CALCULUS III

Multivariate Calculus

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Preface

This booklet contains my notes for CALCULUS III courses at THE HASHEMITE UNIVERSITY. Students are expected to use this booklet during each lecture, filling in the details in the blanks provided during the lecture. This booklet assumes that the student has a good working knowledge of CALCULUS I topics including limits, derivatives and integration. It also assumes that the student has a good knowledge of several CALCULUS II topics including some integration techniques, parametric equations, and polar coordinates. Here is a listing (and brief description) of the material that is in this booklet.

Vectors and the Geometry of Space - In this chapter we will start looking at three-dimensional space and vectors. We will cover the standard three-dimensional coordinate system. We will also discuss how to find the equations of lines and planes in three-dimensional space. We will look at some standard three-dimensional surfaces and their equations.

Vector Functions - In this chapter we will introduce the calculus of vector-valued functions and some of their applications (tangent, normal and binormal vectors, arc length, and curvature).

Partial Derivatives - In this chapter we will take a look at limits of functions of several variables and then move into derivatives of functions of several variables. We will also discuss interpretations of partial derivatives, higher-order partial derivatives and the chain rule as applied to functions of several variables. We will also define and discuss directional derivatives. We will find the equation of tangent planes to surfaces. Finally, we will find relative extrema and absolute extrema of a functions of several variables.

Multiple Integrals - In this chapter we will be looking at double integrals and triple integrals. Included will be double integrals in polar coordinates and triple integrals in cylindrical and spherical coordinates and more generally change in variables in double and triple integrals.

This booklet is based on

1. *Calculus* by James Stewart, 8th Edition, Cengage Learning, 2015.
2. *Thomas' Calculus* by George Thomas Jr. et al., 13th Edition, Pearson, 2013.

The listed chapter numbers correspond to the chapters of the aforementioned books.

No project such as this can be free from errors and incompleteness. I will be grateful to everyone who points out any typos, incorrect statements, or sends any other suggestion on how to improve this booklet.

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12. Vectors and the Geometry of Space

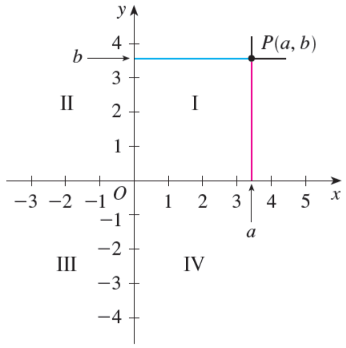
12.1 Three-Dimensional Coordinate Systems

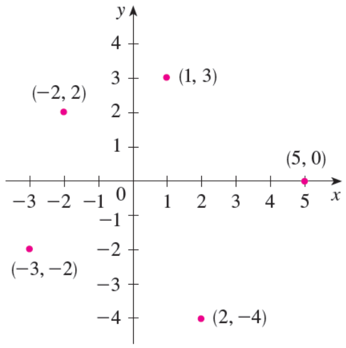
Recall that the **plane** (two-dimensional (2D) rectangular coordinate system) have:

1. Two perpendicular coordinate axes (x-coordinate whose standard equation is $y = 0$ and y-coordinate whose standard equation is $x = 0$).
2. The coordinate axes meet at the origin $(0,0)$. The origin is defined by the letter O.
3. The coordinate axes divide the plane into four parts called quadrants (first quadrant determined by the positive axes).

Locating Points in Plane

Let $P(a,b)$ be any point in the plane. To plot the point P, go a units along the x-axis, then b units in the direction of the y-axis.





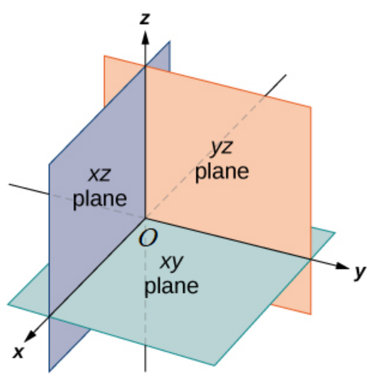
Cartesian Product

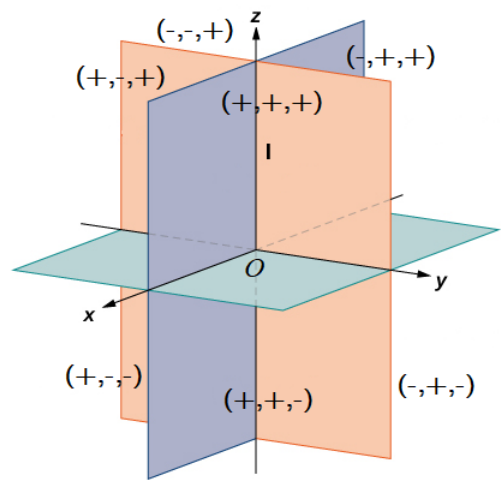
If $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$, then $A \times B = \{(a_1,b_1),(a_1,b_2),(a_2,b_1),(a_2,b_2)\}$.

Remark 12.1 $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x,y \in \mathbb{R}\}$ is the set of all order pairs of real numbers. We denote the plane by $\mathbb{R}^2$.

The situation in **space** (three-dimensional (3D) rectangular coordinate system) is very similar. In the space we have:

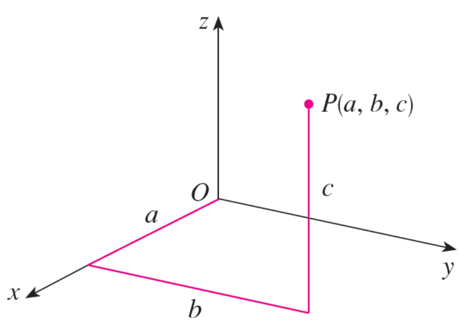
1. Three mutually perpendicular coordinate axes (x-coordinate, y-coordinate, and z-coordinate).
2. The coordinate axes meet at the origin $(0,0,0)$. The origin is defined by the letter O.
3. The planes determined by the coordinates axes are the xy-plane (whose standard equation is $z = 0$), the xz-plane (whose standard equation is $y = 0$), and the yz-plane (whose standard equation is $x = 0$). They meet at the origin O.
4. The three coordinate planes $x = 0$, $y = 0$, and $z = 0$ divide space into eight cells called octanes (first octane determined by the positive axes).

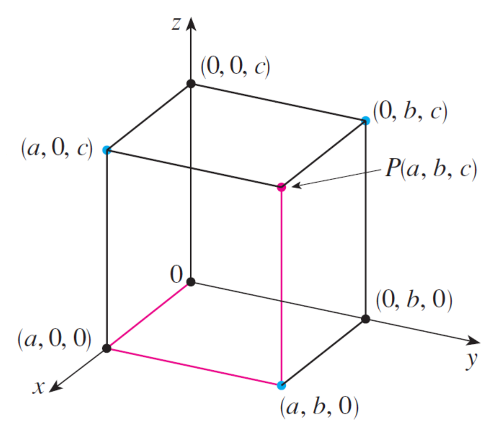




Locating Points in Space

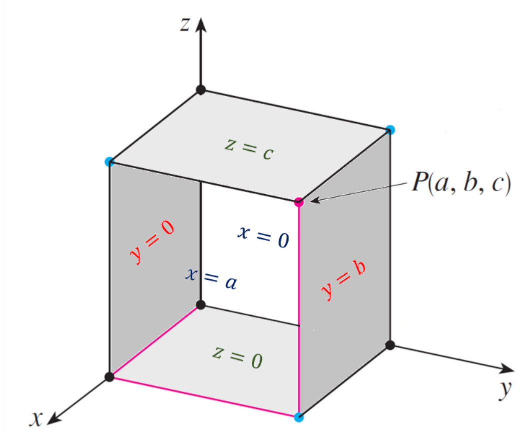
Let $P(a,b,c)$ be any point in space. To plot the point P, go a units along the x-axis, then b units in the direction of the y-axis, then c units in the direction of the z-axis.





Example 12.1 Plot the points $(-4,3,-5)$ and $(3,-2,-6)$.

The point $P(a,b,c)$ gives a rectangular box with 6 faces: $x=0$, $y=0$, $z=0$, $x=a$ (a plane parallel to yz-plane), $y=b$ (a plane parallel to xz-plane), and $z=c$ (a plane parallel to xy-plane).

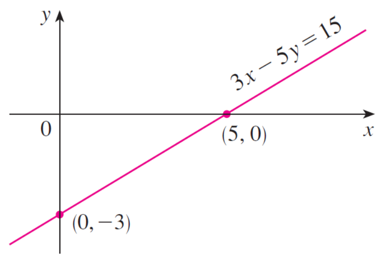


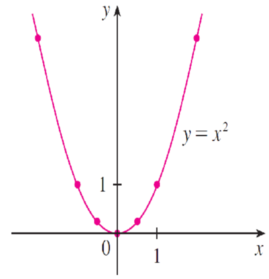
Remark 12.2 $\mathbb{R}^3 := \{(x,y,z) | x,y,z \in \mathbb{R}\}$ is the set of all order triples of real numbers. We denote the space by $\mathbb{R}^3$.

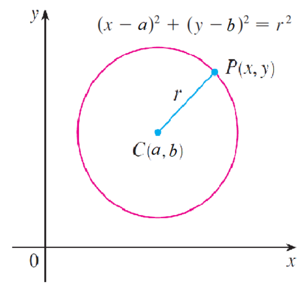
Example 12.2 Find the distance from the point $(2,4,7)$ to the x-axis.

Equations and Inequalities in Space

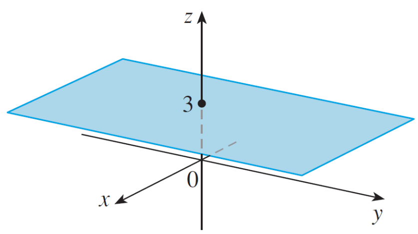
- In the plane: Equations evolving x and y define a curve (collection of points) in the plane.



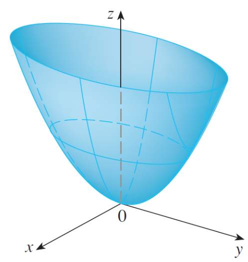




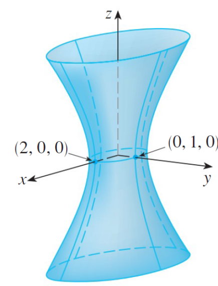
- In space: Equations evolving x, y, and z define a surface (collection of points) in space.



$z=3$



$z=4x^2+y^2$



$\frac{x^2}{4}+y^2-\frac{z^2}{4}=1$

Example 12.3 Describe the sets of points in space whose coordinates satisfy the given equations and inequalities.

1. $y=3$.

2. $x=y$.

3. $y^2+z^2=1, x\leq 1$.

Remark 12.3 Unlike the plane, which consists of a single plane, the xy-plane, the space contains infinitely many planes, just as the plane consists of infinitely many lines.

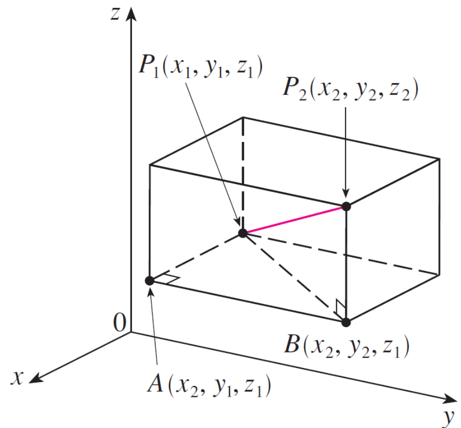
Distance Formula in Space

The distance $|P_1P_2|$ between the points P_1 and P_2 is:

$$|P_1P_2|^2 = \underbrace{|P_1B|^2 + |BP_2|^2 = |P_1A|^2 + |AB|^2 + |BP_2|^2}_{\text{Two Applications of the Pythagorean Theorem}}$$
$$= |x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2$$

So

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$



The Distance Between $P_1\,(x_1, y_1, z_1)$ and $P_2\,(x_2, y_2, z_2)$

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Example 12.4 Find the distance from the point $P(2, -1, 7)$ to the point $Q(1, -3, 5)$.

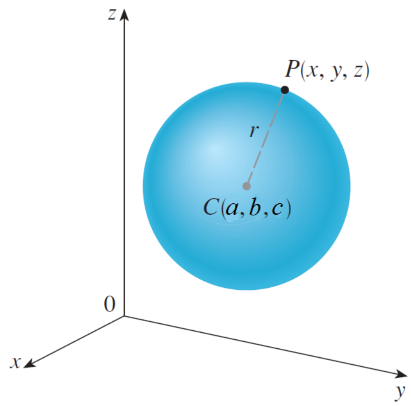
The Mid Point of the Line Segment from $P_1\,(x_1, y_1, z_1)$ to $P_2\,(x_2, y_2, z_2)$

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

Example 12.5 Show that the point $P(3, 1, 2)$ is equidistant from the points $A(2, -1, 3)$ and $B(4, 3, 1)$.

Spheres in Space

The sphere is the set of all points $P(x,y,z)$ whose distance from the center $C(a,b,c)$ is r.



We can use the distance formula to write equations for spheres in space. A point $P(x,y,z)$ lies on the sphere of radius r centered at $C(a,b,c)$ precisely when

$$|CP| = r \quad \text{or} \quad \sqrt{(x-a)^2+(y-b)^2+(z-c)^2} = r.$$

The Standard Equation for the Sphere of Radius r and Center $C(a,b,c)$:

$$(x-a)^2+(y-b)^2+(z-c)^2 = r^2.$$

Example 12.6 Show that $x^2+y^2+z^2+4x-6y+2z+6=0$ is the equation of a sphere, and find its center and radius.

Example 12.7 Find the point on the sphere $x^2+(y-3)^2+(z+5)^2=4$ nearest

1. the xy-plane,
2. the point $(0,7,-5)$.

Example 12.8 What region in $\mathbb{R}^3$ is represented by the inequalities $1 \le x^2+y^2+z^2 \le 4$ and $z \le 0$.

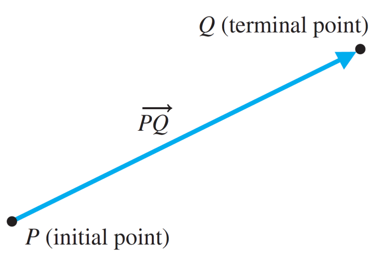
Exercise 12.1

1. Find an equation for the set of all points equidistant from the planes $y = 3$ and $y = -1$.
2. Find the point equidistant from the points $(0,0,0)$, $(0,4,0)$, $(3,0,0)$, and $(2,2,-3)$.
3. Find the volume of the solid that lies inside both of the spheres $x^2+y^2+z^2+4x-2y+4z+5=0$ and $x^2+y^2+z^2=4$.
   Hint: The volume between intersect sphere is $V = \frac{\pi}{12d}(R+r-d)^2\left(d^2+2dr-3r^2+2dR+6rR-3R^2\right)$ where R and r are the radii of the two spheres ($R \geq r$) and d is the distance between the centers of two spheres.
4. Find the distance between the spheres $x^2+y^2+z^2=4$ and $x^2+y^2+z^2=4x+4y+4z-11$.
   Hint: Find the center and radius of each sphere, and then use a simple geometrical argument to show that the distance between the spheres is the distance between the centers minus the radii of both spheres.
5. Find an equation of the sphere passes through the point $(2,3,-1)$ and has center $(-3,-1,-1)$.
6. Show that the equation $x^2+y^2+z^2-2z+6y+4x+14=0$ represents a point.
7. Show that the equation $x^2+y^2+z^2-2z+6y+4x+18=0$ has no graph.
8. Describe the sets of points in space whose coordinates satisfy the given equations and inequalities.
   (a) $z = e^x$.
   (b) $-1 \leq y \leq 1$.
   (c) $y \geq x^2$, $z \geq 0$.

12.2 Vectors

Vectors are a quantity (*displacement, velocity, force, ...*) that has both magnitude (**length**) and direction.

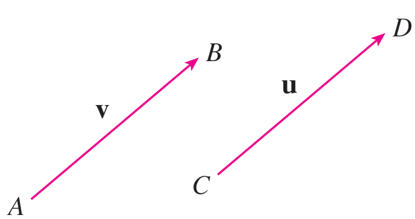
Definition 12.1 The vector represented by the directed line segment $\overrightarrow{PQ}$ has initial point P and terminal point Q and the arrow points in the direction of the terminal point Q.



Remark 12.4 The length of $\overrightarrow{PQ}$ is denoted by symbols $|\overrightarrow{PQ}|$ or $\left\|\overrightarrow{PQ}\right\|$.

Equivalent Vectors

$\vec{u} = \vec{v}$ if the vectors $\vec{u}$ and $\vec{v}$ have the same length and direction, even though their initial points are different.



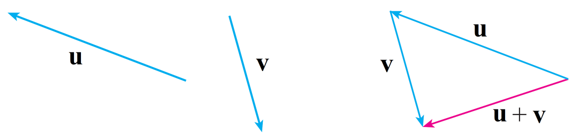
Remark 12.5 We denote a vector by putting an arrow above the letter $\overrightarrow{u}$ or by printing a letter in boldface $\mathbf{u}$.

Zero vector

$\vec{0}$ has length zero with no specific direction.

Definition of Vector Addition

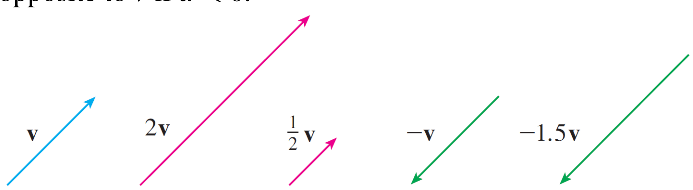
If $\vec{u}$ and $\vec{v}$ are vectors positioned so the initial point of $\vec{v}$ is at the terminal point of $\vec{u}$, then the sum $\vec{u}+\vec{v}$ is the vector from the initial point of $\vec{u}$ to the terminal point of $\vec{v}$.



Scalar Multiplication

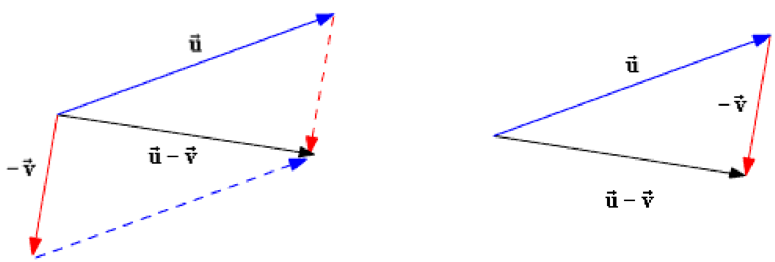
$a\vec{v}$ is the vector whose length is $\underbrace{|a|}_{\text{Absolute Value}}\overbrace{|\vec{v}|}^{\text{Length}}$ and whose direction is

(i) the same as $\vec{v}$ if $a > 0$,
(ii) opposite to $\vec{v}$ if $a < 0$.



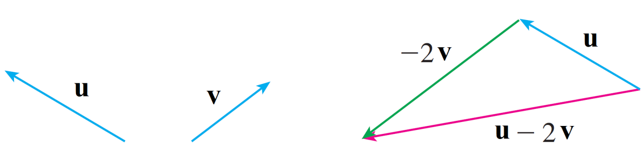
Difference of Two Vectors

The difference $\vec{u}-\vec{v}$ of two vectors is defined by $\vec{u}-\vec{v} = \vec{u}+(-\vec{v})$.



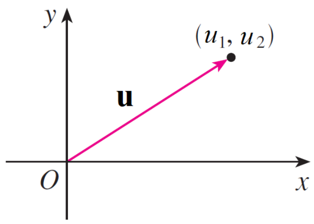
Example 12.9 If $\vec{u}$ and $\vec{v}$ are the vectors shown below. Draw $\vec{u}-2\vec{v}$.

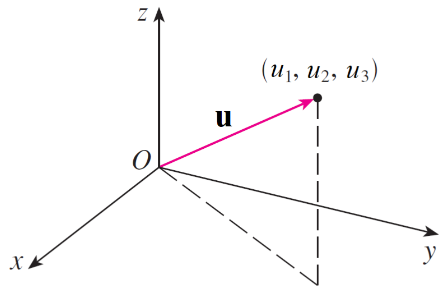
Solution



Components of a Vector

We typically draw vectors with their initial point located at the origin O. Because the initial point is O, we can describe the vector by looking at the coordinates of its terminal point.

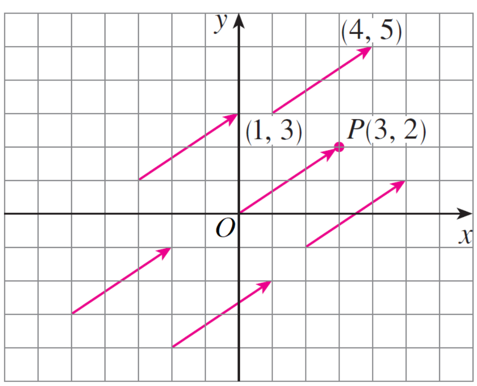




Remark 12.6 If $\vec{u}$ is a vector in the plane with initial point at the origin and terminal point (u_1, u_2). We use the notation $\langle u_1, u_2 \rangle$ for the order pair that refers to the vector $\vec{u}$ as not to confuse it with the order pair (u_1, u_2) that refers to a point in the plane. We call u_1 and u_2 the components of the vector $\vec{u} = \langle u_1, u_2 \rangle$ (i.e., u_1 is the x-component and u_2 is the y-component).

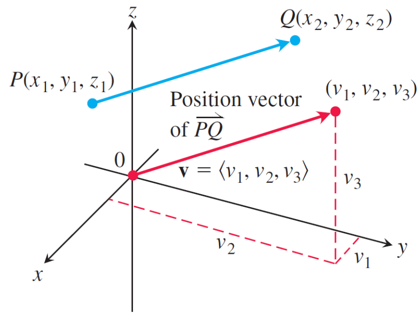
Remark 12.7 If $\vec{u}$ is a vector in space with initial point at the origin and terminal point (u_1, u_2, u_3). We use the notation $\langle u_1, u_2, u_3 \rangle$ for the triple that refers to the vector $\vec{u}$ as not to confuse it with the triple (u_1, u_2, u_3) that refers to a point in the space. We call u_1, u_2, and u_3 the components of the vector $\vec{u} = \langle u_1, u_2, u_3 \rangle$ (i.e., u_1 is the x-component, u_2 is the y-component, and u_3 is the z-component).

Position Vector



The six arrow in the plane (shown above) have the same length and direction. They therefore represent the same vector as $\overrightarrow{OP} = \langle 3, 2 \rangle$. However, because a vector can be placed anywhere in a plane (or space), it may be easier to perform calculations with a vector when its initial point located at the origin O. Such a vector is called a **position vector**.

Example 12.10 Given the point $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, show that the position vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ equal to $\overrightarrow{PQ}$ is $\vec{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$.

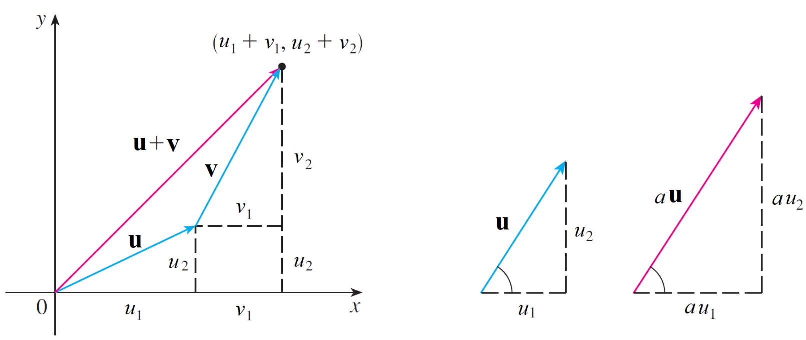


Standard Position Vector $\vec{v}$

Given the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ as points in the plane, the standard position vector $\vec{v}$ with representation $\overrightarrow{PQ}$ is $\vec{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$. Similarly, given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ as points in space, the standard position vector $\vec{v}$ with representation $\overrightarrow{PQ}$ is $\vec{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$.

Example 12.11 Find the standard position vector represented by the directed line segment with initial point $A(2, -3, 4)$ and terminal point $B(-2, 1, 1)$.

Vector Algebra Operations



- If $\vec{u} = \langle u_1, u_2 \rangle$, $\vec{v} = \langle v_1, v_2 \rangle$ and $a \in \mathbb{R}$, then
  1. $\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$.
  2. $a\vec{u} = \langle au_1, au_2 \rangle$.
  3. $\vec{u} - \vec{v} = \langle u_1 - v_1, u_2 - v_2 \rangle$.
- Similarly, if $\vec{u} = \langle u_1, u_2, u_3 \rangle$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $a \in \mathbb{R}$, then
  1. $\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$.
  2. $a\vec{u} = \langle au_1, au_2, au_3 \rangle$.
  3. $\vec{u} - \vec{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$.

Length of a Vector

Rule 12.1

1. If $\vec{u} = \langle u_1, u_2 \rangle$, then $|\vec{u}| = \sqrt{u_1^2 + u_2^2}$. Similarly, if $\vec{u} = \langle u_1, u_2, u_3 \rangle$, then $|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$.
2. $|a\vec{u}| = \underbrace{|a|}_{\text{Absolute Value}} \overbrace{|\vec{u}|}^{\text{Length}}$.

Remark 12.8 The only vector with length 0 is the zero vector.

Example 12.12 If $\vec{u} = \langle 4, 0, 3 \rangle$ and $\vec{v} = \langle -2, 1, 5 \rangle$. Find $|\vec{u}|$, $5\vec{u} + 2\vec{v}$, $|-2\vec{v}|$, and $|2\vec{u} - 3\vec{v}|$.

Remark 12.9 We denote by V_2 the set of all 2D vectors and by V_3 the set of all 3D vectors. In general, V_n is the set of all n-dimensional (nD) vectors. An nD vector is an ordered n-tuple $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$. For example, if $\vec{u} \in V_6$, then $\vec{u}$ is 6D vector and $\vec{u} = \langle u_1, u_2, u_3, u_4, u_5, u_6 \rangle$.

Properties of Vectors

If $\vec{u}$, $\vec{v}$, $\vec{w} \in V_n$ and a, $b \in \mathbb{R}$, then

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$,
2. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$,
3. $\vec{u} + \vec{0} = \vec{u}$,
4. $\vec{u} + (-\vec{u}) = \vec{0}$,
5. $0\vec{u} = \vec{0}$,
6. $1\vec{u} = \vec{u}$.
7. $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$,
8. $(a + b)\vec{u} = a\vec{u} + b\vec{u}$,
9. $a(b\vec{u}) = (ab)\vec{u}$,

Parallel Vectors

Two nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel if they are scalar multiples of one another. Therefore, $\vec{u} \parallel \vec{v}$ if and only if (iff) there exits a number a so that, $\vec{u} = a\vec{v}$.

1. $\vec{u}$ and $\vec{v}$ have the same directions if a is positive.
2. $\vec{u}$ and $\vec{v}$ have opposite directions if a is negative.

Rule 12.2 — Parallel Vectors Formula.

1. $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$ are parallel if $\dfrac{u_1}{v_1} = \dfrac{u_2}{v_2}$.
2. $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ are parallel if $\dfrac{u_1}{v_1} = \dfrac{u_2}{v_2} = \dfrac{u_3}{v_3}$.

Example 12.13 Determine if the sets of vectors are parallel or not.

1. $\vec{u} = \langle 2, -4, 1 \rangle, \quad \vec{v} = \langle -6, 12, -3 \rangle$.
2. $\vec{u} = \langle 4, 10 \rangle, \quad \vec{v} = \langle 2, -9 \rangle$.

Standard Unit Vectors

The standard unit vectors in V_2 are

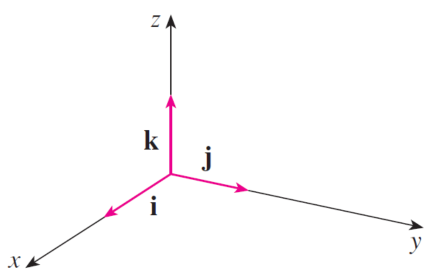
1. $\hat{i} = \langle 1, 0 \rangle \implies |\hat{i}| = 1$.
2. $\hat{j} = \langle 0, 1 \rangle \implies |\hat{j}| = 1$.

Remark 12.10 If $\vec{u} = \langle u_1, u_2 \rangle$, then $\vec{u}$ can be written as a *linear combination* of the standard unit vectors: $\langle u_1, u_2 \rangle = \langle u_1, 0 \rangle + \langle 0, u_2 \rangle = u_1 \langle 1, 0 \rangle + u_2 \langle 0, 1 \rangle$. Therefore,

$$\langle u_1, u_2 \rangle = u_1 \hat{i} + u_2 \hat{j}.$$

The standard unit vectors in V_3 are

1. $\hat{i} = \langle 1, 0, 0 \rangle \implies |\hat{i}| = 1$.
2. $\hat{j} = \langle 0, 1, 0 \rangle \implies |\hat{j}| = 1$.
3. $\hat{k} = \langle 0, 0, 1 \rangle \implies |\hat{k}| = 1$.



Remark 12.11 If $\vec{u} = \langle u_1, u_2, u_3 \rangle$, then $\vec{u}$ can be written as a *linear combination* of the standard unit vectors: $\langle u_1, u_2, u_3 \rangle = \langle u_1, 0, 0 \rangle + \langle 0, u_2, 0 \rangle + \langle 0, 0, u_3 \rangle = u_1 \langle 1, 0, 0 \rangle + u_2 \langle 0, 1, 0 \rangle + u_3 \langle 0, 0, 1 \rangle$. Therefore,

$$\langle u_1, u_2, u_3 \rangle = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k}.$$

Example 12.14 If $\vec{u} = \hat{i} + 2\hat{j} - 3\hat{k}$, $\vec{v} = 4\hat{i} + 7\hat{k}$. Find $2\vec{u} + 3\vec{v}$.

Definition 12.2 — Unit Vector. A unit vector is a vector whose length is 1.

Rule 12.3 If $\vec{u} \neq \vec{0}$, then $\hat{u} = \dfrac{\vec{u}}{|\vec{u}|}$ is a unit vector in the direction of $\vec{u}$.

Example 12.15 Find the unit vector in the direction of the vector $2\hat{\imath} - \hat{\jmath} - 2\hat{k}$.

Example 12.16 Find a vector in the opposite direction of $\langle 6, 2, -3\rangle$ but has length 4.

Example 12.17 Find the unit vector that makes an angle $\theta = \dfrac{2\pi}{3}$ with the positive x-axis.

Exercise 12.2

1. If for nonzero vector $\vec{v}$ the length of the vector $\dfrac{\vec{v}}{\|\vec{v}\|^4}$ is $\dfrac{1}{27}$, then find $\|\vec{v}\|$.
2. Find the vector $\vec{u}$ in the plane of length 2 that makes an angle $\theta = \dfrac{\pi}{4}$ with positive x-axis.
3. Find a vector of magnitude 7 in the direction of $\vec{v} = 12\hat{\imath} - 5\hat{k}$.
4. Find the unit vectors that are parallel to tangent line to the parabola $y = x^2$ at the point $(2, 4)$.
5. If $\vec{r} = \langle x, y, z\rangle$ and $\vec{r}_0 = \langle x_0, y_0, z_0\rangle$. Describe the set of all points (x, y, z) such that $|\vec{r} - \vec{r}_0| = 1$.

12.3 The Dot Products

Definition 12.3

1. The dot product $\vec{u} \cdot \vec{v}$ of vectors $\vec{u} = \langle u_1, u_2\rangle$ and $\vec{v} = \langle v_1, v_2\rangle$ is
$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2.$$
2. The dot product $\vec{u} \cdot \vec{v}$ of vectors $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$ is
$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

The dot product is sometimes called the **scalar product** (or inner product).

Example 12.18 Find

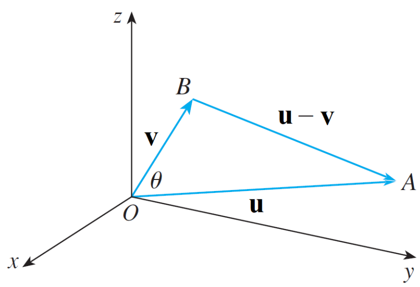
1. $\langle 2, 4\rangle \cdot \langle 3, -1\rangle$.
2. $\left(\hat{\imath} + 2\hat{\jmath} - 3\hat{k}\right) \cdot \left(2\hat{\jmath} - \hat{k}\right)$.

Properties of the Dot Product

If $\vec{u}$, $\vec{v}$, $\vec{w} \in V_n$ and $c \in \mathbb{R}$, then

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.
3. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$.
4. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$.
5. $\vec{0} \cdot \vec{u} = 0$.

Angle Between Vectors



Apply the law of cosines to triangle OAB, we find that

$$
\begin{aligned}
|\vec{u} - \vec{v}|^2 &= |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta, \quad \text{where } \ 0 \le \theta \le \pi. \\
(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) &= |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta \\
|\vec{u}|^2 - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + |\vec{v}|^2 &= |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta \\
|\vec{u}|^2 - 2\vec{u} \cdot \vec{v} + |\vec{v}|^2 &= |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta \\
-2\vec{u} \cdot \vec{v} &= -2|\vec{u}||\vec{v}|\cos\theta \\
\therefore \quad \vec{u} \cdot \vec{v} &= |\vec{u}||\vec{v}|\cos\theta.
\end{aligned}
$$

Theorem 12.1 The angle θ between two nonzero vectors $\vec{u}$ and $\vec{v}$ is given by

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\ |\vec{v}|}\right), \quad 0 \le \theta \le \pi.$$

Example 12.19 If $|\vec{u}| = 4$, $|\vec{v}| = 6$ and the angel between $\vec{u}$ and $\vec{v}$ is $\dfrac{\pi}{3}$, find $\vec{u} \cdot \vec{v}$.

Example 12.20 Find the angle between the vectors $\vec{u} = \langle 2, 2, -1 \rangle$ and $\vec{v} = \langle 5, -3, 2 \rangle$.

Example 12.21 Find the acute angle between the lines $2x - y = 3$ and $3x + y = 7$.

Example 12.22 Find the values of x such that the angle between the vectors $\langle 2, 1, -1 \rangle$ and $\langle 1, x, 0 \rangle$ is $\dfrac{\pi}{4}$.

Orthogonal (Perpendicular) Vectors

Two nonzero vectors $\vec{u}$ and $\vec{v}$ are called **orthogonal (perpendicular)** if the angle between them is $\theta = \pi/2$. Then $\vec{u} \cdot \vec{v} = 0$ because $\cos(\pi/2) = 0$. The converse is also true. If $\vec{u} \cdot \vec{v} = 0$ then $\cos(\theta) = 0$, so $\theta = \cos^{-1}(0) = \pi/2$.

Definition 12.4 Vectors $\vec{u}$ and $\vec{v}$ are orthogonal iff $\vec{u} \cdot \vec{v} = 0$.

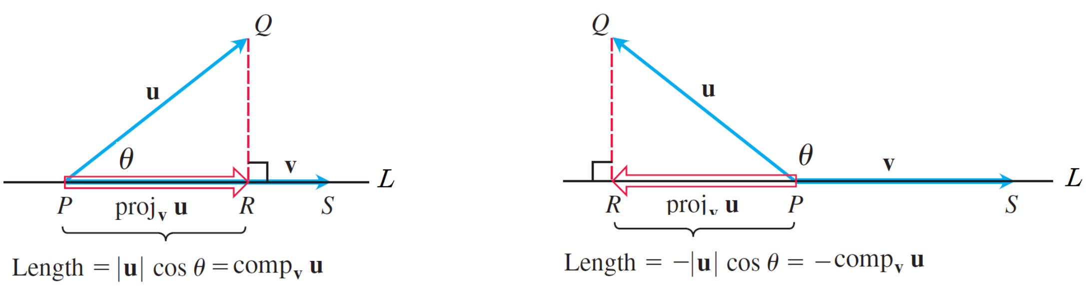
Remark 12.12 The zero vector $\vec{0}$ is considered to be orthogonal to all vectors.

Example 12.23 Show that $2\hat{\imath} + 2\hat{\jmath} - \hat{k}$ is perpendicular to $5\hat{\imath} - 4\hat{\jmath} + 2\hat{k}$.

Example 12.24 Show that the vector $\vec{u} = a\hat{i} + b\hat{j}$ is orthogonal to the line $ax + by = c$.

Projections

We now discuss the problem of projecting one vector onto another. Look at the figures given below:



If a vector $\vec{u} = \overrightarrow{PQ}$ makes an angle θ with a given line L, in the counterclockwise direction, then the vector projection of $\vec{u}$ onto a nonzero vector $\vec{v} = \overrightarrow{PS}$ parallel to L is the vector $\overrightarrow{PR}$ determined by dropping a perpendicular from Q to the line L. The notation for this vector is

$$\text{proj}_{\vec{v}}\vec{u} \qquad \text{(the vector projection of } \vec{u} \text{ onto } \vec{v}\text{)}.$$

If the angle θ between $\vec{u}$ and $\vec{v}$ is acute, $\cos\theta > 0$ and $\text{proj}_{\vec{v}}\vec{u}$ has length $|\vec{u}|\cos\theta$ and direction $\hat{v}$ (where $\hat{v} = \vec{v}/|\vec{v}|$). If θ is obtuse, $\cos\theta < 0$ and $\text{proj}_{\vec{v}}\vec{u}$ has length $-|\vec{u}|\cos\theta$ and direction $-\hat{v}$. In both cases,

$$\text{proj}_{\vec{v}}\vec{u} = (|\vec{u}|\cos\theta)\,\hat{v} = \left(|\vec{u}|\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)\hat{v} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\right)\hat{v} = (\vec{u}\cdot\hat{v})\,\hat{v}.$$

The number $|\vec{u}|\cos\theta \equiv \vec{u}\cdot\hat{v}$ is called the scalar projection of $\vec{u}$ onto $\vec{v}$ (also called the component of $\vec{u}$ along $\vec{v}$). This is denoted by

$$\text{comp}_{\vec{v}}\vec{u} = \vec{u}\cdot\hat{v} \qquad \text{(the scalar projection of } \vec{u} \text{ onto } \vec{v}\text{)}.$$

We summarize these ideas as follows.

Rule 12.4 — Vector Projection. The vector projection of $\vec{u}$ onto $\vec{v}$ is the vector $\text{proj}_{\vec{v}}\vec{u} = (\vec{u}\cdot\hat{v})\,\hat{v}$.

Rule 12.5 — Scalar Projection. The scalar projection of $\vec{u}$ onto $\vec{v}$ is the scalar $\text{comp}_{\vec{v}}\vec{u} = \vec{u}\cdot\hat{v}$.

Example 12.25 Find the scalar projection and vector projection of $\langle 1,1,2\rangle$ onto $\langle -2,3,1\rangle$.

Exercise 12.3

1. If $\vec{u} = \langle 3,0,-1\rangle$, find a vector $\vec{v}$ such that $\text{comp}_{\vec{u}}\vec{v} = 2$.
2. Show that $(\vec{u} - \text{proj}_{\vec{v}}\vec{u}) \cdot \text{proj}_{\vec{v}}\vec{u} = 0$.
3. Use a scalar projection to show that the distance d from a point $P(x_1, y_1)$ to the line $ax+by+c=0$ is $d = \dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$. Use this formula to find the distance from the point $(-2,3)$ to the line $3x-4y+5=0$.
4. Find a unit vector that is orthogonal to both $\hat{i}+\hat{j}$ and $\hat{i}+\hat{k}$.
5. Show that $\vec{v} = a\hat{i}+b\hat{j}$ is parallel to the line $bx-ay=c$.
6. If $\vec{u} = \langle 8,-2,x\rangle$ and $\vec{v} = \langle x,4,-2x\rangle$, then
   (a) Find the value of x such that $\vec{u}$ and $\vec{v}$ are parallel.
   (b) Find the value of x such that $\vec{u}$ and $\vec{v}$ are orthogonal.
7. Show that the vector $(\vec{v} - \text{proj}_{\vec{u}}\vec{v})$ is orthogonal to $\vec{u}$.

12.4 The Cross Products

Recall that, the dot product is a multiplication of two vectors that results in a scalar. In this section, we introduce a product of two nonzero vectors $\vec{u}$ and $\vec{v}$ that generates a third vector orthogonal to both $\vec{u}$ and $\vec{v}$. In th next example we consider how we might find such a vector.

Example 12.26 If $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ two nonzero vectors in space, find a nonzero vector $\vec{n} = \langle n_1, n_2, n_3 \rangle$ that is perpendicular (orthogonal) to both $\vec{u}$ and $\vec{v}$?

Solution If $\vec{n} \perp \vec{u}$ and $\vec{n} \perp \vec{v}$, then $\vec{u} \cdot \vec{n} = 0$ and $\vec{v} \cdot \vec{n} = 0$ and so

$$u_1 n_1 + u_2 n_2 + u_3 n_3 = 0 \quad - - - - - - - (1)$$
$$v_1 n_1 + v_2 n_2 + v_3 n_3 = 0 \quad - - - - - - - (2)$$

Multiply (1) by v_3 and (2) by u_3 and subtract, we get

$$(u_1 v_3 - u_3 v_1) n_1 + (u_2 v_3 - u_3 v_2) n_2 = 0.$$

This equation has an obvious solution $n_1 = (u_2 v_3 - u_3 v_2)$ and $n_2 = -(u_1 v_3 - u_3 v_1)$ (up-to $\pm$ sign). Substituting these values into (1) and (2), we then get $n_3 = (u_1 v_2 - u_2 v_1)$. This means that a vector perpendicular to both $\vec{u}$ and $\vec{v}$ is

$$\vec{n} = \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle .$$

The component terms in the last equation are hard to remember, but they are the same as the terms in the expansion of the symbolic determinant

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} .$$

◀

In Example 12.26, the resulting vector $\vec{n}$ is called the **cross product** of $\vec{u}$ and $\vec{v}$ and is denoted by $\vec{u} \times \vec{v}$.

Definition 12.5 — The cross product. If $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, then the cross product of $\vec{u}$ and $\vec{v}$ is the vector

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} .$$

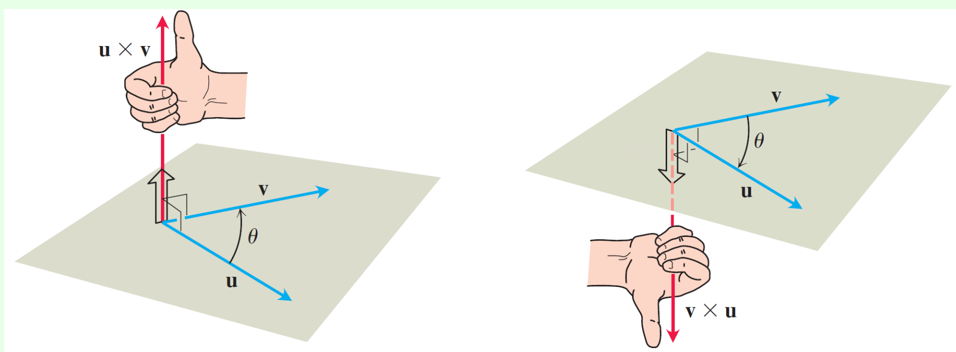
The cross product $\vec{u} \times \vec{v}$ of two vectors $\vec{u}$ and $\vec{v}$, unlike the dot product, is a vector. For this reason it is also called the **vector product**.

Remark 12.13 $\vec{u} \times \vec{v}$ is defined only when $\vec{u}$ and $\vec{v}$ are three dimensional vectors.

Example 12.27 If $\vec{u} = \langle 1,3,4 \rangle$ and $\vec{v} = \langle 2,7,-5 \rangle$. Find $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$.

Rule 12.6 $\vec{u} \times \vec{v} \neq \vec{v} \times \vec{u}$.

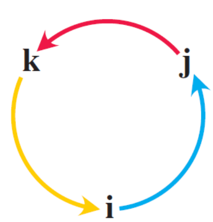
Remark 12.14 If $\vec{u}$ and $\vec{v}$ are not parallel, as we will see in Section 12.5; they determine a plane. We select the vector $\vec{u} \times \vec{v}$ perpendicular to the plane by the right-hand rule. This means that we choose $\vec{u} \times \vec{v}$ to be the (normal) vector that points the way your right thumb points when your fingers curl through the angle θ from $\vec{u}$ to $\vec{v}$.



Example 12.28 Find a vector perpendicular to the plane that passes through the points $P(1,4,6)$, $Q(-2,5,-1)$ and $R(1,-1,1)$.

The Standard Unit Vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$:

$\hat{\imath} \times \hat{\jmath} = \hat{k}$	$\hat{\jmath} \times \hat{k} = \hat{\imath}$	$\hat{k} \times \hat{\imath} = \hat{\jmath}$
$\hat{\jmath} \times \hat{\imath} = -\hat{k}$	$\hat{\imath} \times \hat{k} = -\hat{\jmath}$	$\hat{k} \times \hat{\jmath} = -\hat{\imath}$
$\hat{\imath} \times \hat{\imath} = \vec{0}$	$\hat{\jmath} \times \hat{\jmath} = \vec{0}$	$\hat{k} \times \hat{k} = \vec{0}$



Geometric Interpretation

Recall that, $\vec{u}\times\vec{v}=\langle u_2v_3-u_3v_2,-[u_1v_3-u_3v_1],u_1v_2-u_2v_1\rangle$. Therefore,

$$\begin{aligned}
|\vec{u}\times\vec{v}|^2 &= (\vec{u}\times\vec{v})\cdot(\vec{u}\times\vec{v})\\
&= (u_2v_3-u_3v_2)^2+(u_1v_3-u_3v_1)^2+(u_1v_2-u_2v_1)^2\\
&= u_2^2v_3^2-2u_2u_3v_2v_3+u_3^2v_2^2+u_1^2v_3^2-2u_1u_3v_1v_3+u_3^2v_1^2+u_1^2v_2^2-2u_1u_2v_1v_2+u_2^2v_1^2\\
&= u_1^2v_2^2+u_1^2v_3^2+u_2^2v_1^2+u_2^2v_3^2+u_3^2v_1^2+u_3^2v_2^2-(2u_1u_2v_1v_2+2u_1u_3v_1v_3+2u_2u_3v_2v_3)\\
&= u_1^2v_1^2+u_1^2v_2^2+u_1^2v_3^2+u_2^2v_1^2+u_2^2v_2^2+u_2^2v_3^2+u_3^2v_1^2+u_3^2v_2^2+u_3^2v_3^2\\
&\quad -(u_1^2v_1^2+u_2^2v_2^2+u_3^2v_3^2+2u_1u_2v_1v_2+2u_1u_3v_1v_3+2u_2u_3v_2v_3)\\
&= \left(u_1^2+u_2^2+u_3^2\right)\left(v_1^2+v_2^2+v_3^2\right)-(u_1v_1+u_2v_2+u_3v_3)^2\\
&= |\vec{u}|^2|\vec{v}|^2-(\vec{u}\cdot\vec{v})^2 \quad \text{(Lagrange Identity, see Rule 12.11.)}\\
&= |\vec{u}|^2|\vec{v}|^2-|\vec{u}|^2|\vec{v}|^2\cos^2\theta=|\vec{u}|^2|\vec{v}|^2(1-\cos^2\theta)=|\vec{u}|^2|\vec{v}|^2\sin^2\theta.
\end{aligned}$$

$$\therefore |\vec{u}\times\vec{v}|=|\vec{u}||\vec{v}|\sin\theta,\quad (\sqrt{\sin^2\theta}=\sin\theta \quad \text{since} \quad 0\le\theta\le\pi).$$

Theorem 12.2 If θ is the angle between $\vec{u}$ and $\vec{v}$, then $|\vec{u}\times\vec{v}|=|\vec{u}||\vec{v}|\sin\theta,\quad 0\le\theta\le\pi$.

Two nonzero vectors $\vec{u}$ and $\vec{v}$ are called **parallel** if the angle between them is $\theta=0$ or π. Then $\vec{u}\times\vec{v}=\vec{0}$ because $\sin(0)=\sin(\pi)=0$. The converse is also true. If $\vec{u}\times\vec{v}=\vec{0}$ then $\sin(\theta)=0$, so $\theta=\sin^{-1}(0)=0$ or π.

Corollary 12.1 Two nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel iff $\vec{u}\times\vec{v}=\vec{0}$.

Remark 12.15

1. From the formula $|\vec{u}\times\vec{v}|=|\vec{u}||\vec{v}|\sin\theta$, we can deduce that $\theta=\sin^{-1}\left(\dfrac{|\vec{u}\times\vec{v}|}{|\vec{u}||\vec{v}|}\right)$. This formula do not uniquely determine θ because, $\sin\theta\ge 0$ when $0\le\theta\le\pi$, each value of $\sin\theta$ occurs twice. Therefore, this formula causes some ambiguity and is not a popular formula to use to find angle between vectors.
2. From the two well-known formulas $\vec{u}\cdot\vec{v}=|\vec{u}||\vec{v}|\cos\theta$ and $|\vec{u}\times\vec{v}|=|\vec{u}||\vec{v}|\sin\theta$, we can deduce that
$$\theta=\tan^{-1}\left(\frac{|\vec{u}\times\vec{v}|}{\vec{u}\cdot\vec{v}}\right).$$
Note that, we have to be a bit careful about turning that into a function involving the inverse tangent, since we are looking for an angle in the interval $[0,\pi]$ but the standard inverse tangent has the range $[-\pi/2,\pi/2]$.

Example 12.29 If $\vec{u}\cdot\vec{v}=\sqrt{3}$ and $\vec{u}\times\vec{v}=\langle 1,2,2\rangle$, find the angle between $\vec{u}$ and $\vec{v}$.

Properties of the Cross Product

If $\vec{u}$, $\vec{v}$, $\vec{w} \in V_3$ and $c \in \mathbb{R}$, then

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$,
2. $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$,
3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$,
4. $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$,
5. $\vec{0} \times \vec{u} = \vec{u} \times \vec{0} = \vec{0}$,
6. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$.

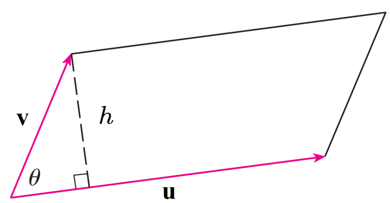
Example 12.30 Since $\hat{\imath} \times (\hat{\imath} \times \hat{\jmath}) = -\hat{\jmath}$ and $(\hat{\imath} \times \hat{\imath}) \times \hat{\jmath} = \vec{0}$, it follows that $\hat{\imath} \times (\hat{\imath} \times \hat{\jmath}) \neq (\hat{\imath} \times \hat{\imath}) \times \hat{\jmath}$.

Rule 12.7 $\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$.

Area of Parallelogram

A **parallelogram** is a special kind of quadrilateral. If a quadrilateral has two pairs of parallel opposite sides, then its called a parallelogram. Rectangle, square, and rhombus are all examples of a parallelogram. The area of a parallelogram can be calculated by multiplying its *base* with the *height* (altitude).

Suppose two vectors $\vec{u}$ and $\vec{v}$ form two sides of a parallelogram (i.e., $\vec{v}$ and $\vec{u}$ are not parallel).



The area of the parallelogram determined by $\vec{u}$ and $\vec{v}$ is

$$\begin{aligned} \text{Area} &= (\text{Base})(\text{Height}) \\ &= |\vec{u}|h \qquad \left(\text{Note: } \sin\theta = \frac{h}{|\vec{v}|} \Rightarrow h = |\vec{v}|\sin\theta.\right) \\ &= |\vec{u}||\vec{v}|\sin\theta = |\vec{u} \times \vec{v}| \end{aligned}$$

Rule 12.8 The area of the parallelogram determined by the vectors $\vec{u}$ and $\vec{v}$ is $|\vec{u} \times \vec{v}|$.

Example 12.31 Find the area of the parallelogram with vertices $A = (-2, 1)$, $B = (0, 4)$, $C = (4, 2)$, and $D = (2, -1)$.

Example 12.32 Find the area of the triangle with vertices $P(1,4,6)$, $Q(-2,5,-1)$, and $R(1,-1,1)$.

Remark 12.16 Three points A, B, and C are **collinear** iff $\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = 0$ (*i.e.*, $\overrightarrow{AB} \parallel \overrightarrow{AC}$ iff $\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = 0$).

Example 12.33 Determine whether the points $A(-3,-2,1)$, $B(1,4,7)$, and $C(4,10,14)$ are collinear.

Triple Product (Box Product)

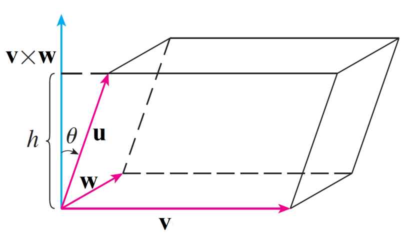
The product $\vec{u}\cdot(\vec{v}\times\vec{w})$ is called the **triple scalar product** of $\vec{u}$, $\vec{v}$ and $\vec{w}$.

Rule 12.9 $\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$

Volume of Parallelepiped

A **parallelepiped** is a 3D shape whose faces are all parallelograms. The volume of the parallelepiped can be calculuated by multiplying its *base area* with the *height*.

Consider the parallelepiped formed by vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ with the same initial point.



The volume of the parallelepiped is given by

$$\begin{aligned} \text{Volume} &= (\text{Base Area})(\text{Height}) \\ &= |\vec{v}\times\vec{w}|\ h \quad \left(\text{Note: } \cos\theta = \frac{h}{|\vec{u}|} \Rightarrow h = |\vec{u}|\cos\theta\right) \\ &= |\vec{v}\times\vec{w}|\ |\vec{u}|\ |\cos\theta| = \big||\vec{u}|\ |\vec{v}\times\vec{w}|\cos\theta\big| = |\vec{u}\cdot(\vec{v}\times\vec{w})| \end{aligned}$$

Remark 12.17 The angle θ between $\vec{u}$ and $\vec{v}\times\vec{w}$ could be larger than $\pi/2$, depending on the order of $\vec{v}$ and $\vec{w}$, so we must take the absolute value in the definition of the volume.

Rule 12.10 The volume of the parallelepiped determined by the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ is $|\vec{u}\cdot(\vec{v}\times\vec{w})|$.

Example 12.34 Find the volume of the parallelepiped determined by the vectors $\hat{\imath}+\hat{\jmath}$, $\hat{\jmath}+\hat{k}$, and $\hat{\imath}+\hat{\jmath}+\hat{k}$.

Remark 12.18 If $\vec{u}\cdot(\vec{v}\times\vec{w})=0$, then the vectors must lie in the same plane, that is, they are **coplanar**.

Example 12.35 Use the scalar triple product to show that the vectors $\langle 1,4,-7\rangle$, $\langle 2,-1,4\rangle$, and $\langle 0,-9,18\rangle$ are coplanar.

Lagrange Identity

The Lagrange identity relates norms, dot products, and cross products.

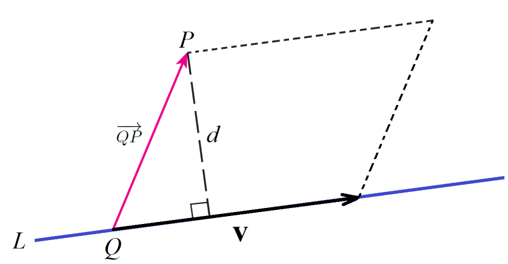
Rule 12.11 If $\vec{u}, \vec{v}\in V_3$, then $|\vec{u}\times\vec{v}|^2 = |\vec{u}|^2\,|\vec{v}|^2-(\vec{u}\cdot\vec{v})^2$.

Example 12.36 If $|\vec{u}-\vec{v}| = \sqrt{5}$, $|\vec{u}| = 2$ and $|\vec{v}| = 3$, then what is $|\vec{u}\times\vec{v}|$ equal to?

Example 12.37 If $|\vec{u}+\vec{v}| = 4$ and $|\vec{u}-\vec{v}| = 3$, find $|\vec{u}|^2+|\vec{v}|^2$ and $\vec{u}\cdot\vec{v}$?

Exercise 12.4

1. Show that $|\vec{u}\times\vec{v}|^2 = |\vec{u}|^2|\vec{v}|^2-(\vec{u}\cdot\vec{v})^2$.
2. Determine whether the points $A(1,2,3)$, $B(3,-1,6)$, $C(5,2,0)$, and $D(3,6,-4)$ lie in the same plane.
3. Find the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$ such that $|\vec{u}+\vec{v}| = \sqrt{8}$, $|\vec{u}| = \sqrt{5}$, and $|\vec{v}| = 1$.
4. Find $|\vec{u}+2\vec{v}|$ if $|\vec{u}| = 2$, $|\vec{v}| = 3$, the angel between $\vec{u}$ and $\vec{v}$ is $\theta = \pi/3$.
5. If $\vec{u}\cdot(\vec{v}\times\vec{w}) = 0$, what can be conclude about the configuration of $\vec{u}$, $\vec{v}$, and $\vec{w}$.
6. Find the volume of the parallelepiped if four of its eight vertices are $A(0,0,0)$, $B(1,2,0)$, $C(0,-3,2)$, and $D(3,-4,5)$.
7. A line L has a direction parallel to $\vec{v} = \hat{i}+3\hat{j}+\sqrt{6}\hat{k}$ and passes through the point Q. Find the distance between the point $P(1,1,-1)$ and the line L if $|\overrightarrow{QP}\times\vec{v}| = 12$.
   Hint: The area of the parallelogram determined by $\vec{v}$ and $\overrightarrow{QP}$ is
   Area $= |\overrightarrow{QP}\times\vec{v}| = (\text{the length of the base})(\text{hight}) = |\vec{v}|d \Longrightarrow d = \dfrac{|\overrightarrow{QP}\times\vec{v}|}{|\vec{v}|}$

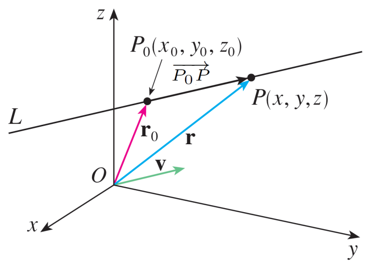


8. Find the value of ρ that makes the vectors $\vec{u} = 2\hat{i}+\rho\hat{j}+\hat{k}$, $\vec{v} = 2\hat{i}-\hat{j}+3\hat{k}$, and $\vec{w} = \hat{i}-2\hat{j}+\hat{k}$ coplanar.
9. If $\vec{u}\neq 0$ and if $\vec{u}\times\vec{v} = \vec{u}\times\vec{w}$ and $\vec{u}\cdot\vec{v} = \vec{u}\cdot\vec{w}$, then does $\vec{v} = \vec{w}$? Give reasons for your answer.

12.5 Equations of Lines and Planes

Lines in Space

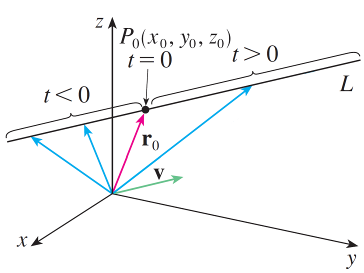
A line L in space is determined by knowing a point on the line and a vector giving the direction of the line L.



Suppose that line L passes through a point $P_0\,(x_0, y_0, z_0)$ is parallel to the nonzero vector $\vec{v} = \langle a, b, c\rangle$. Then L is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to $\vec{v}$. Let $\vec{r}_0 = \overrightarrow{OP_0}$ is the position vector of the point P_0 and $\vec{r} = \overrightarrow{OP}$ is the position vector of the point P. Then the law for vector addition gives $\vec{r} = \vec{r}_0 + \overrightarrow{P_0P}$. Since $\overrightarrow{P_0P}$ and $\vec{v}$ are parallel, then there exist a scalar $t \in \mathbb{R}$ such that $\overrightarrow{P_0P} = t\vec{v}$. Thus

$$\vec{r} = \vec{r}_0 + t\vec{v}.$$

Remark 12.19 The value of t depends on the location of the point P along the line L, and the domain of t is $\mathbb{R}$. If $t > 0$ we move away from $P_0\,(x_0, y_0, z_0)$ in the direction of $\vec{v}$ and if $t < 0$ we move away from $P_0\,(x_0, y_0, z_0)$ in the opposite direction of $\vec{v}$. As t varies over $\mathbb{R}$ we will completely cover the line L.



Vector Equation for a Line

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to the vector $\vec{v} = \langle a, b, c\rangle$ is

$$\vec{r} = \vec{r}_0 + t\vec{v}, \quad t \in \mathbb{R},$$

where $\vec{r}$ is the position vector of a point $P(x, y, z)$ on L and r_0 is the position vector of P_0.

Remark 12.20 If a vector $\vec{v} = \langle a, b, c\rangle$ is used to describe the direction of a line L, then the numbers a, b, and c are called **direction numbers of** L.

The expanded form of the equation $\vec{r} = \vec{r}_0 + t\vec{v}$ is

$$\langle x,y,z\rangle = \langle x_0,y_0,z_0\rangle + t\langle a,b,c\rangle \Longrightarrow \langle x,y,z\rangle = \langle x_0+at, y_0+bt, z_0+ct\rangle.$$

Equating the corresponding components of the two sides gives three scalar equations involving the parameter t: $x = x_0 + at$, $y = y_0 + bt$, and $z = z_0 + ct$.

Parametric Equations for a Line

The standard parametrization of the line through $P_0(x_0,y_0,z_0)$ and parallel to the vector $\vec{v} = \langle a,b,c\rangle$ is

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad t \in \mathbb{R}.$$

Another way of describing the line L is to eliminate the parameter t from the parametric equations. If none of a, b, or c is 0, we can solve each of these equations for t:

$$t = \frac{x-x_0}{a}, \quad t = \frac{y-y_0}{b}, \quad t = \frac{z-z_0}{c} \Longrightarrow \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}.$$

Symmetric Equation for a Line

A symmetric equation for the line through $P_0(x_0,y_0,z_0)$ and parallel to the vector $\vec{v} = \langle a,b,c\rangle$ is

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}.$$

Remark 12.21 If one of the direction numbers of L is equal to 0 we can still eliminate t in the parametric equations of line L. For example, if $a = 0$, we could write the symmetric equation of L as:

$$x = x_0, \quad \frac{y-y_0}{b} = \frac{z-z_0}{c},$$

i.e., the line L with symmetric equation $\dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$ lies in the vertical plane $x = x_0$.

Example 12.38 Find a vector equation, parametric equations, and a symmetric equation for the line that passes through the point $(5,1,3)$ and is parallel to the vector $\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$.

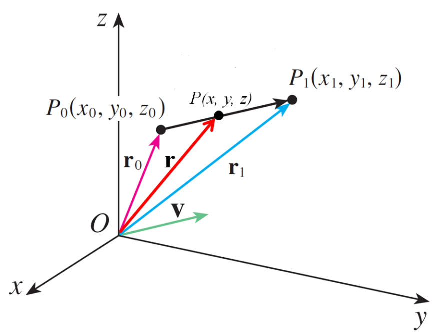
Example 12.39 Find a symmetric equation for the line through $(4,3,-1)$ and parallel to the vector $\langle 2,0,-1\rangle$.

Example 12.40 Find parametric equations and symmetric equation of the line that passes through the points $(2,4,-3)$ and $(3,-1,1)$.

(a) Find two other points on the line.

(b) At what point does the line intersect the xz-plane?

The Line Segment from $P_0(x_0,y_0,z_0)$ to $P_1(x_1,y_1,z_1)$:
How could we describe the line segment joining P_0 and P_1?



To parameterize the line segment P_0P_1, we first parameterize the line through the point $P(x,y,z)$:

$$\vec{r} = \vec{r}_0 + \overrightarrow{P_0P} = \vec{r}_0 + t\vec{v}.$$

Take $\vec{v} = \vec{r}_1 - \vec{r}_0$, we get $\vec{r} = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0)$, and hence

$$\vec{r} = (1-t)\vec{r}_0 + t\vec{r}_1.$$

We then find the t-values for the endpoints P_0 and P_1 and restrict t to lie in the closed interval bounded by these values. If we put $t = 0$ in the equation, we get the endpoint P_0 (the tip of $\vec{r}_0$), and if we put $t = 1$ we get the endpoint P_1 (the tip of $\vec{r}_1$). So the line segment P_0P_1 is given by the vector equation

$$\vec{r} = (1-t)\vec{r}_0 + t\vec{r}_1, \quad 0 \le t \le 1.$$

Parametric Equations for a Line Segment

The standard parametrization of the line segment from $P_0(x_0,y_0,z_0)$ to $P_1(x_1,y_1,z_1)$ is

$$x = x_0(1-t) + x_1t, \quad y = y_0(1-t) + y_1t, \quad z = z_0(1-t) + z_1t, \quad 0 \le t \le 1.$$

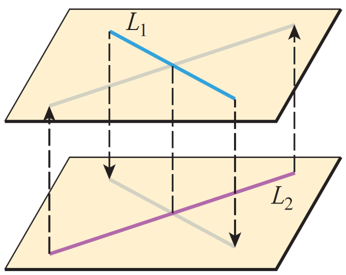
Example 12.41 Find a vector equation for the line segment from $(6,-1,9)$ to $(7,6,0)$.

Example 12.42 Find parametric equations for the line segment from $(-2,18,31)$ to $(11,-4,48)$.

Relative Positions of Lines in Space

Given two lines in space, either they are parallel, they intersect, or they are skew (imagine, for example, the flight paths of two planes in the sky).

Definition 12.6 — Skew lines. Two lines in space that are not parallel and do not intersect are called skew lines. The skew lines lie in parallel planes.



Given two lines in space $\left\{ \begin{array}{l} L_1 : x = x_1 + a_1 t,\ y = y_1 + b_1 t,\ z = z_1 + c_1 t,\ t \in \mathbb{R} \\ L_2 : x = x_2 + a_2 s,\ y = y_2 + b_2 s,\ z = z_2 + c_2 s,\ s \in \mathbb{R} \end{array} \right\}$. We will analyze the necessary and sufficient conditions of the possible cases for the relative positions of L_1 and L_2.

Relative Positions of L_1 and L_2 Determined by Thier Direction Vectors

1. L_1 and L_2 are parallel iff their direction vectors $\vec{v}_1 = \langle a_1, b_1, c_1 \rangle$ and $\vec{v}_2 = \langle a_2, b_2, c_2 \rangle$ are parallel, i.e., $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.
2. If L_1 and L_2 are not parallel. Being P_1 and P_2 two points of the lines L_1 and L_2, respectively. We build the vector $\overrightarrow{P_1P_2}$. Depending on the configration of $\vec{v}_1$, $\vec{v}_2$ and $\overrightarrow{P_1P_2}$, the following cases can be distinguish:
   (i) L_1 and L_2 are intersecting iff $\overrightarrow{P_1P_2} \cdot (\vec{v}_1 \times \vec{v}_2) = 0$ (the vectors are coplanar).
   (ii) L_1 and L_2 are skew iff $\overrightarrow{P_1P_2} \cdot (\vec{v}_1 \times \vec{v}_2) \neq 0$ (the vectors are not coplanar).

Relative Positions of L_1 and L_2 Determined by Thier Parametric Equations

Equating the corresponding components of L_1 and L_2 lead to the following system:

$$\left\{ \begin{array}{l} x_1 + a_1 t = x_2 + a_2 s, \\ y_1 + b_1 t = y_2 + b_2 s, \\ z_1 + c_1 t = z_2 + c_2 s. \end{array} \right. \tag{12.1}$$

1. If L_1 and L_2 are parallel, i.e., $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$, the following cases can be distinguish:
   (i) L_1 and L_2 are distinct iff the system (12.1) has no solution.
   (ii) L_1 and L_2 coincide (identical) iff the system (12.1) has infinitely many solutions.
2. If L_1 and L_2 are not parallel, the following cases can be distinguish:
   (i) L_1 and L_2 are intersecting iff the system (12.1) has a unique solution.
   (ii) L_1 and L_2 are skew iff the system (12.1) has no solution.

Example 12.43 Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

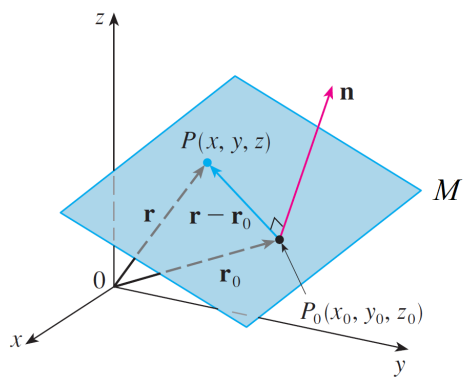
(i) $L_1:\ x = 5-12t,\quad y = 3+9t,\quad z = 1-3t;\qquad L_2:\ x = 3+8s,\quad y = -6s,\quad z = 7+2s.$

(ii) $L_1:\ \dfrac{x-2}{1} = \dfrac{y-3}{-2} = \dfrac{z-1}{-3};\qquad L_2:\ \dfrac{x-3}{1} = \dfrac{y+4}{3} = \dfrac{z-2}{-7}.$

(iii) $L_1:\ x = 1+t,\quad y = -2+3t,\quad z = 4-t;\qquad L_2:\ x = 2s,\quad y = 3+s,\quad z = -3+4s.$

Planes in Space

A plane M in space is determined by knowing a point on the plane and its "tilt" or orientation. This "tilt" is defined by specifying a vector that is perpendicular or normal to the plane M.



Suppose that plane M passes through a point $P_0(x_0, y_0, z_0)$ is normal to the nonzero vector $\vec{n} = \langle a, b, c\rangle$ ($\vec{n}$ is called a **normal vector**). Then M is the set of all points $P(x, y, z)$ for which $\vec{n}$ is orthogonal to $\overrightarrow{P_0P}$. Let $\vec{r}_0 = \overrightarrow{OP_0}$ is the position vector of the point P_0 and $\vec{r} = \overrightarrow{OP}$ is the position vector of the point P. Then the law for vector addition gives

$$\vec{r} = \vec{r}_0 + \overrightarrow{P_0P} \Longrightarrow \overrightarrow{P_0P} = \vec{r} - \vec{r}_0.$$

Since $\vec{n}$ is orthogonal to $\overrightarrow{P_0P}$, we have $\vec{n}$ is orthogonal to $(\vec{r} - \vec{r}_0)$ and hence

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \Longrightarrow \langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0.$$

The expanded for of the above equation is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Equation for a Plane

A vector equation for the plane M through $P_0(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c\rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

By collecting terms in the above equation, we can write the equation for th plane M as

$$ax + by + cz + d = 0, \quad \text{where} \quad d = -\left(ax_0 + by_0 + cz_0\right).$$

Example 12.44 Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $\vec{n} = \langle 1, 3, -3\rangle$. Find the intercepts and sketch the plane.

Example 12.45 Find an equation of the plane that passes through the points $P(1,3,2)$, $Q(3,-1,6)$ and $R(5,2,0)$.

Example 12.46 Find the point at which the line with parametric equations $x = 2+3t$, $y = -4t$, $z = 5+t$ intersects the plane $4x+5y-2z = 18$.

Example 12.47 Find the equation of the plane that contains the line $x = t,\ y = 1,\ z = t,\ t \in \mathbb{R}$ and passes through the point $(1,2,0)$.

Example 12.48 Find the equation of the plane passing through the point $(1,2,1)$ and perpendicular to the line joining the points $(1,4,2)$ and $(2,3,5)$.

Relative Positions of Planes

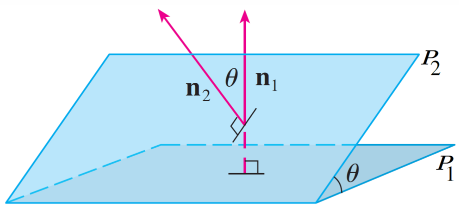
Given two planes in space, either they are parallel (they have no points on common), they intersect in a line, or they coincide.

Rule 12.12 — Parallel Planes. Two planes are parallel iff their normal vectors are parallel.

If two planes P_1 and P_2 are not parallel, then the angle between the two planes is defined as the acute angle between their normal vectors $\vec{n}_1$ and $\vec{n}_2$.

Angles Between Planes

The angle between two intersecting planes is defined to be the acute angle between their normal vectors.



$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}, \quad 0 \le \theta \le \frac{\pi}{2}$$

Example 12.49 Find the angle between the planes $x+y+z=1$ and $x-2y+3z=1$.

Example 12.50 Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them.

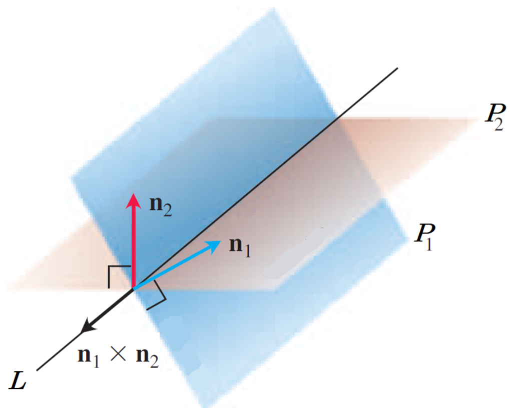
1. $2x+2y-3z-4=0$ and $2x+4y-6z-3=0$.

2. $9x-3y+6z=2$ and $2y=6x+4z-1$.

3. $x+4y-3z=1$ and $-3x+6y+7z-2=0$

Lines of Intersection of Two Planes

Two planes P_1 and P_2 that are not parallel intersect in a line L. Since L lies in both planes, it is perpendicular to both of the normal vectors $\vec{n}_1$ and $\vec{n}_2$. Thus a vector parallel to L is given by the cross product $\vec{n}_1 \times \vec{n}_2$. To find a point on L. For instance, we can find the point where the line intersects the xy-plane by setting $z = 0$ in the equations of both planes. Knowing a point on L and a vector parallel to L, it is easy to write down the parametric equations (or the symmetric equation) of the line L.



Example 12.51 Find the symmetric equation for the line of intersection of the two planes $x+y+z=1$ and $x-2y+3z=1$.

Remark 12.22 Another way to find the line of intersection of two planes is to solve the equations of the planes for two of the variables in terms of the third, which can be taken as the parameter.

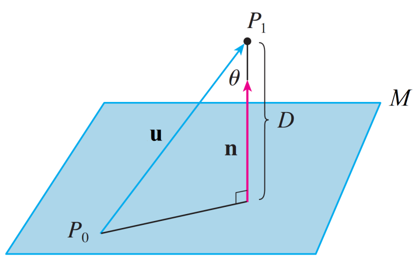
Distance Between Objects in Space

The Distance from a Point to a Plane

Example 12.52 Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane

$$M: \ ax + by + cz + d = 0.$$

Solution



Let $P_0(x_0, y_0, z_0)$ be a point on M. Then $\vec{u} = \overrightarrow{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$ and $\vec{n} = \langle a, b, c \rangle$. From the Figure, it is clear that

$$D = |\text{comp}_{\vec{n}}\vec{u}| = |\vec{u} \cdot \hat{n}| = \frac{|\vec{u} \cdot \vec{n}|}{|\vec{n}|} = \frac{|ax_1 + by_1 + cz_1 \overbrace{-ax_0 - by_0 - cz_0}^{d}|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

◀

Rule 12.13 The distance from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Example 12.53 Find the distance between the point $(1/2, 0, 0)$ and the plane $5x + y - z = 1$.

Example 12.54 Find the radius of the sphere with center $(0, 0, -1)$ and touches the plane $x + 2y + 2z + 11 = 0$.

The Distance Between Parallel Planes

Rule 12.14 The distance between the parallel planes $\left\{ \begin{array}{l} ax + by + cz + d_1 = 0 \\ ax + by + cz + d_2 = 0 \end{array} \right\}$ is

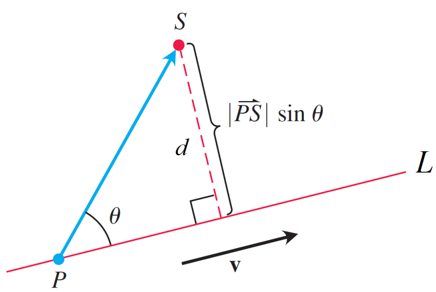
$$D = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

Example 12.55 Find the distance between two planes $2x - 3y + z = 4$ and $4x - 6y + 2z = 3$.

The Distance from a Point to a Line in Space

Example 12.56 Let S be a point not on the line L that passes through a point P parallel to a vector $\vec{v}$. Show that the distance d from the point S to the line L is

$$d = \frac{\left|\overrightarrow{PS} \times \vec{v}\right|}{|\vec{v}|}.$$



Example 12.57 Find the distance from the point $S(1,1,5)$ to the line $L:\ x-1 = 3-y = \frac{z}{2}$.

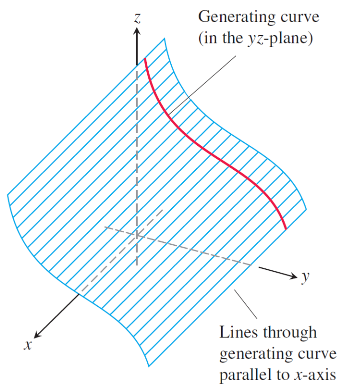
Exercise 12.5

1. Find the parametric equations for the line through $(1,1,1)$ and parallel to the z-axis.
2. Find the symmetric equation for the line through $(2,3,0)$ and perpendicular to the vectors $\vec{u}=\langle 1,2,3\rangle$ and $\vec{v}=\langle 3,4,5\rangle$.
3. Find the parametric equations for the line through $(2,4,5)$ and perpendicular to the plane $3x+7y-5z=21$.
4. Find equation for the plane through $(1,-1,3)$ parallel to the plane $3x+y+z=7$.
5. Show that the line $\dfrac{x-3}{5}=\dfrac{y+1}{5}=\dfrac{z+4}{7}$ lies in the plane $3x+4y-5z=25$.
6. Find the equation for the plane through $A(1,-2,1)$ perpendicular to the vector from the origin to A.
7. Find the plane determined by the intersecting lines
   $L_1: x=-1+t,\ y=2+t,\ z=1-t,\ t\in\mathbb{R}$,
   $L_2: x=1-4s,\ y=1+2s,\ z=2-2s,\ s\in\mathbb{R}$.
8. Find a plane through $(2,1,-1)$ and perpendicular to the line of intersection of the planes $2x+y-z=3, x+2y+z=2$.
9. Find a plane through the points $(1,2,3)$ and $(3,2,1)$, and perpendicular to the plane $4x-y+2z=7$.
10. Find equations of the planes that are parallel to the plane $x+2y-2z=1$ and 2 units away from it.
11. Show that the distance between the parallel planes $\left\{\begin{array}{l} ax+by+cz+d_1=0 \\ ax+by+cz+d_2=0 \end{array}\right\}$ is $D=\dfrac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$.
12. Find the distance between the parallel lines
    $L_1:\ x=1+t,\ y=-2+3t,\ z=4-t,\ t\in\mathbb{R}$,
    $L_2:\ x=-2s,\ y=-2-6s,\ z=-3+2s,\ s\in\mathbb{R}$.
13. Find the distance between the skew lines
    $L_1:\ x=1+t,\ y=-2+3t,\ z=4-t,\ t\in\mathbb{R}$,
    $L_2:\ x=2s,\ y=3+s,\ z=-3+4s,\ s\in\mathbb{R}$.
14. Find the distance from the line $x=2+t, y=1+t, z=-(1/2)-(1/2)t,\ t\in\mathbb{R}$ to the plane $x+2y+6z=10$.
15. Find the equation of the sphere centered at the point $(1,0,0)$ and touches the plane $2x+y-2z=8$. Find the coordinates of the point of tangent.

12.6 Cylinders and Quadratic Surfaces

Cylinders

A cylinder is a surface that consists of all lines (called **rulings**) that are parallel to a given line and pass through a given plane curve



In solid geometry, where cylinder means circular cylinder, the generating curves are circles, but now we allow generating curves of any kind. The cylinders in the next example are generated by a parabola, a line, and a circle (*circular cylinder*).

Example 12.58 Identify and sketch the surfaces.

1. $z = x^2$.

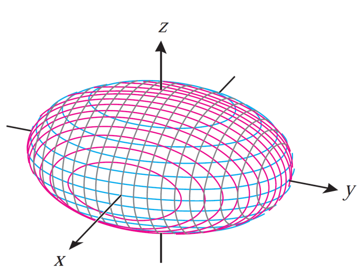
2. $x = 3$.

3. $y^2 + z^2 = 1$.

Remark 12.23 In space, any curve of the form $f(x,y) = a$ or $g(x,z) = b$ or $h(y,z) = c$ defines a cylinder.

Quadric Surface

A quadric surface is the graph in space of a second degree equation in three variables x, y, and z.

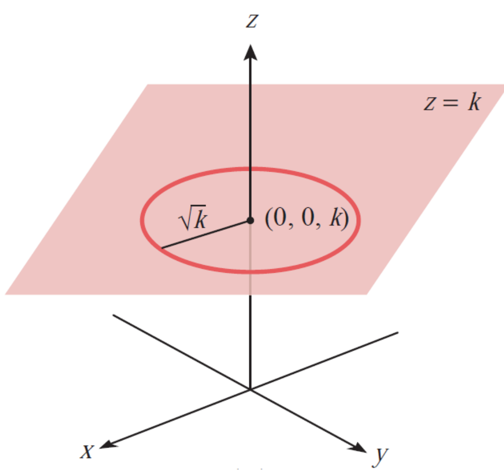


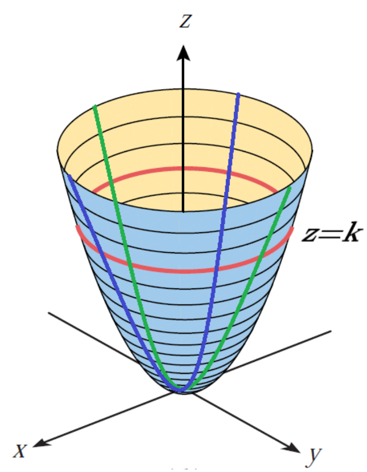
Traces

To sketch the graph of a quadric surface (or any surface), it is useful to determine curves of intersection of the surface with planes parallel to the coordinate planes. These types of curves are called **traces**.

Example 12.59 Use traces to sketch the quadric surface $z = x^2 + y^2$.

Solution The trace in plane $z = k > 0$ are circles of radius $\sqrt{k}$ parallel to the xy-plane and the traces parallel to the xz- and yz-planes are parabolas. Specifically, the trace in the xy-plane is point $x^2 + y^2 = 0$, the trace in the xz-plane is parabola $z = x^2$, and the trace in the yz-plane is parabola $z = y^2$ (see the following Figure).





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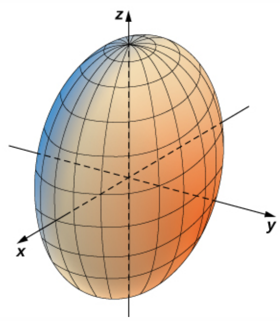
Example 12.60 Use traces to sketch the quadric surfaces $y^2 = x^2 + z^2$.

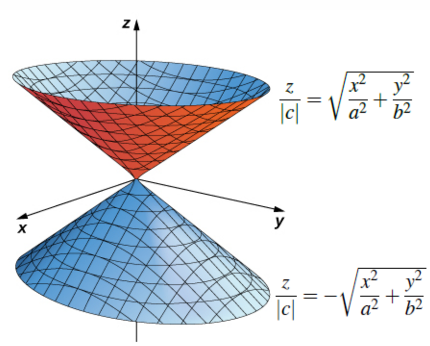
Example 12.61 Find the trace of the surface $S:\ z = x^2 - y^2$ in the plane $z = 2$?

Ellipsoid: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$



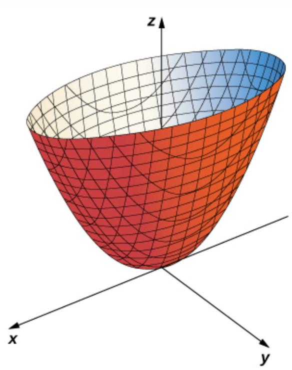
- All traces are ellipses.
- If $a=b=c$, the ellipsoid is a sphere.

Elliptic Cone: $\dfrac{z^2}{c^2}=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$



- Horizontal traces are ellipses.
- Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.

Elliptic Paraboloid: $\dfrac{z}{|c|}=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$



- Horizontal traces are ellipses.
- Vertical traces are parabolas.
- The variable raised to the first power indicates the axis of the paraboloid.

Example 12.62 Classify the surface, and sketch it.

1. $y^2 = x^2 + \frac{1}{9}z^2$.

2. $x^2 + y^2 - 2x - 6y - z + 11 = 0$.

3. $4x^2 + y^2 + z^2 - 24x - 8y + 4z + 55 = 0$.

4. $x^2 + 2z^2 - 6x - y + 10 = 0$.

Example 12.63 Sketch the region bounded by the surfaces $x = \sqrt{y^2 + z^2}$ and $y^2 + z^2 = 1$ for $0 \leq x \leq 2$.

Example 12.64 Sketch the region bounded by the paraboloids $y = x^2 + z^2$ and $y = 2 - x^2 - z^2$.

Exercise 12.6

1. Identify and sketch the surfaces.
   (a) $z = e^y$.
   (b) $x = \sin(z)$.
   (c) $y + z = 2$.
2. Use traces to sketch the quadric surfaces.
   (a) $x^2 + y^2 + z^2 = 1$,
   (b) $x = y^2 + z^2$,
3. Graph the traces of the surface $x^2 - 6x + 2y^2 - z^2 + z = 0$ in the planes $x = 3$, $y = 0$, and $z = k$ where $k = 0,\ 2,\ 4$.
4. Sketch the region bounded by the surfaces $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 = 1$ for $1 \le z \le 2$.
5. An ellipsoid is created by rotating the ellipse $4x^2 + y^2 = 16$ about the x-axis. Find an equation of the ellipsoid (in $\mathbb{R}^3$).
6. Sketch the quadratic surfaces defined by the given equations. On the graph, give the name of the surface, show the intercepts and sketch the trace in the xy-plane.
   (a) $y + 1 = \sqrt{x^2 + z^2}$.
   (b) $x^2 - 4x + y + z^2 = 9$.
   (c) $z + 1 = x^2 + y^2$.
   (d) $x^2 - 4x - y + z^2 = 9$.
   (e) $2x^2 - 16x + 2y^2 + 2z^2 = 9$.

13. Vector Functions

13.1 Vector Functions and Space Curves

Definition 13.1 A vector-valued function, or **vector function**, is simply a function whose domain is a set of real numbers and whose range is a set of vectors. A vector functions of a single variable in V_3 have the form $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ where $f(t)$, $g(t)$, and $h(t)$ are called the **component** functions of the vector $\vec{r}(t)$.

The **domain** of a vector function $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ is the set of all t's for which all the component functions are defined. Equivalently, the domain of $\vec{r}(t)$ is the intersection of the domains of the component functions $f(t)$, $g(t)$, and $h(t)$.

Rule 13.1 If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, then
Domain of $\vec{r}(t) =$ (Domain of $f(t)$) $\cap$ (Domain of $g(t)$) $\cap$ (Domain of $h(t)$).

Example 13.1 Find the domain of $\vec{r}(t) = \left\langle \ln|t-1|, e^t, \sqrt{t}\right\rangle$

Limits

The way we define the limits of a vector functions is similar to the way we define limits of real functions.

Rule 13.2 If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ be a vector function, then

$$\lim_{t\to a}\vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right\rangle$$

provided the limits of the component functions exist.

Example 13.2 Find $\displaystyle\lim_{t\to\infty}\left\langle \frac{1-t^2}{1+t^2}, \tan^{-1} t, \frac{\ln(t)}{t}\right\rangle$.

Continuity

Definition 13.2 The vector function $\vec{r}(t)$ is **continuous at a point** $t = a$ in its domain if

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a).$$

The function is **continuous** if it is continuous at every point on its domain.

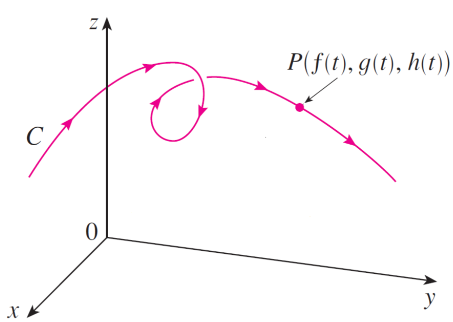
Space Curve

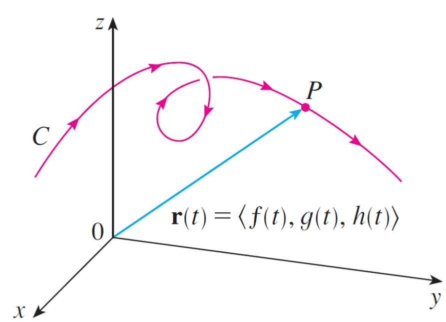
Definition 13.3 Suppose that $f(t)$, $g(t)$, and $h(t)$ are continuous real functions on an interval I. The set C of all points $P(x,y,z)$ in the space, where

$$x = f(t), \quad y = g(t), \quad z = h(t),$$

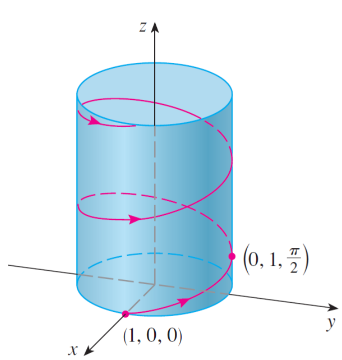
and t varies throughout the interval I, is called a **space curve**. The equations $x = f(t)$, $y = g(t)$ and $z = h(t)$ are called **parametric equations** of C and t is called a **parameter**.

If we now consider the vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, then $\vec{r}(t)$ is the position vector of the point $P(f(t), g(t), h(t))$ on C. Thus any continuous vector function $\vec{r}(t)$ defines a space curve C that is traced out by the tip of the moving vector $\vec{r}(t)$.



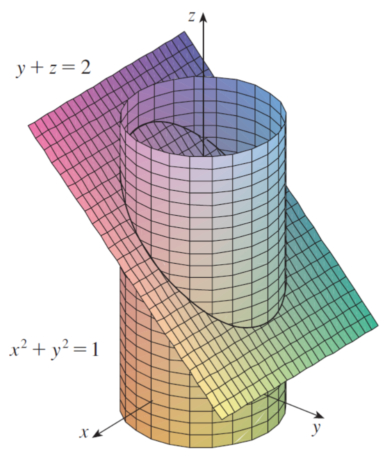


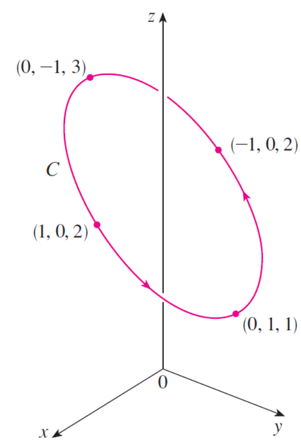
Example 13.3 — Helix. Sketch the space curve whose vector equation is $\vec{r}(t) = \cos t\hat{i} + \sin t\hat{j} + t\hat{k}$ and its surface on which the curve lies. Indicate with an arrow the direction in which t increases.



Example 13.4 Sketch the plane curve whose vector equation is $\vec{r}(t) = (t^2 - 5)\hat{\imath} + t\,\hat{\jmath}$. Indicate with an arrow the direction in which t increases.

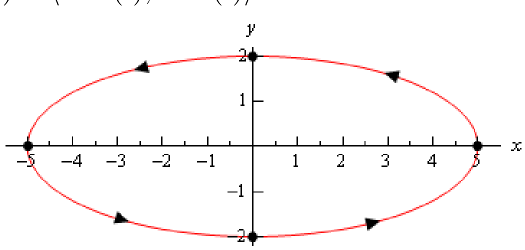
Example 13.5 Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.





Example 13.6 — Choose the correct answer. The graph below, with the indicated direction, represents the vector equation

a) $\vec{r}(t) = \langle 5\cos(t), 2\sin(t)\rangle$.
b) $\vec{r}(t) = \langle 2\cos(t), 5\sin(t)\rangle$.
c) $\vec{r}(t) = \langle 5\sin(t), 2\cos(t)\rangle$.
d) $\vec{r}(t) = \langle 2\sin(t), 5\cos(t)\rangle$.



Exercise 13.1

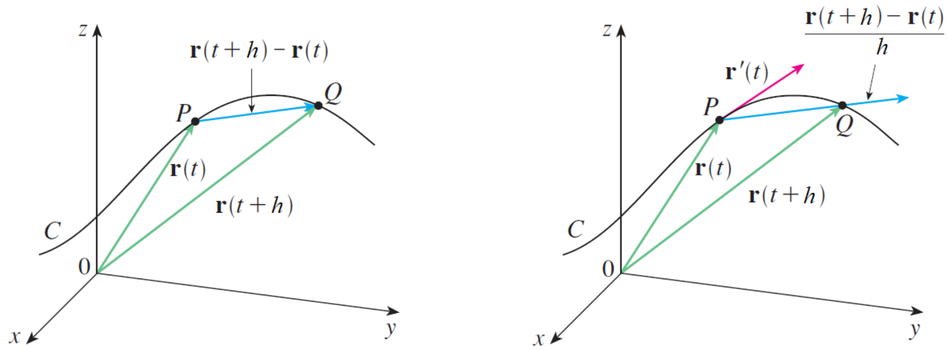
1. Find the domain of the vector function $\vec{r}(t) = \dfrac{t^2-t}{t-1}\hat{\imath} + \sqrt{t+8}\,\hat{\jmath} + \dfrac{\sin(\pi t)}{\ln(t)}\hat{k}$.
2. Find $\lim\limits_{t\to 1}\left(\dfrac{t^2-t}{t-1}\hat{\imath} + \sqrt{t+8}\,\hat{\jmath} + \dfrac{\sin(\pi t)}{\ln(t)}\hat{k}\right)$.
3. Graph $\vec{r}(t) = t\hat{\imath} + (t^2+1)\hat{\jmath}$, $t \in [-1,1]$ (*the resulting curve is a parabola*.) Indicate with an arrow the direction in which t increases.
4. Sketch the space curve $\vec{r}(t) = \left\langle 2,\ \cos^2(t),\ \sin^2(t)\right\rangle$ and its surface on which the curve lies.
5. At what points does the helix $\vec{r}(t) = \langle \sin(t), \cos(t), t\rangle$ intersect the sphere $x^2+y^2+z^2 = 5$.
6. Find a vector function that represents the curve of intersection of the parabolic cylinder $y = x^2$ and the paraboloid $z = 4x^2 + y^2$.
7. Consider the vector function $\vec{r}(t) = \left\langle t, t^2, t\right\rangle$. Find an equation of the plane that contains the graph of $\vec{r}(t)$ (Remark: *the plane is* $x = z$.).
8. Show that the vector function $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ is continuous at $t = a$ iff the component functions f, g, h are continuous functions at $t = a$.
9. Find three different surfaces that contains the curve $\vec{r}(t) = t^2\hat{\imath} + \ln(t)\hat{\jmath} + \dfrac{1}{t}\hat{k}$.
10. Prove that $\lim\limits_{t\to a}[\vec{u}(t)\times\vec{v}(t)] = \lim\limits_{t\to a}\vec{u}(t)\times\lim\limits_{t\to a}\vec{v}(t)$.
11. Match the curve with the given vector function $\vec{r}(t)$ in group A, to its surface in group B **on which the curve lies**.

Group A	Group B
1. $\vec{r}(t) = \langle\cos(t), \sin(t), t\rangle$	A. Cone
2. $\vec{r}(t) = \langle t\cos(t), t\sin(t), t\rangle$	B. Plane
3. $\vec{r}(t) = \left\langle\cos^2(t), \sin^2(t), \cos(2t)\right\rangle$	C. Cylinder
4. $\vec{r}(t) = \left\langle e^t, t^2+e^{2t}, t\right\rangle$, $t \ge 0$	D. Paraboloid

13.2 Derivatives and Integrals of Vector Functions

Derivatives

The derivative $\vec{r}'(t)$ of a vector function $\vec{r}(t)$ is defined in much the same way as for real functions. If the points P and Q have position vectors $\vec{r}(t)$ and $\vec{r}(t+h)$, respectively, then $\overrightarrow{PQ}$ represents the vector $\vec{r}(t+h)-\vec{r}(t)$, which can therefore be regarded as a **secant vector**.



If $h>0$, the scalar multiple $\frac{1}{h}(\vec{r}(t+h)-\vec{r}(t))$ has the same direction as $\vec{r}(t+h)-\vec{r}(t)$. As $h\to 0$, it appears that this vector approaches a vector that lies on the tangent line. For this reason, the vector $r'(t)$ is called the **tangent vector** to the curve defined by $\vec{r}$ at the point P, provided that $\vec{r}'(t)$ exists and $\vec{r}'(t)\neq 0$.

Remark 13.1 If $\vec{r}(t)=\langle f(t),g(t),k(t)\rangle$, where f , g, and h are differentiable functions, then

$$\begin{aligned}
\lim_{h\to 0}\frac{[\vec{r}(t+h)-\vec{r}(t)]}{h} &= \lim_{h\to 0}\frac{[\langle f(t+h),g(t+h),k(t+h)\rangle-\langle f(t),g(t),k(t)\rangle]}{h}\\
&= \lim_{h\to 0}\left\langle \frac{f(t+h)-f(t)}{h},\frac{g(t+h)-g(t)}{h},\frac{k(t+h)-k(t)}{h}\right\rangle\\
&= \left\langle \lim_{h\to 0}\frac{f(t+h)-f(t)}{h},\lim_{h\to 0}\frac{g(t+h)-g(t)}{h},\lim_{h\to 0}\frac{k(t+h)-k(t)}{h}\right\rangle\\
&= \left\langle f'(t),g'(t),k'(t)\right\rangle.
\end{aligned}$$

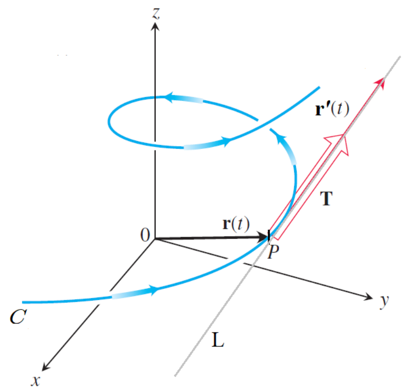
Definition 13.4 The vector function $\vec{r}(t)=\langle f(t),g(t),k(t)\rangle$ has a derivative (is differentiable) at t if f, g, and k have derivatives at t. The derivative is the vector function

$$\vec{r}'(t)=\frac{d}{dt}\vec{r}(t)=\lim_{h\to 0}\frac{\vec{r}(t+h)-\vec{r}(t)}{h}=\left\langle f'(t),g'(t),k'(t)\right\rangle.$$

Example 13.7 Sketch the curve described by the vector function $\vec{r}(t)=t^2\hat{i}+t\hat{j}$, $0\le t\le 2$, and draw the vectors $\vec{r}(1)$ and $\vec{r}'(1)$.

Tangent Vector, Unit Tangent Vector, and Tangent Line

Consider a point P with position vector $\vec{r}(t)$ moves along a curve C.



Definition 13.5 — Tangent Vector $\vec{r}'(t)$. The vector $\vec{r}'(t)$ is called *the tangent vector* to the curve defined by $\vec{r}(t)$ at a point P, provided that $\vec{r}'(t)$ exists and $\vec{r}'(t) \neq 0$.

Definition 13.6 — Unit Tangent Vector $\vec{T}(t)$. The vector $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$ is called *the unit tangent vector* to the curve defined by $\vec{r}(t)$ at a point P.

Definition 13.7 — Tangent Line. *The tangent line* L to the curve defined by $\vec{r}(t)$ at a point P is the line through P parallel to the tangent vector $\vec{r}'(t)$.

Example 13.8 Given $\vec{r}(t) = \left\langle 1+t^3,\ te^{-t},\ \sin 2t \right\rangle$. Find:

1. The derivative of $\vec{r}(t)$, the 2^{nd}-derivative of $\vec{r}(t)$.

2. The tangent vector, the unit tangent vector, and the tangent line at the point where $t = 0$.

3. Sketch the position vector $\vec{r}(0)$ and the tangent vector $\vec{r}'(0)$.

Example 13.9 Find a unit vector parallel to the tangent line to the curve $y = \dfrac{x^2}{2}$ at the point $\left(1, \dfrac{1}{2}\right)$.

Differentiation Rules for Vector Functions

Suppose $\vec{u}$ and $\vec{v}$ are differentiable vector functions, c is a scalar, and f is a real function, then

1. $\frac{d}{dt}[\vec{u}(t) \pm \vec{v}(t)] = \vec{u}'(t) \pm \vec{v}'(t)$.
2. $\frac{d}{dt}[c\vec{u}(t)] = c\vec{u}'(t)$.
3. $\frac{d}{dt}[f(t)\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$.
4. $\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$.
5. $\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$.
6. $\frac{d}{dt}[\vec{u}(f(t))] = f'(t)\vec{u}'(f(t))$.

Smooth Curve

Definition 13.8 The curve traced by $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is **smooth** if $\vec{r}'(t)$ is continuous and never $\vec{0}$, that is, if f, g, and h have continuous first derivatives that are not simultaneously 0.

Remark 13.2 A smooth curve has no sharp corners or cusps; when the tangent vector turns, it does so continuously.

Vector Functions of Constant Length

When we track a particle moving on a sphere centered at the origin, the position vector has a constant length equal to the radius of the sphere.

Rule 13.3 If $\vec{r}(t)$ is a differentiable vector function of t of constant length, then

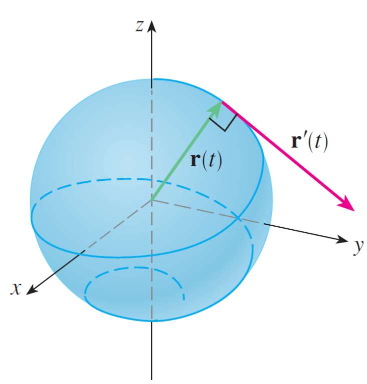
$$\vec{r}'(t) \perp \vec{r}(t) \quad \text{for all} \quad t.$$

Since $|\vec{r}(t)| = c$ is constant, $\vec{r}(t) \cdot \vec{r}(t) = |\vec{r}(t)|^2 = c^2$. Then

$$\frac{d}{dt}[\vec{r}(t) \cdot \vec{r}(t)] = \frac{d}{dt}[c^2] \Rightarrow \vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 0 \Rightarrow 2\vec{r}'(t) \cdot \vec{r}(t) = 0.$$

Therefore, $\vec{r}'(t) \cdot \vec{r}(t) = 0$, which says that $\vec{r}'(t) \perp \vec{r}(t)$.

Geometrically, Rule 13.3 says that if a curve lies on a sphere with center the origin, then the tangent vector $\vec{r}'(t)$ is always perpendicular to the position vector $\vec{r}(t)$.



Integrals

The integral of a continuous vector function $\vec{r}(t)$ can be defined in much the same way as for real functions except that the integral is a vector.

The vector function $\vec{R}(t)$ is called an **anti-derivative** of the vector function $\vec{r}(t)$ whenever $\vec{R}'(t) = \vec{r}(t)$. In component form, if $\vec{R}(t) = \langle F(t),\ G(t),\ H(t)\rangle$ and $\vec{r}(t) = \langle f(t),\ g(t),\ h(t)\rangle$, then

$$\left\langle F'(t),\ G'(t),\ H'(t)\right\rangle = \langle f(t),\ g(t),\ h(t)\rangle$$

Note that the vector function

$$\langle F(t)+C_1,\ G(t)+C_2,\ H(t)+C_3\rangle = \langle F(t),\ G(t),\ H(t)\rangle + \langle C_1,\ C_2,\ C_3\rangle$$

is also an anti-derivative of $\vec{r}(t)$. Therefore, the most general anti-derivative of $\vec{r}(t)$ has the form

$$\vec{R}(t)+\vec{C}$$

where $\vec{C} = \langle C_1,\ C_2,\ C_3\rangle$ is constant vector.

Definition 13.9 — Indefinite integral. If $\vec{R}(t)$ is any anti-derivative of $\vec{r}(t)$, then the indefinite integral of $\vec{r}(t)$ with respect to t is

$$\int \vec{r}(t)\,dt = \vec{R}(t)+\vec{C} \quad \text{for some constant vector } \vec{C} = \langle C_1,C_2,C_3\rangle\,.$$

In component form, the indefinite integral is given by

$$\int \vec{r}(t)\,dt = \left\langle \int f(t)\,dt,\ \int g(t)\,dt,\ \int h(t)\,dt \right\rangle.$$

Example 13.10 Find $\vec{r}(t)$ if $\vec{r}'(t) = \left\langle 2t,\ 3t,\ \sqrt{t}\right\rangle$ and $\vec{r}(1) = \langle 1,1,0\rangle$.

Fundamental Theorem of Calculus

The fundamental theorem of calculus for continuous vector functions says that if a vector function $\vec{r}(t)$ is continuous on $[a,b]$, then

1. $\displaystyle \frac{d}{dt}\left[\int_a^t \vec{r}(x)\,dx\right] = \vec{r}(t)$ at every point t of (a,b).
2. $\displaystyle \int_a^b \vec{r}(t)\,dt = \vec{R}(b)-\vec{R}(a)$ where $\vec{R}'(t) = \vec{r}(t)$ on $[a,b]$.

Example 13.11 Let $\vec{u}(t) = \langle t, e^{-t}, \cos(3t)\rangle$. Find the derivative of $\vec{r}(t) = \displaystyle\int_0^t \vec{u}(x)\,dx,\ 0 \le t \le 1$.

Definition 13.10 — Definite integral. If the components of $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ are integrable over $[a,b]$, and so is $\vec{r}$, then the definite integral of $\vec{r}$ from a to b is

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt,\ \int_a^b h(t)\,dt \right\rangle.$$

Example 13.12 Evaluate $\displaystyle\int_0^\pi \left(\cos t\hat{\imath} + \hat{\jmath} - 2t\hat{k}\right)\,dt$.

Exercise 13.2

1. Evaluate $\displaystyle\int_1^4 \left(\frac{1}{t}\hat{\imath} + \frac{1}{5-t}\hat{\jmath} + \frac{1}{2t}\hat{k}\right)\,dt$.
2. Solve the initial value problem $\dfrac{d\vec{r}}{dt} = -t\hat{\imath} - t\hat{\jmath} - t\hat{k}$, $\vec{r}(0) = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$.
3. Find a vector equation for the tangent line to the curve of intersection of the cylinders $x^2 + y^2 = 25$ and $y^2 + z^2 = 20$ at the point $(3,4,2)$.
4. The curves $\vec{u}(t) = \langle t, t^2, t^3\rangle$ and $\vec{v}(t) = \langle \sin(t), \sin(2t), t\rangle$ intersect at the origin. Find their angle of intersection correct to the nearest degree.
5. Find $f'(2)$, where $f(t) = \vec{u}(t)\cdot\vec{v}(t)$, $\vec{u}(2) = \langle 1,2,-1\rangle$, $\vec{u}'(2) = \langle 3,0,4\rangle$, and $\vec{v}(t) = \langle t, t^2, t^3\rangle$.
6. If $\vec{r}(t) \neq 0$, show that $\dfrac{d}{dt}|\vec{r}(t)| = \dfrac{\vec{r}(t)\cdot\vec{r}'(t)}{|\vec{r}(t)|}$.

   Hint: start with $|\vec{r}(t)| = \sqrt{\vec{r}(t)\cdot\vec{r}(t)}$. This shows that $\dfrac{d}{dt}|\vec{r}(t)| \neq |r'(t)|$, i.e.,

   $$\text{Rate of change for } |\vec{r}(t)| \neq \text{Magnitude of rate of change for } \vec{r}(t).$$

13.3 Arc Length and Curvature

Arc Length

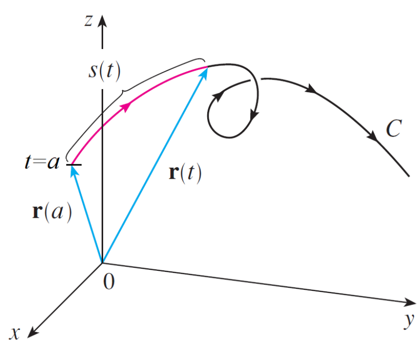
Definition 13.11 The length of a *smooth* curve $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \le t \le b$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

$$L = \int_a^b \left|\vec{r}'(t)\right| \, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} \, dt.$$

Example 13.13 Find the length of the curve with vector equation $\vec{r}(t) = \cos t \hat{i} + \sin t \hat{j} + t\hat{k}$ from the point $(1,0,0)$ to the point $(1,0,2\pi)$.

The Arc Length Function

It is often useful to parameterize a curve with respect to arc length because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system.



Definition 13.12 The arc length function of a *smooth* curve $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, for $t \ge a$, is

$$s(t) = \int_a^t |\vec{r}'(\eta)| \, d\eta.$$

Example 13.14 Determine the arc length function for $\vec{r}(t) = \langle 2t, 3\sin(2t), 3\cos(2t) \rangle$ from the point where $t = 0$ in the direction of increasing t.

Example 13.15 Reparameterize the curve $\vec{r}(t) = \langle 2t, 3\sin(2t), 3\cos(2t) \rangle$ with respect to arc length measured from the point where $t = 0$ in the direction of increasing t.

Solution Take the result of the example above and solve it for t gives $t = \dfrac{s}{2\sqrt{10}}$. Substituting into the original vector function $\vec{r}$ gives the following arc length function parameterization for $\vec{r}$:

$$\vec{r}(s) = \left\langle s/\sqrt{10}, 3\sin\left(s/\sqrt{10}\right), 3\cos\left(s/\sqrt{10}\right) \right\rangle.$$

◀

Why would we want to reparametrize the curve with respect to arc length?

With the reparameterization we can tell where we are on the curve after we have traveled a distance of s along the curve.

Example 13.16 Where on the curve $\vec{r}(t) = \langle 2t, 3\sin(2t), 3\cos(2t) \rangle$ are we after traveling for a distance of $\pi\sqrt{10}/3$?

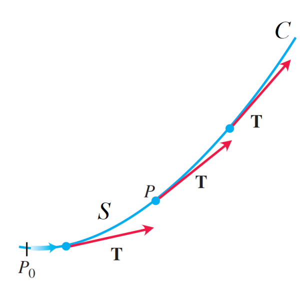
Rule 13.4 Since $s(t) = \displaystyle\int_a^t |\vec{r}'(\eta)|\, d\eta$, it follows that $\dfrac{ds}{dt} = |\vec{r}'(t)|$.

Example 13.17 Show that $\vec{T}(s) = \dfrac{d\vec{r}}{ds}$.

Solution By the Chain Rule, $\dfrac{d\vec{r}}{dt} = \dfrac{d\vec{r}}{ds}\dfrac{ds}{dt}$ and so $\dfrac{d\vec{r}}{ds} = \dfrac{d\vec{r}/dt}{ds/dt} = \dfrac{\vec{r}'(t)}{\underbrace{|\vec{r}'(t)|}_{\text{Rule 13.4}}} = \vec{T}(t) \equiv \vec{T}(s).$ ◀

Curvature

The **curvature** of a smooth curve C at a given point P is a measure of how quickly the curve changes direction at that point. We define it to be *the magnitude of the rate of change of the unit tangent vector $\vec{T}$ with respect to arc length s*. (We use arc length so that the curvature will be independent of the parametrization.)



Remark 13.3 Because $|\vec{T}| = 1$, only changes in direction contribute to the rate of change of T.

Definition 13.13 The curvature of a smooth curve is $\kappa = \left|\dfrac{d\vec{T}}{ds}\right|$, ($\kappa$ reads kappa), where $\vec{T}$ is the unit tangent vector.

Remark 13.4 By the chain rule, $\dfrac{d\vec{T}}{ds} = \dfrac{d\vec{T}}{dt}\dfrac{dt}{ds}$, and so $\dfrac{d\vec{T}}{ds} = \dfrac{\vec{T}'(t)}{ds/dt} = \dfrac{\vec{T}'(t)}{\underbrace{|\vec{r}'(t)|}_{\text{Rule 13.4}}}$.

Rule 13.5 If C is a smooth curve given by $\vec{r}(t)$, then the curvature of C at t is $\kappa = \left|\dfrac{d\vec{T}}{ds}\right| = \dfrac{|\vec{T}'(t)|}{|r'(t)|}$.

Example 13.18 If C is a three-dimensional curve, then show that $\kappa(t) = \dfrac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \dfrac{\left|\vec{r}'(t) \times \vec{r}''(t)\right|}{|\vec{r}'(t)|^3}$.

Solution Start with the formula $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$, then

$$\vec{r}'(t) = |\vec{r}'(t)|\vec{T}(t) = \underbrace{\frac{ds}{dt}}_{\text{Rule 13.4}}\vec{T}(t),$$

and so

$$\vec{r}''(t) = \frac{d^2s}{dt^2}\vec{T}(t) + \frac{ds}{dt}\vec{T}'(t).$$

Using the last two equations, we get

$$\vec{r}'(t) \times \vec{r}''(t) = \frac{ds}{dt}\vec{T}(t) \times \left(\frac{d^2s}{dt^2}\vec{T}(t) + \frac{ds}{dt}\vec{T}'(t)\right) = \frac{ds}{dt}\frac{d^2s}{dt^2}\vec{T}(t) \times \vec{T}(t) + \left(\frac{ds}{dt}\right)^2 \vec{T}(t) \times \vec{T}'(t).$$

Since $\vec{T}(t) \times \vec{T}(t) = 0$, this reduces to

$$\vec{r}'(t) \times \vec{r}''(t) = \left(\frac{ds}{dt}\right)^2 \vec{T}(t) \times \vec{T}'(t).$$

Since $|\vec{T}(t)| = 1$, it follows that $\underbrace{\vec{T}'(t) \perp \vec{T}(t)}_{\text{Rule 13.3}}$. This means that

$$|\vec{r}'(t) \times \vec{r}''(t)| = \left(\frac{ds}{dt}\right)^2 |\vec{T}(t) \times \vec{T}'(t)| = \left(\frac{ds}{dt}\right)^2 |\vec{T}(t)|\, |\vec{T}'(t)| \sin\left(\frac{\pi}{2}\right) = \left(\frac{ds}{dt}\right)^2 |\vec{T}'(t)|.$$

Thus,

$$|\vec{r}'(t) \times \vec{r}''(t)| = |\vec{r}'(t)|^2\, |\vec{T}'(t)| \Longrightarrow |\vec{T}'(t)| = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^2}.$$

Therefore,

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{\dfrac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^2}}{|\vec{r}'(t)|} = \frac{\left|\vec{r}'(t) \times \vec{r}''(t)\right|}{|\vec{r}'(t)|^3}.$$

◀

Rule 13.6 If C is a three dimensional curve given by $\vec{r}(t)$, then the curvature of C at t is given by

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{\left|\vec{r}'(t) \times \vec{r}''(t)\right|}{|\vec{r}'(t)|^3}.$$

Example 13.19 Show that the curvature of a circle of radius a is $\frac{1}{a}$.
Hint: Take $\vec{r}(t) = a\cos(t)\hat{i} + a\sin(t)\hat{j}$.

Example 13.20 Find the curvature of $\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$ at a general region and at $(0,0,0)$.

Example 13.21 Show that if $y = f(x)$ is a twice-differentiable function of x, then

$$\kappa(x) = \frac{|f''(x)|}{\sqrt{[1+(f'(x))^2]^3}}.$$

Solution We start with the assumption that curve C is defined by the function $y = f(x)$. Then, the curve C is given by $\vec{r}(x) = \langle x, f(x), 0\rangle$. Using the previous formula for curvature:

$$\begin{aligned} \vec{r}'(x) &= \left\langle 1, f'(x), 0 \right\rangle \\ \vec{r}''(x) &= \left\langle 0, f''(x), 0 \right\rangle, \\ \vec{r}'(x) \times \vec{r}''(x) &= \left\langle 0, 0, f''(x) \right\rangle. \end{aligned}$$

Therefore, $\kappa(x) = \dfrac{\left|\vec{r}'(x) \times \vec{r}''(x)\right|}{|\vec{r}'(x)|^3} = \dfrac{|f''(x)|}{\sqrt{[1+(f'(x))^2]^3}}$. ◀

Rule 13.7 The curvature of the twice-differentiable plane curve $y = f(x)$ is given by the formula

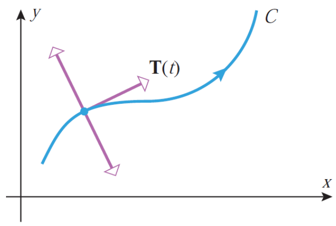
$$\kappa(x) = \frac{|f''(x)|}{\sqrt{[1+(f'(x))^2]^3}}.$$

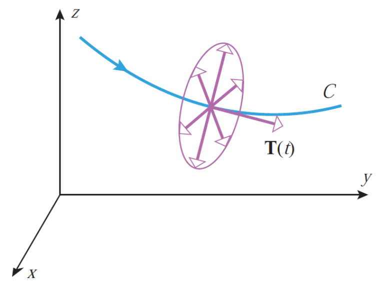
Example 13.22 Find the curvature of the parabola $y = x^2$ at the point $(0,0)$, $(1,1)$ and $(2,4)$

The TNB Frame

Consider a *smooth* curve C has the vector equation $\vec{r}(t)$.

- In the plane there are two unit vectors that are orthogonal to $\vec{T}(t)$.
- In space there are infinitely many unit vectors that are orthogonal to $\vec{T}(t)$.





Remark 13.5 Since $\left|\vec{T}(t)\right| = 1$ (*constant*), the vector $\vec{T}'(t)$ is orthogonal to $\vec{T}(t)$ ($\vec{T}'(t) \perp \vec{T}(t)$).

Typically, $\vec{T}'(t)$ is not a unit vector. For example, let $\vec{r}(t) = \langle \cos t, \sin t, \sqrt{3}t \rangle$, then $\left|\vec{T}'(t)\right| = \frac{1}{2}$.

Unit Normal Vector

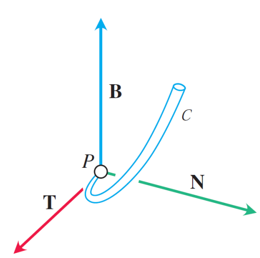
We define the **unit normal vector** as $\vec{N}(t) = \dfrac{\vec{T}'(t)}{\left|\vec{T}'(t)\right|}$ (Note that, $\vec{N}(t) \perp \vec{T}(t)$). We can think of the unit normal vector as indicating the direction in which the curve is turning at each point.

Binormal Vector

We define the **binormal vector** as $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$. The binormal vector is orthogonal to both $\vec{T}(t)$ and $\vec{N}(t)$.

Remark 13.6 $\left|\vec{B}(t)\right| = \left|\vec{T}(t) \times \vec{N}(t)\right| = \left|\vec{T}(t)\right| \left|\vec{N}(t)\right| \underbrace{\sin\left(\frac{\pi}{2}\right)}_{\vec{N}(t) \perp \vec{T}(t)} = 1$. Thus, $\vec{B}(t)$ is a unit vector.

The three mutually orthogonal unit vectors $\vec{T}$, $\vec{N}$, and $\vec{B}$ can be thought of as a movable coordinate system since $\vec{T}(t) = \vec{N}(t) \times \vec{B}(t)$, $\vec{N}(t) = \vec{B}(t) \times \vec{T}(t)$, $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$. This movable coordinate system known as the **TNB frame** *(also called the* **Frenet frame***).*



Example 13.23 Find the unit normal and binormal vectors of the curve defined by

1. $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$.

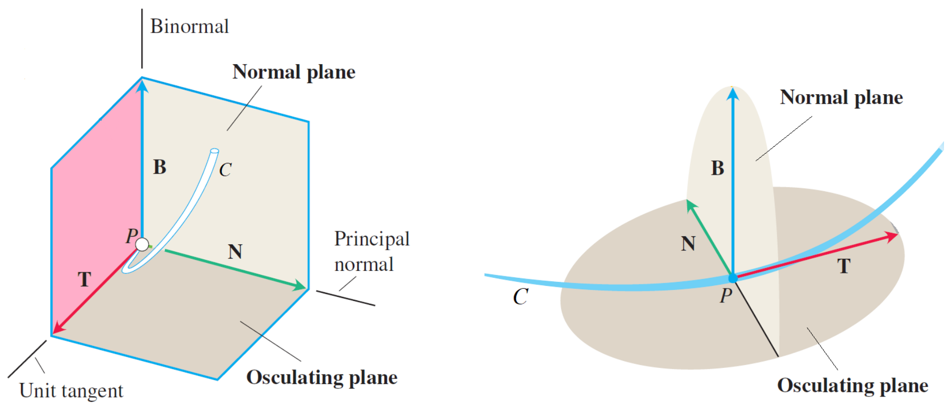
2. $\vec{r}(t) = \frac{t^3}{3}\hat{i} + \frac{t^2}{2}\hat{j}$, $t > 0$.

3. $\vec{r}(t) = t\hat{i} + a\cosh\left(\frac{t}{a}\right)\hat{k}$, $a > 0$.

Normal Plane, Osculating Plane , and Osculating Circle

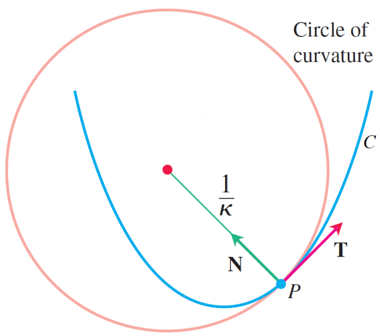
Definition 13.14 — Normal plane. The plane determined by the vectors $\vec{N}$ and $\vec{B}$ at a point P on a curve C is called the **normal plane** of C at P. It consists of all lines that are orthogonal to the unit tangent vector $\vec{T}$.

Definition 13.15 — Osculating plane. The plane determined by the vectors $\vec{T}$ and $\vec{N}$ at a point P on a curve C is called the **osculating plane** of C at P. The name comes from the Latin osculum, meaning "kiss." It is the plane that comes closest to containing the part of the curve near P. (For a plane curve, the osculating plane is simply the plane that contains the curve.)



Example 13.24 Find equations of the normal and osculating planes of the curve defined by $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ at the point $P\left(0, 1, \frac{\pi}{2}\right)$.

Definition 13.16 — Osculating circle for plane curves. The circle that lies in the osculating plane of C at P, has the same tangent as C at P, lies on the concave side of C (toward which $\vec{N}$ points), and has radius $1/\kappa$ is called the **osculating circle** (or the **circle of curvature**) of C at P. It is the circle that best describes how C behaves near P; it shares the same tangent, normal, and curvature at P.



Example 13.25 Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

Exercise 13.3

1. If the arc length of the curve $\vec{r}(t) = \langle \cos(t), \sin(t), 3t \rangle$ from $t = 0$ to $t = b$ is 20 units, find the value of b?
2. Set up an integral to compute the arc length L of the curve C: $\vec{r}(t) = \left\langle \frac{1}{2}t^2, 4t, \sqrt{2}t \right\rangle$ from the point $(0,0,0)$ to the point $\left(\frac{1}{2}, 4, \sqrt{2}\right)$.
3. Let C be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$. Find the exact length of C from the origin to the point $(6, 18, 36)$.
4. Find the vectors $\vec{T}$, $\vec{N}$, and $\vec{B}$ for $\vec{r}(t) = \langle t^2, 2t^3, t \rangle$ at the point $(1,2,1)$.
5. Show that the curvature for any line at any point on the line is $k = 0$.
6. Find the curvature of the curve $\langle t^2, \ln(t), t\ln(t) \rangle$ at the point $(1,0,0)$.
7. Find the curvature of the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at the point $(0, -4)$.
8. Find equations of the normal and osculating planes of the curve of intersection of the parabolic cylinders $x = y^2$ and $z = x^2$ at the point $(1,1,1)$.
9. Find an equation of a parabola that has curvature 4 at the origin.
10. Find equations of the osculating circles of the parabola $y = x^2$ at the points $(0,0)$ and $(1,1)$. Graph both osculating circles and the parabola.

14. Partial Derivatives

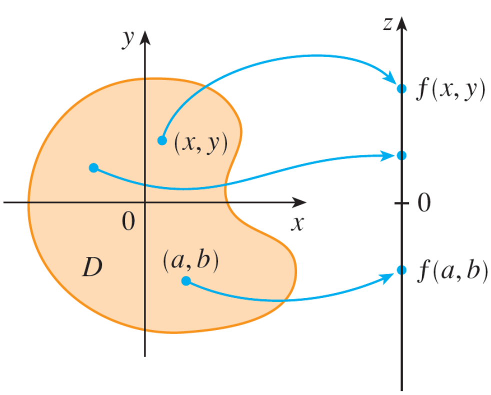
14.1 Functions of Several Variables

Functions of Two Variables

Definition 14.1 Suppose D is a set of ordered pairs of real numbers (x,y). A function f of two variables is a rule that assigns to each element in D a unique real number denoted by $f(x,y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x,y) \mid (x,y) \in D\}$.

We often write $z = f(x,y)$ to make explicit the value taken on by f at the general point (x,y). The function f is said to be a function of the two **independent variables** x and y, and the symbol z is the **dependent variable** of f.

The domain of the function $f(x,y)$ is represented as a region D in the xy-plane and the range is a set of numbers on a real line, shown as a z-axis.



Example 14.1 For each function, evaluate $f(3,2)$? find and sketch the domain?

1. $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$.

2. $f(x,y) = x\ln\left(y^2 - x\right)$.

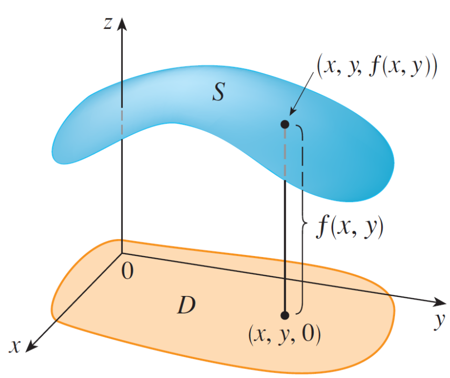
Example 14.2 Find the domain and range of $g(x,y) = \sqrt{9 - x^2 - y^2}$.

Graphs

Another way of visualizing the behavior of a function of two variables is to consider its graph.

> **Definition 14.2** If f is a function of two variables with domain D, then the graph of f is the set of all points (x,y,z) in $\mathbb{R}^3$ such that $z = f(x,y)$ and (x,y) is in D.

The graph of a function f of two variables is a surface S with equation $z = f(x,y)$. We can visualize the graph S of f as lying directly above or below its domain D in the xy-plane.



Example 14.3 Sketch the graph of the following functions? Find the domain and range?

1. $f(x,y) = 6 - 3x - 2y$.

2. $f(x,y) = \ln(1 - |x| - |y|)$.

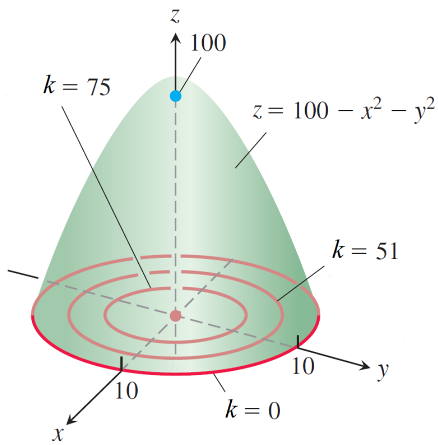
3. $g(x,y) = -2\sqrt{x^2 + y^2}$.

4. $h(x,y) = 4x^2 + y^2$

Level Curves

Definition 14.3 The **level curves** (or Contour curves) of $z = f(x,y)$ are the curves with equations $f(x,y) = k$, where k is a constant in the range of f.

Remark 14.1 The level curves $f(x,y) = k$ are just the **traces** of the graph of f in the horizontal plane $z = k$ **projected down to the** xy**-plane**.



Example 14.4 Sketch the level curves of the functions

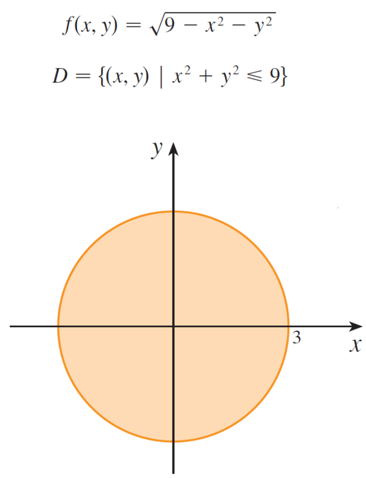
1. $f(x,y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6, 12$,

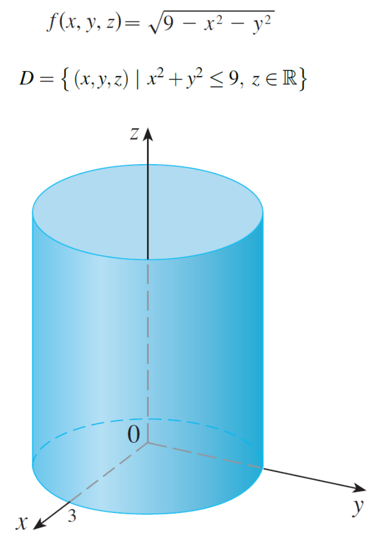
2. $g(x,y) = \sqrt{9 - x^2 - y^2}$ for the values $k = 0, 1, 2$,

3. $z = 4x^2 + y^2 + 1$ for the values $k = 2, 3, 4$.

Functions of Three Variables

Definition 14.4 A function of three variables, f, is a rule that assigns to each ordered triple (x,y,z) in a domain $D \subseteq \mathbb{R}^3$ a unique real number denoted by $f(x,y,z)$.





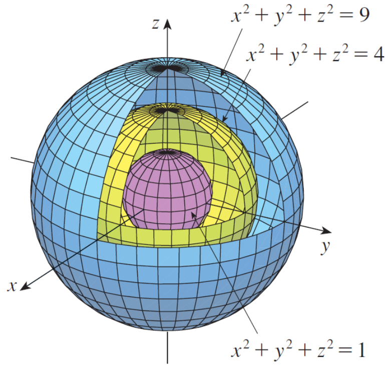
Example 14.5 Find and sketch the domain of $f(x,y,z) = \ln(z-y)\sin x$.

Example 14.6 Find the domain and range of $w = \dfrac{1}{x^2+y^2+z^2}$.

Level Surfaces

Definition 14.5 The **level surfaces** are the surfaces with equations $f(x,y,z) = k$ where k is a constant. If the point (x,y,z) moves along a level surface, the value of $f(x,y,z)$ remains fixed.

Example 14.7 Find the level surfaces of the function $f(x,y,z) = x^2 + y^2 + z^2$.

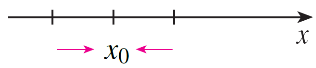


Exercise 14.1

1. Let $G(x,y) = 1 + \sqrt{4 - y^2}$.
   (a) Evaluate $G(3,1)$.
   (b) Find and sketch the domain of G.
   (c) Find the range of G.
2. Find and sketch the domain for each functions:
   (a) $f(x,y) = \dfrac{(x-1)(x+2)}{(y-x)(y-x^3)}$.
   (b) $g(x,y) = \cos^{-1}\left(y - x^2\right)$.
   (c) $h(x,y) = \ln(xy + x - y - 1)$.
   (d) $f(x,y) = \sqrt{2x - y}$.
   (e) $f(x,y) = \ln(9 - x^2 - 9y^2)$.
   (f) $f(x,y,z) = xy\ln(z)$.
   (g) $f(x,y,z) = \sqrt{|x| + |y| - 1}$.
   (h) $f(x,y,z) = \dfrac{1}{x}$.
3. Find the function range:
   (a) $f(x,y) = 4x^2 + 9y^2$,
   (b) $g(x,y) = \sqrt{y - x}$,
   (c) $h(x,y) = \tan^{-1}\left(\dfrac{y}{x}\right)$,
   (d) $f(x,y,z) = \ln\left(16 - 4x^2 - 4y^2 - z^2\right)$,
   (e) $z = \ln(9 - x^2 - y^2)$,
   (f) $w = \sqrt{4 - x^2} + \sqrt{9 - y^2} + \sqrt{1 - z^2}$.
4. Find an equation for the level surface of the function $f(x,y,z) = \sqrt{x - y} - \ln(z)$ through the point $(3, -1, 1)$.
5. Find the range of the function $f(x,y) = 4 - \sqrt{9 - x^2 - y^2}$.

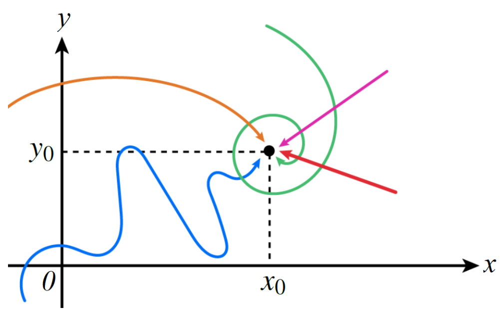
14.2 Limits and Continuity

Recall that, for the limit of a function of a single variable, $\lim\limits_{x\to x_0} f(x) = L$ iff $\lim\limits_{x\to x_0^+} f(x) = L$ and $\lim\limits_{x\to x_0^-} f(x) = L$. Note that, there are only two directions from which x can approach x_0.



Limit of Functions of Two Variables

For functions of two variables, a point $(x_0, y_0) \in \mathbb{R}^2$, can be approached from any direction along any path (linear and nonlinear paths).



If all paths led to L on the surface $z = f(x,y)$ as (x,y) approaches (x_0,y_0), then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$.

Properties of Limits of Functions of Two Variables

The following rules hold if L, M, and k are real numbers and $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = M$.

1. $\lim\limits_{(x,y)\to(x_0,y_0)} [f(x,y) \pm g(x,y)] = L \pm M$.
2. $\lim\limits_{(x,y)\to(x_0,y_0)} k \cdot f(x,y) = k \cdot L$.
3. $\lim\limits_{(x,y)\to(x_0,y_0)} [f(x,y) \cdot g(x,y)] = L \cdot M$.
4. $\lim\limits_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$ for $M \neq 0$.
5. $\lim\limits_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, $n \in \mathbb{Z}^+$ (n a positive integer).
6. $\lim\limits_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$, $n \in \mathbb{Z}^+$, and if n is even, we assume that $L > 0$.

Strategy in Finding $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$

In general, if (x_0,y_0) is not at the edge of the domain of $f(x,y)$ (which can happen with, say, square root functions) we can outline a process to calculate finite limits:

1. Limit Using Direct Substitution

Substitute the limit point (x_0,y_0) into $f(x,y)$. By doing this, we will end up with one of three "forms":

Form 1

$\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = \dfrac{c}{d}$ with $d \neq 0$. This is the answer to the limit.

Example 14.8 Find

1. $\lim\limits_{(x,y)\to(0,1)} \dfrac{x-xy+3}{x^2y+5xy-y^3}$,

2. $\lim\limits_{(x,y)\to(3,-4)} \sqrt{x^2+y^2}$.

3. $\lim\limits_{(x,y)\to(1,0)} \dfrac{y^2\sin(x)}{x}$.

Form 2

$\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = \dfrac{c}{0}$, with $c \neq 0$. The limit does not exist (undefined).

Example 14.9 Find $\lim\limits_{(x,y)\to(2,2)} \dfrac{x+3}{xy-4}$.

Form 3

$\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = \dfrac{0}{0}$. More investigation is needed (undetermined)!

Example 14.10 Find

1. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}$,

2. $\lim\limits_{(x,y)\to(1,-1)} \dfrac{x^2y+xy^2}{x^2-y^2}$.

3. $\lim\limits_{(x,y)\to(0,1)} \dfrac{y^2\sin(x)}{x}$.

2. Limit Using (Non)Linear Paths

If direct substitution does not help in an attempt to evaluate the limit, we may alternatively try to show that the limit does not exist, our strategy for doing this draws inspiration from the following fact:

If $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists, then $f(x,y)$ must approach the same limit no matter how (x,y) approaches (x_0,y_0). Thus, if we can find two different paths of approach along which the function $f(x,y)$ has different limits, then it follows that $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

Example 14.11 Find the limit, if it exists, or show that the limit does not exist.

1. $\lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$.

2. $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$.

3. $\lim_{(x,y)\to(1,0)} \dfrac{xy-y}{(x-1)^2+y^2}$.

4. $\lim_{(x,y)\to(0,0)} \dfrac{5y^4\cos^2 x}{x^4+y^4}$.

5. $\lim_{(x,y)\to(0,0)} \dfrac{5y^4\sin^2 x}{x^4+y^4}$!!?

Remark 14.2 However, showing a limit is equal along all linear paths, or along all parabolas, or so on is not a proof that the limit does exist, since we could always just consider another path not of this form approaching (x_0,y_0).

3. Limit Using the Squeeze Theorem

If substitution and (non)linear paths do not help in an attempt to evaluate the limit, we may try to use the Squeeze Theorem to prove the limit does exists.

Theorem 14.1 If $g(x,y) \leq f(x,y) \leq h(x,y)$ for all $(x,y) \neq (x_0,y_0)$ in a disk centered at (x_0,y_0) and if g and h have the same finite limit L as $(x,y) \to (x_0,y_0)$, then $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$.

Example 14.12 Find $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$.

1. $f(x,y) = \dfrac{5y^4 \sin^2 x}{x^4+y^4}$.

2. $f(x,y) = \dfrac{3x^2 y}{x^2+y^2}$.

3. $f(x,y) = y \sin\left(\dfrac{1}{x}\right)$.

4. Limit Using Change of Variables

Sometimes, it is possible to try change of variables to make the limit of one variable.

Example 14.13 Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\tan^{-1}(xy)}{xy}$.

Solution It is possible to substitute $t = xy$, where the new variable $t \to 0$ as $(x,y) \to (0,0)$, to get

$$\lim_{(x,y)\to(0,0)} \frac{\tan^{-1}(xy)}{xy} = \lim_{t\to 0} \frac{\tan^{-1}(t)}{t} \overset{\substack{\text{L'Hopital's}\\ \text{Rule}}}{=\!=} \lim_{t\to 0} \frac{\frac{1}{1+t^2}}{1} = \frac{1}{1} = 1.$$

◀

5. Limit Using Polar Coordinates

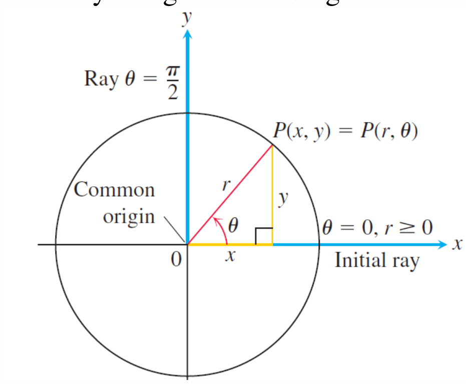
If you cannot make any headway with $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ in rectangular coordinates, try changing to **polar coordinates**. However, before we describe how to make this change, we need to review the concept of polar coordinates.

Relations Between Reactangular and Polar Coordinates

1. If the point in rectangular coordinates $P(x,y)$, then the polar coordinates $P(r,\theta)$ can be found by using the following conversions

$$r=\sqrt{x^2+y^2}\ (\text{or } r^2=x^2+y^2), \quad \theta=\tan^{-1}(y/x).$$

2. If the point in polar coordinates $P(r,\theta)$, then the rectangular coordinates $P(x,y)$ can be found by using the following conversions $x=r\cos\theta$, $y=r\sin\theta$.



To change $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ to polar coordinate, substitute: $x=r\cos\theta$, $y=r\sin\theta$, and investigate the limit of the resulting expression as $r\to 0^+$ (Because $r\geq 0$).

Example 14.14 Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$, if it exists, or show that the limit does not exist.

1. $f(x,y)=\dfrac{x^3}{x^2+y^2}$,
2. $f(x,y)=\dfrac{x^2}{x^2+y^2}$.

Solution

1. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3}{x^2+y^2} = \lim\limits_{r\to0^+} \dfrac{r^3\cos^3\theta}{r^2} = \lim\limits_{r\to0^+} r\underbrace{\cos^3\theta}_{\text{bounded function}} = 0.$

2. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2} = \lim\limits_{r\to0^+} \dfrac{r^2\cos^2\theta}{r^2} = \cos^2\theta$, takes on all values from 0 to 1 regardless of how small r is, so that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2}$ does not exist.

◀

Example 14.15 Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$, if it exists, or show that the limit does not exist.

1. $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$.

2. $f(x,y) = \dfrac{e^{-x^2-y^2}-1}{x^2+y^2}$.

3. $f(x,y) = \left(x^2+y^2\right)\ln\left(x^2+y^2\right)$.

4. $f(x,y) = \dfrac{2xy}{x^2+y^2}$.

Remark 14.3 Changing to polar coordinates does not always help, however, and may even tempt us to false conclusions. For example, the limit may exist along every straight line (or ray) $\theta =$ constant and yet fail to exist in the broader sense. To illustrates this point, consider the problem of finding $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^4+y^2}$ in polar coordinates. $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^4+y^2}$ becomes

$$\lim_{r\to 0^+} \frac{2r^3\cos^2\theta\sin\theta}{r^4\cos^4\theta + r^2\sin^2\theta} = \lim_{r\to 0^+} \frac{2r\cos^2\theta\sin\theta}{r^2\cos^4\theta + \sin^2\theta} = \frac{0}{\sin^2\theta}.$$

If we hold θ constant and let $r \to 0^+$, the limit is 0 except at $\theta = 0,\ \pi,\ 2\pi$. So we conclude nothing about the existence of the limit. On the path

$$y = mx^2 \implies \sin\theta = mr\cos^2\theta,$$

however, we have

$$\lim_{r\to 0^+} \frac{2r\cos^2\theta\sin\theta}{r^2\cos^4\theta + \sin^2\theta} = \lim_{r\to 0^+} \frac{2r\cos^2\theta\left(mr\cos^2\right)}{r^2\cos^4\theta + \left(mr\cos^2\right)^2} = \lim_{r\to 0^+} \frac{\cancel{2r^2\cos^4\theta}\ (m)}{\cancel{r^2\cos^4\theta}\ \left(1+m^2\right)} = \frac{2m}{1+m^2}.$$

Since the answer depends on m, the limit does not exist.

Continuity

As with function of a single variable, continuity is defined in terms of limits.

Definition 14.6 A function $f(x,y)$ is continuous at the point (x_0,y_0) if **(i)** f is defined at (x_0,y_0), **(ii)** $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists, and **(iii)** $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$. We say f is continuous if f is continuous at every point of its domain.

Remark 14.4 The algebraic combinations of continuous functions are continuous at every point at which all the functions involved are defined. This means that sums, differences, constant multiples, products, quotients, and powers of continuous functions are continuous where defined. In particular, polynomials and rational functions of two variables are continuous at every point at which they are defined.

Example 14.16 Determine the set of points at which the function is continuous.

1. $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0), \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$

2. $g(x,y) = \begin{cases} \dfrac{x^2+2xy+y^2}{x+y} - 2 & \text{if} \quad (x,y) \neq (0,0), \\ -1 & \text{if} \quad (x,y) = (0,0). \end{cases}$

3. $h(x,y) = \tan^{-1}\left(\dfrac{y}{x}\right)$.

Example 14.17 Find k that makes the function f continuous at the point $(2,2)$.

$$f(x,y) = \begin{cases} \dfrac{x^2+2xy+y^2-16}{x+y-4} & \text{if} \quad x+y \neq 4, \\ k & \text{if} \quad x+y = 4. \end{cases}$$

Functions of More Than Two Variables

Everything that we have done for functions of two variables can be extended to functions of three or more variables.

Example 14.18 Find the limit, if it exists, or show that the limit does not exist.

1. $\lim\limits_{(x,y,z)\to(\pi,0,1/3)} e^{y^2}\tan(xz)$.

2. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xy+yz}{x^2+y^2+z^2}$.

3. $\lim\limits_{(x,y,z)\to(1,-1,1)} \dfrac{xz+5x-yz+5y}{x+y}$.

4. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{yz^2}{x^2+y^2+z^4}$.

Example 14.19 Determine the set of points at which the function $f(x,y,z)$ is continuous.

1. $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2}$.

2. $f(x,y,z) = \sin^{-1}\left(x^2+y^2+z^2\right)$.

3. $f(x,y,z) = \sqrt{y-x^2}\ln z$.

Exercise 14.2

1. Evaluate the limit or show the limit does not exit.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\cos(y)\sin(x)}{x}$. (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$. (c) $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x-xy+3}{x^2y+5xy-y^3}$.

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+2y^2}$. (e) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2-2xy+y^2}{x-y}$. (f) $\displaystyle\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4}$.

(g) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$. (h) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4-4y^2}{x^2+2y^2}$. (i) $\displaystyle\lim_{(x,y,z)\to(\pi,0,1/3)} e^{y^2}\tan(xz)$.

(j) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^4}$. (h) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy-3x-2y+6}{y-3}$.

2. Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\tan^{-1}(xy)}{xy}$. (Hint: $1-\dfrac{x^2y^2}{3} < \dfrac{\tan^{-1}(xy)}{xy} < 1$.)

3. Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y\cos(y)}{x^4+y^2}$ along the curve $y=x^2$.

4. Find $\displaystyle\lim_{(x,y)\to(1,0)} \frac{y^2-6xy}{3x-y-3}$ along the curve $x=e^y$.

5. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$ does not exist by finding the limit along the path $y=-\sin(x)$.

6. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{|xy|}$ does not exist.

7. Find the limit, if it exists, or show that the limit does not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+3y^2}$. (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{4x^2+y^2}$.

(c) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{2xy-2y}{x^2+y^2-2x+1}$. (d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3\cos(x)}{x^2+y^6}$.

(e) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2}$. (f) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xz^2+3y^2}{x^2+y^2+z^4}$.

(g) $\displaystyle\lim_{(x,y,z)\to(0,3,1)} \left[e^{\sin(\pi x)}+\ln(\cos(\pi(y-z)))\right]$. (h) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$.

(i) $\displaystyle\lim_{(x,y)\to(0,1,0)} \frac{x(y-1)z}{3x^4-3y^2+6y-\frac{1}{2}z^4-3}$. (j) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$.

(k) $\displaystyle\lim_{(x,y)\to(0,1)} \frac{\sin(xy^2)\tan(2xy)}{x^2}$. (l) $\displaystyle\lim_{(x,y)\to(0,0)} x^4\sin\left(\frac{1}{x^2+|y|}\right)$.

8. Find k that makes the function f continuous at the point $(0,0)$.

$$f(x,y)=\begin{cases} \dfrac{1-\cos\left(\sqrt{x^2+y^2}\right)}{x^2+y^2} & \text{if } (x,y)\neq(0,0),\\ k & \text{if } (x,y)=(0,0).\end{cases}$$

9. Find k that makes the function f continuous at the point $(1,2)$.

$$f(x,y)=\begin{cases} \dfrac{2x^2+(2k-1)xy-ky^2}{2x-y} & \text{if } y\neq 2x,\\ 7 & \text{if } y=2x.\end{cases}$$

10. Determine the set of points at which the function is continuous.

(a) $f(x,y)=\begin{cases} \dfrac{x^2y^3}{2x^2+y^2} & \text{if } (x,y)\neq(0,0),\\ 1 & \text{if } (x,y)=(0,0).\end{cases}$

(b) $g(x,y,z)=\dfrac{1}{x^2+z^2-1}$.

14.3 Partial Derivatives

The calculus of several variables is similar to single-variable calculus applied to several variables one at a time.

Functions of Two variables

Let f is a function of two variables x and y and (x_0, y_0) is a point in the domain of f. Suppose we let only x vary while keeping y fixed $(y = y_0)$. Then we are really considering a function of a single variable x, namely, $g(x) = f(x, y_0)$. If g has a derivative at x_0, then we call it the **partial derivative of f with respect to** x at (x_0, y_0) and denote it by $f_x(x_0, y_0)$. Thus

$$f_x(x_0, y_0) = g'(x_0) \text{ where } g(x) = f(x, y_0).$$

By the definition of a derivative, we have

$$f_x(x_0, y_0) = g'(x_0) = \lim_{h \to 0} \frac{g(x_0 + h) - g(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

Definition 14.7 The partial derivative of $f(x, y)$ with respect to x at the point (x, y) is

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

provided the limit exists.

Similarly, the **partial derivative of f with respect to** y at (x_0, y_0), denoted by $f_y(x_0, y_0)$, is obtained by keeping x fixed $(x = x_0)$ and finding the ordinary derivative at y_0 of the function $G(y) = f(x_0, y)$:

$$f_y(x_0, y_0) = G'(y_0) = \lim_{h \to 0} \frac{G(y_0 + h) - G(y_0)}{h} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

Definition 14.8 The partial derivative of $f(x, y)$ with respect to y at the point (x, y) is

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

provided the limit exists.

Example 14.20 If $f(x, y) = x^3 + x^2 y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Solution Define $g(x) := f(x, 1) = x^3 + x^2 - 2$. Then $f_x(x, 1) = g'(x) = 3x^2 + 2x$. Therefore, $f_x(2, 1) = 16$. Define $G(y) := f(2, y) = 4y^3 - 2y^2 + 8$. Then $f_y(2, y) = G'(y) = 12y^2 - 4y$. Therefore, $f_y(2, 1) = 8$. ◀

Example 14.21 Let $f(x,y)=\begin{cases} x+y^2 & \text{if } x \geq y^2 \\ x^2+y^2 & \text{if } x<y^2 \end{cases}$, find $f_x(0,0)$?

Solution Define $g(x) := f(x,0) = \begin{cases} x & \text{if } x \geq 0 \\ x^2 & \text{if } x<0 \end{cases}$. Then

$$f_x(x,0)=g'(x)=\begin{cases} 1 & \text{if } x>0 \\ 2x & \text{if } x<0 \end{cases} \Longrightarrow f_x(0,0)=g'(0)=\begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x<0 \end{cases}$$

Therefore, $f_x(0,0)$ does not exist because $f_x^+(0,0) \neq f_x^-(0,0)$. ◀

Example 14.22 If $f(x,y)=\left(y^3+e^{x-1}-1\right)^2 \ln\left(y^3 \sin\left(\dfrac{\pi y(x-1)}{2}\right)+e^{3y}\right)+4(x-1)y^3$, find $f_y(1,2)$?

Solution Define $G(y) := f(1,y) = \left(y^3\right)^2 \ln\left(e^{3y}\right) = 3y^7 \Longrightarrow f_y(1,2)=G'(2)=(3)(7)(2)^6=1344$. ◀

Notations for Partial Derivatives

We use several notations for the partial derivative:

$$\frac{\partial f}{\partial x}(x_0,y_0), \text{ or } f_x(x_0,y_0), \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)}, \text{ and } \frac{\partial f}{\partial x}, f_x, \frac{\partial z}{\partial x} \text{ or } z_x.$$

$$\frac{\partial f}{\partial y}(x_0,y_0), \text{ or } f_y(x_0,y_0), \left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)}, \text{ and } \frac{\partial f}{\partial y}, f_y, \frac{\partial z}{\partial y} \text{ or } z_y.$$

Remark 14.5 To distinguish partial derivatives from ordinary derivatives we use the symbol "∂" rather than the "d" previously used.

Rule for Finding Partial Derivatives of $f(x,y)$

- To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.
- To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

Example 14.23 If $f(x,y)=\sin\left(\dfrac{x}{1+y}\right)$, calculate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

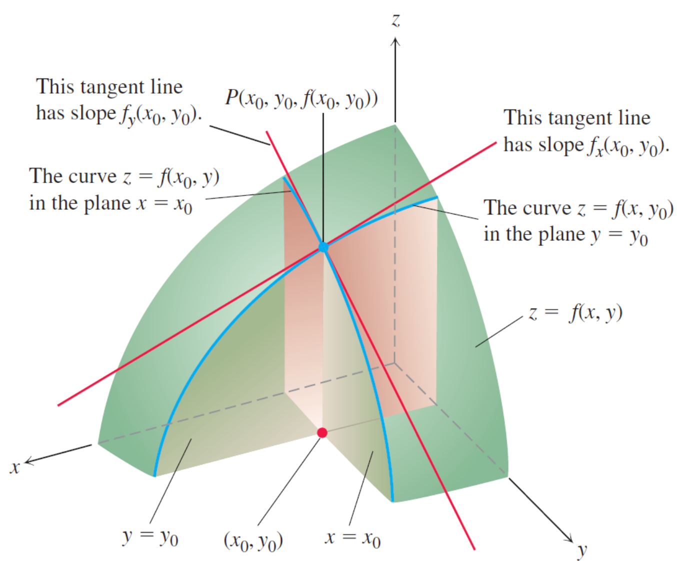
Functions of More Than Two Variables

Partial derivatives can also be defined for functions of three or more variables. For example, if f is a function of three variables x, y, and z, then its partial derivative with respect to x is found by regarding y and z as constants and differentiating $f(x,y,z)$ with respect to x.

Example 14.24 Find f_x, f_y and f_z if $f(x,y,z) = e^{xy}\ln z$.

Interpretations of Partial Derivatives

If (x_0, y_0) is a point in the domain of a function $z = f(x,y)$.



The vertical plane (i.e., the trace) $y = y_0$ intersects the surface $z = f(x,y)$ in the curve $z = f(x,y_0)$. The **slope** of the tangent line to the curve $z = f(x,y_0)$ at the point $P(x_0,y_0,f(x_0,y_0))$ is the value of f_x at (x_0,y_0). The partial derivative f_x at (x_0,y_0) gives the rate of changes of z with respect to x when y held fixed at the value y_0.

Similarly, the slope of tangent line to the curve $z = f(x,y)$ at the point P is the value of f_y at (x_0,y_0). The partial derivative f_y at (x_0,y_0) gives the rate of changes of z with respect to y when x held fixed at the value x_0.

Example 14.25 The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at the point $(1,2,5)$.

Second-Order Partial Derivatives

When we differentiate a function $f(x,y)$ twice, we produce its second-order partial derivatives. These derivatives are usually denoted by

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}, \qquad f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x},$$

$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}, \qquad f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}.$$

Example 14.26 If $f(x,y) = x\cos(y) + ye^x$, find the second-order partial derivatives.

Theorem 14.2 — Clairaut's Theorem. If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined on an open region containing a point (x_0,y_0) and are all continuous at (x_0,y_0), then

$$f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0).$$

Example 14.27 Find f_{yx} if $f(x,y) = xy + \dfrac{e^y}{y^2+1}$.

Higher-Order Partial Derivatives

Partial derivatives of order three or higher can also be defined in similar way. For instance, $f_{xyy} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right)$. Using **Theorem 14.2** it can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$ if these functions are continuous.

Example 14.28 If $f(x,y,z) = xy^2z^3 + \sin^{-1}(x\sqrt{z})$, find f_{xzy}.

Example 14.29 If $f(x,y,z) = x^4 z\sin(xyz)$, find f_{xxxzy} at the point $(1,0,-1)$.

Partial Derivatives by Limit Definition

Example 14.30 If $f(x,y) = \sqrt[3]{x^3+y^3}$, find $f_x(0,0)$ and $f_y(0,0)$.

Example 14.31 Let $f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$

1. Find $f_x(x,y)$ and $f_y(x,y)$, when $(x,y) \neq (0,0)$.

2. Find $f_x(0,0)$ and $f_y(0,0)$.

3. Find $f_{xy}(x,y)$ and $f_{yx}(x,y)$, when $(x,y) \neq (0,0)$.

4. Show that $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$.

5. Does the result of the previous part contradict Theorem 14.2? Use graphs of f_{xy} and f_{yx} to illustrate your answer.

Differentiability

Recall that in the case of a function of a single-variable $f(x)$, differentiability at a point $(x_0, f(x_0))$ means that there is a unique tangent line to the graph of $f(x)$ at $(x_0, f(x_0))$ *.

The concept of differentiability for functions of several variables $f(x,y)$ is more complicated than for single-variable functions because a point (x_0, y_0) in the domain of $f(x,y)$ can be approached along more than one path. The following Theorem tell us that functions with **continuous** first partial derivatives f_x and f_y at (x_0, y_0) are differentiable there.

Theorem 14.3 If the partial derivatives f_x and f_y exist near (x_0, y_0) and are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

Remark 14.6 Existence alone of partial derivatives does not imply differentiability.

Theorem 14.4 If f is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

We remark that the converse of the above Theorem is not true (see Exercise 14).

Corollary 14.1 If f is not continuous at (x_0, y_0), then f is not differentiable at (x_0, y_0).

Example 14.32 Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

1. Show that f_x and f_y exist at $(0,0)$.
2. Show that f_x and f_y are not continuous at $(0,0)$.
3. Show that f is not continuous at $(0,0)$.
4. Show that f is not differentiable at $(0,0)$.

Solution

1. We can find $f_x(0,0)$ and $f_y(0,0)$ using the limit definition of the partial derivative, which are given by $f_x(0,0) = 0$ and $f_y(0,0) = 0$.
2. The partial derivatives are $f_x = \frac{y\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2}$ and $f_y = \frac{x\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}$. Since $\lim\limits_{(x,y)\to(0,0)} f_x$ and $\lim\limits_{(x,y)\to(0,0)} f_x$ are not exist, then f_x and f_y are not continuous at $(0,0)$. So even though f_x and f_y exist at every point in $\mathbb{R}^2$, they are not continuous at $(0,0)$.
3. Take the limit of f as $(x,y) \to (0,0)$ along the line $y = mx$; this gives a result depend on m and hence the limit does not exist.
4. Using Corollary 14.1, since f is not continuous at $(0,0)$, f is not differentiable at $(0,0)$.

Note that, existence alone of the partial derivatives f_x and f_y at $(0,0)$ is not enough, but continuity of the partial derivatives f_x and f_y at $(0,0)$ guarantees differentiability. ◀

*Except for one case: *if the tangent line is vertical, then the function is not differentiable at that point.*

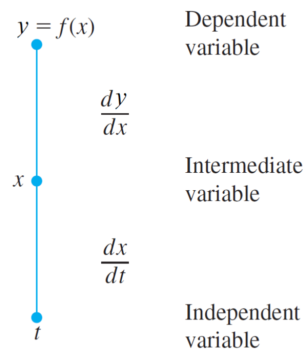
Exercise 14.3

1. If $f(x,y,z)=\sqrt{\sin^2(x+y)+\sin^2(z)}$, find $f_z\left(0,0,\frac{\pi}{4}\right)$.
2. If $f(x,y)=x^{\sqrt{y}}$, find f_x, f_y, f_{xy}, f_{yy}, f_{yx}, and f_{xx}.
3. Find the first partial derivatives:
   (a) $f(x,y)=\ln(x+\sqrt{x^2+y^2})$.
   (b) $F(x,y)=\int_y^x \cos(e^t)\,dt$.
4. Find the partial derivative $f_{xxyyy}(0,3\pi/2)$ if $f(x,y)=\sin\left(x^2+y\right)$
5. Find all the second partial derivative of $w=\sqrt{u^2+v^2}$
6. $f(x,y,z)=e^{xyz^2}$, find f_{xyz}.
7. If $f(x,y,z)=z^2\mathrm{e}^{\sin\left(xy+\pi y^2\right)}+x^2z^2y+10$, find $f_{yyxzxyzz}(x,y,z)$.
8. Find $f_x(0,0)$ and $f_y(0,0)$.
   (a) $f(x,y)=\sqrt{x^2+y^2}$.
   (b) $f(x,y)=\sqrt[3]{x^3-y^3}$.
9. Let $f(x,y)=\begin{cases}\dfrac{xy\left(y^2-x^2\right)}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=(0,0)\end{cases}$. Use the limit definition of the partial derivative to show that $f_{xy}(0,0)=1$ and $f_{yx}(0,0)=-1$.
10. Find $f_{xy}(0,0)$, $f_{yx}(0,0)$ where $f(x,y)=x\sqrt[3]{8x^3+27y^3}$.
11. If $f(x,y,z)=x^3y^3z^2+x^3yz^3$, find $f_{yzxzxzz}(1,1,1)$.
12. Give an example of a function $f(x,y)$ such that $f_x(0,0)=f_y(0,0)=0$ but f is not continuous at $(0,0)$. Hence, the existence of the first partial derivatives does not ensure continuity.
13. Let $f(x,y)=\begin{cases}0 & \text{if } x^2<y<2x^2\\ 1 & \text{if otherwise}\end{cases}$. Show that $f_x(0,0)$ and $f_y(0,0)$ exist, but f is not differentiable at $(0,0)$.
14. Let $f(x,y)=\begin{cases}\dfrac{y\sin(3x)}{\sqrt{x^2+y^2}} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$
   (a) Show that $f(x,y)$ is continuous at $(0,0)$.
   (b) Show that f_x and f_y exist at $(0,0)$, but they are not continuous at $(0,0)$.
   (c) Show that f is not differentiable at $(0,0)$.

14.4 The Chain Rule

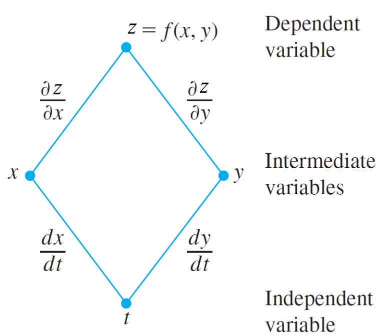
Recall that, the Chain Rule for a function of a single variable gives the rule for differentiating a composite function: *If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t and $\frac{dy}{dt}$ can be calculated by the formula $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$.*

For this composite function $y(t) = f(g(t))$, we can think of t as the "independent variable" and $x = g(t)$ as the "intermediate variable," because t determines the value of x which in turn gives the value of y from the function f.



For functions of more than one variable, the Chain Rule has several forms. The form depends on how many intermediate and independent variables are involved, but once this is taken into account, it works like the Chain Rule for functions of a single variable.

Example 14.33 — Functions of One Independent Variable and Two Intermediate Variables. Suppose that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t.

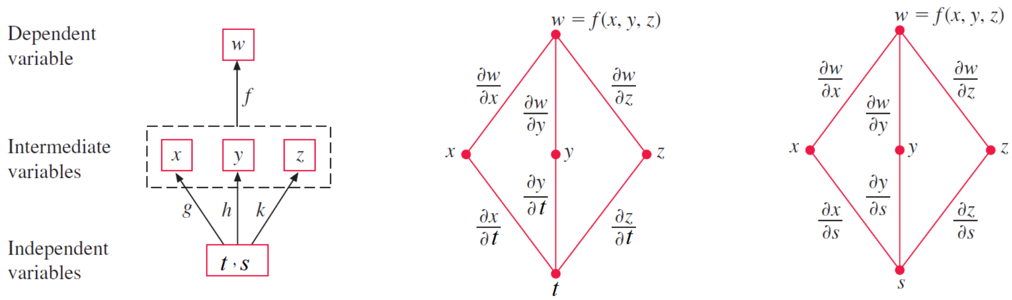


Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Example 14.34 If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$

Example 14.35 — Functions of Two Independent Variables and Three Intermediate Variables. Suppose that $w = f(x,y,z)$ where $x = g(t,s)$, $y = h(t,s)$ and $z = k(t,s)$.



If all four functions are differentiable, then w has partial derivatives with respect to t and s, given by the formulas

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t},$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

Example 14.36 Express $\partial w/\partial t$ and $\partial w/\partial s$ in terms of t and s if $w = x + 2y + z^2$, $x = t/s$, $y = t^2 + \ln(s)$, and $z = 2te^s$.

Example 14.37 Write out the Chain Rule for the case where $z = f(x,y)$, $x = g(t,s)$, and $y = h(s)$.

Example 14.38 If $u = x^4 y + y^2 z^3$, where $x = rse^t$, $y = rs^2 e^{-t}$, and $z = r^2 s \sin t$, find the value of $\partial u / \partial s$ when $r = 2$, $s = 1$, $t = 0$.

Example 14.39 If $g(s,t) = f\left(s^2 - t^2, t^2 - s^2\right)$ and f is differentiable, show that g satisfies the equation $t \dfrac{\partial g}{\partial s} + s \dfrac{\partial g}{\partial t} = 0$.

Example 14.40 If $z = f(x,y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$, find:

1. $\partial z / \partial r$.

2. $\partial^2 z / \partial r^2$.

Implicit Differentiation

Suppose that an equation of the form

$$F(x,y)=0$$

defines y implicitly as a differentiable function of x, that is, $y=f(x)$, where $F(x,f(x))=0$ for all x in the domain of f. If F is differentiable, then

$$F_x \underbrace{\frac{dx}{dx}}_{=1} + F_y \frac{dy}{dx} = 0 \Longrightarrow F_y \frac{dy}{dx} + F_x = 0.$$

So if $F_y \neq 0$ we solve for $\frac{dy}{dx}$ and obtain $\frac{dy}{dx} = -\frac{F_x}{F_y}$.

Rule 14.1 $F(x,y)=0 \Longrightarrow \dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$.

Example 14.41 Find y' if $x^3+y^3=6xy$.

Example 14.42 Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation $x^2+y^3+z^3+6xyz=1$.

Now we suppose that an equation of the form

$$F(x,y,z)=0$$

defines z implicitly as a differentiable function of (x,y), that is, $z=f(x,y)$, where $F(x,y,f(x,y))=0$ for all (x,y) in the domain of f. If F is differentiable, then

$$F_x \underbrace{\frac{\partial x}{\partial x}}_{=1} + F_y \underbrace{\frac{\partial y}{\partial x}}_{=0} + F_z \frac{\partial z}{\partial x} = 0 \Longrightarrow F_x + F_z \frac{\partial z}{\partial x} = 0.$$

If $F_z \neq 0$, we solve for $\frac{\partial z}{\partial x}$ and obtain $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$. The formula for $\frac{\partial z}{\partial y}$ is obtained in a similar manner.

Rule 14.2 $F(x,y,z)=0 \Longrightarrow \quad \dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} \quad$ and $\quad \dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}$.

Example 14.43 Find $\partial z/\partial x$ and $\partial z/\partial y$ if $x^3+y^2=1-z^3-6xyz$.

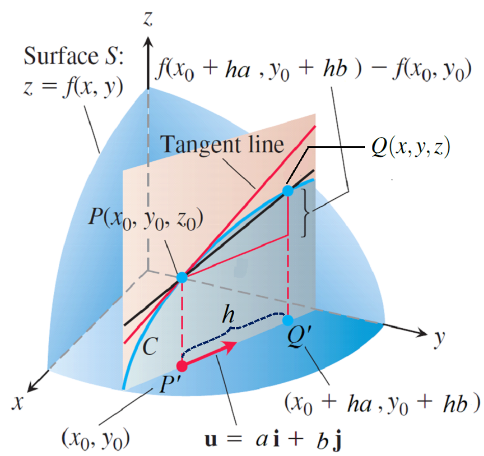
Exercise 14.4

1. If $z=\sqrt{1+x^2+y^2}$, $x=\ln(t)$, $y=\cos(t)$, find $\dfrac{dz}{dt}$.
2. If $z=\sin(\theta)\cos(\phi)$, $\theta=st^2$, $\phi=s^2t$, find $\dfrac{\partial z}{\partial s}$, and $\dfrac{\partial z}{\partial t}$.
3. If $z=f(x,y)$, where f is differentiable, and $x=g(t)$, $y=h(t)$, $g(3)=2$, $g'(3)=5$, $h(3)=7$, $h'(3)=-4$, $f_x(2,7)=6$, $f_y(2,7)=-8$. Find $\dfrac{dz}{dt}$ when $t=3$.
4. Given the implicitly defined function $\sin(x^2y^2)+y^3=x+y$, find y'.
5. If $z^3+\ln\left(\mathrm{e}^{3x}-\mathrm{e}^{y}+\mathrm{e}^{z}\right)=0$, find $\dfrac{\partial z}{\partial x}$ at the point $(0,0)$.
6. Find $\partial z/\partial x$ if $\mathrm{e}^z=xyz$.
7. If $\mathrm{e}^{zx}\sin(y)=xz^2+xy^2-4$, then find $\dfrac{\partial z}{\partial y}$ at $(1,0,2)$.
8. Use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.
   (a) $x^2+2y^2+3z^2=1$.
   (b) $x\sin(z)-z^2y=1$.
9. If $z=\ln(\sqrt{x^2+y^2})$, show that $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=1$.
10. Let $f(x,y)$ be a function such that $f_x(0,0)=3$, $f_x(1,1)=-2$, $f_y(0,0)=4$, and $f_y(1,1)=6$. If $x(s,t)=\mathrm{e}^s-t\mathrm{e}^t$ and $y(s,t)=\mathrm{e}^t-s\mathrm{e}^s$, find the value of $\dfrac{\partial f}{\partial s}$ at $(s,t)=(0,0)$.

14.5 Directional Derivatives and Gradient Vectors

To this point we've only looked at the two partial derivatives f_x and f_y. Recall that these derivatives represent the rate of changes of $z = f(x,y)$ in a direction parallel to the x-axis (when y held fixed) and in a direction parallel to the y-axis (when x held fixed) respectively. We now discuss how to find the rate of change of $z = f(x,y)$ in the direction of an arbitrary unit vector $\hat{u} = \langle a,b\rangle$.

Consider the surface S with the equation $z = f(x,y)$ and let $z_0 = f(x_0,y_0)$. Then the point $P(x_0,y_0,z_0)$ lies on S.



The vertical plane that passes through P in the direction of $\hat{u}$ intersects S in a curve C. The slope of the tangent line to C at the point P is the rate of change of $z = f(x,y)$ in the direction of $\hat{u}$.

If $Q(x,y,z)$ is another point on C, and the points P' and Q' are the projections of P and Q onto the xy-plane, then the vector $\overrightarrow{P'Q'}$ is parallel to $\hat{u}$ and so there exists a scalar h such that $\overrightarrow{P'Q'} = h\hat{u} = \langle ha,hb\rangle$. **Because $\hat{u}$ is a unit vector, the value of h is precisely the distance along the line from** $P'(x_0,y_0)$ **to** $Q'(x,y)$, so $x = x_0 + ha$, $y = y_0 + hb$, and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0,y_0)}{h}.$$

If we take the limit as $h \to 0$, we obtain the slope of the tangent line to C at the point P which is the rate of change of $z = f(x,y)$ (with respect to distance) in the direction of $\hat{u}$, which is called the **directional derivative of f in the direction of** $\hat{u}$.

Definition 14.9 The directional derivative of f at (x_0,y_0) in the direction of the unit vector $\hat{u} = \langle a,b\rangle$ is the number

$$D_{\hat{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0,y_0)}{h},$$

provided the limit exits.

Remark 14.7 The partial derivatives of f with respect to x and y are just special cases of the directional derivative (**Note:** *if $\hat{u} = \hat{\imath} = \langle 1,0\rangle$, then $D_{\hat{\imath}} f = f_x$, and if $\hat{u} = \hat{\jmath} = \langle 0,1\rangle$, then $D_{\hat{\jmath}} f = f_y$*).

Calculation and Gradients

We now develop an efficient formula to calculate the directional derivative for a differentiable function f. Define a composite function g by $g := f(x,y)$ where $x = x_0 + ha$ and $y = y_0 + hb$. Then by the Chain Rule we find

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh} = f_x(x,y)a + f_y(x,y)b.$$

If we put $h = 0$, then $x = x_0$, $y = y_0$ and

$$\begin{aligned} g'(0) &= f_x(x_0,y_0)a + f_y(x_0,y_0)b \\ g'(0) &= \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle \cdot \hat{u} ---------- (1) \end{aligned}$$

On the other hand, the function g can be written as $g(h) = f(x_0 + ha, y_0 + hb)$, then, by the definition of a derivative, we have

$$\begin{aligned} g'(0) &= \lim_{h\to 0} \frac{g(h) - g(0)}{h} \\ &= \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0,y_0)}{h} \\ g'(0) &= D_{\hat{u}} f(x_0,y_0) --------------- (2) \end{aligned}$$

Comparing Equations (1) and (2), we obtain that

$$D_{\hat{u}} f(x_0,y_0) = \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle \cdot \hat{u}.$$

Definition 14.10 If f is a differentiable function of x and y, then f has a derivative in the direction of any unit vector $\hat{u} = \langle a, b\rangle$ and $D_{\hat{u}} f = \underbrace{\langle f_x, f_y\rangle}_{\text{Gradiant of } f} \cdot \hat{u}$.

Remark 14.8 The vector $\langle f_x, f_y\rangle$ is called **the gradient vector of** f and it is denoted by

$$\overrightarrow{\nabla f} \quad (\text{“ del} f \text{ ”}).$$

Example 14.44 Find the directional derivative $D_{\hat{u}} f(x,y)$ if $f(x,y) = x^3 - 3xy + 4y^2$ at the point $(1,2)$ in the direction of the vector $\vec{u} = 2\hat{i} - \hat{j}$.

Example 14.45 Find the gradient vector of the function $f(x,y) = \sin x + e^{xy}$ at the point $(2,0)$.

Example 14.46 Find the directional derivative $D_{\hat{u}}f(x,y)$ if $f(x,y) = x^3 - 3xy + 4y^2$ and $\vec{u}$ is the vector given by the angle $\theta = \pi/6$. What is $D_{\hat{u}}f(1,2)$?

Functions of More Than Two Variables

If $f(x,y,z)$ is differentiable function and $\hat{u} = \langle a,b,c\rangle$ is a unit vector, then

1. The directional derivative of f at (x_0,y_0,z_0) in the direction of $\hat{u}$ is

$$D_{\hat{u}}f(x_0,y_0,z_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb, z_0+hc) - f(x_0,y_0,z_0)}{h} \quad \text{(if this limit exists)}.$$

2. The gradient vector is $\overrightarrow{\nabla f} = \langle f_x, f_y, f_z\rangle$.
3. The directional derivative of f in the direction of a unit vector $\hat{u}$ is $D_{\hat{u}}f = \overrightarrow{\nabla f} \cdot \hat{u}$.

Example 14.47 Consider $f(x,y,z) = x\sin(yz)$. Find the gradient vector of f. Find the directional derivative of f at $(1,3,0)$ in the direction of $\hat{i}+2\hat{j}-\hat{k}$.

Maximizing the Directional Derivative

Evaluating the dot product in the formula of the directional derivative,

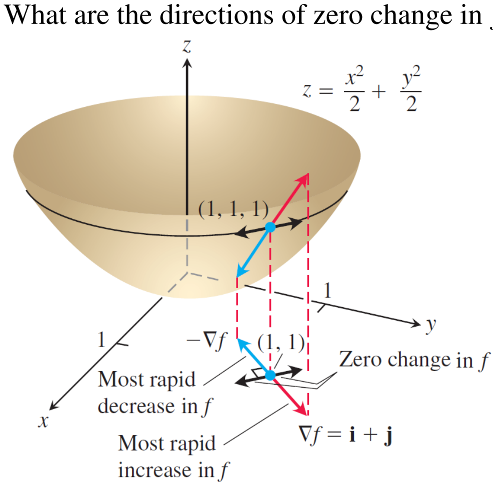
$$D_{\hat{u}}f = \overrightarrow{\nabla f} \cdot \hat{u} = |\overrightarrow{\nabla f}||\hat{u}|\cos(\theta) = |\overrightarrow{\nabla f}|\cos(\theta),$$

revels the following properties:

1. The function f increases most rapidly when $\hat{u}$ has the same direction as $\overrightarrow{\nabla f}$. The directional derivative in this direction is $D_{\hat{u}}f = |\overrightarrow{\nabla f}|\cos(0) = |\overrightarrow{\nabla f}|$.
2. The function f decreases most rapidly when $\hat{u}$ has the opposite direction as $\overrightarrow{\nabla f}$. The directional derivative in this direction is $D_{\hat{u}}f = |\overrightarrow{\nabla f}|\cos(\pi) = -|\overrightarrow{\nabla f}|$.
3. The maximum value of the directional derivative $D_{\hat{u}}f$ is $|\overrightarrow{\nabla f}|$.
4. The minimum value of the directional derivative $D_{\hat{u}}f$ is $-|\overrightarrow{\nabla f}|$.
5. Any direction $\hat{u}$ orthogonal to $\overrightarrow{\nabla f}$ is a direction of zero change in f (Because $\theta = \pi/2$.).

Example 14.48 Consider $f(x,y) = \left(x^2+y^2\right)/2$.

1. Find the rate of change of f at the point $P(1,1)$ in the direction from P to $Q(2,1)$.
2. In what direction does f have the maximum rate of change at $P(1,1)$? What is this maximum rate of change?
3. What are the directions of zero change in f at $P(1,1)$?



The Second Directional Derivative

If we want to take the second directional derivative of $f(x,y)$ in the direction of a unit vector $\hat{u} = \langle a,b \rangle$, then we have that

$$\begin{aligned}
D^2_{\hat{u}} f(x,y) &= D_{\hat{u}}\left[D_{\hat{u}} f(x,y)\right] \\
&= D_{\hat{u}}\left[\langle f_x(x,y), f_y(x,y)\rangle \cdot \hat{u}\right] \\
&= D_{\hat{u}}\left[a f_x(x,y) + b f_y(x,y)\right] \\
&= \left\langle \frac{\partial}{\partial x}\left(a f_x(x,y) + b f_y(x,y)\right),\ \frac{\partial}{\partial y}\left(a f_x(x,y) + b f_y(x,y)\right) \right\rangle \cdot \hat{u} \\
&= \langle a f_{xx}(x,y) + b f_{yx}(x,y),\ a f_{xy}(x,y) + b f_{yy}(x,y)\rangle \cdot \langle a,b\rangle \\
&= a^2 f_{xx}(x,y) + ba f_{yx}(x,y) + ab f_{xy}(x,y) + b^2 f_{yy}(x,y).
\end{aligned}$$

If the second partial derivatives of $f(x,y)$ are continuous, then by Clairaut's Theorem 14.2 we get

$$D^2_{\hat{u}} f(x,y) = a^2 f_{xx}(x,y) + 2ab f_{xy}(x,y) + b^2 f_{yy}(x,y).$$

Rule 14.3 If $\hat{u} = \langle a,b \rangle$ is a unit vector and $f(x,y)$ has continuous second partial derivatives, then

$$D^2_{\hat{u}} f(x,y) = a^2 f_{xx}(x,y) + 2ab f_{xy}(x,y) + b^2 f_{yy}(x,y).$$

Remark 14.9 A formula for the second directional derivative of functions of three variables can be obtained in a similar manner.

Example 14.49 If $f(x,y) = x^3 + 5x^2y + y^3$ and $\vec{u} = 3\hat{\imath} + 5\hat{\jmath}$, calculate $D^2_{\hat{u}} f(2,1)$.

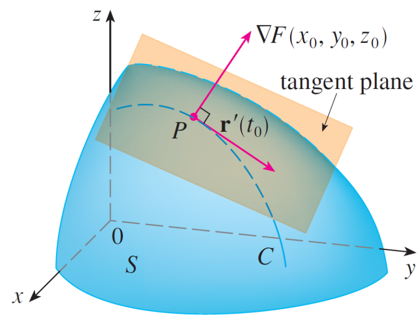
Example 14.50 Find the second directional derivative of $f(x,y) = xe^{2y}$ in the direction $\vec{v} = 4\hat{\imath} + 6\hat{\jmath}$.

Exercise 14.5

1. If $f(x,y,z) = x^2yz - xyz^3$, $P(2,-1,1)$, $\vec{u} = \left\langle 0, \frac{4}{5}, -\frac{3}{5} \right\rangle$.
   (a) Find $\overrightarrow{\nabla f}$.
   (b) Evaluate $\overrightarrow{\nabla f}$ at P.
   (c) Find the directional derivative of f at P in the direction $\vec{u}$.
2. Prove:
   (a) $\overrightarrow{\nabla(f+g)} = \overrightarrow{\nabla f} + \overrightarrow{\nabla g}$, (b) $\overrightarrow{\nabla(fg)} = g\overrightarrow{\nabla f} + f\overrightarrow{\nabla g}$, (c) $\overrightarrow{\nabla(f/g)} = \left(g\overrightarrow{\nabla f} - f\overrightarrow{\nabla g}\right)/g^2$.
3. Find a unit vector $\hat{u}$ in which the directional derivative of $f(x,y) = x^2 + y^2 - xy + 5$ at $(1,-1)$ equals 0.
4. Find the maximum rate of change of $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ at the point $(3,6,-2)$ and the direction in which it occurs.
5. Find the unit vector $\hat{u}$ in which the directional derivative of $f(x,y) = x^2\sin(y)$ at $(2,\pi)$ is maximum.
6. Let $f(x,y,z) = x^2+y^2+z^2$. Find the maximum rate of change at the point $(1,1,1)$ and direction it occurs.
7. Let $f(x,y) = x^2+y^2-2x-4y$. Find all points at which the direction of the fastest change of the function f is $\hat{\imath}+\hat{\jmath}$.
   Hint: We have to find all points $(x,y) \in \mathbb{R}^2$ where $\overrightarrow{\nabla f}$ is parallel to $\hat{\imath}+\hat{\jmath}$.
8. Let $f(x,y,z) = x^2y + x\sqrt{1+z}$.
   (a) Find the directional derivative of f at $(1,2,3)$ in the direction $\vec{v} = \langle 2,1,-2\rangle$.
   (a) Find the maximum rate of change of f at the point $(1,2,3)$ and direction it occurs.
   (b) Find equation for the tangent plane to the surface given by $f(x,y,z) = 4$ at the point $(1,2,3)$.
   (c) Find parametric equations of the normal line to the surface given by $f(x,y,z) = 4$ at the point $(1,2,3)$.

14.6 Tangent Planes

Suppose S is a level surface with equation $F(x,y,z) = k$, and let $P(x_0,y_0,z_0)$ be a point on S. Let C be any curve that lies on S and passes through P.



Assume that the curve C is described by a continuous vector function $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$. Let t_0 be the parameter value corresponding to P; that is, $\vec{r}(t_0) = \langle x_0, y_0, z_0\rangle$. Since C lies on S, any point $(x(t), y(t), z(t))$ must satisfy the equation of S, that is,

$$F(x(t), y(t), z(t)) = k.$$

Differentiating both sides of this equation with respect to t leads to

$$\frac{d}{dt}F(x(t),y(t),z(t)) = \frac{d}{dt}[k] \Longrightarrow \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0 \text{ (Chain Rule)} \Longrightarrow$$
$$\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle = 0 \Longrightarrow \langle F_x, F_y, F_z\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle = 0 \Longrightarrow \overrightarrow{\nabla F} \cdot \vec{r}'(t) = 0.$$

In particular, when $t = t_0$ we have

$$\overrightarrow{\nabla F}(x_0, y_0, z_0) \cdot \vec{r}'(t_0) = 0.$$

This equation says that the gradient vector $\overrightarrow{\nabla F}$ at P is perpendicular to the tangent vector $r'(t_0)$ to any curve C on S that passes through P. Therefore it is natural to define the tangent plane to the level surface $F(x,y,z) = k$ at P as the plane that passes through P and has normal vector $\overrightarrow{\nabla F}(x_0,y_0,z_0)$.

Definition 14.11 The equation of the tangent plane at the point $P(x_0,y_0,z_0)$ to the level surface $F(x,y,z) = k$ can be written as

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0.$$

Remark 14.10 The normal line to the level surface $F(x,y,z) = k$ at the point $P(x_0,y_0,z_0)$ is the line passing through P and perpendicular to the tangent plane. The direction of the normal line is therefore given by the gradient vector $\overrightarrow{\nabla F}(x_0,y_0,z_0)$. Its symmetric equation is given by

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}.$$

Example 14.51 Find the equations of the tangent plane and normal line at $(-2,1,-3)$ to the ellipsoid $\frac{x^2}{4}+y^2+\frac{z^2}{9}=3$

Surfaces Given in the Form $z=f(x,y)$

In the special case in which the equation of a surface S is of the form $z=f(x,y)$, we can rewrite the equation as

$$F(x,y,z)=f(x,y)-z=0$$

and regard S as a level surface (with $k=0$) of F. Then

$$\overrightarrow{\nabla F}(x_0,y_0,z_0)=\langle f_x(x_0,y_0), f_y(x_0,y_0), -1\rangle,$$

and the equation of the tangent plane at the point $P(x_0,y_0,z_0)$ to the surface $\boxed{z=f(x,y)}$ becomes

$$\boxed{z-z_0=f_x(x_0,y_0)\,(x-x_0)+f_y(x_0,y_0)\,(y-y_0)\,,\quad \text{where } z_0=f(x_0,y_0).}$$

Example 14.52 Find the tangent plane to the elliptic paraboloid $z=2x^2+y^2$ at $(1,1)$.

Exercise 14.6

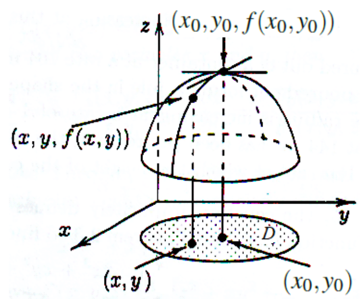
1. Find an equation of the tangent plane to $z=\ln(x-2y)$ at point $(3,1,0)$.
2. Find the equation of tangent plane and the normal line to the surface: $2(x-1)^2+(y-1)^2+(z-3)^2=10$, at $(3,3,5)$.
3. Show that the equation of the tangent plane to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, at $P(x_0,y_0,z_0)$ can be written as $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}+\frac{zz_0}{c^2}=1$.
4. At what points does the normal line through the point $(1,2,1)$ on the ellipsoid $4x^2+y^2+4z^2=12$ intersect the sphere $x^2+y^2+z^2=102$.
5. At what point the plane $2x-2y+3z-9=0$ touches the paraboloid $z=\frac{x^2}{4}+\frac{y^2}{2}$.
6. Find a point on the surface $S:\ 3x^2+2y^2+z^2+4z=14$ at which the tangent plane to the surface S is parallel to xz-plane.

14.7 Maximum and Minimum Values

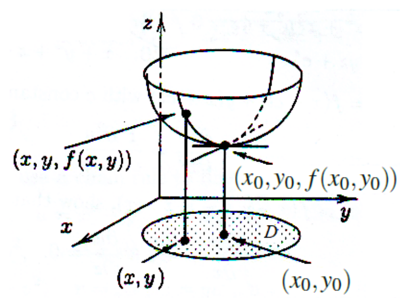
In this section we see how to use partial derivatives to locate maxima and minima of functions of two variables.

Definition 14.12

1. A function $f(x,y)$ has a **local maximum** at (x_0,y_0) if $f(x,y) \le f(x_0,y_0)$ when (x,y) is near (x_0,y_0) (*i.e.,* $f(x,y) \le f(x_0,y_0)$ *for all points* (x,y) *in some disk D with center* (x_0,y_0).). The number $f(x_0,y_0)$ is called a *local maximum value*. If the inequality holds for all points (x,y) in the domain of f, then f has an **absolute maximum** at (x_0,y_0).
2. A function $f(x,y)$ has a **local minimum** at (x_0,y_0) if $f(x,y) \ge f(x_0,y_0)$ when (x,y) is near (x_0,y_0) (*i.e.,* $f(x,y) \ge f(x_0,y_0)$ *for all points* (x,y) *in some disk D with center* (x_0,y_0).). The number $f(x_0,y_0)$ is called a *local minimum value*. If the inequality holds for all points (x,y) in the domain of f, then f has an **absolute minimum** at (x_0,y_0).

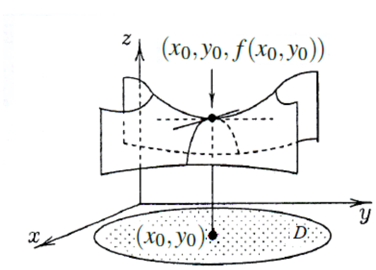


local maximum



local minimum

Definition 14.13 A function $f(x,y)$ has a **saddle point** at (x_0,y_0) if in every open disk D centered at (x_0,y_0) there are domain points (x,y) where $f(x,y) > f(x_0,y_0)$ and domain points (x,y) where $f(x,y) < f(x_0,y_0)$. The point (x_0,y_0) is called a saddle point.



saddle point

Theorem 14.5 If f has a local maximum or minimum at (x_0, y_0) and the first-order partial derivatives of f exist there, then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, i.e., $\overrightarrow{\nabla f}(x_0, y_0) = \langle 0, 0 \rangle$.

Remark 14.11 If we put $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ in the equation of a tangent plane at the point (x_0, y_0), we get $z = f(x_0, y_0)$. Thus the geometric interpretation of Theorem 14.5 is that if the graph of f has a tangent plane at a local maximum or minimum, then the tangent plane must be horizontal.

Definition 14.14 A point (x_0, y_0) is called a **critical point** of f if $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, or if one of these partial derivatives does not exist.

Thus if f has a local maximum or minimum at (x_0, y_0), then (x_0, y_0) is a critical point of f. However, not all critical points give rise to maxima or minima. At a critical point, a function could have a local maximum or a local minimum or neither.

Example 14.53 Find the critical points of f.

1. $f(x, y) = x^2 + y^2 - xy - x - 2$.

2. $f(x, y) = ye^{-2x^2 - 2y^2}$.

3. $f(x,y) = x^3 + y^5 - 3x - 10y + 4$.

4. $f(x,y) = x^3 + x^2y - y^2 - 4y$.

Second Derivatives Test

Suppose the second partial derivatives of $f(x,y)$ are continuous on a disk with center (x_0,y_0), and suppose that (x_0,y_0) is a critical point of f. Let

$$D := D(x_0,y_0) = f_{xx}(x_0,y_0)\, f_{yy}(x_0,y_0) - \left(f_{xy}(x_0,y_0)\right)^2.$$

1. If $D > 0$ and $f_{xx}(x_0,y_0) > 0$, then $f(x_0,y_0)$ is a local minimum.
2. If $D > 0$ and $f_{xx}(x_0,y_0) < 0$, then $f(x_0,y_0)$ is a local maximum.
3. If $D < 0$, then (x_0,y_0) is a saddle point of f.

Remark 14.12 If $D(x_0,y_0) = 0$, the test gives no information: $f(x,y)$ could have a local maximum or local minimum at (x_0,y_0), or (x_0,y_0) could be a saddle point of f.

Example 14.54 Find the local maximum and minimum values and saddle point(s) of the function.

1. $f(x,y) = y^2 - x^2$.

2. $f(x,y) = x^4 + y^4 - 4xy + 1$.

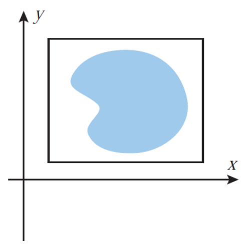
3. $f(x,y) = y(e^x - 1)$.

Absolute Maximum and Minimum Values

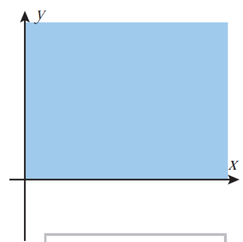
Definition 14.15 A **bounded** set in $\mathbb{R}^2$ is one that is contained within some rectangle (circle). A region is **unbounded** if it is not bounded.

Examples of *bounded* sets in the plane include triangles, interiors of triangles, rectangles, circles, and disks. Examples of *unbounded* sets in the plane include lines, coordinate axes, the graph of functions defined on infinite intervals, quadrants, half-planes, and the plane itself.

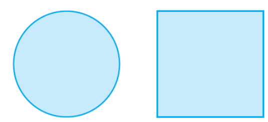
Definition 14.16 A **closed** set in $\mathbb{R}^2$ is one that contains all its boundary points.

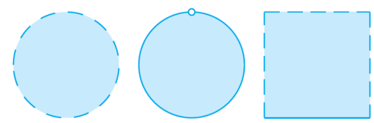


A bounded set in 2-space



An unbounded set in 2-space (the first quadrant)


(a) Closed sets


(b) Sets that are not closed

Theorem 14.6 — Extreme Value Theorem. If $f(x,y)$ is continuous on a **bounded** and **closed** set D, then f has both an absolute maximum and an absolute minimum on D.

Absolute Maxima and Minima on Bounded Closed Sets

To find the absolute extrema of a continuous function $f(x,y)$ on a **bounded**, **closed** set D:

1. Determine the critical points of f in D.
2. Calculate f at each of these critical points.
3. Determine the maximum and minimum values of f on the boundary points of D.
4. The absolute maximum and minimum values of f will occur at one of the values obtained in steps 2 and 3.

Example 14.55 Find the absolute maximum and minimum values of the function.

1. $f(x,y) = x^2 + y^2 - 2x$, D is the closed triangular region with vertices $(2,0)$, $(0,2)$, and $(0,-2)$.

2. $f(x,y) = 2x^3 + y^4$, $D = \{ (x,y) | x^2 + y^2 \leq 1 \}$.

3. $f(x,y) = xy^2$, $D = \{ (x,y) | x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq 3 \}$.

Exercise 14.7

1. Find the local maximum and minimum values and saddle point(s) of the functions.
   (a) $f(x,y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$.
   (b) $f(x,y) = \cos(y)e^{2x}$.
   (c) $f(x,y) = 2\ln(x) + \ln(y) - 4x - y$.
2. Find the absolute maximum and minimum values of the functions on the given domains.
   (a) $f(x,y) = x^2 - 2xy + 2y$, $D = \{(x,y) | 0 \le x \le 3, \quad 0 \le y \le 2\}$.
   (b) $f(x,y) = x^2 + y^2 - x - y + 1$, $D = \left\{(x,y) | x^2 + y^2 \le 1\right\}$.
   Hint: on the boundary let $x = \cos(t)$ and $y = \sin(t)$, so the restriction of f to the boundary is the function $f(t) = 2 - \cos(t) - \sin(t)$, $t \in [0, 2\pi]$.
3. Find the critical point of $f(x,y) = xy + 2x - \ln\left(x^2y\right)$ in the open fist quadrant $(x, y > 0)$ and show that f takes on a minimum there.
4. Find the minimum value of $f(x,y) = x^2 + y^2 - 4y + 8$ on $\left\{(x,y) | x^2 + y^2 \le 5,\ y \ge 1\right\}$.
5. Let $f(x,y) = x^2 + y^2 - xy - x - 2$. Find the local extrema of f.
6. Find the absolute maximum and minimum values of $f(x,y) = e^x\cos(y)$ on the set D, where D is the closed triangular region with vertices $(2,0)$, $(0,2)$ and $(0,-2)$.
7. Find the absolute maximum and minimum values of $f(x,y) = x^2 + 2x + y^2 + 2y$, constrained to the region bounded by the circle $x^2 + y^2 = 4$.
8. Find the absolute maximum and minimum values of $f(x,y) = x^2 + xy - 2x - \frac{y}{2}$ defined in the region bounded by the lines $y = x^2$ and $y = 2 - x^2$.
9. Find the shortest distance between the lines
$$\mathrm{L}_1 : \frac{x-2}{4} = \frac{y+1}{-7} = \frac{z+1}{1}, \qquad \mathrm{L}_2 : \frac{x-2}{-2} = \frac{y-1}{1} = \frac{z-2}{-3}.$$
10. Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.
11. A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

15. Multiple Integrals

Recall that, the partial derivatives of a function of several variables are calculated by holding one of the variables fixed and differentiating with respect to the other variable. Consider the reverse of differentiation, partial integration.

- The symbol $\int f(x,y)\,dx$ is a partial integral of $f(x,y)$ with respect to x. It is evaluated by holding y fixed and integrating with respect to x. For example,

$$\int 6x^2y\,dx = 2x^3y + C(y).$$

  Note that, the "constant" of integration here is any (differentiable) function of y, denoted by $C(y)$, since any such function would vanish upon partial differentiation with respect to x.
- The symbol $\int f(x,y,z)\,dz$ is a partial integral of $f(x,y,z)$ with respect to z. It is evaluated by holding $x,\ y$ fixed and integrating with respect to z. For example,

$$\int 6x^2y\,dz = 6x^2yz + C(x,y).$$

Remark 15.1 Since integration with respect to one variable treats others as constants, the limit of integration may be functions of the non-integrating variables.

Example 15.1 Evaluate:

1. $\displaystyle\int_1^5 \left(x^2y + \cos(xy)\right)\,dy.$

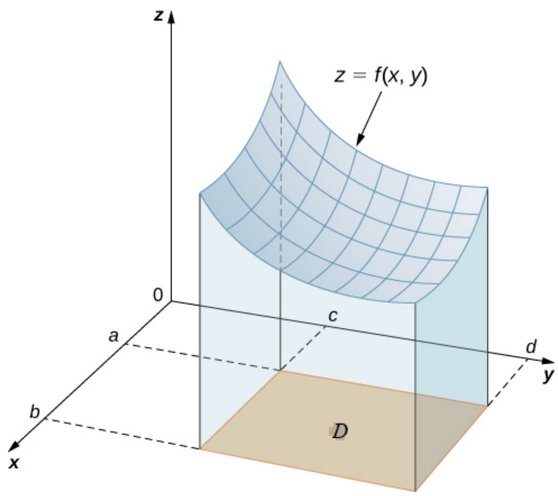
2. $\displaystyle\int_{y^2}^{3y+z} \left(\mathrm{e}^z y^2 x\right)\,dx.$

Remark 15.2 The variable of integration cannot appear as a limit for integration. For example, an expression like $\displaystyle\int_{x^2}^{2x+1} f(x,y)\,dx$ is not valid.

In this chapter we consider the integral of a function of two variables $f(x,y)$ over *a region in the plane* and the integral of three variables $f(x,y,z)$ over *a solid in space*.

15.1 Double Integrals Over Rectangles

Suppose that $f(x,y) \geq 0$ is a function of two variables that is integrable on the rectangular region $D = \left\{(x,y) \in \mathbb{R}^2 \ : \ a \leq x \leq b,\ c \leq y \leq d\right\}$.



Theorem 15.1 — Fubini's Theorem. Suppose that $f(x,y)$ is continuous on the rectangular region $D = \left\{(x,y) \in \mathbb{R}^2 \ : \ a \leq x \leq b,\ c \leq y \leq d\right\}$, then

$$\begin{aligned}\iint_D f(x,y)\,dA &= \int_a^b \int_c^d f(x,y)\,dydx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx \\ &= \int_c^d \int_a^b f(x,y)\,dxdy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy\end{aligned}$$

Remark 15.3 The rectangular region D can be expressed as $D = [a,b] \times [c,d]$.

Example 15.2 Evaluate $\displaystyle\iint_D (x+y)\,dA$ where $D = [0,3] \times [1,2]$.

Volume

We can interpret the double integral

$$V = \iint\limits_D f(x,y)\,dA$$

as the volume V of the solid that lies above D and under the surface $z = f(x,y)$.

Example 15.3 The integral $\iint\limits_D \sqrt{9-y^2}\,dA$, where $D = \left\{(x,y) \in \mathbb{R}^2 \;:\; 0 \leq x \leq 4,\; 0 \leq y \leq 2\right\}$, represents the volume of a solid. Sketch the solid.

Rule 15.1 If $f(x,y)$ is continuous on the rectangular region D and $f(x,y)$ can be factored as the product of a function of x only and a function of y only, that is $f(x,y) = g(x)h(y)$, then

$$\iint\limits_D f(x,y)\,dA = \int\limits_c^d \int\limits_a^b g(x)h(y)\,dxdy = \left[\int\limits_a^b g(x)\,dx\right]\left[\int\limits_c^d h(y)\,dy\right], \text{ where } D = [a,b]\times[c,d].$$

Example 15.4 Evaluate $\iint\limits_D \sin(x)\cos(y)\,dA$ where $D = [0,\pi/2]\times[0,\pi/3]$.

Remark 15.4 When we evaluate double integrals it is wise to choose the order of integration that gives simpler integrals.

Example 15.5 Evaluate $\iint_D y\sin(xy)\,dA$ where $D = [1,2]\times[0,\pi]$.

Exercise 15.1

1. Evaluate:
   (a) $\int_0^3\int_1^2 e^{x+2y}\,dxdy$.
   (b) $\int_0^{\pi/2}\int_{-3}^3 \left(y+y^2\cos(x)\right)\,dydx$.
   (c) $\iint_R \left(y+xy^{-2}\right)\,dA$, $R = [0,2]\times[1,2]$.
   (d) $\int_0^1\int_0^1 xy\sqrt{x^2+y^2}\,dydx$.
   (e) $\int_1^3\int_1^5 \frac{\ln(y)}{xy}\,dydx$.
   (f) $\int_0^{\pi/2}\int_0^1 \frac{\cos(x)}{1+y^2}\,dydx$.
2. The integral $\iint_D \sqrt{5-x^2-y^2}\,dA$, where $D = [0,1]\times[0,2]$, represents the volume of a solid. Sketch the solid.
3. The integral $\iint_D \left(4-\frac{2}{3}x-\frac{1}{3}y\right)\,dA$, where $D = [0,4]\times[0,1]$, represents the volume of a solid. Sketch the solid.
4. Find the volume of the solid below the function $f(x,y) = 9xy$ and above the region $R = [0,1]\times[0,1]$.
5. Find the volume of the solid enclosed by the paraboloid $z = 2+x^2+(y-2)^2$ and the planes $z = 1$, $x = 1$, $x = -1$, $y = 0$, and $y = 4$.
6. Find the volume of the solid S that is bounded by the elliptic paraboloid $x^2+2y^2+z = 16$, the planes $x = 2$, $y = 3$ and the three coordinates.
7. Find the volume of the solid in the first octant bounded by the cylinder $z = 4-x^2$ and the plane $y = 1$.

15.2 Double Integrals Over General Regions

For single integrals, the region over which we integrate is always an interval. But for double integrals, we want to be able to integrate a function f not just over rectangles but also over regions D of more general shape. Recall that, abounded region in the plane is one that is contained within some rectangle (or circle). A region is **unbounded** if it is not bounded.

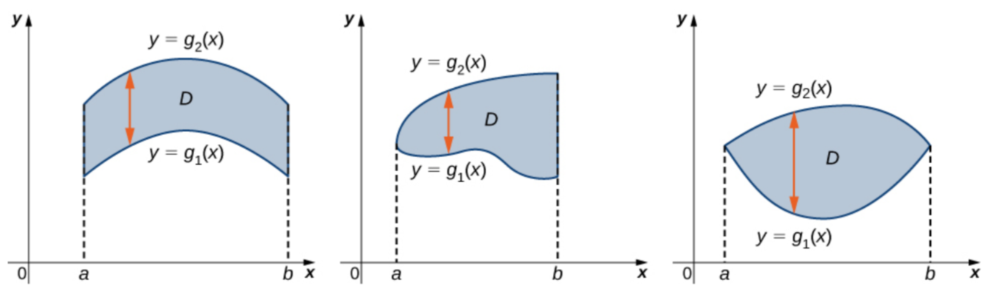
Consider a continuous function $f(x,y) \geq 0$ defined over a **bounded**, **nonrectangular region** D. We define the volume of the solid region between D and the surface $z = f(x,y)$ to be

$$V = \iint\limits_D f(x,y)\,dA.$$

To evaluate this integral, we consider two types of region D:

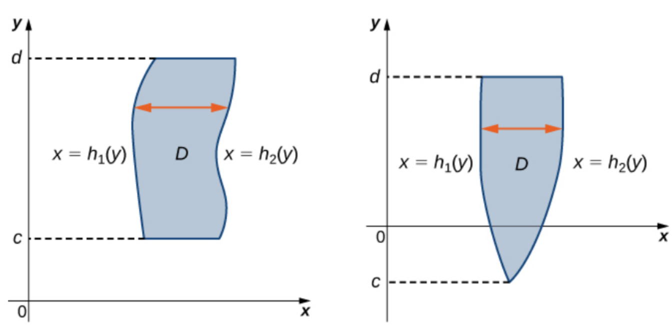
1. <u>**Type I region:**</u>
   If D is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a,b]$, then

$$\iint\limits_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx$$



2. <u>**Type II region:**</u>
   If D is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c,d]$, then

$$\iint\limits_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right] dy$$

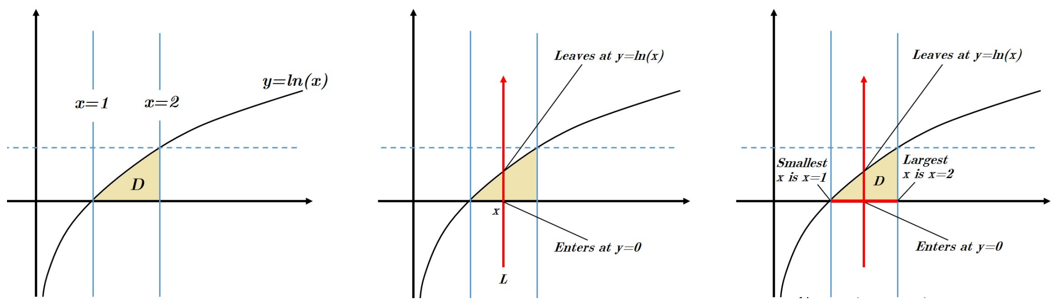


Finding Limits of Integration

Example 15.6 Evaluate $\iint\limits_D f(x,y)\,dA$ where D is the region in the first quadrant bounded by the x-axis, the line $x=1$, the line $x=2$ and the curve $y=\ln(x)$.

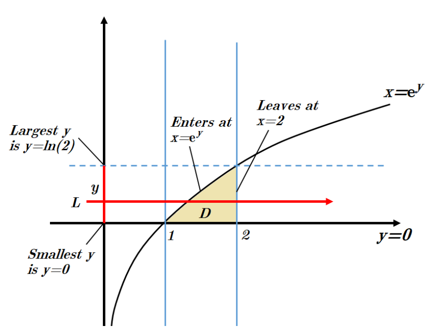
Solution

1. **Type I region** (*using vertical cross-sections*):
   To integrate first with respect to y and then with respect to x, do the following 3 steps:
   (a) **Sketch.** Sketch the region of integration D and label the bounding curves.
   (b) **Find the y-limits of integration.** Imagine a vertical line L cutting through D in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants).
   (c) **Find the x-limits of integration**. Choose x-limits that include all the vertical lines through D.



The integral is $\iint\limits_D f(x,y)\,dA = \int\limits_{x=1}^{x=2}\int\limits_{y=0}^{y=\ln(x)} f(x,y)\,dydx.$

2. **Type II region** (*using horizontal cross-sections*):
   To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in Steps (b) and (c).



The integral is $\iint\limits_D f(x,y)\,dA = \int\limits_{y=0}^{y=\ln(2)}\int\limits_{x=e^y}^{x=2} f(x,y)\,dxdy.$

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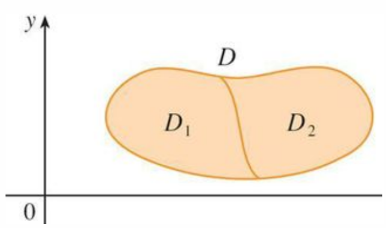
Properties of Double Integrals

If $f(x,y)$ and $g(x,y)$ are continuous functions on the bounded region D, then following properties hold:

1. Sum and Difference: $\iint_D [f(x,y) \pm g(x,y)]\, dA = \iint_D f(x,y)\, dA \pm \iint_D g(x,y)\, dA$.
2. Constant Multiple: $\iint_D c f(x,y)\, dA = c \iint_D f(x,y)\, dA$ where $c \in \mathbb{R}$.
3. Domination:
   (a) If $f(x,y) \geq 0$ for all (x,y) in D, then $\iint_D f(x,y)\, dA \geq 0$.
   (b) If $f(x,y) \geq g(x,y)$ for all (x,y) in D, then $\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA$.
4. Additivity: If $D = D_1 \cup D_2$ and $D_1 \cap D_2 = \phi$ except an overlap on the boundaries, then

$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA.$$



Example 15.7 Evaluate $\iint_D xy\, dA$ where D is the region bounded by the line $y = x - 1$ and parabola $y^2 = 2x + 6$.

To Switch the Order of Integration:

Sketch the region determined by the limits and use the graph to help you write new limits.

Example 15.8 Evaluate $\iint_D (3e^{x+y} + 2xy)\,dA$ where D is the region in the first quadrant bounded by $x^2 + y^2 = 1$ and $x + y = 1$.

Example 15.9 Sketch the region of integration for the integral $\int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,dxdy$ and write an equivalent integral with the order of integration reversed.

Example 15.10 Combine the sum of the two double integrals $\int_0^3 \int_0^{4x/3} e^{xy}\,dydx + \int_3^5 \int_0^{\sqrt{25-x^2}} e^{xy}\,dydx$

How Do We Decide $dxdy$ or $dydx$?

Consider both: (1) the shape of the region, and (2) the integrand. Usually one order of integration is easier than the other

(1) The shape of the region:

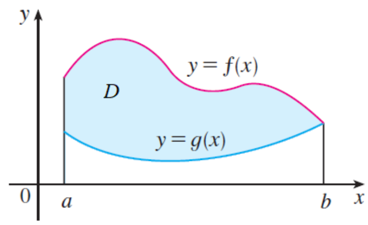
Example 15.11 Evaluate $\iint\limits_D (x+2y)\,dA$ where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1+x^2$.

(2) The integrand:

Example 15.12 Evaluate $\int_0^1 \int_x^1 \sin(y^2)\,dydx$.

Length and Area

$$\text{Length} = \int_a^b dx$$

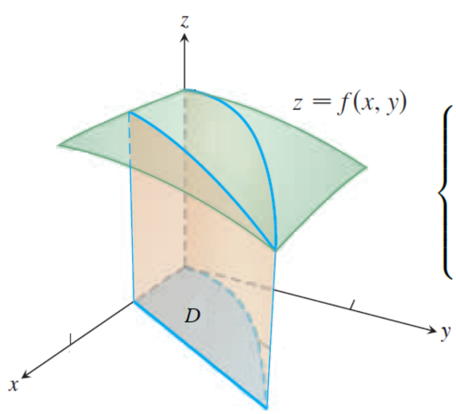


$$\begin{cases} \text{Area} = \displaystyle\int_a^b [f(x) - g(x)]\, dx \\ \text{Area} = \displaystyle\iint_D dA \end{cases}$$

Example 15.13 Find the area of the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Volume

Rule 15.2 The volume under a surface $z = f(x,y)$ over a bounded region D is $\displaystyle\iint_D f(x,y)\, dA$.



$$\Big\{ \text{Volume} = \iint_D f(x,y)\, dA$$

Example 15.14 Find the volume of the solid lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Rule 15.3 If $m \leq f(x,y) \leq M$ for all (x,y) in D, then $m(A(D)) \leq \iint\limits_D f(x,y)\,dA \leq M(A(D))$ where $A(D)$ is the area of the region D.

Example 15.15 Estimate the integral $\iint\limits_D \mathrm{e}^{\sin(x)\cos(y)}\,dA$ where $D = \{(x,y) \mid x^2+y^2 \leq 4\}$.

Exercise 15.2

1. Evaluate:
   (a) $\int_0^1 \int_x^1 e^{x/y}\,dydx$.
   (b) $\int_0^1 \int_x^1 \cos(y^2)\,dydx$.
   (c) $\int_0^3 \int_y^3 e^{-x^2}\,dxdy$.
   (d) $\iint\limits_D x\cos(y)\,dA$, where D is bounded by $y=0$, $y=x^2$, and $x=1$.
   (e) $\int_0^1 \int_{\sqrt{x}}^1 \sqrt{a+y^3}\,dydx$ where $a \in \mathbb{R}$.
2. Find the area of the triangle with vertices at $(1,1)$, $(3,1)$ and $(5,5)$.
3. Sketch the region of integration and change the order of integration of $\int_0^{\pi/2} \int_0^{\cos(x)} f(x,y)\,dydx$.
4. Find the area of the region enclosed by $y=2x$ and $y=x^2$.
5. Find the volume of the solid under the plane $3x+2y-z=0$ and above the region enclosed by the parabolas $y=x^2$ and $x=y^2$.
6. Find the volume of the solid enclosed by the planes $x=0$, $y=0$, $z=0$, $x+y=2$, and the paraboloid $z=x^2+y^2$.
7. Combine the sum of the two double integrals
   (a) $\int_0^1 \int_0^{2y} f(x,y)\,dxdy + \int_1^3 \int_0^{3-y} f(x,y)\,dxdy$.
   (b) $\int_0^2 \int_{-2x}^0 \mathrm{e}^{x^2}\,dydx + \int_2^4 \int_{2x-8}^0 \mathrm{e}^{x^2}\,dydx$.
   (c) $\int_0^2 \int_0^{2y} (2x^2y)\,dxdy + \int_2^4 \int_0^{-2y+8} (2x^2y)\,dxdy$.

15.3 Double Integrals in Polar Coordinates

Integrals are sometimes easier to evaluate if we change rectangular coordinates to polar coordinates.

Changing Rectangular Integrals Into Polar Coordinates

To change a rectangular integral $\iint\limits_D f(x,y)\,dA$ into a polar integral:

1. Substitute $x = r\cos\theta$, $y = r\sin\theta$, and replace dA by $rdrd\theta$ in the rectangular integral.
2. Supply polar limits of integration for the boundary of D.

The Rectangular integral then becomes:

$$\iint\limits_D f(x,y)\,dA = \iint\limits_R f(r\cos\theta, r\sin\theta)rdrd\theta$$

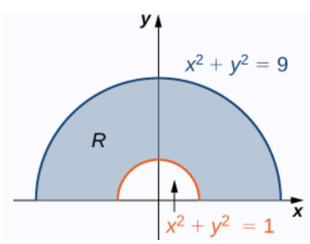
where R denotes the same region D now described in polar coordinates.

Remark 15.5 The area differential dA is not replaced by $drd\theta$ but by $rdrd\theta$.

Note that all the properties listed in page (109) hold true for the double integral in polar coordinates as well, so we can use them without hesitation.

Example 15.16 Sketch the polar rectangular region $R = \{(r,\theta)\ :\ 1 \le r \le 3,\ \ 0 \le \theta \le \pi\}$.

Solution



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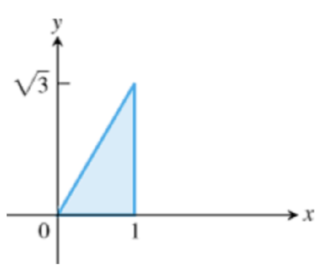
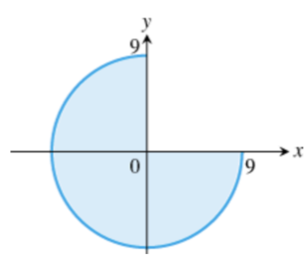
Example 15.17 Describe the given region in polar coordinates.





Example 15.18 Evaluate $\displaystyle\iint_D (3x+4y)\,dA$ where $D = \{(x,y) \mid 1 \le x^2+y^2 \le 4,\ x \ge 0,\ y \ge 0\}$.

Example 15.19 Evaluate $\displaystyle\iint_D \left(4-x^2-y^2\right) dA$ where D is the disk of radius 2 on the xy-plane.

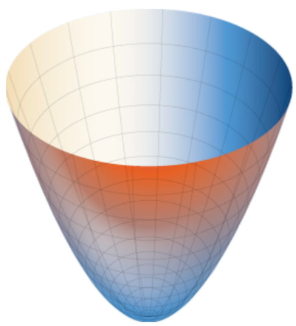
Example 15.20 Find the volume of the solid that lies under the paraboloid $z = 4 - x^2 - y^2$ and above the disk $(x-1)^2 + y^2 = 1$ on the xy-plane.

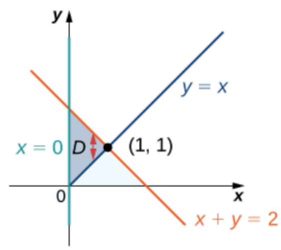
Example 15.21 Evaluate $\int_0^\infty e^{-x^2}\, dx$.

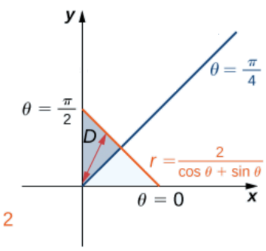
Remark 15.6 If the region D has a more natural expression in polar coordinates (*circular symmetry*) or if f has a simpler anti-derivative in polar coordinates, then the change in polar coordinates is appropriate; otherwise, use rectangular coordinates.

Example 15.22 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.

Solution







- In polar coordinates:

$$V = \int_{\pi/4}^{\pi/2} \int_0^{2/(\cos\theta+\sin\theta)} r^2\, r\, dr\, d\theta = 4\int_{\pi/4}^{\pi/2} \frac{1}{(\cos\theta+\sin\theta)}\, d\theta! \quad \text{Very complicated integral.}$$

- In rectangular coordinates: $V = \displaystyle\int_0^1 \int_x^{2-x} \left(x^2+y^2\right) dy\, dx = \frac{4}{3}.$

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Exercise 15.3

1. Evaluate the iterated integral by converting to polar coordinates.
   (a) $\displaystyle\int_{-3}^{3}\int_0^{\sqrt{9-x^2}} \sin\left(x^2+y^2\right) dy\, dx.$ (b) $\displaystyle\int_0^2\int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx.$
   (c) $\displaystyle\int_0^a\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (2x+y)\, dx\, dy.$ (d) $\displaystyle\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}\, dx\, dy.$
   (e) $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}} \frac{2}{1+x^2+y^2}\, dy\, dx.$
2. Evaluate $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx.$
3. Find the area of a circle with radius 2.
4. Find the area of the region inside the circle $(x-1)^2 + y^2 = 1$ and outside the circle $x^2 + y^2 = 1$.
5. Find the volume of the solid below the sphere $x^2 + y^2 + z^2 = 1$ and above the cone $z = \sqrt{x^2+y^2}$.
6. Combine the two double integrals into a single integral in polar coordinates, then evaluate. Sketch the region from each double integral.

$$\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \frac{x}{x^2+y^2}\, dy dx + \int_0^{\sqrt{3}} \int_1^{\sqrt{4-y^2}} \frac{x}{x^2+y^2}\, dx dy.$$

15.4 Triple Integrals

Just as we defined single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables. Let's first deal with the simplest case where f is defined on a rectangular box.

Triple Integrals Over Rectangular Box

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

Theorem 15.2 — Fubini's Theorem for Triple Integrals. If $f(x,y,z)$ is continuous on the box $S=\{(x,y,z): a\le x\le b, c\le y\le d, e\le z\le f\}$, then

$$\iiint\limits_S f(x,y,z)\,dV=\int_a^b\int_c^d\int_e^f f(x,y,z)\,dz dy dx.$$

There are five other possible orders in which we can integrate, all of which give the same value. For instance, if we integrate with respect to y, then z, and then x, we have

$$\iiint\limits_S f(x,y,z)\,dV=\int_a^b\int_e^f\int_c^d f(x,y,z)\,dy dz dx$$

Remark 15.7 Triple integrals have the same algebraic properties as double integrals. Simply replace the double integrals in the five properties given in page (109) with triple integrals.

Example 15.23 Evaluate $\iiint\limits_S \left(x^2+y^2+z^2\right)\,dV$ where $S=[1,3]\times[0,2]\times[2,4]$.

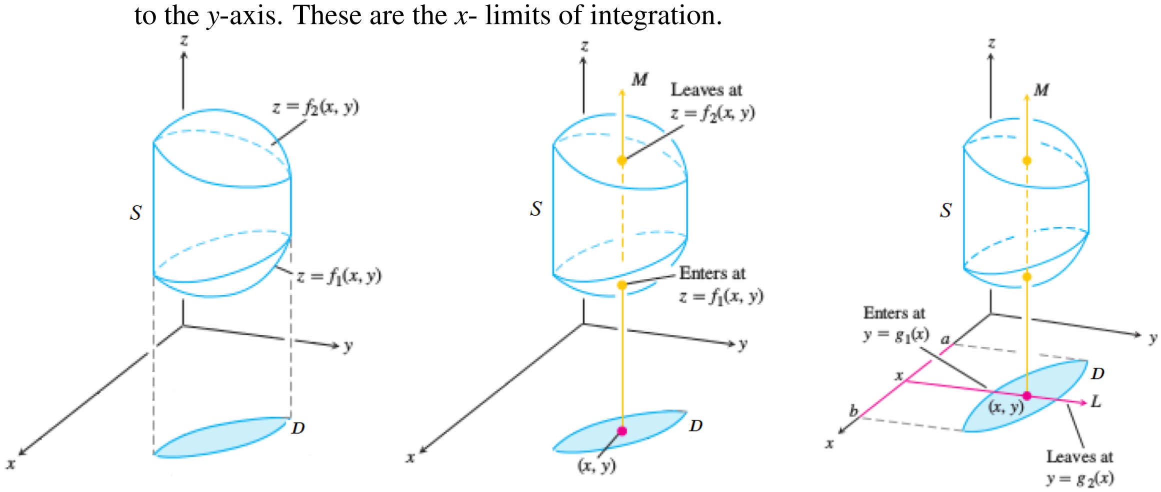
Example 15.24 Evaluate $\iiint\limits_S xyz^2\,dV$ where $S=[0,1]\times[-1,2]\times[0,3]$.

Triple Integrals Over General Region

We define the triple integral over a general bounded region S in three dimensional space (a solid) by much the same procedure that we used for double integrals.

To evaluate $\iiint\limits_S f(x,y,z)\,dV$ over a general bounded region S:

1. **Sketch.** Sketch the region S along with its "shadow" D (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of S. The region D in the xy-plane may be of Type I or Type II as described in Section 15.2.
2. **Find the z-limits of integration.** Draw a line M passing through a typical point (x,y) in D parallel to the z-axis. As z increases, M enters S at $z = f_1(x,y)$ and leaves $z = f_2(x,y)$. These are the $z-$limits of integration.
3. If D in the xy-plane is of Type I, then:
   (a) **Find the y-limits of integration.** Draw a line L through (x,y) parallel to the y-axis. As y increases, L enters D at $y = g_1(x)$ and leaves $y = g_2(x)$. These are the y-limits of integration.
   (b) **Find the x-limits of integration**. Choose x-limits that include all lines through D parallel to the y-axis. These are the x- limits of integration.



Hence the triple integral becomes $\iiint\limits_S f(x,y,z)\,dV = \int\limits_{x=a}^{x=b}\int\limits_{y=g_1(x)}^{y=g_2(x)}\int\limits_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,dzdydx.$

Remark 15.8 If D in the xy-plane is of Type II, then the triple integral becomes

$$\iiint\limits_S f(x,y,z)\,dV = \int\limits_{y=c}^{y=d}\int\limits_{x=G_1(y)}^{x=G_2(y)}\int\limits_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,dzdxdy.$$

In general

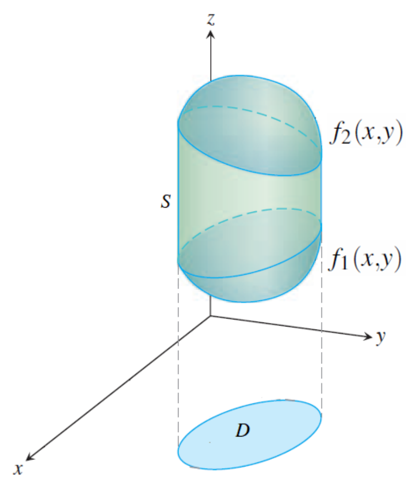
$$\iiint\limits_S f(x,y,z)\,dV = \iint\limits_D \left[\int\limits_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,dz\right] dA$$

where D could be rectangular or general region.

Volume

The volume V under a surface $z = f(x,y)$ over a region D is $V = \iint\limits_D f(x,y)\,dA$ (see Section 15.2). It follows that if $f_2(x,y) \geq f_1(x,y)$ on D, then the volume between $f_2(x,y)$ and $f_1(x,y)$ on D is

$$V = \iint\limits_D f_2(x,y)\,dA - \iint\limits_D f_1(x,y)\,dA = \iint\limits_D [f_2(x,y) - f_1(x,y)]\,dA = \iint\limits_D \left[\int\limits_{f_1(x,y)}^{f_2(x,y)} dz\right] dA.$$



Rule 15.4 The volume V of a general bounded region S is denoted by a triple integral

$$\text{Volume} = \iiint\limits_S dV = \iint\limits_D \left[\int\limits_{f_1(x,y)}^{f_2(x,y)} dz\right] dA.$$

Example 15.25 Find the volume of the solid tetrahedron bounded by the four planes $x = 2y$, $x = 0$, $z = 0$ and $x + 2y + z = 2$:

1. using a double integrals,
2. using a triple integrals.

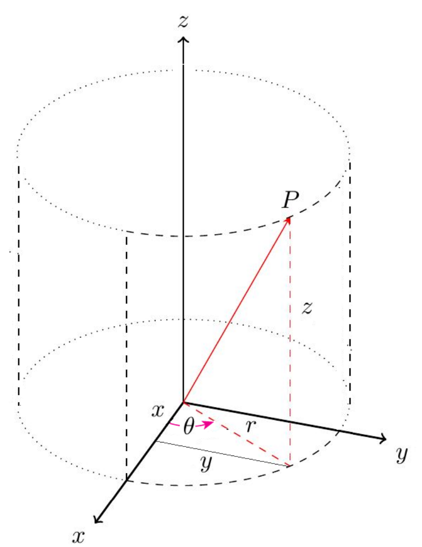
Example 15.26 Setup the integral to find the volume of the solid enclosed by the cylinders $y = 1 - x^2$, $y = x^2 - 1$ and the planes $x + y + z = 2$ and $2x + 2y - z = -10$.

Exercise 15.4

1. Evaluate $\displaystyle\iiint_S \sqrt{x^2+z^2}\,dV$, where S is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.
2. Evaluate $\displaystyle\iiint_S xy\,dV$, where S is bounded by the parabolic cylinders $y = x^2$ and $x = y^2$ and the planes $z = 0$ and $z = x + y$.
3. Find the volume of the solid enclosed by the cylinder $x^2 + z^2 = 4$ and the planes $y = -1$ and $y + z = 4$.
4. Setup the integral to find the volume of the solid enclosed by the cylinder $y = 1 - x^2$, $y = x^2 - 1$, and the planes $x + y + z = 2$, $2x + 2y - z + 10 = 0$.

15.5 Triple Integrals in Cylindrical Coordinates

We obtain cylindrical coordinates for space by combining polar coordinates in the xy-plane with the usual z-axis. This assigns to every point $P(x,y,z)$ in space one or more coordinate triples of the form $P(r,\theta,z)$, $r \geq 0$, $0 \leq \theta \leq 2\pi$.



Relations Between Reactangular and Cylinderical Coordinates

1. If the point in rectangular coordinates $P(x,y,z)$ the cylindrical coordinates $P(r,\theta,z)$ can be found by using the following conversions:

$$r = \sqrt{x^2+y^2} \text{ (or } r^2 = x^2+y^2), \quad \theta = \tan^{-1}\left(\frac{y}{x}\right), \quad z = z.$$

2. If the point in cylindrical coordinates $P(r,\theta,z)$ the rectangular coordinates $P(x,y,z)$ can be found by using the following conversions:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

Example 15.27

1. Plot the point with cylindrical coordinates $(2,\ 2\pi/3,\ 1)$. Find its rectangular coordinates.

2. Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

3. Identify the surface whose equation in cylindrical coordinates is
   (a) $z = r$.

   (b) $r = k$, (k is positive constant).

   (c) $\theta = c$, (c is constant).

   (d) $z = 4 - r^2$.

   (e) $2r^2 + z^2 = 1$.

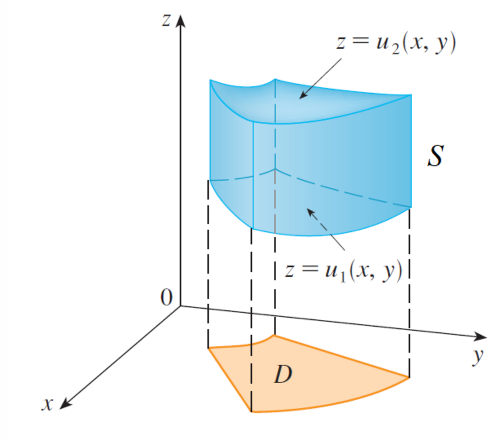
Triple Integrals Over Solid in Cylindrical Coordinates

To change a rectangular integral

$$\iiint_S f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

into a cylindrical integral:

1. Substitute $x = r\cos\theta$, $y = r\sin\theta$, and replace dV by $r dz dr d\theta$ in the rectangular integral.
2. Supply polar limits of integration for the boundary of D.



The rectangular integral then becomes:

$$\iiint_S f(x,y,z)\,dV = \iint_R \left[\int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,dz \right] r\,dr\,d\theta,$$

where R denotes the same region D now described in polar coordinates.

Remark 15.9 The volume differential dV is not replaced by $dz dr d\theta$ but by $r dz dr d\theta$.

Example 15.28 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

Example 15.29 Evaluate $\iiint\limits_S \sqrt{x^2+y^2}\,dV$ where S is the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

Example 15.30 Find the volume of the solid S lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$.

Example 15.31 Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}\left(x^2+y^2\right)dz\,dy\,dx.$

Example 15.32 Find the volume of the solid that lies within both the cylinder $x^2+y^2=4$ and the sphere $x^2+y^2+z^2=49$.

Example 15.33 Sketch the solid whose volume is given by the integral $\displaystyle\int_{0}^{4}\int_{0}^{2\pi}\int_{0}^{4-r} r\,dz\,d\theta\,dr$

Example 15.34 Use (i) rectangular coordinates, and (ii) cylindrical coordinates to evaluate

$$\iiint\limits_B \mathrm{e}^{\left(x^2+y^2+z^2\right)^{1/2}}\,dV \text{ where } B \text{ is the unit ball.}$$

.

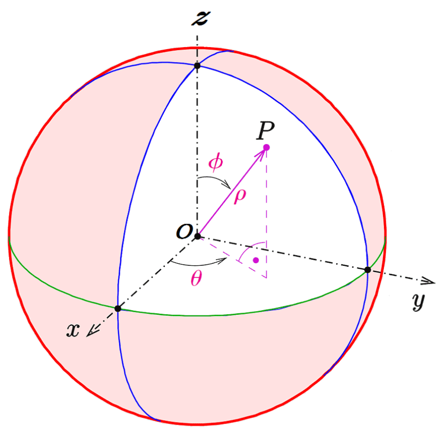
Exercise 15.5

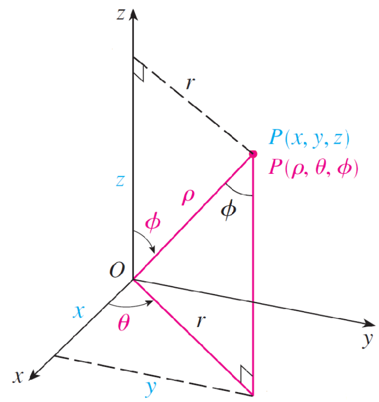
1. (a) Plot the point with cylindrical coordinates $\left(\sqrt{2}, \dfrac{3\pi}{4}, 2\right)$ and find its rectangular coordinates.

   (b) Find cylindrical coordinates of the point with rectangular coordinates $(-\sqrt{2}, \sqrt{2}, 1)$.
2. Find the volume of the solid S that lies within the cylinder $x^2+y^2=1$ and the sphere $x^2+y^2+z^2=4$.
3. Find the volume of the solid S in the first octant bounded by $z=x^2+y^2$ and $z=36-3x^2-3y^2$.
4. Evaluate $\displaystyle\iiint\limits_E xe^{x^2+y^2+z^2}\,dV$, where E is the portion of the unit ball $x^2+y^2+z^2\leq 1$ that lies in the first octant.
5. Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xz\,dz\,dy\,dx$.
6. Evaluate $\displaystyle\iiint\limits_E x\,dV$, where E is the solid enclosed by the xy-plane, $z=x+y+5$, and the cylinders $x^2+y^2=4$, $x^2+y^2=9$.
7. Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{0}\int_{\sqrt{x^2+y^2}}^{2} xz\,dz\,dx\,dy$.
8. Find the volume of the solid that lies between the paraboloid $z=24-x^2-y^2$ and the cone $z=2\sqrt{x^2+y^2}$.
9. Sketch the solid whose volume is given by the integral $\displaystyle\int\limits_{-\pi/2}^{\pi/2}\int\limits_0^2\int\limits_0^{r^2} r\,dz\,dr\,d\theta$.
10. Find the volume V of the solid that lies below the paraboloid $x^2+y^2+z=a$ and above the plane $z=b$ where $a>b$.

15.6 Triple Integrals in Spherical Coordinates

Another useful coordinate system in three dimensions is the spherical coordinate system. It simplifies the evaluation of triple integrals over regions bounded by spheres or cones. Spherical coordinates locate points in space with two angles and one distance, as shown in the Figure below.





Spherical coordinates represent a point $P(x,y,z)$ in space by ordered triples $P(\rho,\theta,\varphi)$ in which

1. $\rho = |OP|$ is the distance from P to the origin $(\rho \geq 0)$,
2. θ is the same angle as in cylindrical coordinates, and
3. φ is the angle between the positive z-axis and the line segment OP. Note that $0 \leq \varphi \leq \pi$.

Relations Between Reactangular and Spherical Coordinates

- If the point in rectangular coordinates (x,y,z) the spherical coordinates (ρ,θ,φ) can be found by using the following conversions:

$$\rho = \sqrt{x^2+y^2+z^2},\ \theta = \tan^{-1}(y/x),\ \varphi = \cos^{-1}\left(z/\sqrt{x^2+y^2+z^2}\right).$$

- If the point in spherical coordinates (ρ,θ,φ) the rectangular coordinates (x,y,z) can be found by using the following conversions:

$$x = \rho \sin\varphi \cos\theta,\ y = \rho \sin\varphi \sin\theta,\ z = \rho \cos\varphi.$$

Example 15.35 The point $(2,\pi/4,\pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

Relations Between Cylindrical and Spherical Coordinates

- If the point in spherical coordinates (ρ, θ, φ) the cylinderical coordinates (r, θ, z) can be found by using the following conversions:

$$r = \rho \sin\varphi, \quad \theta = \theta, \quad z = \rho \cos\varphi.$$

- If the point in cylindrical coordinates (r, θ, z) the spherical coordinates (ρ, θ, φ) can be found by using the following conversions:

$$\rho = \sqrt{r^2 + z^2}, \quad \theta = \theta, \quad \varphi = \cos^{-1}\left(z/\sqrt{r^2 + z^2}\right).$$

Example 15.36 The point $(2\sqrt{3}, \pi/2, -2)$ is given in cylindrical coordinates. Find spherical coordinates for this point.

Example 15.37

1. Identify the surface whose spherical equations is

   a) $\varphi = c$, (c is constant).

   (b) $\rho = k$, (k is positive constant).

   (c) $\rho = \sin(\varphi)\sin(\theta)$.

   (d) $\rho \leq \csc(\varphi)$.

   (e) $\rho\left(\sin^2(\varphi)\sin^2(\theta) + \cos^2(\varphi)\right) = \sin(\varphi)\cos(\theta)$.

2. Write the following equations in spherical coordinates:
   (a) $x^2 - 2x + y^2 + z^2 = 0$.

   (b) $z^2 = x^2 + y^2$.

Triple Integrals Over Solid in Spherical Coordinates

To change a rectangular integral $\displaystyle\iiint_S f(x,y,z)\,dV$ into a spherical integral:

1. Substitute $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, $z = \rho\cos\varphi$ and replace dV by $\rho^2\sin\varphi d\rho d\theta d\varphi$ in the rectangular integral.
2. Supply spherical limits of integration for the boundary of the solid S.

The rectangular integral then becomes:

$$\iiint_S f(x,y,z)\,dV = \iiint_E f(\rho\sin\varphi\cos\theta,\, \rho\sin\varphi\sin\theta,\, \rho\cos\varphi)\,\rho^2\sin\varphi\; d\rho d\theta d\varphi,$$

where E denotes the same solid region S now described in spherical coordinates.

Remark 15.10 The volume differential dV is not replaced by $d\rho d\theta d\varphi$ but by $\rho^2\sin(\varphi)\,d\rho d\theta d\varphi$.

Example 15.38 Evaluate $\displaystyle\iiint_E \mathrm{e}^{\left(x^2+y^2+z^2\right)^{1/2}}\,dV$ where E is the unit ball.

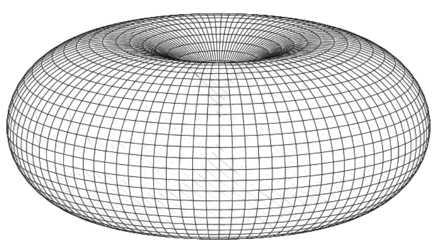
Example 15.39 Find the volume of the solid that lies between the spheres $x^2+y^2+z^2=9$ and $x^2+y^2+z^2=16$.

Example 15.40 Find the volume of the solid that lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$.

Example 15.41 Convert $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} \left(x^2+y^2+z^2\right) dz\,dx\,dy$ into spherical coordinates.

Exercise 15.6

1. (a) Plot the point with spherical coordinates $\left(3, \frac{\pi}{2}, \frac{3\pi}{4}\right)$ and find its rectangular coordinates.
   (b) Find spherical coordinates of the point with rectangular coordinates $(0,-2,0)$.
2. Evaluate the integral by changing to spherical coordinates $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx$.
3. Find the volume of a sphere of radius a.
4. Find the volume of the torus $\rho = \sin\phi$.



5. Evaluate $\iiint_E y\,dV$, where E is the portion in the first octant of the first hemisphere $x^2+y^2+z^2 \leq 9$, $y \geq 0$.
6. Find the volume of the solid S that enclosed by the cone $z = -\sqrt{3x^2+3y^2}$ and the sphere $x^2+y^2+z^2 = 16$.
7. Find the volume of the solid S that lies between the two spheres $x^2+y^2+z^2 = 4$ and $x^2+y^2+z^2 = 9$ and above the cone $z = -\sqrt{\dfrac{x^2+y^2}{3}}$
8. Sketch the solid whose volume is given by the integral $\int_0^{\pi/6} \int_0^{\pi/2} \int_0^1 \rho^2 \sin(\varphi)\,d\rho\,d\theta\,d\varphi$.

15.7 Finding Equivalent Triple Integrals (Skipped)

For triple integrals, there can be as many as six, since there are six ways of ordering dx, dy, and dz. Each ordering leads to a different description of the region of integration in space, and to different limits of integration.

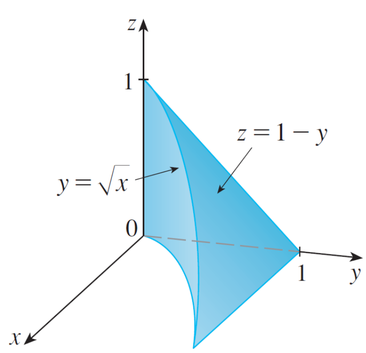
Example 15.42 Express the iterated integral $\int_0^1 \int_0^{x^2} \int_0^y dzdydx$ as a triple integral and then rewrite it as an iterated integral in a different order $dxdydz$, $dxdzdy$, $dydxdz$, $dydzdx$, $dzdxdy$.

As we have already seen in double integrals over general bounded regions in Section 15.2, changing the order of the integration is done quite often to simplify the computation. With a triple integral over a rectangular box, the order of integration does not change the level of difficulty of the calculation. However, with a triple integral over a general bounded region, choosing an appropriate order of integration can simplify the computation quite a bit. Sometimes making the change to polar coordinates can also be very helpful.

Example 15.43 Evaluate $\iiint\limits_S \sqrt{x^2+z^2}\,dV$ where S is the region bounded by the paraboloid $y = x^2+z^2$ and the plane $y = 4$.

Exercise 15.7

1. The figure shows the region of integration for the integral $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dz\,dy\,dx$. Rewrite this integral as an equivalent iterated integral in the five other orders.



2. Write five other iterated integrals that are equal to the following iterated integral.

$$\int_0^1 \int_y^1 \int_0^z f(x,y,z)\,dx\,dz\,dy.$$