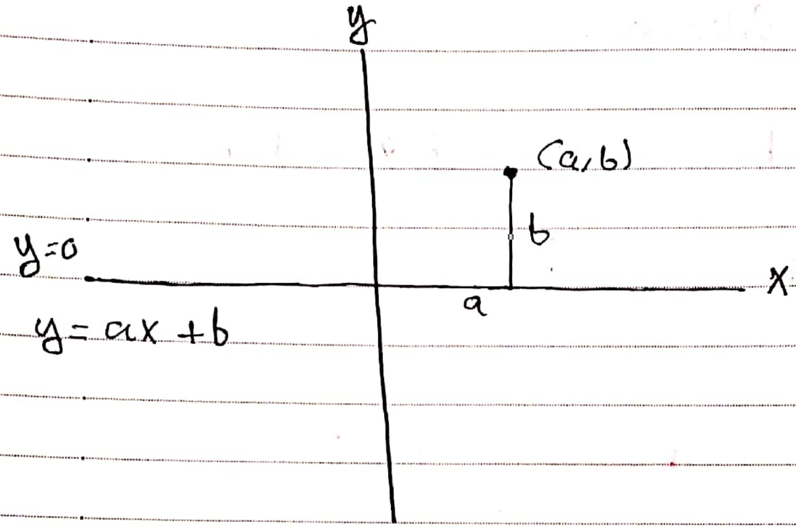
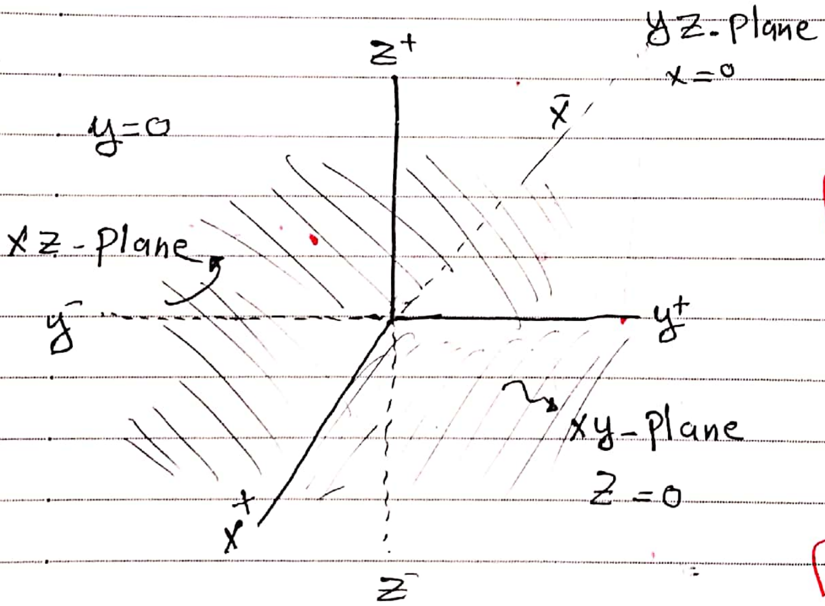
Calculus 3 محاضرة 1 يوم الثلاثاء ٢٤/٩/٢٠١٩

12.1 Three Dimentionel space.

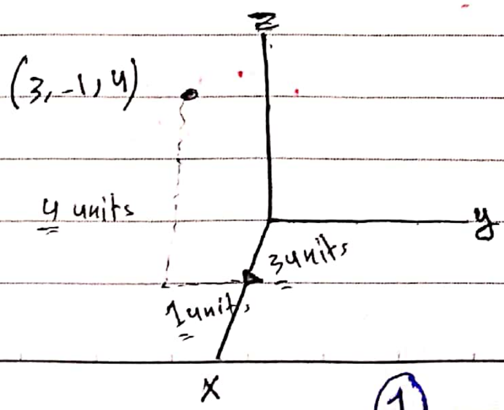




$\boxed{\text{x-axis}}$: $x = t$
$y = 0$
$z = 0$

$-\infty < t < \infty$

$\boxed{\text{y-axis}}$: $x = 0$
$y = t$
$z = 0$



$\boxed{\text{z-axis}}$: $x = 0$
$y = 0$
$z = t$

example:

Sketch the following :-

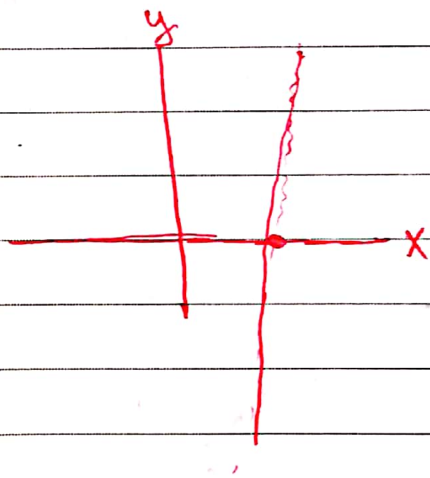
(a) $x=1$ in $\mathbb{R}$ (b) $x=1$ in $\mathbb{R}^2$

(c) $x=1$ in $\mathbb{R}^3$

(a):

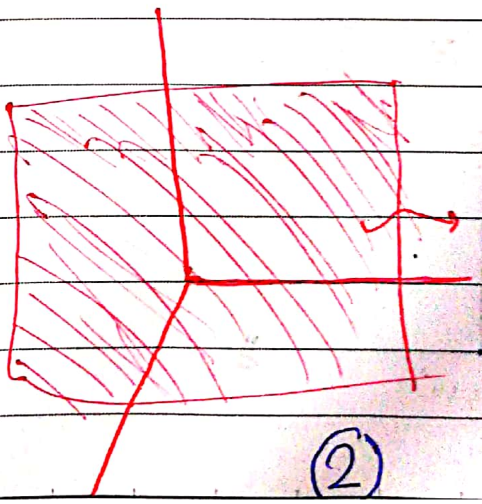
Point

(b)



line Carve

(c)

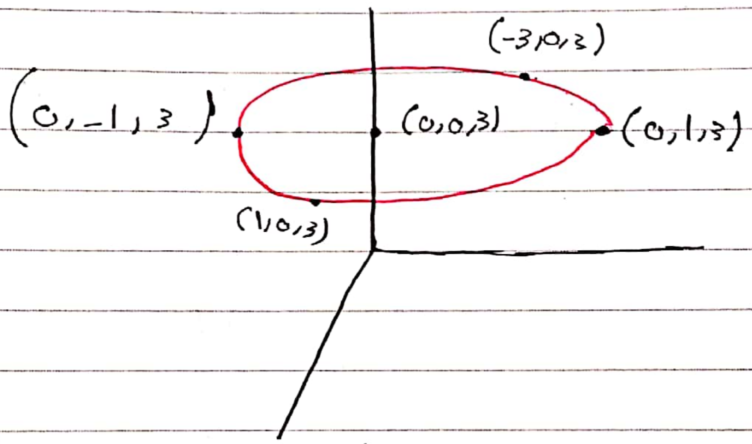


surface
plane

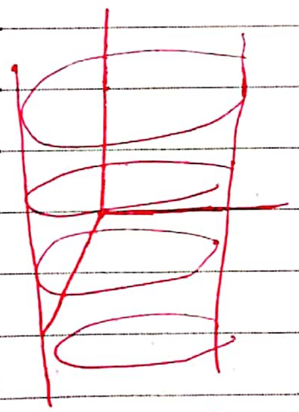
Example 2 which points (x, y, z) satisfiy the equations

ⓐ

دائرة $x^2+y^2=1$ and $z=3$ منحنى



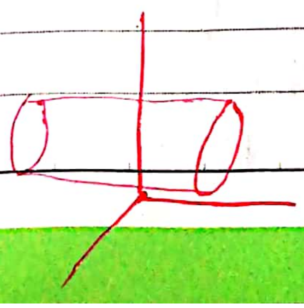
ⓑ what does equation $x^2+y^2=1$ represent as surface in R^3?

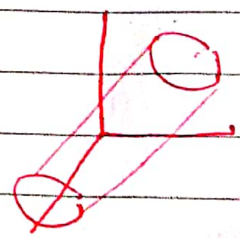


Cylinder اسطوانة

$x^2+z^2=1$

$y^2+z^2=1$



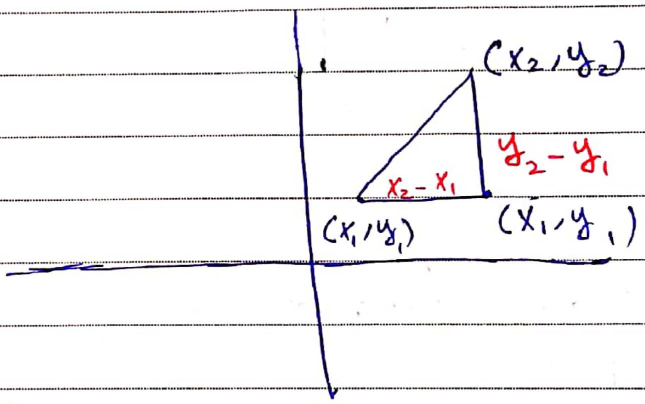


$A(x_1, y_1)$, $B(x_2, y_2)$

$$d = |AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

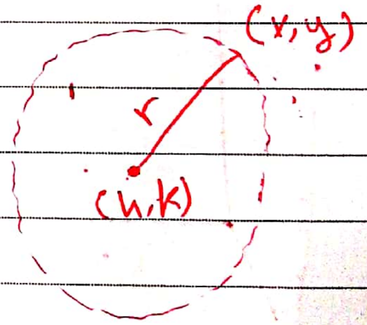
$A(x_1, y_1, z_1)$ $B(x_2, y_2, z_2)$

$$d = |AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



* equation of sphere with center (h, k, L) and radius r is :

$$(x-h)^2 + (y-k)^2 + (z-L)^2 = r^2$$



$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

$$r^2 = (x-h)^2 + (y-k)^2$$

* Note

$$x^2 + y^2 = 1$$

in 3D (surface) in 2D (circle)

$$y = x$$

in 2D (line) in 3D (plane)

example: Show that

$$x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$$

is the equation of sphere (جسم كروي)

Sol. (بإكمال مربع)

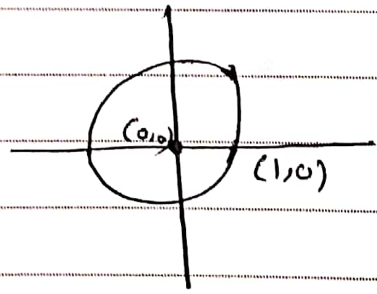
$$x^2 + 4x \underline{\underline{+4}} + y^2 - 6y \underline{\underline{+9}} + z^2 + 2z + 1 = -6 + 4 + 9 + 1$$

$$(x+2)^2 + (y-3)^2 + (z+1)^2 = (\sqrt{8})^2$$

Center $(-2, 3, -1)$ $\quad r = \sqrt{8}$

$x^2 + y^2 \leq 1$

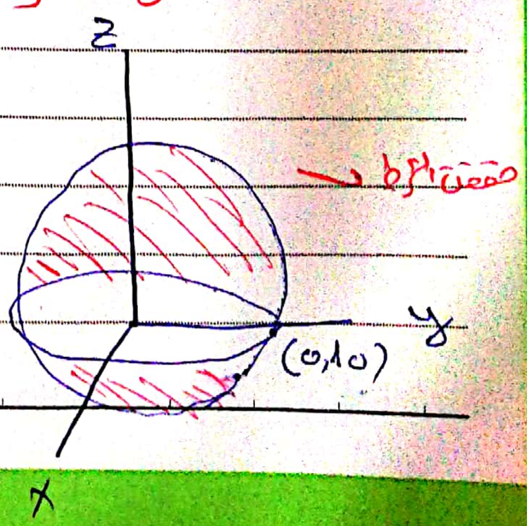
$x^2 + y^2 = 1$



* اذا تحقق الشرط بتكون داخل الدائرة أما اذا لم تحقق تكون خارج الدائرة

* example: sketch $x^2 + y^2 + z^2 \leq 1$

$x^2 + y^2 + z^2 = 1$



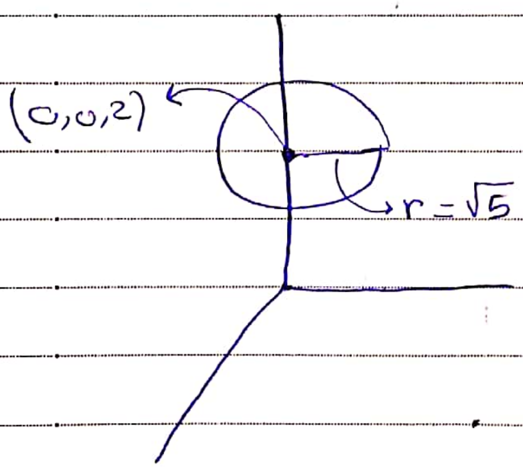
Example: Sketch the curve of intersection between

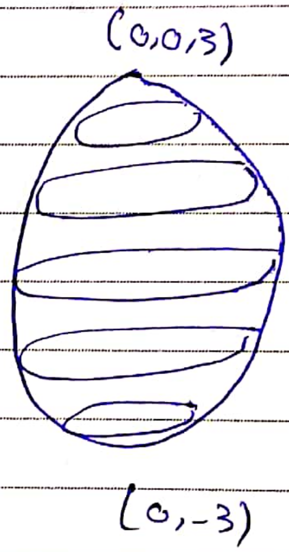
$$x^2+y^2+z^2=9 \quad \text{and} \quad z=2$$

Sub $z=2$ in $x^2+y^2+z^2=9$

$$x^2+y^2+4=9$$

$$\boxed{x^2+y^2=5}$$





example 7 (بالكتاب)

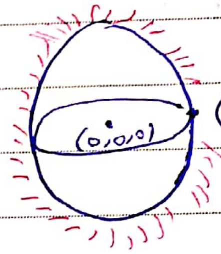
what Region in R^3 represented by

$$1 \leq x^2+y^2+z^2 \leq 4 \quad \text{and} \quad z \leq 0$$

Sol.

① $1 \leq x^2+y^2+z^2$

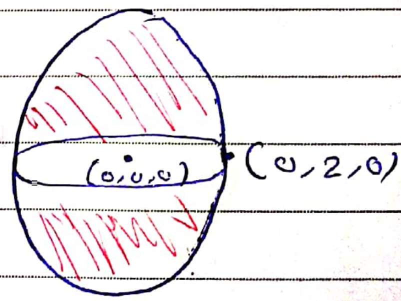
$x^2+y^2+z^2=1$



(نقاط برا ال1)

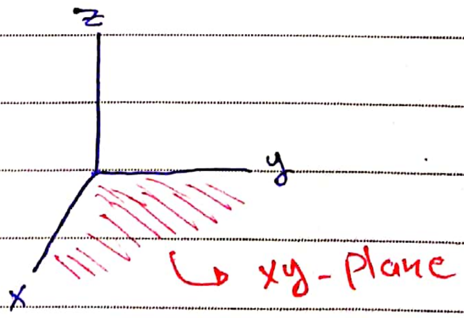
outside the unit sphere

② $x^2+y^2+z^2 \leq 4$



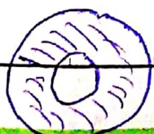
(نقاط الداخل)

③ $z=0$



Lower half space

* بالنهاية الشكل النهائي حيكون مثل CD



12.2 Vectors

Defn.

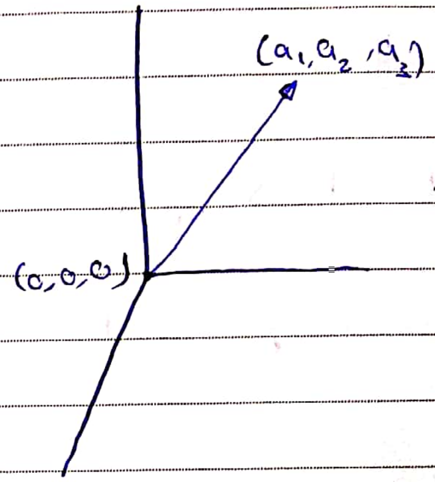
$\vec{A} = \langle a_1, a_2, a_3 \rangle$

رمز ال Vector

للتمييز بينه وبين النقطة

أو

$\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

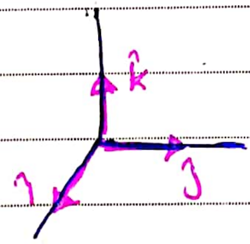


where

$\hat{i} = \langle 1, 0, 0 \rangle$

$\hat{j} = \langle 0, 1, 0 \rangle$

$\hat{k} = \langle 0, 0, 1 \rangle$



Defn.

A vector with initial point $A(a_1, b_1, c_1)$ and terminal point $B(a_2, b_2, c_2)$ is given by

$$\overrightarrow{AB} = (a_2 - a_1)\hat{i} + (b_2 - b_1)\hat{j} + (c_2 - c_1)\hat{k}$$

Defn. the length of vector $\vec{A} = \langle a_1, a_2, a_3 \rangle$

$$|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Defn.

$$\vec{A} = \langle a_1, a_2, a_3 \rangle$$
$$\vec{B} = \langle b_1, b_2, b_3 \rangle$$

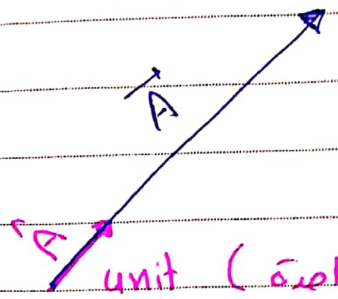
$$\vec{A} \pm \vec{B} = \langle a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3 \rangle$$

$$c\vec{A} = \langle ca_1, ca_2, ca_3 \rangle$$

scalar vector

* unit vector in the Direction of $\vec{A}$ is given by

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|}$$



Example :

$\vec{A} = \hat{i} - 3\hat{j} + \hat{k}$

$\vec{B} = -2\hat{i} + 4\hat{j} - 3\hat{k}$

find a vector in the direction of $\vec{A} - 3\vec{B}$ whose length equals 5

Sol.

$\vec{C} = \vec{A} - 3\vec{B} = 7\hat{i} - 15\hat{j} + 10\hat{k} \longrightarrow \hat{C} = 1$ بالوضع العادي

$\underline{5}$ بس انا بدي

$$5\hat{C} = 5\left(\frac{7\hat{i} - 15\hat{j} + 10\hat{k}}{\sqrt{7^2 + (15)^2 + (10)^2}}\right)$$

12.3 The Dot Product

$c\vec{A} = \text{Vector}$

Scalar ↲ ↳ vector

Defn. let $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{B} = \langle b_1, b_2, b_3 \rangle$

} الاثنان Vector

$$\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad \text{(scalar)}$$

vector

vector

example

let $\vec{A} = \langle 3, 0, 1 \rangle$

$\vec{B} = \langle -1, -3, 0 \rangle$

find $\vec{A} \cdot \vec{B} = (3)(-1) + (0)(-3) + (1)(0) = -3$

let $\vec{A} = 3\hat{i} + k$

$\vec{B} = -\hat{i} - \hat{j}$

$\vec{A} \cdot \vec{B} = -3$

* Thm (1)

(1) $\vec{a}\cdot\vec{a} = |\vec{a}|^2$

(2) $\vec{0}\cdot\vec{a} = 0$

↳ zero vector $\vec{0} = \langle 0,0,0\rangle$ / $0\hat{i}+0\hat{j}+0\hat{k}$

(3) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$

(4) $(c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(c\vec{b})$

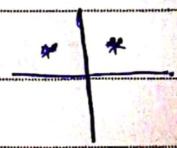
*proof (البرهان)

(1) let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

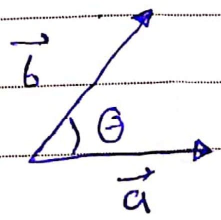
$$\vec{a}\cdot\vec{a} = a_1^2+a_2^2+a_3^2 = \left(\sqrt{a_1^2+a_2^2+a_3^2}\right)^2 = |\vec{a}|^2$$

* Thm (2) :-

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta \qquad 0 \le \theta \le \pi$



لـ الزاوية الصغرى بين الـ 2 vector



يتبع

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$\vec{a}\cdot\vec{b} > 0 \longrightarrow 0 < \theta < \frac{\pi}{2}$ حادة

$\vec{a}\cdot\vec{b} < 0 \longrightarrow \frac{\pi}{2} < \theta < \pi$

Dfn.

let $\vec{a}$ and $\vec{b}$ are two vector.

$\vec{a}$ and $\vec{b}$ are arthogonal (perpendicular)

$$\vec{a}\cdot\vec{b} = 0$$

---

example 3

Find the angle between

$\vec{a} = \langle 2,2,-1\rangle$ and $\vec{b}\ \langle 5,-3,2\rangle$

Sol.:

$\vec{a}\cdot\vec{b} = 10 + -6 + -2 = 2$

$|\vec{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3$

$|\vec{b}| = \sqrt{25 + 9 + 4} = \sqrt{38}$

$\cos\theta = \frac{2}{3\sqrt{38}} \longrightarrow \theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right)$

(زاوية حادة)

Scanned by CamScanner

example 4

Show that $2\hat{i}+2\hat{j}-\hat{k}$ is perpendicular to $5\hat{i}-4\hat{j}+2\hat{k}$

Sol.

$(2\hat{i}+2\hat{j}-k) \cdot (5\hat{i}-4\hat{j}+2\hat{k}) = 10-8-2 = \underline{\underline{0}}$

$\longrightarrow$ orthogonal perpendicular $\perp$

للتوضيح

$\langle 3, -2, 5 \rangle$

مثلاً بدي ال vector العامودي عليه

$3x - 2y + 5z = 0$

نثبت اثنان ونخلي الثالث
مثلاً

$z = 1$

$y = 1$

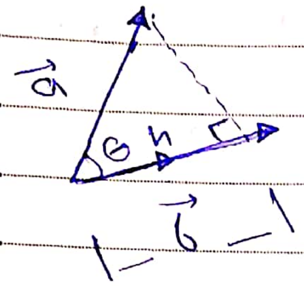
$3x - 2 + 5 = 0$

$\boxed{x = -1}$

اذاً ال vector العامودي

$\langle -1, 1, 1 \rangle$

Vector projection



(كأنها اسقاط عامودي)

$\vec{X}$ is called the vector projection of $\vec{a}$ onto $\vec{b}$ $\left(\text{Proj}_{\vec{b}}\,\vec{a}\right)$

$|\vec{X}|$ is called the scalar projection of $\vec{a}$ onto $\vec{b}$ $\left(\text{Comp}_{\vec{b}}\,\vec{a}\right)$

$$\cos\theta = \frac{h}{|\vec{a}|} \longrightarrow h = |\vec{a}|\cos\theta \longrightarrow h = \frac{|\vec{a}||\vec{b}|\cos\theta}{|\vec{b}|}$$

$$\text{Comp}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

$$\text{projection}_{\vec{b}}\,\vec{a} = h\,\vec{b} \qquad \text{unit vector } \frac{\vec{b}}{|\vec{b}|}$$

(إسقاط $\vec{a}$ على $\vec{b}$)

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\,\frac{\vec{b}}{|\vec{b}|}$$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\,\vec{b}$$

example 6

Find the scalar projection and vector projection of $\vec{b} = \langle 1,1,2 \rangle$ onto $\vec{a} = \langle -2,3,1 \rangle$

① Scalar Projection

$$\text{Comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

$$= \frac{-2+3+2}{\sqrt{4+9+1}} = \frac{+3}{\sqrt{14}}$$

Vector projection

$$\text{proj}_{\vec{a}}\vec{b} = \left(\frac{\text{Comp}\,\vec{b}}{\vec{a}}\right)(\hat{a}) = \frac{3}{\sqrt{14}} \cdot \frac{\langle -2,3,-1 \rangle}{\sqrt{14}}$$

$$= \frac{3}{14}\langle -2,3,-1 \rangle$$

12.4 Cross product

$C\vec{A} \parallel \vec{A}$ بيعطيني vector موازي

جواب vector

$\vec{A}\cdot\vec{B}$ = Scalcar

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

vector x vector = vector

Q 13 (الكتاب)

ⓐ $\vec{A}\cdot(\vec{B} \times \vec{C})$ = Scalar

ⓑ $\vec{A} \times (\vec{B}\cdot\vec{C})$ ✗ لا يمكن

vector x scalar

ⓒ $(\vec{A}\cdot\vec{B}) \times (\vec{C}\cdot\vec{D})$ ✗ لا يمكن

scalar x scalar

example 1

if $\vec{A} = \langle 1, 3, 4 \rangle$, $\vec{B} = \langle 2, 7, -5 \rangle$

find $\vec{A} \times \vec{B}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \overset{+}{\hat{i}} & \overset{-}{\hat{j}} & \overset{+}{\hat{k}} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \hat{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \hat{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \hat{k}$$

$-15 - 28$

$$= -43\,\hat{i} + 13\,\hat{j} + 1\hat{k}$$

example 2

Show that $\vec{A} \times \vec{A} = \vec{0}$

برهان

$$\vec{A} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$(a_2 a_3 - a_2 a_3)\hat{i} - (a_1 a_3 - a_1 a_3)\hat{j} + (a_1 a_2 - a_1 a_2)\hat{k}$$

$$0\hat{i} - 0\hat{j} + 0\hat{k}$$

zero vector

ملاحظة

$$\begin{vmatrix} a & b & c \\ 3a & 3b & 3c \end{vmatrix} = \vec{0} \quad \text{zero vector}$$

$$\begin{vmatrix} a & b & c \\ -a & -b & -c \end{vmatrix} = \vec{0}$$

$$\begin{vmatrix} a & b & c \\ x & y & z \\ u_1 & u_2 & u_3 \end{vmatrix} = 3 \longrightarrow \begin{vmatrix} a & b & c \\ u_1 & u_2 & u_3 \\ x & y & z \end{vmatrix} = \underline{\underline{-3}}$$

Thm:

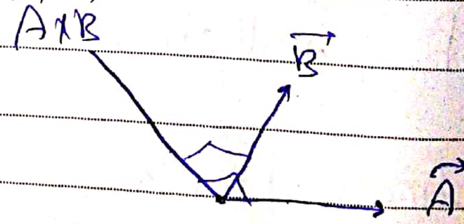
عامودي

The vector $\vec{A} \times \vec{B}$ is orthogonal to both $\vec{A}$ and $\vec{B}$

برهان.

$$\vec{A} \cdot (\vec{A} \times \vec{B}) = 0$$

$$\vec{B} \cdot (\vec{A} \times \vec{B}) = 0$$



Thm:

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta$$

$$\vec{A} \parallel \vec{B} \iff \vec{A} \times \vec{B} = \vec{0} \iff \vec{B} = c\vec{A}$$

$$3\hat{i} - 5\hat{j} + 6\hat{k} \parallel \hat{i} - \frac{5}{3}\hat{j} + 2\hat{k}$$

$$\times \quad = \vec{0}$$

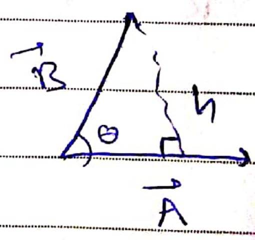
Thm:

$$|\vec{A} \times \vec{B}|^2 = |\vec{A}|^2 |\vec{B}|^2 - (\vec{A} \cdot \vec{B})^2$$

$$(\vec{A} \cdot \vec{B})^2 + |\vec{A} \times \vec{B}|^2 = |\vec{A}|^2 |\vec{B}|^2$$

Thm: the length of $\vec{A} \times \vec{B}$ equal the area of متوازي الاضلاع Parallelogram determinal by $\vec{A}$ and $\vec{B}$

$$\text{Area} = |\vec{A}|\, h$$
$$= |\vec{A}||\vec{B}| \sin\theta$$
$$= |\vec{A} \times \vec{B}|$$



$$\sin\theta = \frac{h}{|\vec{B}|}$$

$$h = |\vec{B}| \sin\theta$$

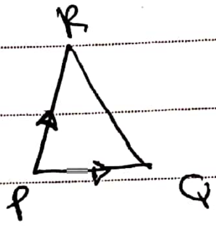
example find the area of triangle with vertices

$P(1,4,6)$, $Q(-2,5,-1)$ and $R(1,-1,1)$

Sol.

$$\text{Area of } \Delta = \frac{1}{2} \text{ Area of } \overset{\text{متوازي}}{\square}$$

$$= \frac{1}{2} \left| \overline{PQ} \times \overline{PR} \right|$$



$\overline{PQ} = \langle -3, 1, -7 \rangle$

$\overline{PR} = \langle 0, -5, -5 \rangle$

$$\overline{PQ} \times \overline{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= -40\hat{i} - 15\hat{j} + 15\hat{k} = 5(-8\hat{i} - 3\hat{j} + 3\hat{k})$$

للتأكد يجب ان تكون عامودية على

هنا استخدم خاصية

$|c\vec{A}| = |c|\,|\vec{A}|$

$$\text{Area} = \frac{1}{2} \left| \overline{PQ} \times \overline{PR} \right| = \frac{1}{2} 5 \sqrt{64+9+9}$$

example 3

Find a vector perpendicular to the plane passes the points

$P(1,4,6)$, $Q(-2,5,-1)$ and $R(1,-1,1)$

- A vector ⊥ the to plane ≡ vector ⊥ any يساوي vector lies on the plane.

∴ vector $\vec{PQ} \times \vec{PR} \perp \vec{PQ}$ and $\vec{PR}$ at the same time

∴ $\vec{PQ} \times \vec{PR} \perp$ Plane that Contains $\vec{PQ}$ and $\vec{PR}$

$$\vec{n} = -40\hat{i} - 15\hat{j} + 15\hat{k}$$

* Scalar Triple product.

$$\vec{A} \cdot \vec{B} \times \vec{C}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Show that

$\vec{A} \perp \vec{A} \times \vec{B}$

$$\vec{A} \cdot (\vec{A} \times \vec{B}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$$

محاضرة الخميس ٣/١٠/ ٢٠١٩ م

12.5

Equation of line and planes in the space

$x\ y\ \langle 1,0,0\rangle$ $\quad (0,0,0)$

x-axis : $\quad y=0\ ,\ z=0 \quad -\infty < t < \infty$

$x = t$

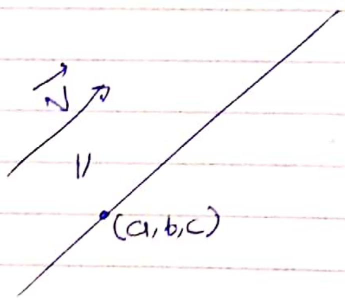
The Parametric equations of line Passes through a point (a,b,c) and vector

$\boxed{\vec{v} = \langle v_1, v_2, v_3 \rangle}$ parallel to this line

$$x = a + v_1 t$$

$$y = b + v_2 t$$

$$z = c + v_3 t$$



The symmetric equation of the line Passes through (a,b,c) parallel to $\langle v_1, v_2, v_3\rangle$

$$\frac{x-a}{v_1} = \frac{y-b}{v_2} = \frac{z-c}{v_3} = t$$

examples:

example:

Find the Parametric and Symmetric equations for the line passes through $(5,1,3)$ parallel $i+4j-2k$

$x = 5 + 1t$

$y = 1 + 4t \qquad -\infty < t < \infty \qquad \frac{x-1}{1} = \frac{y-1}{4} = \frac{z-3}{-2}$

$z = 3 + -2t$

find two other points

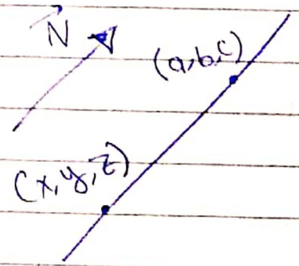
set $t=1 \rightarrow x=6,\ y=5,\ z=1\ (6,5,1)$

set $t=-1 \rightarrow x=4,\ y=-3,\ z=5\ (-1,-3,-5)$

c.q. V & A

$\overrightarrow{AB} \parallel \vec{v}$

$\langle -2,-8,4 \rangle = -2 \langle 1,4,-2 \rangle$



$\langle x-a, y-b, z-c \rangle = t \langle v_1, v_2, v_3 \rangle$

example 2

(a) Find the symmetric equations of the line passes through $A(2,4,-3)$ and $B(3,-1,1)$

Sol.

* اولاً نجيب ايجاد فيكتور موازي

$v \parallel \vec{AB} = \langle 1, -5, 4 \rangle$

$x = 2 + 1t$

$y = 4 + -5t \qquad -\infty < t < \infty$

$z = -3 + 4t$

(b) at what point does the line intersect xy plane

معادلة line	معادلة plane $z = 0$
$x = 2 + t$	
$y = 4 - 5t$	
$z = -3 + 4t$	

$z = 0$ and $z = -3 + 4t$

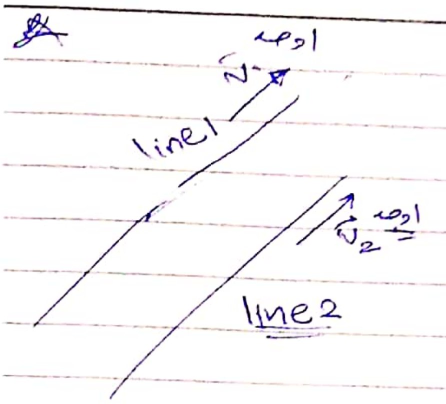
$0 = -3 + 4t \rightarrow \boxed{t = \frac{3}{4}}$

intersection point set $t = \frac{3}{4}$

$x = 2 + \frac{3}{4}$

$y = 4 - \frac{15}{4} \qquad \left(\frac{11}{4}, \frac{1}{4}, 0\right) \qquad t \langle v_1, v_2, v_3 \rangle$

$z = 0$



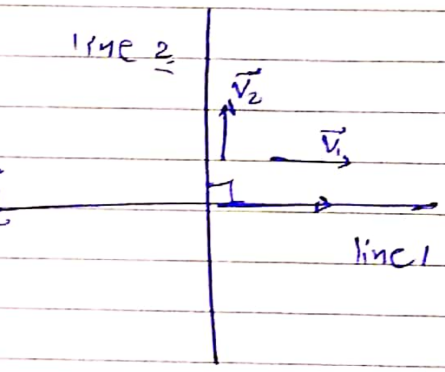
Notes

① line1 // line2 $\Leftrightarrow$ $\vec{v}_1 \parallel \vec{v}_2$

* طريقة اختبار التوازي
انو الـ vector لازم
متوازيين

② line1 ⊥ line 2 $\Leftrightarrow$ $\vec{v}_1 \perp \vec{v}_2$

* طريقة اختبار التعامد



lines
- parallel
- intersect (تقاطع) → (⊥)
- skew lines (they do not intersect and they are not parallel)

example 3

Determine whether the following lines are Parallel, intrsect (perpendicular), or skew

line 1	line 2
$x = 1 + t$	$x = 2s$
$y = -2 + 3t$	$y = 3 + s$
$z = 4 - t$	$z = -3 + 4s$

Sol. ① parallel $\vec{v}_1 = \langle 1, 3, -1 \rangle$ $\quad v_2 = \langle 2, 1, 4 \rangle$

$\vec{v}_1 \neq c\,\vec{v}_2 \longrightarrow \vec{v}_1, \vec{v}_2$ are not parallel

$\therefore$ skew lines.

② intersect :-

$x = x, \quad y = y, \quad z = z$

$1 + t = 2s \quad \dots ①$

$-2 + 3t = 3 + s \quad \dots ②$

$4 - t = -3 + 4s \quad ③$

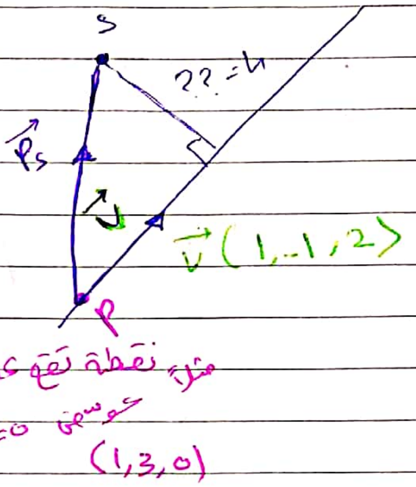
حل معادلة ① مع ③ بعدين حل بيز مشترك ② مع ③

Kashin Color

example:

Find the distance from the point $s(1,1,5)$ to line $x=1+t,\ y=3-t,\ z=2t$

* النقطة اما ستقع على الخط او بعيد عنه لمعرفة اذا كانت تقع على الخط نعوض t اذا تحققت النقطة تقع على الخط



$$\sin\theta = \frac{h}{|\vec{Ps}|}$$

$$h = |\vec{Ps}|\sin\theta = \frac{|\vec{Ps}||\vec{v}|\sin\theta}{|\vec{v}|}$$

$$D = \frac{|\vec{Ps}\times\vec{v}|}{|\vec{v}|}$$

$\vec{Ps} = \langle 0, -2, 5\rangle$

$\vec{v} = \langle 1, -1, 2\rangle \quad , |\vec{v}| = \sqrt{1+1+4} = \sqrt{6}$

$$\vec{Ps}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix}$$

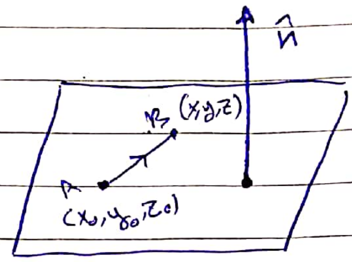
$$= \hat{i} + 5\hat{j} + 2\hat{k} \quad , |\vec{Ps}\times\vec{v}| = \sqrt{30}$$

$$D = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}$$

Defn

The equation of plane passes through a point (x_0, y_0, z_0) and arthoganal (perpendicular) vector

$\vec{n} = \langle n_1, n_2, n_3 \rangle$



$\vec{AB} \cdot \vec{n} = 0$

$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle n_1, n_2, n_3 \rangle = 0$

$$\boxed{n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0}$$

$\vec{n}$ = normal vector

example: (a) find an equation of the plane through $(2, 4, -1)$ with normal vector $\langle 2, 3, 4 \rangle$

$2(x-2) + 3(y-4) + 4(z+1) = 0$

$2x + 3y + 4z - 12 = 0$

(b) find the intersecepts and sketch plane.

يعني وين بتقطع المحاور الاساسية الثلاثة

x-intercept (x-axis)

$x = t \quad y = 0 \quad z = 0$ line ← ثبّت اثنين وخلي الثالث

$2t + 0 + 0 - 12 = 0 \rightarrow t = 6$

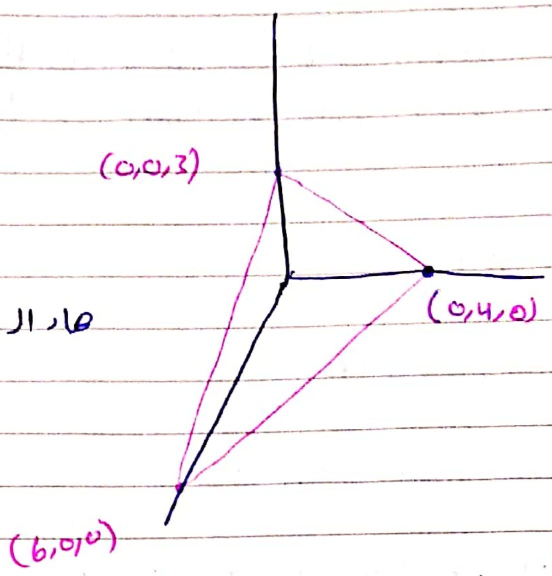
$(6, 0, 0)$

KeshinColor

تابع

y_intercept $(0,4,0)$

Z_intercept $(0,0,3)$



صار ال plane سطح من سطوح الهرم الثلاثي

Note *العلاقات

① $\text{Plane}_1 \parallel \text{plane}_2$ نستطيع إيجاد $\vec{n}_1 \parallel \vec{n}_2$

② $\text{Plan}_1 \perp \text{plane}_2$ نستطيع ايجاد vector عامودي $\vec{n}_1 \perp \vec{n}_2$

③ line $\parallel$ plane $\vec{v} \perp \vec{n}$

④ line $\perp$ plane $\vec{v} \parallel \vec{n}$

موازي لل line

عامودي على ال plane

⑤ the angle between two Planes = the angle between $\vec{n}_1$ and $\vec{n}_2$, $\left(0 \le \theta \le \frac{\pi}{2}\right)$ شرط

Example 5 (بالكتاب)

Find the equation of the plane passes through
$P(1,3,2)$, $Q(3,-1,6)$, $R(5,2,0)$

$\vec{n} = \vec{PQ} \times \vec{PR}$

$= 12\hat{i} + 20\hat{j} + 14\hat{k}$

equation of this plane is

$12(x-1) + 20(y-3) + 14(z-2) = 0$

Example 6. (بالكتاب)

Find the point of ~~interseption~~ intersection between
$x = 2+3t$, $y = -4t$, $z = 5+t$

عوضي with $4x + 5y - 2z = 18$

$4(2+3t) + 5(-4t) + -2(5+t) = 18$

$-10t - 2 = 18$

$-10t = 20$

$\boxed{t = -2}$

عوض $(-4, 8, 3)$

نقطة تقاطع اذا فهي تحقق معادلتين

Example 7. (بالتبادل)

(a) Find the angle between $x+y+z=1$ and $x-2y+3z=1$

Sol.

$\vec{n}_1 = \langle 1,1,1 \rangle$

$\vec{n}_2 = \langle 1,-2,3 \rangle$

angle between two planes = angle between ($\vec{n}_1$ and $\vec{n}_2$)

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1|\,|\vec{n}_2|}$$

$$\cos\theta = \frac{2}{\sqrt{3}\,\sqrt{14}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right)$$

(b) Find the equation of the line intersection of the planes

* بدك أول شي (نقطة + vector موازي)

Sol.

بدي

Point :.

Set $z=1$

$x+y+1=1$
$x-2y+3=1$

$x+y=0$ ... (1)
$x-2y=-2$

$3y=2 \rightarrow y=\frac{2}{3}$ $\rightarrow$ $\left(\frac{-2}{3}, \frac{2}{3}, 1\right)$ ع.ب

$x=\frac{-2}{3}$

12.5 p

تكملة

* vector // line

$\vec{V} = ?. ?$

line // plane 1 $\longrightarrow \vec{V} \perp \vec{n_1}$

line // plane 2 $\longrightarrow \vec{V} \perp \vec{n_2}$

$\vec{V} = \vec{n_1} \times \vec{n_2}$

$$\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\hat{i} - 2\hat{j} - 3\hat{k}$$

line of intersect :

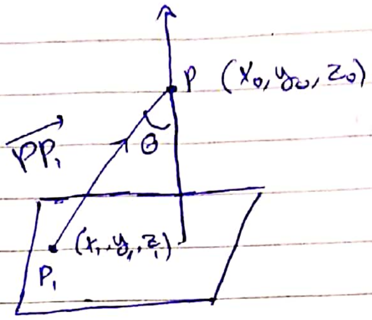
$x = \frac{-2}{3} + 5t$

$y = \frac{2}{3} + -2t$

$z = 1 + -3t$

example

Find the distance between (x_0, y_0, z_0) and $ax + by + cz + d = 0$



$\vec{n} = \langle a, b, c \rangle$

$$\cos\theta = \frac{h}{|\vec{PP_1}|}$$

$$h = \frac{|\vec{PP_1}| \cos\theta \, |\vec{n}|}{|\vec{n}|}$$

$$h = \frac{\vec{n} \cdot \vec{PP_1}}{|\vec{n}|}$$

$$\vec{PP_1} = \langle x_0 - x_1, y_0 - y_1, z_0 - z_1 \rangle$$

$$\vec{n} = \langle a, b, c \rangle$$

$$h = \frac{a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)}{\sqrt{a^2 + b^2 + c^2}}$$

$$h = \frac{ax_0 + by_0 + cz_0 - [ax_1 + by_1 + cz_1]}{\sqrt{a^2 + b^2 + c^2}}$$

$P_1(x_1, y_1, z_1)$ ~~the~~ lies on the plane

تحقق المعادلة

$$ax_1 + by_1 + cz_1 + d = 0$$

$$ax_1 + by_1 + cz_1 = -d$$

$$D = \frac{|ax_0 + by_0 + cz_0 - (-d)|}{\sqrt{a^2 + b^2 + c^2}}$$

يلزم كل القانون إيجاد
point و Normal
تقع عليه

KeshinColor

example:

Find the distance between

$3x + 2y - z = 9$, $\vec{n}_1 = \langle 3, 2, -1 \rangle$

and

$6x + 4y - 2z = 15$, $\vec{n}_2 = \langle 6, 4, -2 \rangle$

$\vec{n}_2 \parallel \vec{n}_1 \rightarrow \text{plane}_1 \parallel \text{plane}_2$

اوجد نقطة على ال plane الأول ثم أطبق على
ال plane الثاني

Choose Point on Plane 1 → Set $x=0, y=0$ then $z=9$ $P(0,0,-9)$

$D(\text{plane}_1, \text{plane}_2) = D(P, \text{plane}_2)$

$$= \frac{|6(0) + 4(0) - 2(-9) - 15|}{\sqrt{6^2 + 4^2 + (-2)^2}}$$

$$= \frac{3}{2\sqrt{4+4+1}}$$

KeshinColor

12.6 Quadric surfaces:

① sphere. $(x-h)^2+(y-k)^2+(z-l)^2=r^2$

② cylinder $x^2+y^2=9$

$y^2+z=16$

③ planes

$ax+by+cz+d=0$

Defn.

Trace: intersection (curve) between plane and surface.

example:

$z=4x^2+9y^2$

find the following traces:

① trace in xy-plane ② trace in $z=3$ ③ trace in $z=9$ ④ trace in $y=1$
⑤ trace in $x=1$ ⑥ $z=1$ ⑦ $z=4$ ⑧ $y=0$

Sol.

① trace in xy-plane "$z=0$".

$0=4x^2+9y^2 \rightarrow (x,y)=(0,0)$

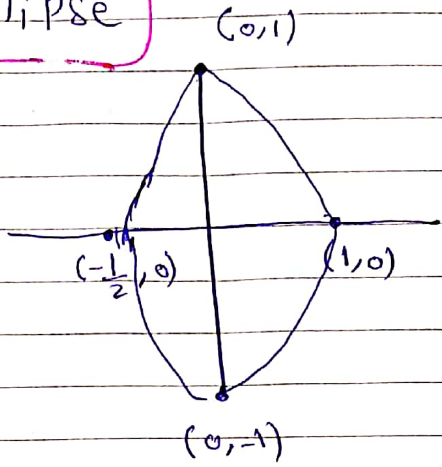
② trace in $z=3$

$3=4x^2+9y^2 \longrightarrow 1=\frac{x^2}{\frac{3}{4}}+\frac{y^2}{\frac{3}{4}}$

⑥ $z=1$

$1 = 4x^2 + y^2$ ellipse

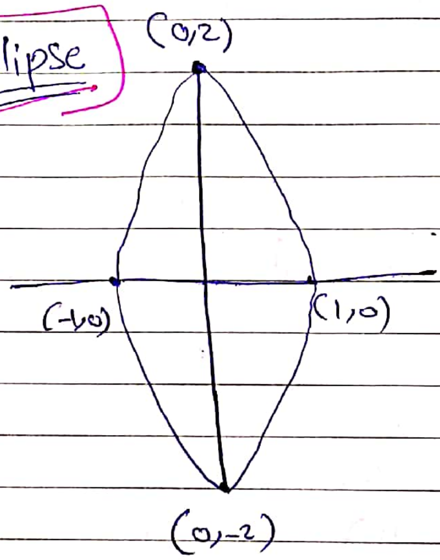
$1 = \frac{x^2}{\frac{1}{4}} + y^2$



⑦ $z=4$ ellipse

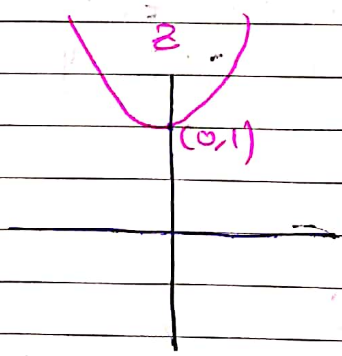
$4 = 4x^2 + y^2$

$1 = \frac{x^2}{1} + \frac{y^2}{4}$



④ Sub $y=1$

$z = 4x^2 + 1$

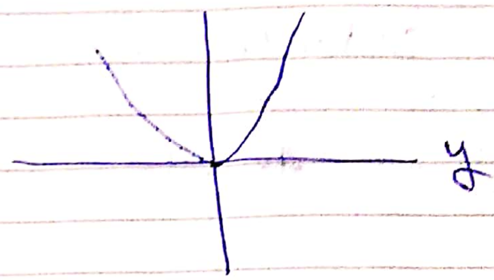


⑧ $y=0$ (Parabola)

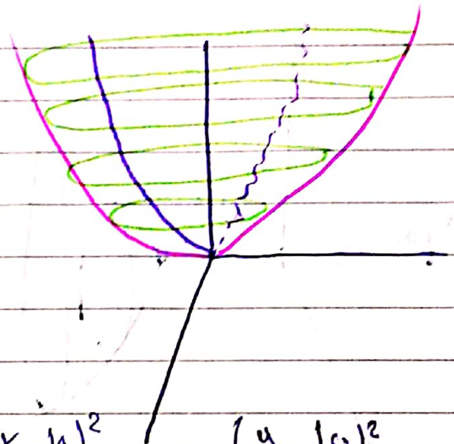
$z = 4x^2 + 0$

9. $X = 0$

$Z = y^2$ Parabola



الشكل النهائي (elliptic paraboliod)



elipse $1 = \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2}$

hyperoba $1 = \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2}$

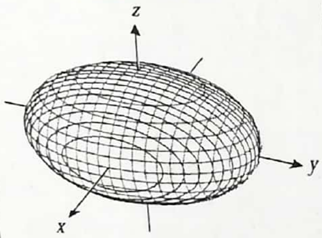
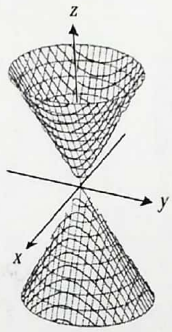
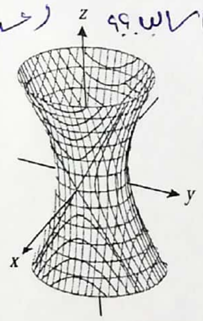
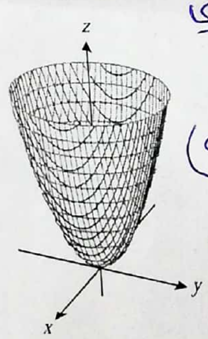
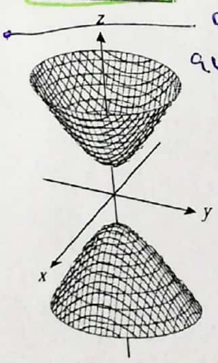
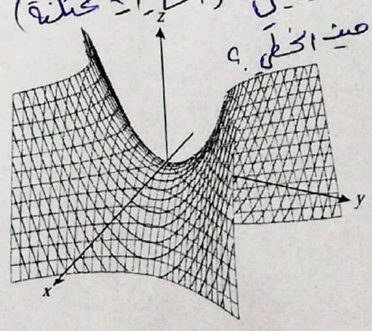
parabola $(y-k) = a(x-L)^2$

* تم توزيع ورقة خاصة بالاشكال المطلوبة في الحارة

which is called a ***second-degree equation in x, y, and z***. The graphs of such equations are called ***quadric surfaces*** or sometimes ***quadrics***.

Six common types of quadric surfaces are shown in Table 11.7.1—*ellipsoids, hyperboloids of one sheet, hyperboloids of two sheets, elliptic cones, elliptic paraboloids,* and *hyperbolic paraboloids.* (The constants a, b, and c that appear in the equations in the table are assumed to be positive.) Observe that none of the quadric surfaces in the table have cross-product terms in their equations. This is because of their orientations relative

Table 11.7.1

SURFACE	EQUATION	SURFACE	EQUATION
ELLIPSOID 	٣ موجبات $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ The traces in the coordinate planes are ellipses, as are the traces in those planes that are parallel to the coordinate planes and intersect the surface in more than one point.	ELLIPTIC CONE 	$z^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ The trace in the xy-plane is a point (the origin), and the traces in planes parallel to the xy-plane are ellipses. The traces in the yz- and xz-planes are pairs of lines intersecting at the origin. The traces in planes parallel to these are hyperbolas.
HYPERBOLOID OF ONE SHEET (عشان نحدد المحور الرئيسي) حيث السالب؟؟ 	٢ موجب / ١ سالب $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ The trace in the xy-plane is an ellipse, as are the traces in planes parallel to the xy-plane. The traces in the yz-plane and xz-plane are hyperbolas, as are the traces in those planes that are parallel to these and do not pass through the x- or y-intercepts. At these intercepts the traces are pairs of intersecting lines.	ELLIPTIC PARABOLOID تربيعي أمين الخطي؟ ٢ ا،ح متشابهة 	$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ The trace in the xy-plane is a point (the origin), and the traces in planes parallel to and above the xy-plane are ellipses. The traces in the yz- and xz-planes are parabolas, as are the traces in planes parallel to these.
HYPERBOLOID OF TWO SHEETS (عشان نحدد المحور الرئيسي) حيث الموجب؟ 	٢ سالب ١ موجب $\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ There is no trace in the xy-plane. In planes parallel to the xy-plane that intersect the surface in more than one point the traces are ellipses. In the yz- and xz-planes, the traces are hyperbolas, as are the traces in those planes that are parallel to these.	HYPERBOLIC PARABOLOID تربيعي (اشارات مختلفة) حيث الخطي؟ 	$z=\frac{y^2}{b^2}-\frac{x^2}{a^2}$ The trace in the xy-plane is a pair of lines intersecting at the origin. The traces in planes parallel to the xy-plane are hyperbolas. The hyperbolas above the xy-plane open in the y-direction, and those below in the x-direction. The traces in the yz- and xz-planes are parabolas, as are the traces in planes parallel to these.

٣ ساوو ١

٤ = صفر ← ٣ تربيعات ساوو صفر

٤ تربيعات

A TECHNIQUE FOR IDENTIFYING QUADRIC SURFACES

The equations of the quadric surfaces in Table 11.7.1 have certain characteristics that make it possible to identify quadric surfaces that are derived from these equations by reflections. These identifying characteristics, which are shown in Table 11.7.2, are based on writing the equation of the quadric surface so that all of the variable terms are on the left side of the equation and there is a 1 or a 0 on the right side. These characteristics do not change when the surface is reflected about a coordinate plane or planes of the form $x = y$, $x = z$, or $y = z$, thereby making it possible to identify the reflected quadric surface from the form of its equation.

Table 11.7.2

IDENTIFYING A QUADRIC SURFACE FROM THE FORM OF ITS EQUATION

EQUATION	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$z^2-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	$z-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$	$z-\frac{y^2}{b^2}+\frac{x^2}{a^2}=0$
CHARACTERISTIC	No minus signs	One minus sign	Two minus signs	No linear terms	One linear term; two quadratic terms with the same sign	One linear term; two quadratic terms with opposite signs
CLASSIFICATION	Ellipsoid	Hyperboloid of one sheet	Hyperboloid of two sheets	Elliptic cone	Elliptic paraboloid	Hyperbolic paraboloid

▶ **Example 10** Identify the surfaces

(a) $3x^2 - 4y^2 + 12z^2 + 12 = 0$ (b) $4x^2 - 4y + z^2 = 0$

Solution (a). The equation can be rewritten as

$$\frac{y^2}{3} - \frac{x^2}{4} - z^2 = 1$$

This equation has a 1 on the right side and two negative terms on the left side, so its graph is a hyperboloid of two sheets.

Solution (b). The equation has one linear term and two quadratic terms with the same sign, so its graph is an elliptic paraboloid. ◀

✔ QUICK CHECK EXERCISES 11.7

(See page 832 for answers.)

1. For the surface $4x^2 + y^2 + z^2 = 9$, classify the indicated trace as an ellipse, hyperbola, or parabola.
(a) $x = 0$ (b) $y = 0$ (c) $z = 1$

2. For the surface $4x^2 + z^2 - y^2 = 9$, classify the indicated trace as an ellipse, hyperbola, or parabola.
(a) $x = 0$ (b) $y = 0$ (c) $z = 1$

3. For the surface $4x^2 + y^2 - z = 0$, classify the indicated trace as an ellipse, hyperbola, or parabola.
(a) $x = 0$ (b) $y = 0$ (c) $z = 1$

4. Classify each surface as an ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, elliptic cone, elliptic paraboloid, or hyperbolic paraboloid.
(a) $\frac{x^2}{36} + \frac{y^2}{25} - z = 0$ (b) $\frac{x^2}{36} + \frac{y^2}{25} + z^2 = 1$
(c) $\frac{x^2}{36} - \frac{y^2}{25} + z = 0$ (d) $\frac{x^2}{36} + \frac{y^2}{25} - z^2 = 1$
(e) $\frac{x^2}{36} + \frac{y^2}{25} - z^2 = 0$ (f) $z^2 - \frac{x^2}{36} - \frac{y^2}{25} = 1$

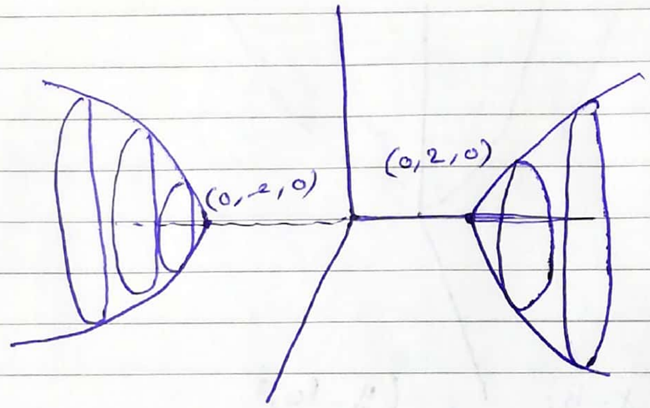
example 7 (بالشكل)

Indintify and Sketch the Surface.

$$4x^2 - y^2 + 2z^2 + 4 = 0$$

$$\left(4 = -4x^2 + y^2 - 2z^2\right) \div 4$$

$$1 = -x^2 + \frac{y^2}{4} - \frac{z^2}{2}$$



example 8 (بالشكل)

Classify $x^2 + 2z^2 - 6x - y + 10 = 0$

* اول خطوة يجب القيام بعملية إكمال مربع ليبقى عنا متغير خطي واحد

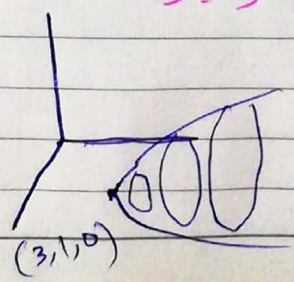
$$x^2 - 6x + 9 + 2z^2 - y + 10 - 9 = 0$$

$$(x-3)^2 + 2z^2 + y + 1 = 0$$

$$(x-3)^2 + 2z^2 - (y-1) = 0$$

$$(y-1) = (x-3)^2 + 2z^2$$

elliptic paraboliod $\xleftarrow{\text{يعطي}}$ خطي + 2 تربيعيات نفس الاشارة



الأحد ١٣/١٠/٢٠١٩

Ch. 13.1 Vector functions and Space Curves

$f : R \to R$

$f(x) = 3\sin x + 1$ (scalar function)

Defn. vector function

$\vec{r}(t) : R \to R^3$

$\vec{r}(t) = \langle f(t), h(t), g(t) \rangle$

Notes

[1] $D(\vec{v}(t)) = D(f) \cap D(h) \cap D(g)$

(2) $\lim\limits_{t \to t_0} \vec{V}(t) = \left\langle \lim\limits_{t \to t_0} f(t), \lim\limits_{t \to t_0} h(t), \lim\limits_{t \to t_0} g(t) \right\rangle$

(3) $\dfrac{d\vec{v}}{dt} = \left\langle \dfrac{df}{dt}, \dfrac{dh}{dt}, \dfrac{dg}{dt} \right\rangle$

(4) $\int \vec{v}(t)\,dt = \left\langle \int f(t)\,dt, \int h(t)\,dt, \int g(t)\,dt \right\rangle$

example 1

Find the Domain for

$\vec{v}(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$

$t^3 : \mathbb{R}$

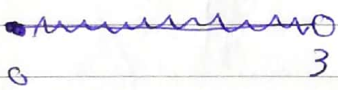
$\ln(3-t)$: $3-t > 0$

$t < 3$

$\sqrt{t}$:

$t \geq 0$

$D(\vec{v}(t))$



$D(\vec{v}(t)) = [0, 3)$

example

Find $\lim\limits_{t \to 0} \vec{v}(t)$, where $\vec{v}(t) = (1+t^3)\,\hat{i} + t\,e^{-t}\,\hat{j} + \dfrac{\sin t}{t}\,k$

$$\lim_{t \to 0} \vec{v}(t) = \lim_{t \to 0} (1+t^3)\,\hat{i} + \lim_{t \to 0} t\,e^{-t}\,\hat{j} + \lim_{t \to 0} \frac{\sin t}{t}\,\hat{k}$$

$$= 1\hat{i} + 0\hat{j} + 1k$$

example 1 (بالكتاب)

13.2

Find the Derivative of $\vec{v}(t) = (1+t^3)\,\hat{i} + t\,e^{-t}\,\hat{j} + \sin(2t)\,\hat{k}$

$$\frac{d\vec{v}}{dt} = \frac{d}{dt}[1+t^3]\,\hat{i} + \frac{d}{dt}[t e^{-t}]\,\hat{j} + \frac{d}{dt}(\sin 2t)\,k$$

$$= 3t^2\,\hat{i} + (-t e^{-t} + e^{-t})\,\hat{j} + 2\cos 2t\,k$$

example 5 : if $\vec{v}(t) = 2\cos t\,\hat{i} + \sin t\,\hat{j} + 2t\,k$

13.2 Evalute $\int \vec{v}(t)\,dt$, $\int_0^{\frac{\pi}{2}} v(t)\,dt$

~~① $\int \vec{v}(t)\,dt = 2\sin t\,\hat{i} - \cos t\,\hat{j}$~~

① $\int \vec{v}(t)\,dt = (2\sin t + c_1)\,\hat{i} + (-\cos t + c_2)\,\hat{j} + (t^2 + c_3)$

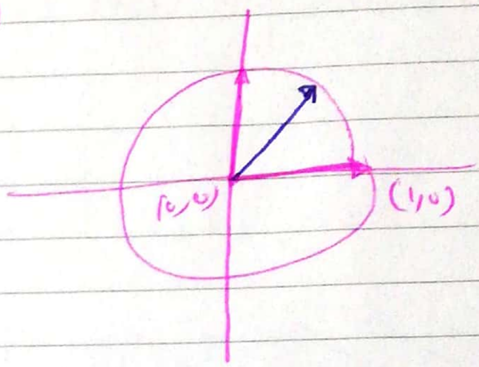
$= 2\sin t\,\hat{i} - \cos t\,\hat{j} + t^2 k + \vec{C}$

② $\int_0^{\frac{\pi}{2}} \vec{v}(t)\,dt = 2\hat{i} + \hat{j} + \dfrac{\pi^2}{4}k$

example : ① sketch the curve of $\vec{v}(t) = \langle \cos t, \sin t \rangle$

$x = \cos t$ $\qquad \cos^2 t + \sin^2 t = 1$

$y = \sin t$ $\qquad x^2 + y^2 = 1$



$\vec{r}(0) = \langle 1, 0 \rangle$

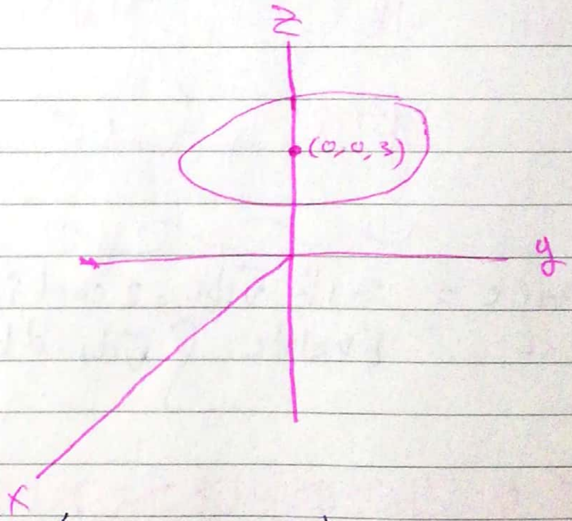
$\vec{v}\left(\frac{\pi}{2}\right) =$

② Sketch the curve of $\langle \cos t, \sin t, 3 \rangle$

$x = \cos t$

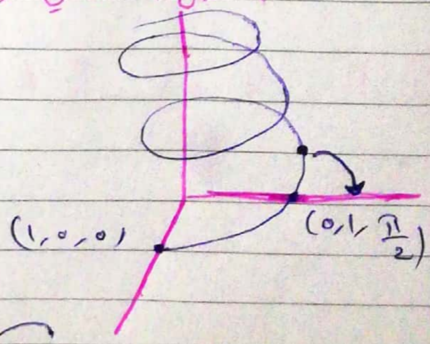
$y = \sin t$

$z = 3$



③ Sketch the curve of $\langle \cos t, \sin t, t \rangle$

لما كل ما نغير t بتغير فقط على الـ z بالغاية
الشكل النهائي، ح يطلع حلزوني



line $\langle x_0+v_1t, y_0+v_2t, z_0+v_3t \rangle$

<u>example 6</u>

Find vector function for the curve of interesection of $x^2+y^2=1$ and $y+z=2$

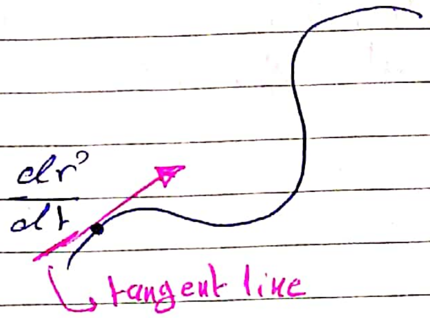
* بالبداية بدي أكتب ٣ معادلات بدلالة t
بحيث تحقق لي المعادلتين المعطاة بالسؤال.

$$\left.\begin{aligned} x(t) &= \cos t \\ y(t) &= \sin t \\ z(t) &= 2-\sin t \end{aligned}\right\} \rightarrow \vec{r}(t) = \langle \cos t, \sin t, 2-\sin t \rangle$$

أو مثلاً:

$$\left.\begin{aligned} x &= t \\ y &= \sqrt{1-t^2} \\ z &= 2-\sqrt{1-t^2} \end{aligned}\right\} \quad \vec{r}(t) = \langle t, \sqrt{1-t^2}, 2-\sqrt{1-t^2} \rangle$$

13.2 $\left.\frac{d\vec{r}}{dt}\right|_{t=t_0}$: Vector parallel to tangen Line "tangent vector"



Def. unit tangent vector $\vec{T}$

$$\rightarrow \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}$$

Ex.:- for $\vec{r}(t) = (1+t^3)\hat{i} + te^{-t}\hat{j} + \sin 2t\,\hat{k}$
find the unit tangent vector of point where $t=0$

Sol: $\frac{d\vec{r}}{dt} = 3t^2\hat{i} + [-te^{-t} + e^{-t}]\hat{j} + 2\cos 2t\,\hat{k}$

$$\vec{T}_{(0)} = \frac{0\hat{i} + 1\hat{j} + 2\hat{k}}{\sqrt{1+4}} = \frac{\hat{j}}{\sqrt{5}} + \frac{2k}{\sqrt{5}}$$

* Ex. find the parametric equations for the tangent Line to the helix $x = 2\cos t$, $y = \sin t$, $z = t$ at the Point $(0, 1, \frac{\pi}{2})$

Sol.

$\vec{r}(t) = 2\cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$

* tangent vector $= \frac{d\vec{r}}{dt} = -2\sin t\,\hat{i} + \cos t\,\hat{j} + 1\hat{k}$

at the tangent point $\rightarrow t = \frac{\pi}{2}$.

$\vec{V} \Rightarrow$ tangent vector $= \left.\frac{d\vec{r}}{dt}\right|_{t=\frac{\pi}{2}} = -2\hat{i} + 0\hat{j} + 1k$

تابع $\rightarrow$

the parametric equations for the tangent line are :-

$x = 0 + -2t$
$y = 1 + 0t$
$z = \frac{\pi}{2} + 1t$

* Thm:

$\boxed{1}$ $\frac{d}{dt}\left[f(t)\,.\,\vec{r}(t)\right] = f(t)\,\frac{d\vec{r}}{dt} + \vec{r}(t)\,.\,f(t)$

$\boxed{2}$ $\frac{d}{dt}\left[\vec{r}_1(t)\,.\,\vec{r}_2(t)\right] = \vec{r}_1(t)\,.\,\frac{d\vec{r}_2}{dt} + \vec{r}_2\,.\,\frac{d\vec{r}}{dt}$

$\boxed{3}$ $\frac{d}{dt}\left[\vec{r}_1(t) \times \vec{r}_2(t)\right] = \vec{r}_1(t) \times \frac{d\vec{r}_2}{dt} + \vec{r}_2 \times \frac{d\vec{r}_1}{dt}$

$r_1(t) \times \frac{d\vec{r}_2}{dt} + \frac{d\vec{r}_1}{dt} \times \vec{r}_2$

Ex:- Show that if $|\vec{r}(t)| = c$ (constant) then $\frac{d\vec{r}}{dt}$ and $\vec{r}(t)$ are orthogonal.

$\frac{d\vec{r}}{dt}\,.\,\vec{r} = 0$

البرهان * proof $\vec{r}.\vec{r} = |\vec{r}|^2 = c^2$ constant

$\frac{d}{dt}\left[\vec{r}.\vec{r}\right] = \frac{d}{dt}\left[c^2\right] = 0$

$\longrightarrow \vec{r}.\frac{d\vec{r}}{dt} + \vec{r}.\frac{d\vec{r}}{dt} = 0$

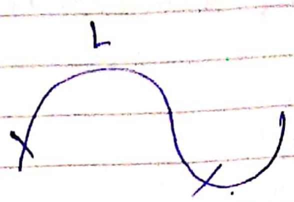
$2r.\frac{d\vec{r}}{dt} = 0$

13.3 Arc length ; Curvature.

$\boxed{\begin{matrix} x = x(t) \\ y = y(t) \end{matrix}}$ $\quad \boxed{y = f_{(x)}}$

$$L = \int_a^b \sqrt{1 + (f'_{(x)})^2}\, dx$$

L



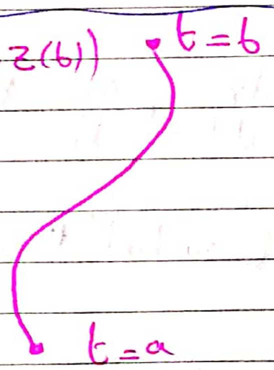
$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

3D

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$L = \int_a^b \left| \frac{d\vec{r}}{dt} \right| dt$$

$(x(b), y(b), z(b))$ $\quad t = b$



$t = a$

$(x(a), y(a), z(a))$

example : Find the Arc length of the curve

$$\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$$

from $(0,1,0)$ to $(0,1,\frac{\pi}{2})$

Sol.

$$\frac{d\vec{r}}{dt} = \langle -\cos t, \sin t, 1 \rangle$$

$$\left| \frac{d\vec{r}}{dt} \right| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

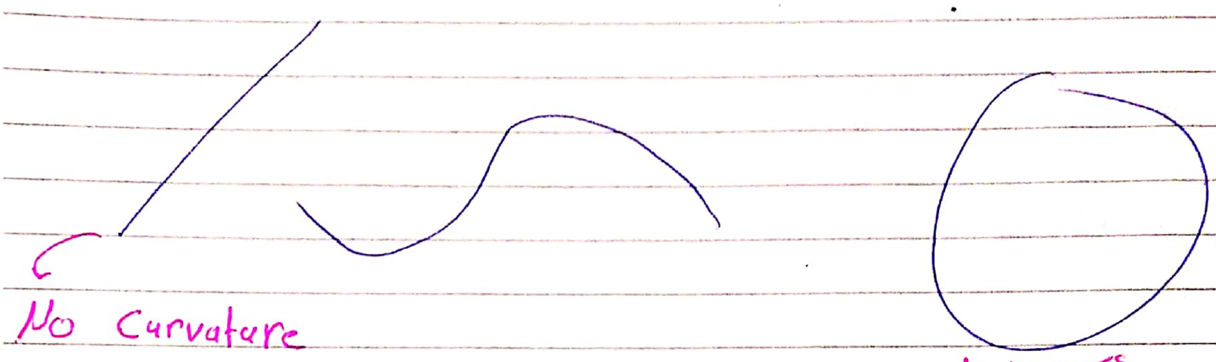
at $(0,1,0) \longrightarrow t = 0$

$(0,1,2\pi) \rightarrow t = 2\pi$

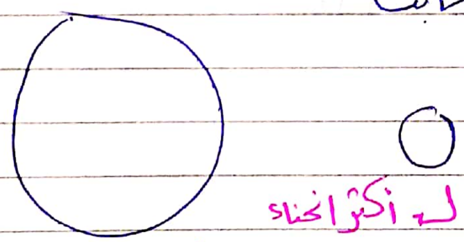
يتبع

$$L = \int_0^{2\pi} \sqrt{2}\, dt = (2\pi)\sqrt{2}$$

* Curvature $K(t)$
(مدى إنحناء المنحنى) → Kappa



* ال Curvature للدائرة عدد ثابت



* كلما صغرت الدائرة زاد ال Curvature

علاقة عكسية مع نصف القطر (r)

ال Curvature له قانونان.

unit tangent vector

① $$K(t) = \frac{\left|\frac{dT}{dt}\right|}{\left|\frac{d\vec{r}}{dt}\right|} \qquad \boxed{\text{where } T = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}}$$

② $$K(t) = \frac{\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|}{\left|\frac{dr}{dt}\right|^3}$$

* اذا كان طول المشتقة ← ثابت نطبق على القانون الاول أسهل

example Show that the Curvature of a circle with radius a is $\boxed{\frac{1}{a}}$

Sol.

$$x^2 + y^2 = a^2$$

$$x = a\cos t$$

$$y = a\sin t$$

$$\vec{r}(t) = a\cos t\, \hat{i} + a\sin t\, \hat{j}$$

$$\frac{d\vec{r}}{dt} = -a\sin t\, \hat{i} + a\cos t\, \hat{j}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2} = a$$

هون الجواب طلع عدد ثابت فالافضل نستخدم
القانون الأول

$$\vec{T} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = -\sin t\, \hat{i} + \cos t\, \hat{j}$$

$$\frac{dT}{dt} = -\cos t\, \hat{i} + -\sin t\, \hat{j}$$

$$\left|\frac{dT}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t} = 1$$

$$k(t) = \frac{\left|\frac{d\vec{T}}{dt}\right|}{\left|\frac{d\vec{r}}{dt}\right|} = \frac{1}{a} \quad \#$$

* example   * مثال نستخدم عليه القانون الثاني

Find the curvature of $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at $(0,0,0)$

Sol.

$$\frac{d\vec{r}}{dt} = \langle 1, 2t, 3t^2 \rangle$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{1 + 4t^2 + 9t^4}$$

* نلجأ للقانون الثاني اختصاراً للاشتقاقات في القانون الأول كثيرة.

$$\frac{d^2\vec{r}}{dt^2} = \langle 0, 2, 6t \rangle$$

$$\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix}$$

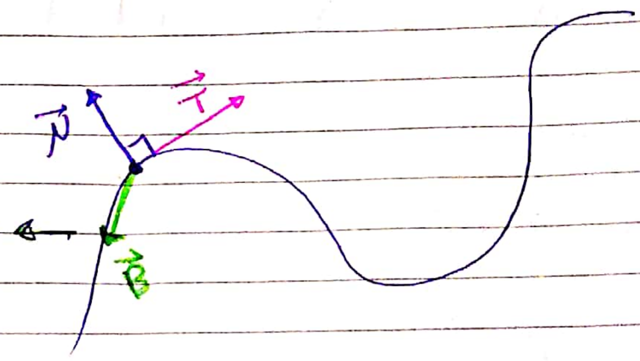
$$= (6t^2)i - (6t)j + 2k$$

$$\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = \sqrt{36t^4 + 36t^2 + 4}$$

$$K(t) = \frac{\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|}{\left|\frac{d\vec{r}}{dt}\right|^3} = \frac{\sqrt{36t^4 + 36t^2 + 4}}{\left(\sqrt{1 + 4t^2 + 9t^4}\right)^3}$$

at $(0,0,0) \longrightarrow t = 0$, $K(0) = 2$

* unit tangent vector, normal unit vector, Binormal vector



عامودي على $\vec{N}$ و $\vec{T}$

$$\vec{T} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}$$

$$\vec{N} = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$$

+ تذكير محاضرة سابقة

$|\vec{r}| = c$

then $\frac{d\vec{r}}{dt} \perp \vec{r}$

$$\vec{B} = \frac{\vec{T} \times \vec{N}}{\cancel{|\vec{T} \times \vec{N}|}}$$

بدي vector عامودي على الاثنين (Cross)

لماذا يجي

$$|\vec{T} \times \vec{N}| = |\vec{T}||\vec{N}| \sin\theta$$

$1 \cdot 1 \cdot 1$

لان الجواب 1

example

الأحد ٢٠/١٠/٢٠١٩

For the Vector function

$\vec{r}(t) = \cos t\, \hat{i} + \sin t\, \hat{j} + t\hat{k}$.

Find the Binormal unit vector.

$$\vec{T} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} , \quad \frac{d\vec{r}}{dt} = -\sin t\, \hat{i} + \cos t\, \hat{j} + 1\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

① $\vec{T} = \frac{-1}{\sqrt{2}} \sin t\, \hat{i} + \frac{1}{\sqrt{2}} \cos t\, \hat{j} + \frac{1}{\sqrt{2}} k$

② $\vec{N} = \dfrac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$

$$\frac{d\vec{T}}{dt} = \frac{-1}{\sqrt{2}} \cos t\, \hat{i} - \frac{1}{\sqrt{2}} \sin t\, \hat{j} + 0\hat{k}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{1}{2}\cos^2 t + \frac{1}{2}\sin^2 t} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$\vec{N}(t) = -\cos t\, \hat{i} - \sin t\, \hat{j} + 0k.$

تابع

$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{-1}{\sqrt{2}} \sin t & \frac{1}{\sqrt{2}} \cos t & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}} \sin t \, \hat{i} - \left(\frac{1}{\sqrt{2}} \cos t\right) \hat{j} + \underbrace{\left(\frac{1}{\sqrt{2}} \sin^2 t + \frac{1}{\sqrt{2}} \cos^2 t\right)}_{1} k$$

$$\vec{B} = \frac{1}{\sqrt{2}} \sin t \, \hat{i} - \frac{1}{\sqrt{2}} \cos t \, \hat{j} + \frac{1}{\sqrt{2}} k$$

Ch 13 تم انتهاء *

example

التاريخ ٢٠/١٠/٢٠١٩

For the Vector function

$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$.

Find the Binormal unit vector.

$$\vec{T} = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} \quad , \quad \frac{d\vec{r}}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j} + 1\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

① $\vec{T} = \frac{-1}{\sqrt{2}} \sin t\,\hat{i} + \frac{1}{\sqrt{2}} \cos t\,\hat{j} + \frac{1}{\sqrt{2}} k$

② $\vec{N} = \dfrac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$

$$\frac{d\vec{T}}{dt} = \frac{-1}{\sqrt{2}} \cos t\,\hat{i} - \frac{1}{\sqrt{2}} \sin t\,\hat{j} + 0\hat{k}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \sqrt{\frac{1}{2}\cos^2 t + \frac{1}{2}\sin^2 t} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\vec{N}(t) = -\cos t\,\hat{i} - \sin t\,\hat{j} + 0k.$$

$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} i & j & k \\ \frac{-1}{\sqrt{2}}\sin t & \frac{1}{\sqrt{2}}\cos t & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}}\sin t\, i - \left(\frac{1}{\sqrt{2}}\cos t\right) j + \overbrace{\left(\frac{1}{\sqrt{2}}\sin^2 t + \frac{1}{\sqrt{2}}\cos^2 t\right)}^{1} k$$

$$\vec{B} = \frac{1}{\sqrt{2}}\sin t\, i - \frac{1}{\sqrt{2}}\cos t\, j + \frac{1}{\sqrt{2}}\, k$$

Ch 13 تم بانتهاء *

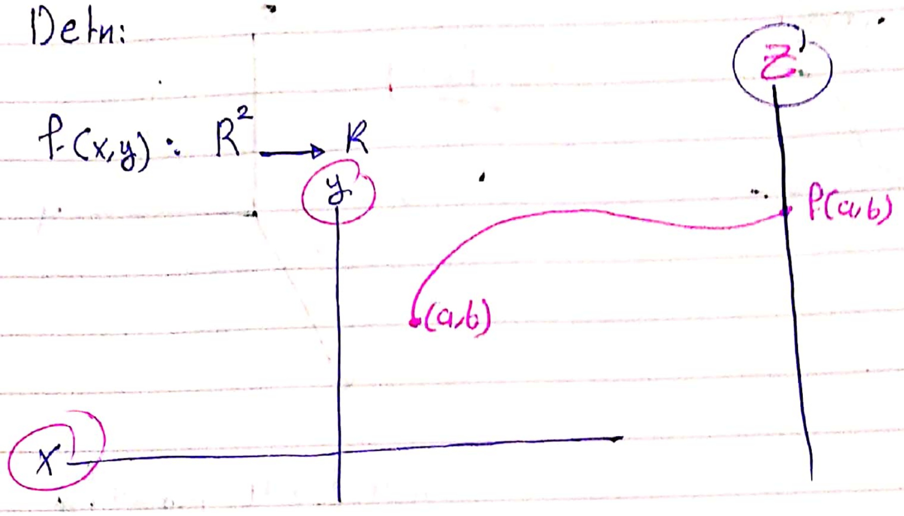
Ch. 14.1 Functions of Several Variables

$\vec{r}(t): R \to R^2$ or R^3

هون صورها كانت على ال y

Defn:

$f(x,y): R^2 \to R$



*** for example**

let $f(x,y) = 2xy^2 + x$

Find

$f(1,0) = 1$

$f(0,3) = 0$

الجواب هو صورها اللي بتكون على ال z

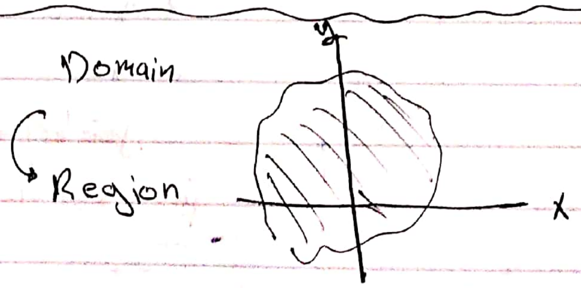
(نقطة, قيمة)

function of one variable	function of two variable
$y = f(x): R \to R$	$z = f(x,y): R^2 \to R$

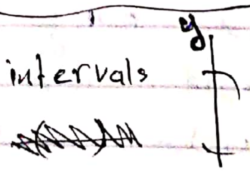
Domain — اما كل R او جزء منها

intervals

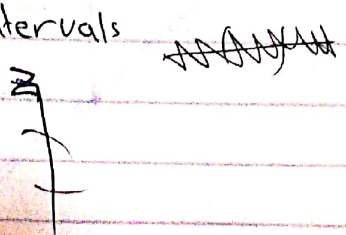
Domain — Region



Range — intervals

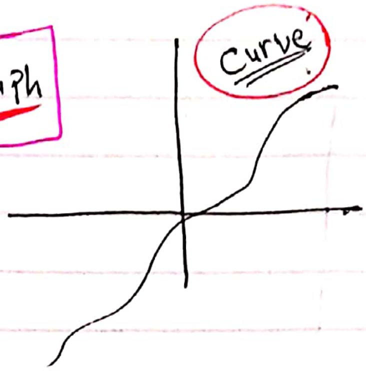


intervals



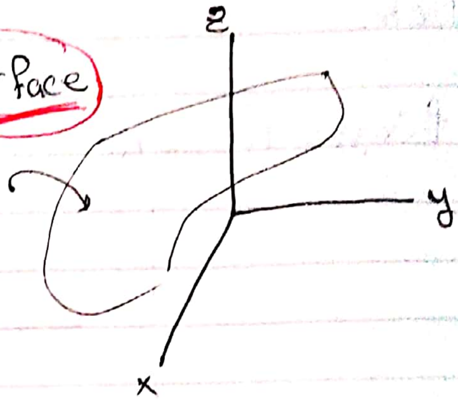
Function of one variable

graph



Function of two variable

surface



* يجب تذكر السطوح الستة اللي اخذناها في ورقة العمل

---

example: For each functions evalute $f(3,2)$, Find and sketch the Domain for f.

ⓐ $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$

ⓑ $f(x,y) = x\ln(y^2 - x)$

$f(3,2) = \dfrac{\sqrt{3+2+1}}{3-1} = \dfrac{\sqrt{6}}{2}$

$D(f) = \{ (x,y) \in \mathbb{R}^2 : x+y+1 \geq 0 \text{ and } x-1 \neq 0 \}$

$x-1=0$

$\boxed{x=1}$

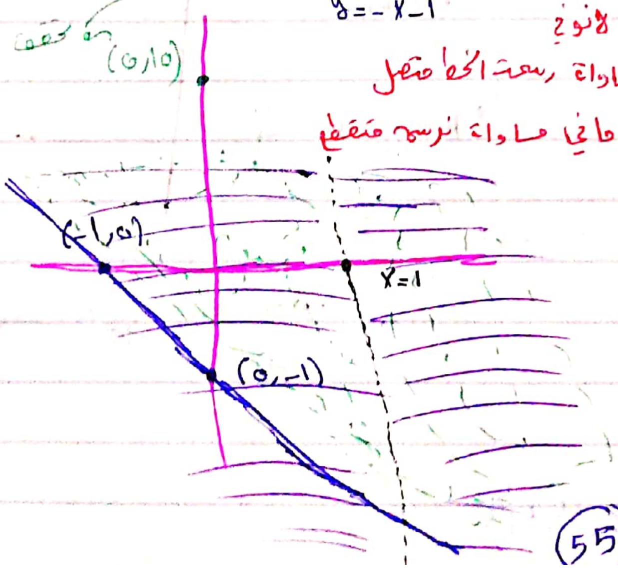
$x+y+1=0$

$y=-x-1$

لأنو ≥ مساواة رسمت الخط متصل

لو ما في مساواة نرسمه متقطع

لأنو ما في مساواة رسمنا الخط متقطع



التعويض يحقق $x \neq 1$

الأصفر تحقق $x+y+1 \geq 0$

المشترك بينهم هو الـ Domain

(كلش فوق الخط الأزرق ما عدا $x=1$)

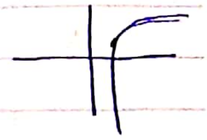
(b) $f(3,2) = 3\ln(4-3) = 0$

$$D(f) = \{ (x,y) \in \mathbb{R}^2 : y^2 - x > 0 \}$$

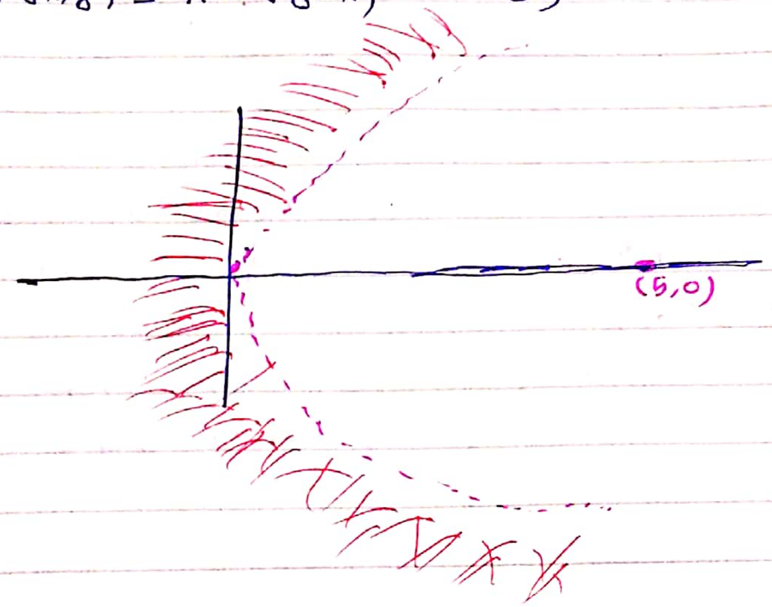
$$y^2 - x = 0$$

+ تذكير

رسمة ال ln



(6) $f(x,y) = x\ln(y^2 - x)$ نرسم



*مثلا نأخذ اي نقطة (5,0) ما حققت المعادلة

اذا نظلل خارج ال parabola

$\alpha \quad 0^2 - 5 > 0$

example

Sketch the graph of $g(x,y) = \sqrt{9 - x^2 - y^2}$

Sol.

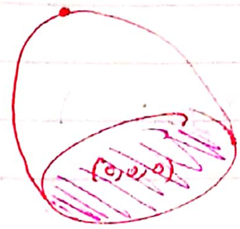
$z = \sqrt{9 - x^2 - y^2}$

$z^2 = 9 - x^2 - y^2$

$x^2 + y^2 + z^2 = 9$

~~(start~~ هاي المعادلة تمثل sphere (كرة)

مركزها (0,0,0) نصف قطرها (3)



عرف ال Domain
عبارة عن Region
وهو ما بداخل
الدائرة التي
مركزها (0,0,0)
ونصف قطرها 3

الثلاثاء ٢٢/١/٢٠١٩ (مثال على المحاضرة السابقة) بالكتاب example:

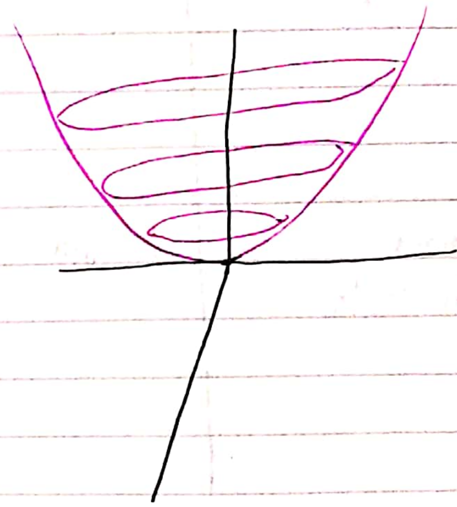
Find the domain and Range of $h(x,y) = 4x^2 + y^2$, Then sketch the graph of h

Sol. let $z = 4x^2 + y^2$

$\{(x,y) \in \mathbb{R}^2\}$

$D(h) = \mathbb{R}^2$

$Range(h) = [0, \infty)$



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* Level Curves : Trace on $z =$ Constant

example:- sketch the level curves of $g(x,y) = \sqrt{9 - x^2 - y^2}$

Sol. $z = \sqrt{9 - x^2 - y^2}$

$z = K$ ثابت

$K = \sqrt{9 - x^2 - y^2}$, K constant

مثلاً $K = -1, 0, 1, 2, 3, \ldots$

$\boxed{K = -1}$

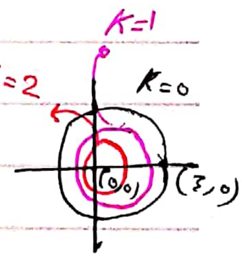
$-1 = \sqrt{9 - x^2 - y^2}$ no graph

(لما تكون k سالبة ← لا يوجد رسمة)

$\boxed{K = 0}$

$0 = \sqrt{9 - x^2 - y^2}$

$9 = x^2 + y^2$



$\boxed{K = 1}$

$1 = \sqrt{9 - x^2 - y^2}$

$1 = 9 - x^2 - y^2$

$x^2 + y^2 = 8$

$\boxed{K = 2}$

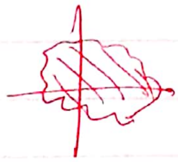
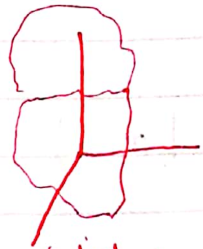
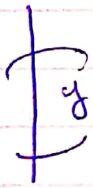
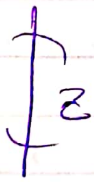
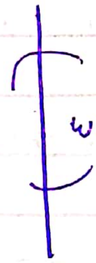
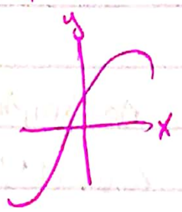
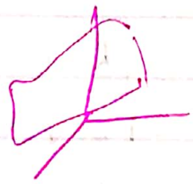
$2 = \sqrt{9 - x^2 - y^2}$

$4 = 9 - x^2 - y^2$

$5 = x^2 + y^2$

$K = 3 \rightarrow$ z تصير نقطة

* Function of 3 Variables

one Variable	two variables	3 variables
$f: R \to R$ $f(x)$	$f: R^2 \to R$ $f(x,y)$	$f: R^3 \to R$ $f(x,y,z)$
Domain intervals	Region 	 Solid Region يعني الكرة وما بداخلها
Range  intervals	intervals 	intervals 
graph Curve 	surface  level Curves	4^{th} D No graph level surfaces

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Defn.
Level surfaces for $w=f(x,y,z)$ are $f(x,y,z)=$ constant

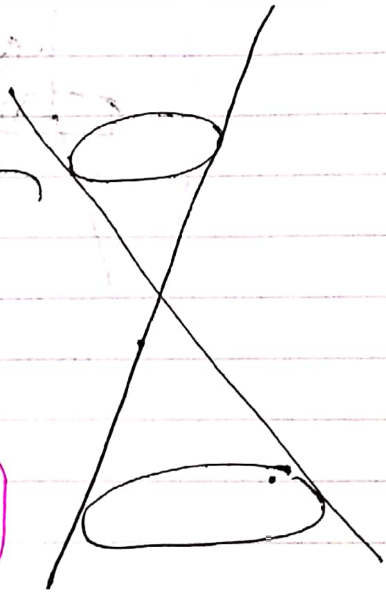
* example
find and sketch the domain for $f(x,y,z)=\ln\left(z-\sqrt{x^2+y^2}\right)+xyz$

$$D=\{(x,y,z)\in R^3 : z-\sqrt{x^2+y^2} > 0$$

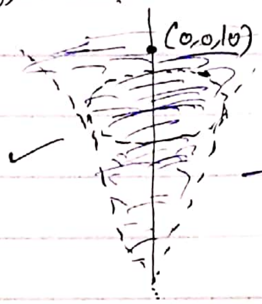
$$z=\sqrt{x^2+y^2}$$

$$z^2=x^2+y^2$$

$$x^2+y^2-z^2=0$$



لأنه جذر موجب خذ ح اخذ الجزء العلوي فقط



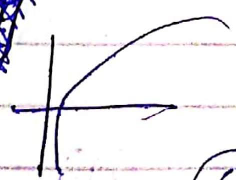
متقطع لأنه لا يوجد مساواة

* ملاحظة
هنا رسم ال Domain
اما ال function لا يمكن رسم

اخذ نقطة داخل ال Corn.
مثلاً (0,0,10) اذا حققت المتباينة (نظلل)
اذا لم تحقق نظلل [redacted] الداخل [redacted] الخارج

هل (0,0,10) تحقق $z-\sqrt{x^2+y^2} > 0$ ؟

نعم تحقق اذاً نظلل الداخل



Range $(-\infty,\infty)$

example

find and sketch some level surfaces of $f(x,y,z) = x^2 + y^2 + z^2$

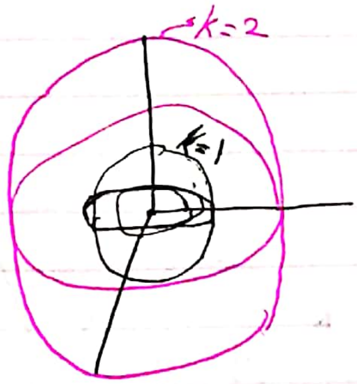
Sol.

let $w = x^2 + y^2 + z^2$

level surfaces

$x^2 + y^2 + z^2 = k$ Constant $k = 0, 1, 2, 3, 4, \ldots$

* لعد مانعوض رح يطلع شكل كرة كلما زادت ال k كبرت الكرة

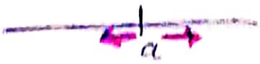


14.2 limits

الاثنين ٢٢/١٠/٢٠١٩

One variable

$$\lim_{x \to a} f(x)$$



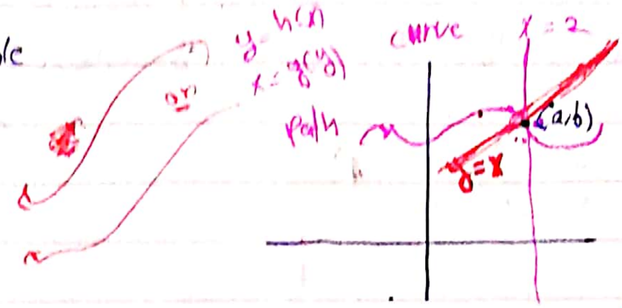
Example $f(x) = x^2$

$\lim_{x \to 2} f(x) =$

x	1.999	2.0001
f_x		

two variable

$$\lim_{(x,y) \to (a,b)} f(x,y)$$



Example

$$\lim_{(x,y) \to (2,2)} (x^2 + 3y^2)$$

x/y	2, 2.0001	2.0001, 2.0001
$f(x,y)$		

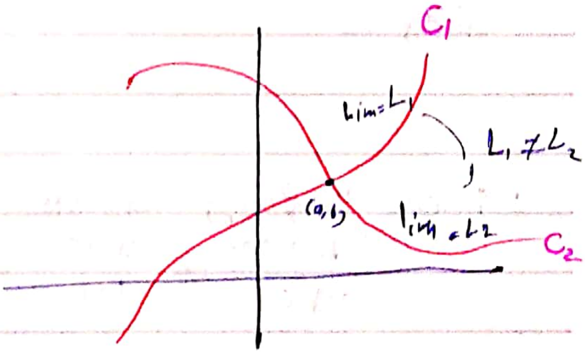
Notes

on each path the $\lim_{(x,y) \to (a,b)} f(x,y)$ becomes limit of one variable.

- if $\lim_{(x,y) \to (a,b)} f(x,y) \neq \lim_{(x,y) \to (a,b)} f(x,y)$

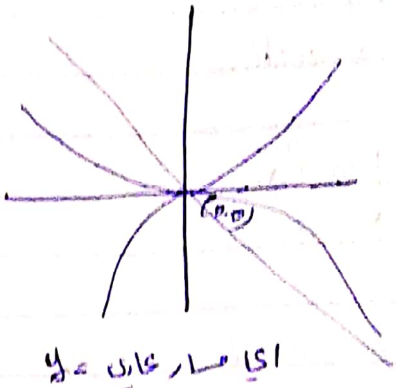
along path c_1 along path c_2

then $\lim_{(x,y) \to (a,b)} f(x,y)$ d.n.e.



example 1

Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ D.n.E

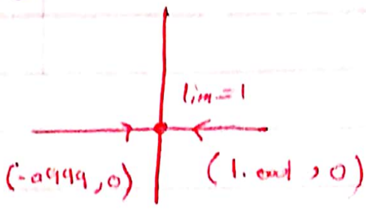


اي مسار عادي y
بنرسمه

Sol.

take path C_1 x-axis "$y=0$"

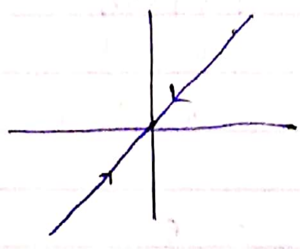
$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_1}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-0^2}{x^2+0^2} = 1$$



ناخذ مسار آخر

take path C_2 "$y=x$"

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_2}} \frac{x^2-y^2}{x^2+y^2} = \lim_{y\to 0} \frac{y^2-y^2}{y^2+y^2} = \lim_{y\to 0} 0 = 0$$



$\therefore$ Since $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } C_1}} f(x,y) \neq \lim\limits_{\substack{(x,y)\to(0,0)\\ \text{along } C_2}} f(x,y)$

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y)$ D.n.E

Q 14 (واجب)

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$$

y-axis $\quad \lim_{y\to 0} \frac{-y^4}{y^2} = 0$

x-axis $\quad \lim_{x\to 0} \frac{x^4}{x^2} = 0$

$y=x \quad \lim_{x\to 0} \frac{x^4-x^4}{x^2+x^2} = 0$

$y=x^2 \quad \lim_{x\to 0} \frac{x^4-x^8}{x^2+x^4} \longrightarrow \frac{(x^2-x^4)(x^2+x^4)}{x^2+x^4} = 0$

اذا نحلل

$$= \frac{(x^2-y^2)\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}} = 0$$

★ Example 2 (واجب)

Does the $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ exist?

take C_1 : x-axis

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_1}} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x(0)}{x^2+0^2} = 0$$

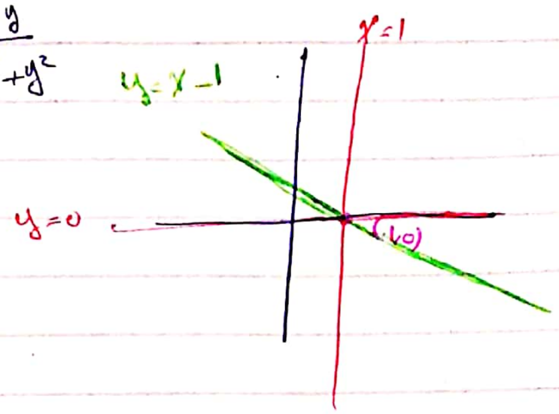
take C_2 "$y=x$"

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_2}} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

$\therefore \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ D.N.E

example

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)y}{(x-1)^2+y^2}$$



$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

$\boxed{x=0}$ $\boxed{y^2=x}$

محاضرة الأحد ٢٧/١٠/٢٠١٩

Ex. $f(x,y) = \dfrac{xy^2}{x^2+y^4}$; does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist ?

*درجة البسط أقل من درجة المقام

C_1 : X-axis "$y=0$"

لو أخذنا
along
X-axis
يطلع صفر

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } c_1}} f(x,y) = \lim \frac{x(0)^2}{x^2+(0)^4} = \boxed{0}$$

along

C_2 : $y = mX$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along}}} f(x,y) = \frac{x(mx)^2}{x^2+(mx)^4} = \frac{m^2x^3}{x^2+m^4x^4} = \frac{\cancel{x^2}}{\cancel{x^2}}\left(\frac{mx}{1+m^4x^2}\right) = \boxed{0}$$

أخذنا عدد لا نهائي من المسارات

C_3 : $X=y^2$ → كأساس أساوي درجة البسط بالمقام
عشان يطلع عندي Constant ← ثاني مسار , هو موجودة
ال limit عليه

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along}\\ x=y^2}} \frac{y^2y^2}{y^4+y^4} = \frac{y^4}{2y^4} = \boxed{\frac{1}{2}}$$

∴ the limit does not exist

Ex: does $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ exist?

* درجة البسط أكبر من درجة المقام

Let: $x = r\cos\theta$ → polar coordinates بال

$y = r\sin\theta$ في تمثيل لانهائي

لل (origin) / (0,0)

أي زاوية بس لازم المسافة صغ ← عشان أمثل ال (origin) بال Polar

$$r = \sqrt{x^2+y^2}$$

* Continuity if

$\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ then

$f(x,y)$ is continuos at (a,b)

$(x,y)\to(0,0)$ then $r\to 0^+$

$0 < \theta < 2\pi$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{\substack{r\to0^+ \\ 0<\theta<2\pi}} \frac{3(r\cos\theta)^2(r\sin\theta)}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \lim_{r\to0^+} \frac{3r^3(\cos^2\theta \times \sin\theta)}{r^2}$$

$$= \lim_{r\to0^+} 3r(\cos^2\theta\,\sin\theta) = 0$$

* لوكانت درجة البسط تساوي درجة المقام ال (r) ح تنشطب مع بعض

حتى لو اختلفت الزوايا هتكون فهو مضروب ب (r) وتعويضه (0)

∴ the limit exists

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Ex Find $\lim_{(x,y)\to(1,3)} x^2 + x^2y - x^3y$

$= 1^2 + 1^2(3) - 1^3(3) = 1$

واد كثير صدود خال lim قسا ويالصورة

$\because$ Polynomial

$\therefore \lim_{(x,y)\to(1,3)} f(x,y) = f(1,3)$

$\therefore$ it is Cont.

Ex where is the function $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ Cont.

$f(x,y)$ is not cont. at $(0,0)$
since $f(0,0)$ is not defined

Ex: where is the function $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} , & (x,y) \neq (0,0) \\ 0 , & (x,y) = (0,0) \end{cases}$

① $f(0,0) = 0$ it's defined

② $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

الـ (Lim) هاي حلينا زيها

درجة البسط = درجة المقام

D.N.E

Ex $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ نفس المثال ص67 بس بالتعويض

- $\lim_{r\to 0^+} \frac{r\cos\theta \, r\sin\theta}{r^2}$

$0 < \theta < 2\pi$

$= \lim_{r\to 0^+} \cos\theta \sin\theta$ ← كون ال limit بتعتمد على (θ) ومع اختلاف (θ) رح تختلف قيم ال (Lim) ← ال (lim) غير موجود

$0 < \theta < 2\pi$

Lim D.N.E

بعد هيك بنبحث عن مساريت مختلفين بيعطوني أجوبة مختلفة عشان يكتمل الحل .

$\lim_{(x,y)\to(1,1)}$ ← عشان نستخدم ال Polar لازم يكون يؤول لـ (0,0) ف بعوضنا

$u = x-1$

$v = y-1$

C_1 : x-axis → lim → 1

C_2 : y-axis → lim → -1

f(x,y) isn't Cont. at (0,0)

f(x,y) Cont. at $R^2 - \{0,0\}$

R^2 ← 2 Variables

R ← 1 Variable

Ex Find the value of (a) that makes $h(x,y) = \begin{cases} \frac{3xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ a, & (x,y) = (0,0) \end{cases}$ Cont. on R^2.

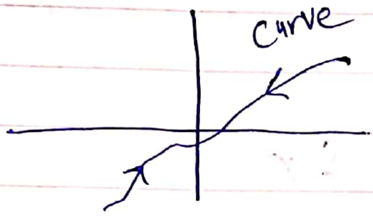
lim حلينا ال lim قبل $\lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+y^2} = a = 0$

$\boxed{a = 0}$

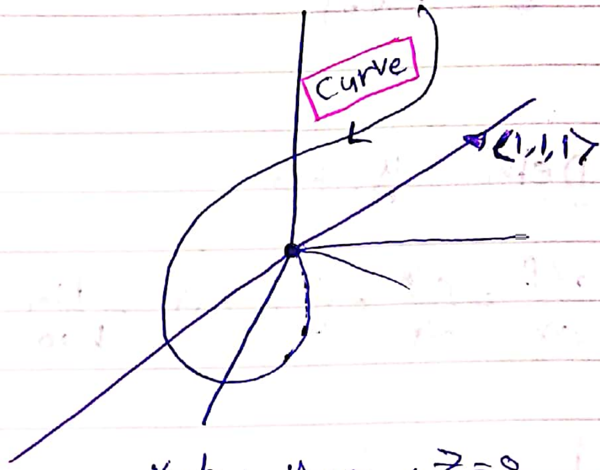
محاضرة الثلاثاء ، ٢٩/١٠/٢٠١٩

14.2 limits

2D في طبيعة المسار



3D



$x=t$, $y=0$, $z=0$

أو $x=0$, $y=t$, $z=0$

$x=t$, $y=t^2$, $z=t^3$

$\boxed{x=t \ , \ y=t \ , \ z=t}$

Q20 , page (899)

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz}{x^2+y^2+z^2}$$ D.N.E ← درجة البسط = درجة المقام

لازم أثبت

take C_1 : x_ axis $x=t$, $y=0$, $z=0$

$$\lim_{\substack{(x,y,z)\to(0,0,0)\\ \text{along } C_1}} \frac{xy+yz}{x^2+y^2+z^2} = \lim_{t\to 0} \frac{0+0}{t^2+0+0^2} = 0$$

take C_2 : $x=t$, $y=t$, $z=t$

$$\lim_{\substack{(x,y,z)\to(0,0,0)\\ \text{along } C_2}} = \lim_{t\to 0} \frac{t(t)+t(t)}{t^2+t^2+t^2} = \frac{2}{3}$$

$\therefore$ D.N.E

14.3 partial Derivatives

$z = f(x,y)$

* **Defn.** $y = f(x)$

$$\frac{df}{dx} = \frac{dy}{dx} = y' = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

* **Defn.** The partial Derivatives of $z = f(x,y)$

partial

لأن التغير حصل على x

$$\boxed{a}\ \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = z_x = f_x = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

لأن التغير حصل على y

$$\boxed{b}\ \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = z_y = f_y = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

Example:

If $f(x,y) = x^3 + x^2y^3 - 2y^2$. find $f_x(2,1)$ and $f_y(2,1)$

Sol.

$f_x(x,y) = 3x^2 + 2y^3x - 0$

$f_x(2,1) = 3(2)^2 + 2(2)1 = 16$

$f_y(x,y) = 0 + 3x^2y^2 - 4y$

$f_y(2,1) = 0 + 12 - 4 = 8$

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* الطريقة الأخرى ممكن لما الطلب $f_x(2,1)$
إني أعوض $y=1$ بعدين اشتق
خاني يكون one variable
والعكس صحيح

طريقة أخرى [or] set $y=1$

$m(x) = f(x,1) = x^3 + x^2 - 2$

$m'(x) = f_x(x,1) = 3x^2 + 2x$

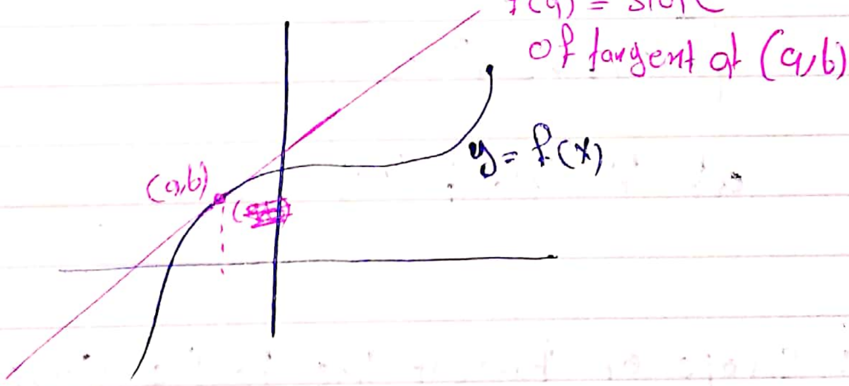
$m'(2) = f_x(2,1) = 3(2)^2 + 2(2) = 16$

set $x=2$

$g(y) = f(2,y) = 8 + 4y^3 - 2y^2$

$g'(y) = f_y(2,y) = 12y^2 - 4y$

$g'(1) = f_y(2,1) = 8$



Example

if $f(x,y) = 4 - x^2 - 2y^2$ find $f_x(1,1)$ and $f_y(1,1)$ and interpret these numbers as slope

تفسير

Sol.

$f_x(x,y) = 0 - 2x$

$f_x(1,1) = -2$

$f_y(x,y) = -4y$

$f_y(1,1) = -4$

$\boxed{-2}$: slope of tangent line of Curve of intersection between the plane $y=1$ and $z = 4 - x^2 - 2y^2$

$$\left[m(x) = 4 - x^2 - 2 \right]$$

at the point $(1,1)$

$\boxed{-4}$: slope of tangent line to the Curve of intersection between plane $x=1$ and $z = 4 - x^2 - 2y^2$

$$\left[g(y) = 3 - 2y^2 \right]$$

at the Point $(1,1)$

example

if $f(x,y) = \sin\left(\frac{x}{1+y}\right)$

$$\frac{\partial f}{\partial x} = \frac{1}{1+y} \cos\left(\frac{x}{1+y}\right) \qquad \frac{\partial f}{\partial y} = \frac{-x}{(1+y)^2} \cos\left(\frac{x}{1+y}\right)$$

<u>example</u> الخميس ٣١/١٠/٢٠١٩

find f_x, f_y and f_z . if $w = f(x,y,z) = e^{xy} \ln z$

$f_x = y\, e^{xy} \ln z$

$f_y = x\, e^{xy} \ln z$

$f_z = e^{xy} \dfrac{1}{z}$.

---

* الإشتقاق الضمني

$xy\, e^{xy} \sin(xy) + y^2 x^3 = 0$ find $\dfrac{dy}{dx}$

$y = f(x)$ أي اشتقاق ضمني

<u>Ex</u>

$3y + 2z + x = 0$ find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

<u>example</u>... if $x^3 + y^3 + z^3 + 6xyz = 1$

find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

<u>Sol.</u>

for $\dfrac{\partial z}{\partial x}$:- $3x^2 + 0 + 3z^2 \dfrac{\partial z}{\partial x} + 6xy \dfrac{\partial z}{\partial y} + 6yz = 0$

(تفاضل الـ y ← constant)

$$\frac{\partial z}{\partial x} = \frac{-3x^2 - 6yz}{3z^2 + 6xy}$$

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for $\frac{\partial z}{\partial y}$

$$0 + 3y^2 + 3z^2 \frac{\partial z}{\partial y} + 6xy \frac{\partial z}{\partial y} + 6xz = 0$$

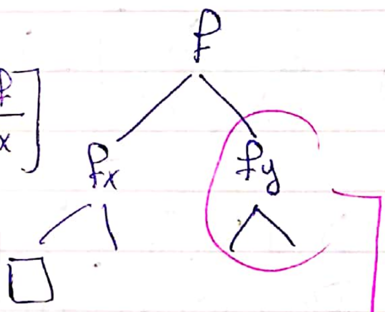
$$\frac{\partial z}{\partial y} = \frac{-3y^2 - 6xz}{3z^2 + 6xy}$$

---

* المشتقات العليا

$$y^{(2)} = \frac{d^2 f}{dx^2} = f'' = \ddot{y} = \frac{d^2 y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$$

The 2^{nd} partial Derivatives of $z = f(xy)$ are



$\boxed{1}$ $\frac{\partial^2 f}{\partial x^2} = f_{xx} = z_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]$

$\boxed{2}$ $\frac{\partial^2 f}{\partial y \partial x} = f_{xy} = z_{xy} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$

$\boxed{3}$ $\frac{\partial^2 f}{\partial x \partial y} = f_{yx} = z_{yx} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]$

4 $\frac{\partial^2 f}{\partial y^2} = f_{yy} = z_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]$

Example.

Find the 2^{nd} partial Derivatives

$$f_{(x,y)} = x^3 + x^2y^3 - 2y^2$$

Sol.

$$f_x = 3x^2 + 2xy^3$$
$$f_y = 3x^2y^2 - 4y$$

$$f_{xx} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial x}\left[3x^2 + 2xy^3\right] = 6x + 2y^3$$

$$f_{xy} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}\left[3x^2 + 2xy^3\right] = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[3x^2y^2 - 4y\right] = 6xy^2$$

$$f_{yy} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial y}\left[3x^2y^2 - 4y\right] = 6x^2y - 4$$

Th.m (نظرية)

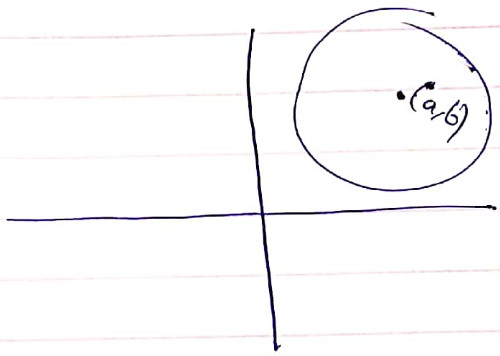
Suppose f defined on a disk D

that Contains (a,b)

f_{xy} and f_{yx} are Conts متصل

on D then

$f_{xy}(a,b) = f_{yx}(a,b)$



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example :

Let $f(x,y,z) = e^{xyz} + \sin^{-1}(x)\, e^{x^2 y^3 \sin(xy)}$, find f_{xyz}

Find $f_{xyz} = f_{zxy}$

$f_z = xy\, e^{xyz}$

$f_{zx} = (xy)(yz)\, e^{xyz} + y\, e^{xyz}$

$= xy^2 z\, e^{xyz} + y e^{xyz} = [\quad\quad]\, e^{xyz}$

14.4 (سيتم شرحه فيما بعد)

14.5- Chain Rule :

٢٠١٩/١١/٥ الثلاثاء

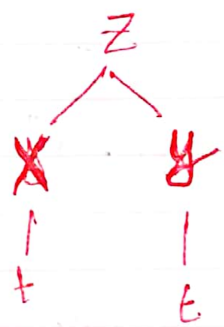
* if $y = f(x)$, $x = h(t)$

$y = f(h(t)) = g(t)$

$$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt}$$

* if $z = f(x,y)$; $x = h(t)$, $y = g(t)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$



Example: if $z = x^2 y + 3xy^4$, $x = \sin 2t$
$y = \cos t$

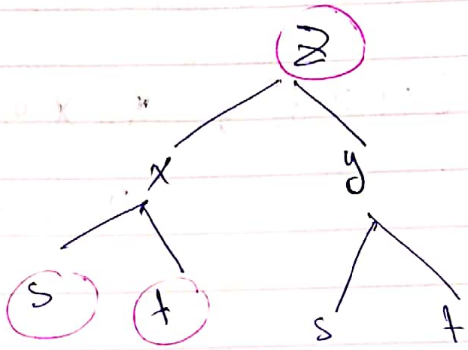
find $\frac{dz}{dt}$

Sol.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2xy + 3y^4)\,2\cos(2t) + (x^2 + 12xy^3)(-\sin 2t)$$

* if $z = f(x,y)$, $x = g(s,t)$
$y = g(s,t)$



هنالك مسارين

① $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$

z ← تعتمد على متغيرين لذلك
نستخدم ← partial div.

[2] $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$

---

example

if $z = e^x \sin y$, $x = st^2$; $y = s^2 t$

find $\frac{\partial z}{\partial t}$.

Sol.

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$

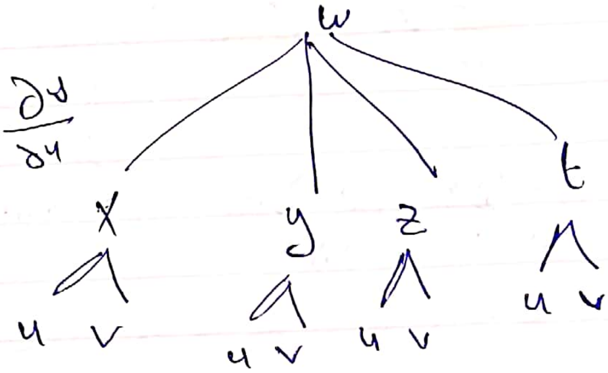
$$= \left(e^{st^2} \sin(s^2 t)\right)(2st) + \left(e^{st^2} \cos(s^2 t)\right)(s^2)$$

example 4 (بيتي)

write out the Chain Rule for the Case

$w = f(x, y, z, t)$; $x = x(u, v)$; $y = y(u, v)$

$z = z(u, v)$; $t = t(u, v)$



① $$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u}$$

② $$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}$$

example

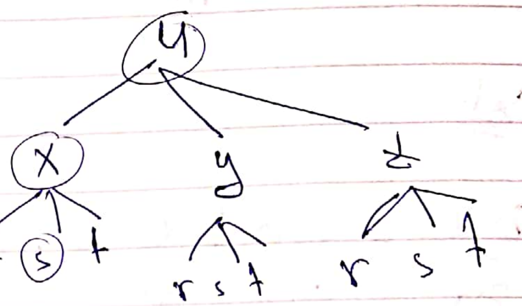
if $u = x^4 y + y^2 z^3$ where $x = rse^{t}$

$y = r s^2 e^{-t}$, $z = r^2 s \sin t$

Find $\frac{\partial u}{\partial s}$ where $r = 2$

$s = 1$

$t = 0$



Sol.

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$= (4x^3 y)(re^{t}) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2 z^2)(r^2 \sin t)$$

when $r = 2$ / $s = 1$ / $t = 0$ the $x = (2)(1)e^{(0)} = 2$

$y = (2)(1)e^{2-0} = 2$

$z = 4(1)\sin(0) = 0$

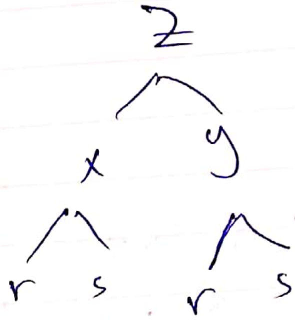
$$\left.\frac{\partial u}{\partial s}\right|_{\substack{r=2\\ s=1\\ t=0}} = \text{......}$$

example

if $z = f(x,y)$; $x = r^2 + s^2$, $y = 2rs$

find ⓐ $\frac{\partial z}{\partial r}$ ⓑ $\frac{\partial^2 z}{\partial r^2}$.

Sol.



$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

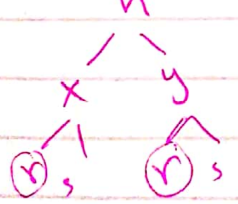
$$= \frac{\partial f}{\partial x}(2r) + \frac{\partial f}{\partial y}(2s)$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left[\frac{\partial f}{\partial x}(2r) + \frac{\partial f}{\partial y}(2s)\right]$$

$$= \frac{\partial}{\partial r}\left[\frac{\partial f}{\partial x}(2r)\right] + 2s\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial f}{\partial x}(2) + \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right)2r + 2s\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right)$$

$\boxed{\text{نفرض}}$ $\frac{\partial f}{\partial x} = h(x,y)$

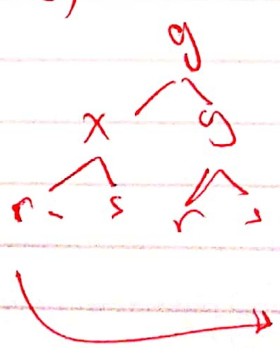


$$\frac{\partial h}{\partial r} = \frac{\partial h}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial h}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(2r) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(2s)$$

$$\frac{\partial^2 f}{\partial x^2}(2r) + \frac{\partial^2 f}{\partial y \partial x}(2s)$$

$\frac{\partial f}{\partial y} = g(x,y)$



$$\frac{\partial f}{\partial y} = g$$

$$\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial g}{\partial r}$$

$$= \frac{\partial g}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial^2 f}{\partial x \partial y} \quad (2r) \qquad \frac{\partial^2 f}{\partial y^2} \quad (2s)$$

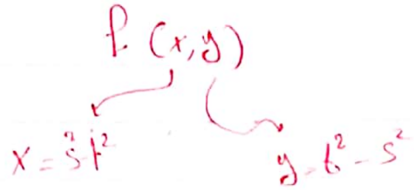
example :

if $g(s,t) = f(s^2 - t^2, t^2 - s^2)$

show that g satisfice $t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$

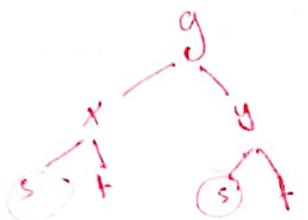
Sol.

let $x = s^2 - t^2$ $y = t^2 - s^2$



$$\boxed{1}\ \frac{\partial g}{\partial s} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial s}$$

$$= \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)$$



$$\boxed{2}\ \frac{\partial g}{\partial t} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial x}$$

$$= \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)$$

multiply ① by t and ② by s then add them

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t}$$

$$t\left[\frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s)\right] + s\left[\frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)\right]$$

$$= (2st - 2st)\frac{\partial f}{\partial x} + (-2st + 2st)\frac{\partial f}{\partial y} = 0$$

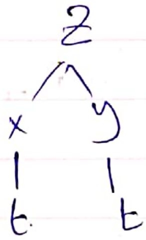
14.6 Direction derivatives

let $z = f(x,y)$ then the directional Derivative of $f(x,y)$ at (a,b) in the direction of unit vector $\hat{u} = u_1 \hat{i} + u_2 \hat{j}$ is given by

The equation of this Direction is given by

$x = a + u_1 t$

$y = b + u_2 t$

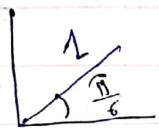


$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\boxed{D_{\hat{u}} f(a,b) = f_x \, u_1 + f_y \, u_2}$$

example :-
Find the Directional Derivative of $f(x,y) = x^3 - 3xy + 4y^2$ at $(1,2)$ in the Direction of unit vector makes an angle of $\theta = \frac{\pi}{6}$ with x-axis

Sol. →



$\hat{u} = 1 \cos\frac{\pi}{6} \hat{i} + 1 \sin\frac{\pi}{6} \hat{j}$

$\hat{u} = \frac{\sqrt{3}}{2} \hat{i} + \frac{1}{2} \hat{j}$

$f_x = 3x^2 - 3y$

$f_y = -3x + 8y$

$D_{\hat{u}} f(x,y) = f_x u_1 + f_y u_2$

$= (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\frac{1}{2}$

$D_{\hat{u}}(1,2) = -\frac{3\sqrt{3}}{2} + \frac{13}{2}$

$$D_{\hat{u}} f(x,y) = f_x u_1 + f_y (u_2) \qquad \hat{u} = \langle u_1, u_2 \rangle$$

$$D_{\hat{u}} f(x,y) \ \boxed{\langle f_x, f_y \rangle} \cdot \langle u_1, u_2 \rangle$$

Defin.

The gradient vector of f is given by

$$\nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$$

$$\boxed{D_{\hat{u}} f = \nabla f \cdot \hat{u}}$$

Example ...

if $f(x,y) = \sin x + e^{xy}$
find $\vec{\nabla} f$ and $\nabla f(0,1)$.

Sol

$$\vec{\nabla} f = f_x \hat{i} + f_y \hat{j}$$

$$= (\cos x + y e^{xy}) \hat{i} + (x e^{xy}) \hat{j}$$

example

find the Directional Derivative of $f(x,y) = x^2 y^3 - 4y$ at $(2,-1)$ in the Direction of $\vec{v} = 2\hat{i} + 5\hat{j}$

Sol:

$$\vec{\nabla} f = f_x \hat{i} + f_y \hat{j}$$

$$= 2xy^3\, \hat{i} + (3x^2y^2 - 4)\, \hat{j}$$

$$\nabla f(2,-1) = -4\hat{i} + 8\hat{j}$$

$$\cancel{D_{\hat{u}} f = \nabla f \cdot \vec{v} = (-4\hat{i} + 8\hat{j}) \cdot (2\hat{i} + 5\hat{j}) =}$$

$$\hat{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{2\hat{i}}{\sqrt{29}} + \frac{5}{\sqrt{29}}\hat{j}$$

$$D_{\hat{u}} f(2,-1) = \nabla f \cdot \hat{u}$$

$$= (-4\hat{i} + 8\hat{j}) \cdot \left(\frac{2}{\sqrt{29}}\hat{i} + \frac{5}{\sqrt{29}}\hat{j}\right)$$

$$= \frac{-8}{\sqrt{29}} + \frac{40}{\sqrt{29}}$$

$$= \frac{32}{\sqrt{29}}$$

Example :

if $f(x,y,z) = x \sin(yz)$.

ⓐ find the gradient of f at $(1,3,0)$

ⓑ find the Directional Derivative of f at $(1,3,0)$ in the Direction of $\vec{v} = \hat{i} + 2\hat{j} - k$.

Sol.

$$\nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$$
$$= \sin(yz)\hat{i} + xz\cos(yz)\hat{j} + xy\cos(yz)\, k$$

ⓐ $\nabla f(1,3,0) = 0\hat{i} + 0\hat{j} + 3\hat{k}$

ⓑ $\hat{v} = \frac{\hat{i}}{\sqrt{6}} + \frac{2}{\sqrt{6}}\hat{j} - \frac{k}{\sqrt{6}}$

$$D_{\hat{v}} f(1,3,0) = \nabla f(1,3,0) \cdot \hat{v}$$
$$= (3k) \cdot \left(\frac{\hat{i}}{\sqrt{6}} + \frac{2}{\sqrt{6}}\hat{j} - \frac{k}{\sqrt{6}} \right)$$
$$= \frac{-3}{\sqrt{6}}$$

$\nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$

$\vec{A}\cdot\vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$
$= |\vec{A}||\vec{B}| \cos\theta$

$D_{\hat{u}} f(a,b,c) = \nabla f \cdot \hat{u}$

$= |\nabla f| |\hat{u}| \cos\theta$

$= |Df| \cos\theta$

$Max(D_{\hat{u}} f) = |Df|$ occurse when $\hat{u} = \frac{Df}{|Df|}$

$Min(D_{\hat{u}} f) = -|Df|$ occurse when $\hat{u} = \frac{-Df}{|Df|}$

example

(a) if $f(x,y) = xe^y$ find therate of change at $P(2,0)$ in the direction from P to $Q(\frac{1}{2}, 2)$?

rate of change = slope = $D_{\hat{u}} f(2,0) = \nabla f(2,0) \cdot \hat{u} = (\hat{i} + 2\hat{j}) \cdot (\frac{-3}{5}\hat{i} + \frac{4}{5}\hat{j})$

$\nabla f = e^y \hat{i} + xe^y \hat{j}$ $\quad \nabla f(2,0) = \hat{i} + 2\hat{j}$ $\quad = \frac{-3}{5} + \frac{8}{5} = 1$

$\hat{u} = \frac{\vec{PQ}}{|PQ|} = \frac{\frac{-3}{2}\hat{i} + 2\hat{j}}{\sqrt{\frac{9}{4} + 4}} = \frac{-3}{5}\hat{i} + \frac{4}{5}\hat{j}$

(b) what is the Max Rate of change at P in what Direction does f have Max Rate of change.

Sol.

$$\text{Max Rate of Change} = \text{Max}\left(D_{\hat{u}} f(2,0)\right) = \left|\nabla f(2,0)\right|$$

$$= \left|\hat{i} + 2\hat{j}\right|$$

$$= \sqrt{5}$$

occurse when $\hat{u} = \dfrac{\nabla f}{|\nabla f|} = \dfrac{1}{\sqrt{5}}\hat{i} + \dfrac{2}{\sqrt{5}}\hat{j}$

Example: suppose the temp. at (x, y, z) in space given by $T = \dfrac{80}{1 + x^2 + y^2 + z^2}$

Ex: In which direction does the temp. increase fastest at $(1, 1, -2)$? what is the Max rate of increase

$$\text{Max Rate} = \text{Max}\left(D_{\hat{u}} T(1,1,-2)\right)$$

$$= \left|\nabla T(1,1,-2)\right|$$

Occurse when $\hat{u} = \dfrac{\nabla T}{|\nabla T|}$

$$\nabla T = T_x\hat{i} + T_y\hat{j} + T_z\hat{k} = \frac{-160x}{(1+x^2+y^2+z^2)^2}\hat{i} + \frac{-160y}{(1+x^2+y^2+z^2)^2}\hat{j} + \frac{-160z}{(1+x^2+y^2+z^2)^2}\hat{k}$$

تكملة →

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$$\nabla T(1,1,-2) = \frac{-160}{49}\hat{i} - \frac{160}{49}\hat{j} + \frac{320}{49}\hat{k}$$

$$\frac{-160}{49}\left(\hat{i} + \hat{j} + 2k\right)$$

$$|\nabla T| = \left|\frac{-160}{49}\right| \left|\hat{i} + \hat{j} - 2k\right|$$

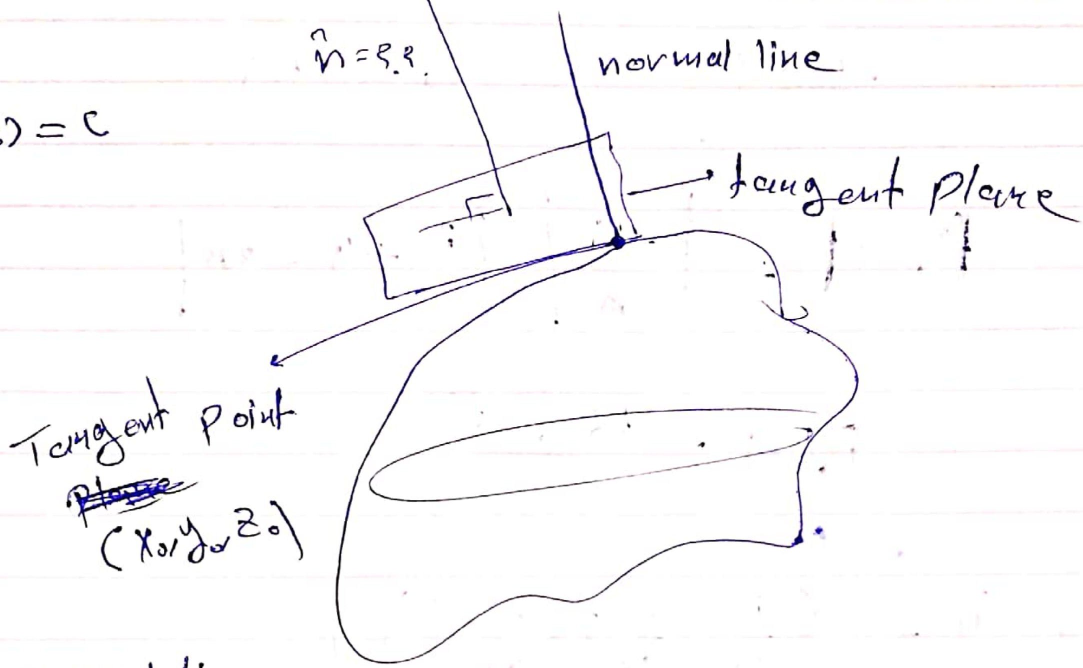
$$|\nabla T| = \frac{160\sqrt{6}}{49}$$

$$\boxed{\hat{u} = -\left(\frac{\hat{i}}{\sqrt{6}} + \frac{1}{\sqrt{6}}\hat{j} - \frac{2}{\sqrt{6}}\hat{k}\right)}$$

$\boxed{14.6}$ الخميس ١٢/١١/19

The tangent plane and normal line to the Level Surface

$f(x,y,z) = c$



$\vec{n} \parallel \nabla f \parallel$ normal line

Tangent plane.

$$\vec{n} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$\frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) + \frac{\partial f}{\partial z}(z - z_0) = 0$$

normal line :-

$$x = x_0 + \frac{\partial f}{\partial x} t$$

$$y = y_0 + \frac{\partial f}{\partial y} t$$

$$z = z_0 + \frac{\partial f}{\partial z} t$$

Example

find the equations of tangent plane and normal line to the surface

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3 \qquad \text{at } (-2, 1, -3)$$

Sol.

Let $f(x,y,z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$

$\nabla f = \frac{x}{2} \hat{i} + 2y \hat{j} + \frac{2z}{9} k$

$\nabla f(-2, 1, -3) = -\hat{i} + 2\hat{j} - \frac{6}{9} k$

The equation of tangent plane is

$$-1(x - -2) + 2(y - 1) + \frac{-6}{9}(z - -3) = 0$$

The equations of normal line.

$x = -2 + -1t$

$y = 1 + 2t$

$z = -3 + \frac{-6}{9} t$

14.4

$z = f(x,y)$

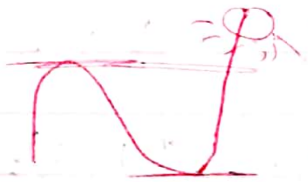
let $F(x,y,z) = f(x,y) - z = 0$

$f_x (x - x_0) + f_y (y - y_0) - (z - z_0) = 0$

---

* 14.7 Maximum and Minimum Values:

local relative   Absolute



* Defn. (a) $f(x_0, y_0)$ is local Max iff

$f(x_0, y_0) \geq f(x,y)$ for all (x,y) near (x_0, y_0)

(b) $f(x_0, y_0)$ is an absolute Max for $f(x,y)$ on D $f(x_0, y_0) \geq f(x,y)$ for all $(x,y) \in D$

* Defn. The critical points of $f(x,y)$ is given by

$D_{\hat{u}} f(x,y) = 0$

$\nabla f \cdot \hat{u} = 0$

$\nabla f = \vec{0}$ $\langle f_x, f_y \rangle = 0$

$f_x = 0$ —— (1)

$f_y = 0$ —— (2)

*** Thm** (2nd Derivative Test)

let (a,b) be critical point of $f(x,y)$ $\longrightarrow$ $\left(\begin{array}{l} f_x(a,b)=0 \\ f_y(a,b)=0 \end{array} \right)$

$$D = f_{xx} f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$$f_{xx} f_{yy} > f_{xy}^2 > 0$$

$D(a,b)$

- $D>0$
  - $f_{xx}>0$: local Min
  - $f_{xx}<0$: local Max
- $D<0$: Saddle point
- $D=0$: test fail

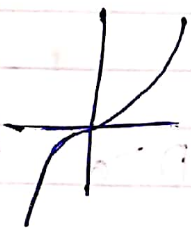
$\boxed{\text{if } D>0 \text{ then } f_{xx}, f_{yy} \text{ have the same sign}}$

0.19/11/17

(a,b) critical point $\quad f_x(a,b)=0$
$\quad f_y(a,b)=0$

$$D = f_{xx} f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

- $D>0$
  - $f_{xx}>0$ or $(f_{yy}>0)$ → local min
  - $f_{xx}<0$ or $(f_{yy}<0)$ → local max
- $D<0$ → saddle point
- $D=0$ → Test fail



example: find all critical point(s) of $f(x,y)=x^4+y^4-4xy+1$ then classify them as local max, local min or saddle point

Sol. $f_x = 4x^3-4y \qquad 4x^3-4y=0$ —①

$f_y = 4y^3-4x \longrightarrow 4y^3-4x=0$ —②

$y=x^3 \qquad$ ① sub in $x=y^3$

$x=(x^3)^3$

$x=x^9 \longrightarrow x^9-x=0$

$x(x^8-1)=0$

$x(x^4-1)(x^4+1)=0$

$x(x-1)(x+1)(x^4+1)=0$

$x=0 \;/\; x=1 \;/\; x=-1$

تابع

at $x=0 \rightarrow y=0$ $(0,0)$

at $x=1 \rightarrow y=1$ $(1,1)$

at $x=-1 \rightarrow y=-1$ $(-1,-1)$

$f_{xx} = 12x^2$

$f_{yy} = 12y^2$

$f_{xy} = -4$

$D = f_{xx} f_{yy} - f_{xy}^2$

$D>0$: local Max, local min

$D<0$: saddle

$\cancel{D=0}$

critical point	$D = (12x^2)(12y^2) - 16$	sign $(12x^2)$ f_{xx}	conclusion
$(0,0)$	$-16 < 0$	$\longrightarrow$	Saddle
$(1,1)$	$(12)(12) - 16 > 0$	$12(1)^2 > 0$	local Min.
$(-1,-1)$	$(12)(12) - 16 > 0$	$12(-1)^2 > 0$	local Min.

$f(1,1) = -1$

$f(-1,-1) = -1$

are local Min for f

$(0,0)$ is Saddle Point

example locate the local extrema and saddle point(s) for
$f(x,y) = 2x^3 + 6xy^2 - 3y^3 - 150x$

Sol.

$f_x = 6x^2 + 6y^2 - 150$

$f_y = 12xy - 9y^2$

$6x^2 + 6y^2 - 150 = 0$ ---- (1)

$12xy - 9y^2 = 0$ ---- (2)

$x^2 + y^2 = 25$ ---- (1)

$3y(4x - 3y) = 0$

$\boxed{y=0}$ or $4x - 3y = 0 \rightarrow \boxed{x = \frac{3y}{4}}$

Case (1) $y=0$

Sub $y=0$ in (1) $x^2 + 0^2 = 25$

$x = \pm 5$

$(5,0)$, $(-5,0)$

case (2) $x = \frac{3}{4}y$ in (1)

sub

$\frac{9}{16}y^2 + y^2 = 25$

$\frac{9y^2 + 16y^2}{16} = 25$

$\frac{25}{16} y^2 = 25 \rightarrow y^2 = 16$

$y = \pm 4$

تابع →

when $y=4 \longrightarrow x = \frac{3}{4}(4) = 3$

$(3,4)$

when $y=-4 \longrightarrow x = \frac{3}{4}(-4) = -3$

$(-3,4)$

$f_{xx} = 12x$ / $f_{yy} = 12x - 18y$ / $f_{xy} = 12y$

Critical points	$D = 12x(12x-18y)-(12y)^2$	$f_{xx} = 12x$	Conclusion
$(5,0)$	$12(5)[12(5)] > 0$	$12(5) > 0$	local min
$(-5,0)$	$12(-5)[12(-5)] > 0$	$12(-5) < 0$	local max
$(3,4)$	$12(3)[12(3)-18(4)]-(12(4))^2 < 0$		saddle
$(-3,-4)$	$12(-3)[12(-3)-18(-4)]-(12(-4))^2 < 0$		saddle

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example

الثلاثاء 19/11/2019م

Find the shortest distance from $(1,0,-2)$ to the plane
$x+2y+z=4$

Sol:

lies on the plane

$$d\big((x,y,z),(1,0,-2)\big)=\sqrt{(x-1)^2+y^2+(z+2)^2}$$

but $x+2y+z=4$

$$z=4-x-2y$$

$$d=\sqrt{(x-1)^2+y^2+(4-x-2y+2)^2}$$

للتخلص من الجذر ← للتسهيل

$$F(x,y)=d^2=(x-1)^2+y^2+(4-x-2y+2)^2$$

$$f_x=2(x-1)+-2(6-x-2y)$$

$$f_y=2y-2(2)(6-x-2y)$$

$$2(x-1)-2(6-x-2y)=0$$

$$2y-4(6-x-2y)=0$$

$$4x+4y=14 \quad (1)$$

$$4x+10y=24 \quad (2)$$

تابع

توضيح

مربع مسافة: (4) (36)

2 4 10

مسافة: 2 6

اصغر مربع مسافة (4)

اصغر مسافة (2)

بس النقطة لم تتغير الي هي 4

$6y = 10 \rightarrow y = \frac{10}{6} = \frac{5}{3}$

$x = \frac{11}{6}$

critical point $\left(\frac{11}{6}, \frac{5}{3}\right)$

$D = f_{xx} f_{yy} - f_{xy}^2 > 0$

$f_{xx} > 0$

$f_{xx} > 0 \rightarrow$

at $\left(\frac{11}{6}, \frac{5}{3}\right)$ there is a local min for

$\underline{\underline{\text{min } d}} = \sqrt{f\left(\frac{11}{6}, \frac{5}{3}\right)} = \boxed{}$

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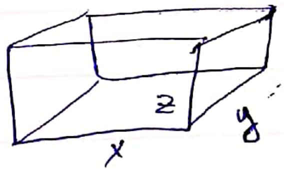
example (5 أوجه فقط) يعني مالها سقف

A rectangular box without lid is to be made from $12\ cm^2$ of Cardboard. Find the Maximum volume of such box.

$V = xyz$



Surface Area $= 12 = xy + yz + yz + xz + xz$

$$12 = xy + 2yz + 2xz$$

$$12 = xy + (2y + 2x)z$$

$$z = \frac{12 - xy}{2x + 2y}$$

$$V(x,y) = xy\left(\frac{12 - xy}{2x + 2y}\right)$$

$$= \frac{12xy - x^2y^2}{2x + 2y}$$

$$V_x = \frac{(2x+2y)(12y - 2xy^2) - 2(12xy - x^2y^2)}{(2x+2y)^2}$$

$$V_y = \frac{(x+y)(12x - 2yx^2) - (12xy - x^2y^2)}{2(x+y)^2}$$

تابع

$$(x+y)(12y-2xy^2)-(12xy-x^2y^2)=0 \quad \text{——} \quad (1)$$

$$(x+y)(12x-2yx^2)-(12xy-x^2y^2)=0 \quad \text{——} \quad (2)$$

نفكك

$$-x^2y^2$$

$$\cancel{12xy}-\cancel{2x^2y^2}+12y^2-2xy^3-\cancel{12xy}+\cancel{x^2y^2}=0$$

$$y^2(-x^2+12-2xy)=0$$

$$x^2(-y^2+12-2xy)=0$$

$$x^2-y^2=0 \longrightarrow x=+y$$

$$-x^2+12-2x^2=0$$

$$12-3x^2=0 \longrightarrow x=+2=y$$

نجد z

$$z=\frac{12-(2)(2)}{2(2)+(2)(2)}=1$$

$$(2,2,1)$$

$$V=4\ cm^3$$

الخميس ٢٠١٩/١١/٢١

Thm. let $f(x,y)$ be conts on Closed bounded Region D then f has both absolute max and min values to find the absolute max and min for f ?

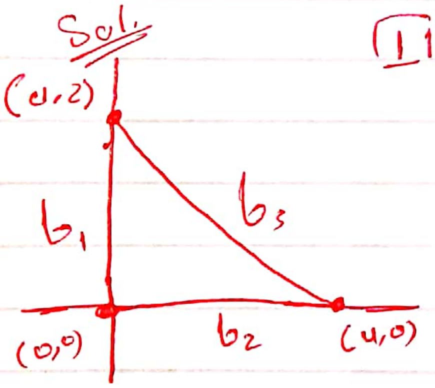
[1] find all critical point(s) inside D, $f_x = 0$ and $f_y = 0$

[2] find all critical point(s) on the boundary of D on the boundary $f(x,y)$ beacause a function of one variable ?

[3] Compute and compare.

* example let $f(x,y) = x + y - xy$ and D is the Closed trianglular region with vertices $(0,0), (0,2), (4,0)$

Sol.



[1] $f_x = 1 - y$, $f_y = 1 - x$

$1 - y = 0$, $1 - x = 0$

$y = 1$, $x = 1$

$(1,1) \in D$

[2] on b_1 " $x = 0$ " then $f(x,y)$ becomes $m(y) = f(0,y) = y$

$0 \le y \le 2$

$m'(y) = 2 \ne 0$, $y = 0$ and $y = 2$

$(0,0)$, $(0,2)$

on b_2 " $y = 0$ then $f(x,y)$ becomes $k(x) = f(x,0) = x$

$0 \le x \le 4$

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$k'(x) = 1 \neq 0$, $x = 0$, $x = 4$

$(0,0)$, $(4,0)$

In b_3 $y - 2 = (x - 0)$

slope $= \frac{-1}{2}$, $y = \frac{-1}{2}x + 2$

$f(x,y)$ becomes $f(x, \frac{-1}{2}x + 2)$

$L(x) = x - \frac{1}{2}x + 2 - x(-1/2\, x + 2)$

$L(x) = \frac{1}{2}x + 2 + \frac{1}{2}x^2 - 2x$

$= \frac{1}{2}x^2 - \frac{3}{2}x + 3$

$L'(x) = x - 3/2 = 0 \longrightarrow x = 3/2 \in (0,4)$

$x = 0$, $x = \frac{3}{2}$, $x = 4$

$(0,2)$ $(\frac{2}{2}, \frac{5}{4})$ $(4,0)$

critical point	$f(x,y) = x + y - xy$
$(0,0)$	$0 \longrightarrow$ min
$(4,0)$	$4 \longrightarrow$ max
$(1,1)$	1
$(0,2)$	2
$(\frac{3}{2}, \frac{5}{4})$	$\frac{3}{2} + \frac{5}{4} - \frac{15}{8} =$

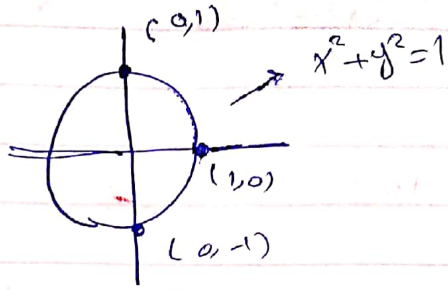
$0 \le f(x,y) \le 4$

اسهل طرقة حساب المدى

ex: $f(x,y) = x^2 y$

$D = \{(x,y) : x^2+y^2 \leq 1\}$

closed



(1) $f_x = 2xy \Rightarrow f_y = x^2$

$2xy = 0$, $x^2 = 0$

$(0,0) \in D$

(2) on the boudary $x^2 = 1-y^2$ $f(x,y)$, becomes

$f(x,y) = g(x) = (1-y^2)y$

$-1 \leq y \leq 1$

$g(y) = y - y^3$, $g'(y) = 1-3y^2 = 0$

$y = \pm \frac{1}{\sqrt{3}} \in (-1,1)$

$y=-1$	$y = \frac{-1}{\sqrt{3}}$	$y = \frac{1}{\sqrt{3}}$	$y=1$
$x^2 = 1-y^2$	$x^2 = \frac{2}{3}$	$x = \pm\sqrt{\frac{2}{3}}$	$(0,1)$
$x^2 = 1-(-1)^2$	$x = \pm\sqrt{\frac{2}{3}}$	$\left(\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right)$	
$x = 0$			
$(0,-1)$	$\left(\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}\right)$	$\left(-\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right)$	
	$\left(-\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}\right)$		

critical	x^2y
$(0,0)$	0
$(0,1)$	0
$(0,-1)$	0
$\left(\sqrt{\frac{2}{3}}, \frac{-1}{\sqrt{3}}\right)$	$\frac{-2}{3\sqrt{3}}$

critical	
$\left(-\sqrt{\frac{2}{3}}, \frac{-1}{\sqrt{3}}\right)$	$\frac{-2}{3\sqrt{3}}$
$\left(\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right)$	$\frac{2}{3\sqrt{3}}$
$\left(-\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right)$	$\frac{2}{3\sqrt{3}}$

إلى
هون
مادة
السكند