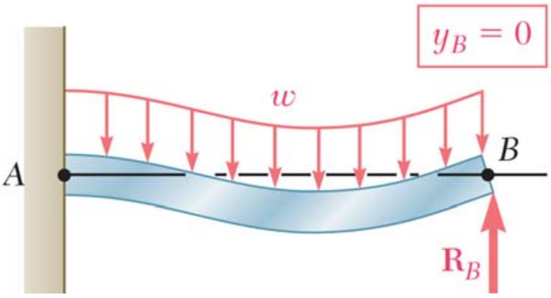


$$\theta_D = (\theta_D)_P + (\theta_D)_w$$
$$y_D = (y_D)_P + (y_D)_w$$



20

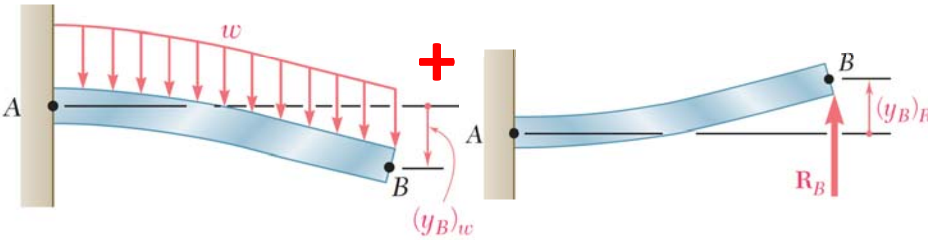
9.8 APPLICATION OF SUPERPOSITION TO STATICALLY INDETERMINATE BEAMS



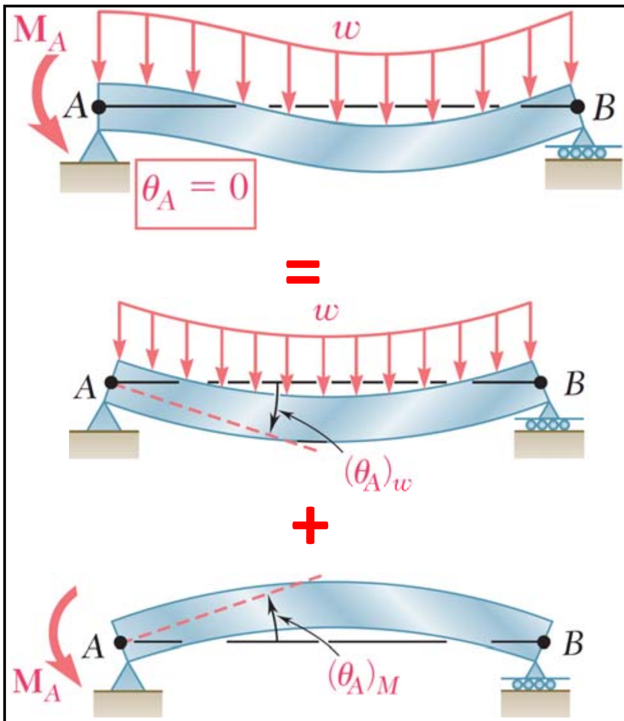
$$y_B = (y_B)_w + (y_B)_R = 0$$

$$= -\frac{wL^4}{8EI} + \frac{R_B L^3}{3EI} = 0$$

$$R_B = \frac{3wL}{8}$$



21



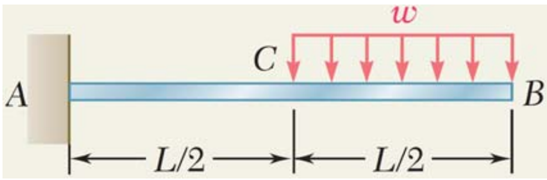
$$\theta_A = (\theta_A)_w + (\theta_A)_R = 0$$

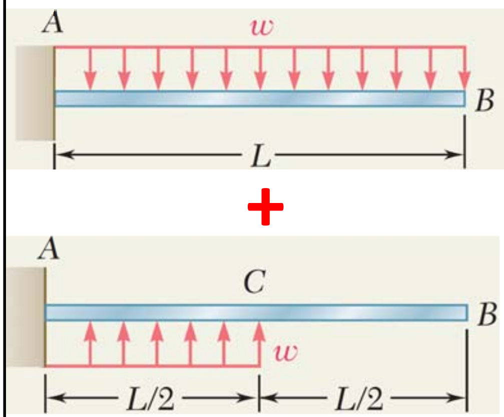
$$= -\frac{wL^3}{25EI} + \frac{M_A L}{3EI} = 0$$

$$\mathbf{M_A} = \frac{wL^2}{8}$$

22

Example:
For the beam and loading shown, determine the slope and deflection at point *B*.





$$\theta_B = (\theta_B)_1 + (\theta_B)_2$$

$$\theta_B = (\theta_B)_1 + (\theta_C)_2$$

$$= -\frac{wL^3}{6EI} + \frac{w(L/2)^3}{6EI} = -\frac{7wL^3}{48EI}$$

$$y_B = (y_B)_1 + (y_B)_2$$

$$y_B = (y_B)_1 + \left[(y_C)_2 + \left(\frac{L}{2}\right)(\theta_C)_2\right]$$

$$= -\frac{wL^4}{8EI} + \frac{wL^4}{128EI} + \frac{wL^3}{48EI}\left(\frac{L}{2}\right)$$

$$= -\frac{41wL^4}{384EI}$$

23

END OF CHAPTER 9

24