Complex Numbers – Part 1

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Numbers

- Historical succession of discovering classes of numbers:

- Natural numbers: counting.
- Integers: we added zero and the –ve numbers.
- Rational numbers: fraction of two integers m/n.
- Real numbers: rational and irrational ($\pi, e, \sqrt{2}$).
- Complex numbers: $\sqrt{-1}$.

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The numbers zero and 10

- The number zero and the decimal numbers were first defined by Brahmagupta (Indian 598 – 668 AD).
- Al-Khwarizmi (Persian: 780 – 850 AD) (father of algebra) documented this work and introduced the Arabic numerals.
- Older numbering systems such as the Roman numerals were used: I = 1, V = 5, X = 10, L = 50, C = 100, D = 500, M = 1,000.

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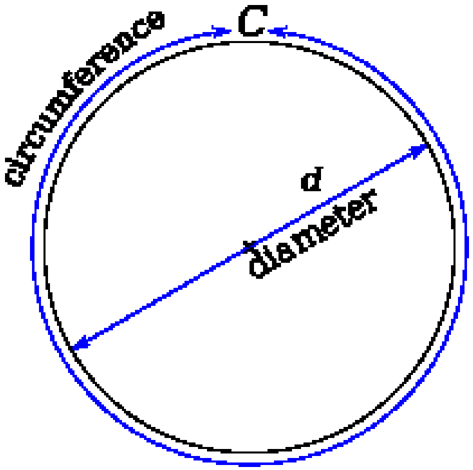
The number π

- The Egyptians and Babylonians used approximate values.
- Archimedes (Greek: 287 – 212 BC) created an algorithm for calculating π and defined the area of a circle, the surface area and volume of a sphere.
- $\pi = C/d$
- $\pi = 3.14159\ldots$
- $\pi \approx 22/7$, $333/106$, and $355/113$



The number π

- Angles can be measured in radians or rad.
- $180^o \equiv \pi$
- Angle (degrees) x $\pi/180$ = angle (rad)

The number e

- Euler's number e = 2.71828… was Introduced by Euler (German: 1707 – 1783 AD).

- Natural growth. Compound interest.

1 JD (100% per year) → 2 JD

1 JD (50% per 0.5 year) → 1JD x $(1+50\%)^2$ → 2.25 JD

1 JD (1/12 per month) → 1JD x $(1+1/12)^{12}$ → 2.61 JD

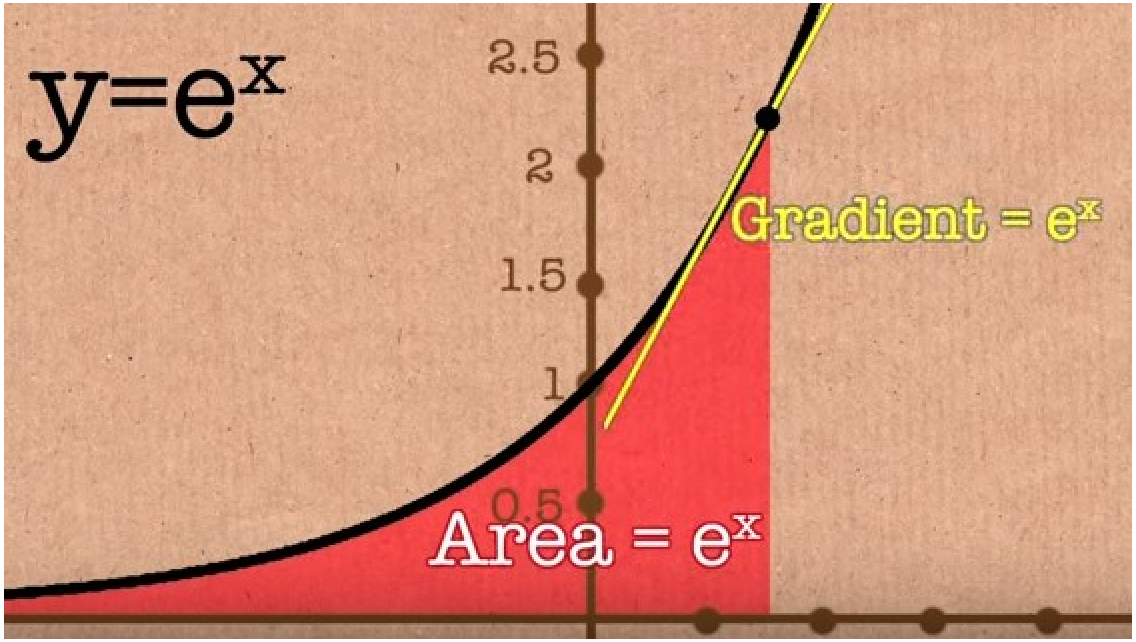
1 JD (1/52 per week) 1JD x $(1+1/52)^{52}$ → 2.69 JD

1 JD (1/365 per day) 1JD x $(1+1/365)^{365}$ → 2.71 JD

lim of $(1+1/n)^n$ as $n \to \infty$ is e = 2.71828…

The number e

- Euler's formula for e $$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots$$
- For y=e^x: value = rate of change = area



Youtube: e (Euler's Number) – Numberphile

- Later we will show that: $e^{i\pi} + 1 = 0$

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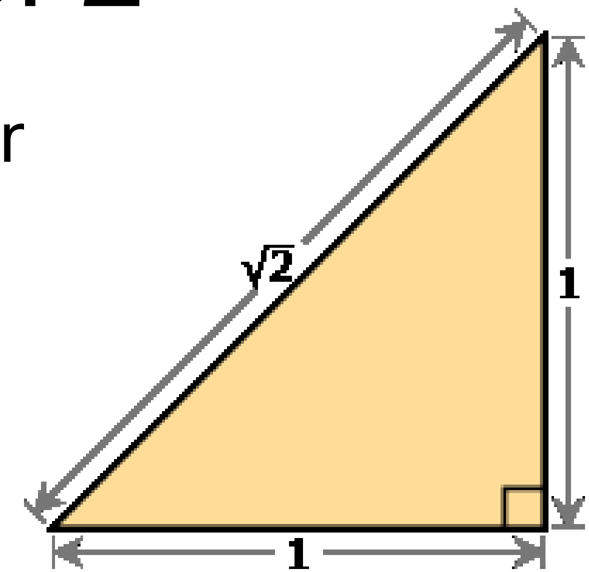
Square root of 2

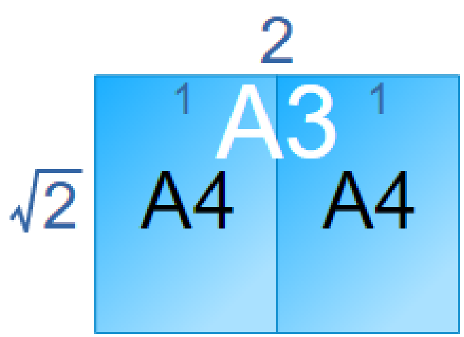


- Hypotenuse of isosceles right triar

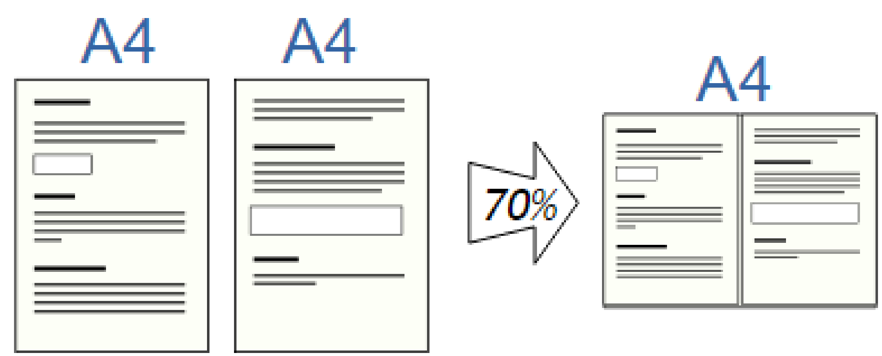
$= \sqrt{1^2 + 1^2} = \sqrt{2} = 1.41421\ldots$

- $\frac{1}{\sqrt{2}} = \sin(45) = \cos(45)$

- Paper sizes



Two **A4**s make an **A3**
and have the same proportions

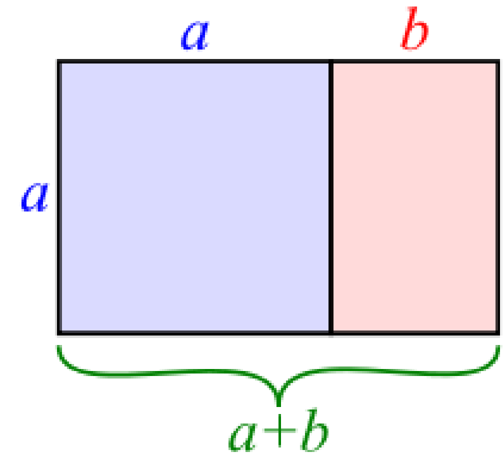


$1/\sqrt{2}$ = **0.7071...** which is close to **70%**

F.Y.I.

The golden ratio (φ)

- $$\frac{a+b}{a} = \frac{a}{b} \stackrel{\text{def}}{=} \varphi,$$

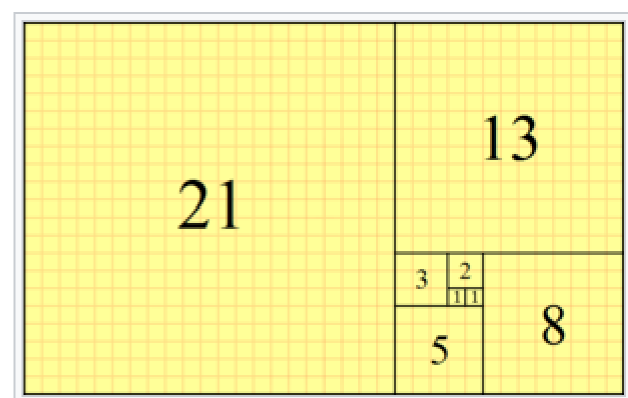


- Solution of $x^2 - x - 1 = 0$

$$\varphi = \frac{2+\sqrt{5}}{2} = 1.61803$$

- Euclid (Greek: 365 – 300 BC) (father of geometry) studied the properties of φ

- Relationship with Fibonacci sequence



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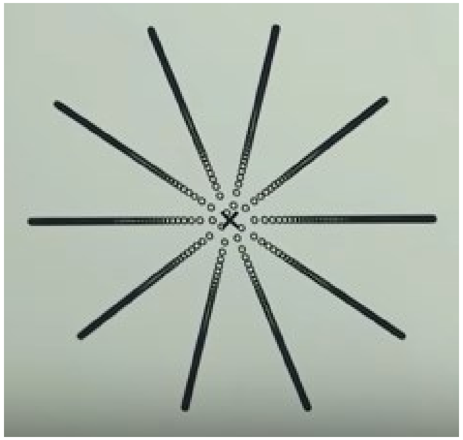
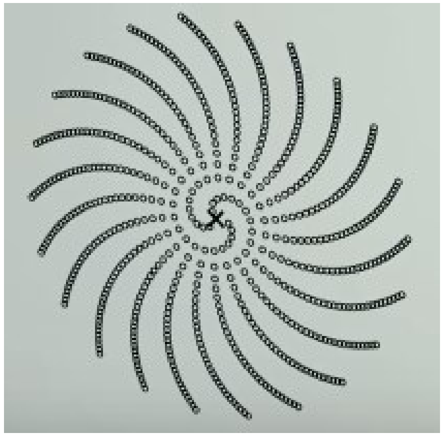
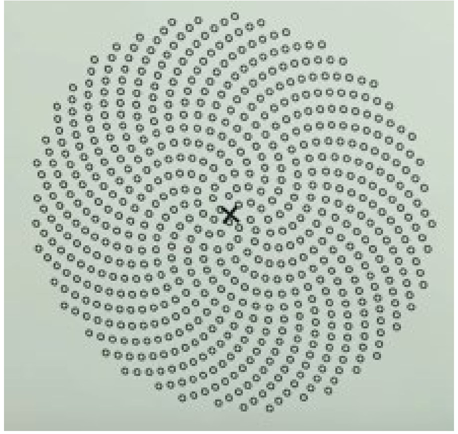
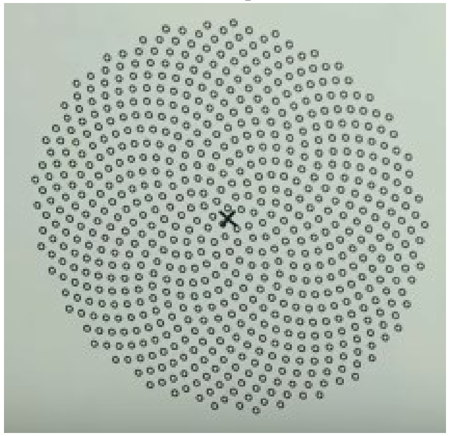
Why the golden ratio (φ) is so irrational

- Plants (such as sunflower) arrange their seeds according to φ
- If we arrange seeds according to fractions of 360^o

3/10	$1/\pi$	$1/\sqrt{2}$	$1/\varphi$
			

Youtube: The Golden Ratio (why it is so irrational) - Numberphile

F.Y.I.

Why the golden ratio (φ) is so irrational

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}}}}$$

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}}.$$

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

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Square Roots of Negative Numbers?

- The most commonly occurring application problems that require people to take square roots of numbers are problems which result in a quadratic equation.

- e.g. Determine the dimensions of a square with an area of 9 cm^2, 25 cm^2.

- Thus, having to take the square root of a negative number in this context means that such a rectangle does not exist.

Motivation: The Cubic Equation

- The solution of a general cubic equation that contains the square roots of negative numbers led to the introduction of complex numbers by Cardano (Italian: 1501 – 1576) after several attempts by other mathematicians before him in the 16th century.

- The term "complex number" was introduced by Gauss (German: 1777 – 1855) who also paved the way for a general use of complex numbers.

Imaginary Numbers

- The solutions to the quadratic equation:

$$x^2 - 1 = 0$$

are:

$$x = +1 \ \text{ and } \ x = -1$$

- The solutions to the quadratic equation:

$$x^2 + 1 = 0$$

are:

$$x = +\sqrt{-1} \ \text{ and } \ x = -\sqrt{-1}$$

- Hence, the imaginary number i was introduced: $i = \sqrt{-1}$

Complex Numbers

- By definition, a complex number z is an ordered pair (x, y) of real numbers x and y, written as:

$$z = (x, y)$$

- x is called the real part and y the imaginary part of z, written as:

$$x = Re\, z, \qquad y = Im\, z$$

- (0, 1) is called the imaginary unit and is denoted by i:

$$i = (0,1)$$

Complex Numbers Notations

- Ordered pair notation: $z = (x, y)$
- Some references use the notation:

$$z = x + iy$$

$$z = x + yi$$

- In some disciplines, in particular electrical engineering, j is used instead of i, since i is frequently used for electric current:

$$z = x + jy$$

$$z = x + yj$$

Addition & Multiplication of Complex Numbers

Notation: $z=x+iy$

- Addition of two complex numbers :

$$(x_1 + i\, y_1) + (x_2 + i\, y_2) = (x_1 + x_2) + i(y_1 + y_2)$$

- Multiplication is defined by:

$$(x_1 + i\, y_1)(x_2 + i\, y_2) = (x_1 x_2 + i x_1 y_2 + i x_2 y_1 + i^2\, y_1 y_2).$$

$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

Examples: Addition & Multiplication

- Add and multiply z_1 and z_2:

Let $z_1 = 8 + 3i$ and $z_2 = 9 - 2i$.

$$z_1 + z_2 = (8 + 3i) + (9 - 2i) = 17 + i,$$

$$z_1 z_2 = (8 + 3i)(9 - 2i) = 72 + 6 + i(-16 + 27) = 78 + 11i.$$

Subtraction & Division of Complex Numbers

- Subtraction of two complex numbers :

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2).$$

- Division is defined by:

$$z = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.$$

Examples: Subtraction & Division

- Subtract and divide z_1 and z_2:

For $z_1 = 8 + 3i$ and $z_2 = 9 - 2i$

$$z_1 - z_2 = (8 + 3i) - (9 - 2i) = -1 + 5i$$

$$\frac{z_1}{z_2} = \frac{8 + 3i}{9 - 2i} = \frac{(8 + 3i)(9 + 2i)}{(9 - 2i)(9 + 2i)} = \frac{66 + 43i}{81 + 4} = \frac{66}{85} + \frac{43}{85}\,i.$$

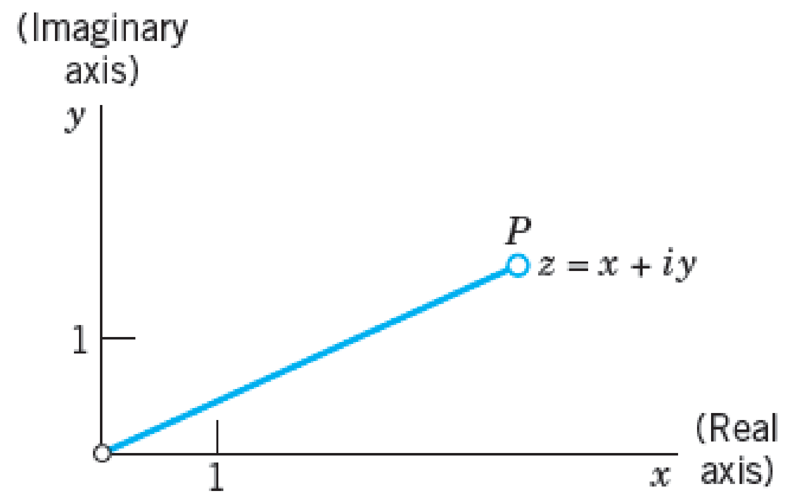
Complex Plane

- Remember: a complex number z can be written as:

$$z = (x, y)$$

- x is called the real part and y the imaginary part of z, written as:

$x = Re\, z, \quad y = Im\, z$



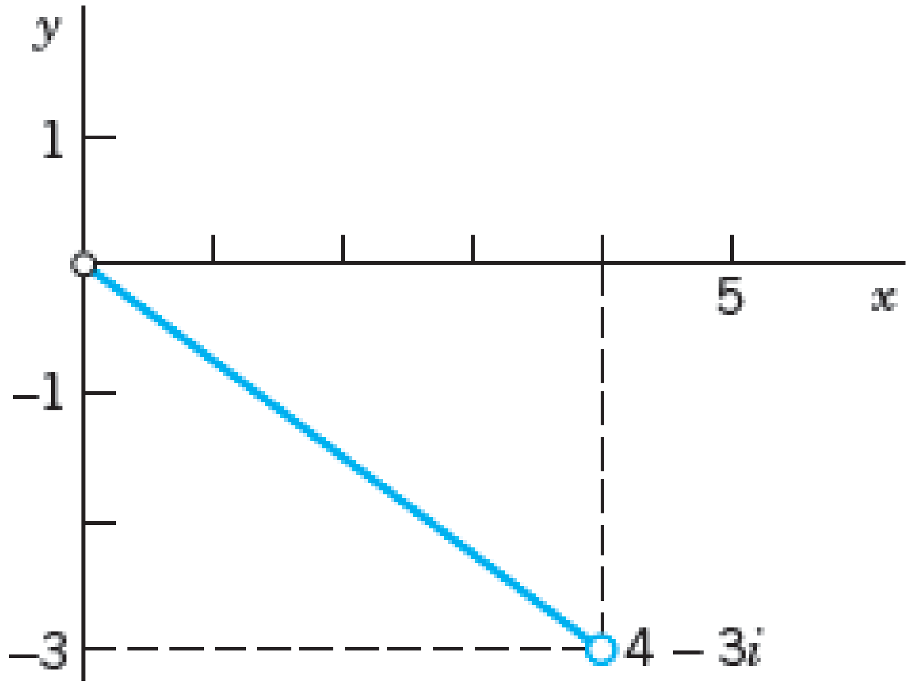
- Hence, it is possible to present z on an xy-plane called the complex plane.

- This is called the Cartesian coordinate system

(as opposed to the polar coordinate system that will be explained later)

Example: Complex Plane

- Plot $4 - 3i$ on the complex plane



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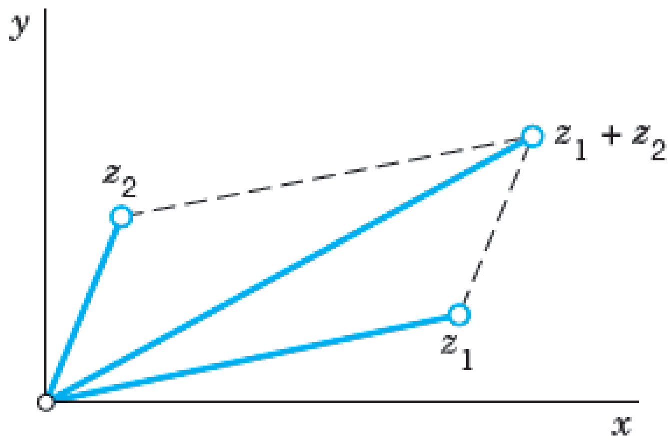
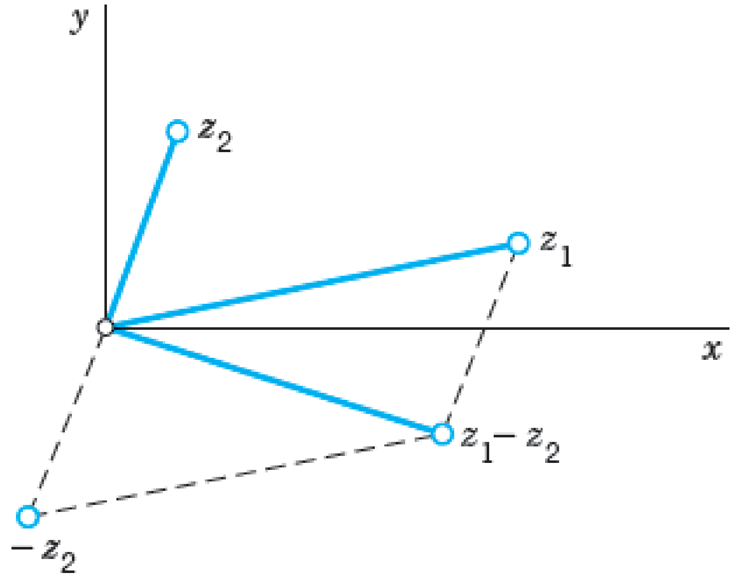
Complex Plane: Addition & Subtraction





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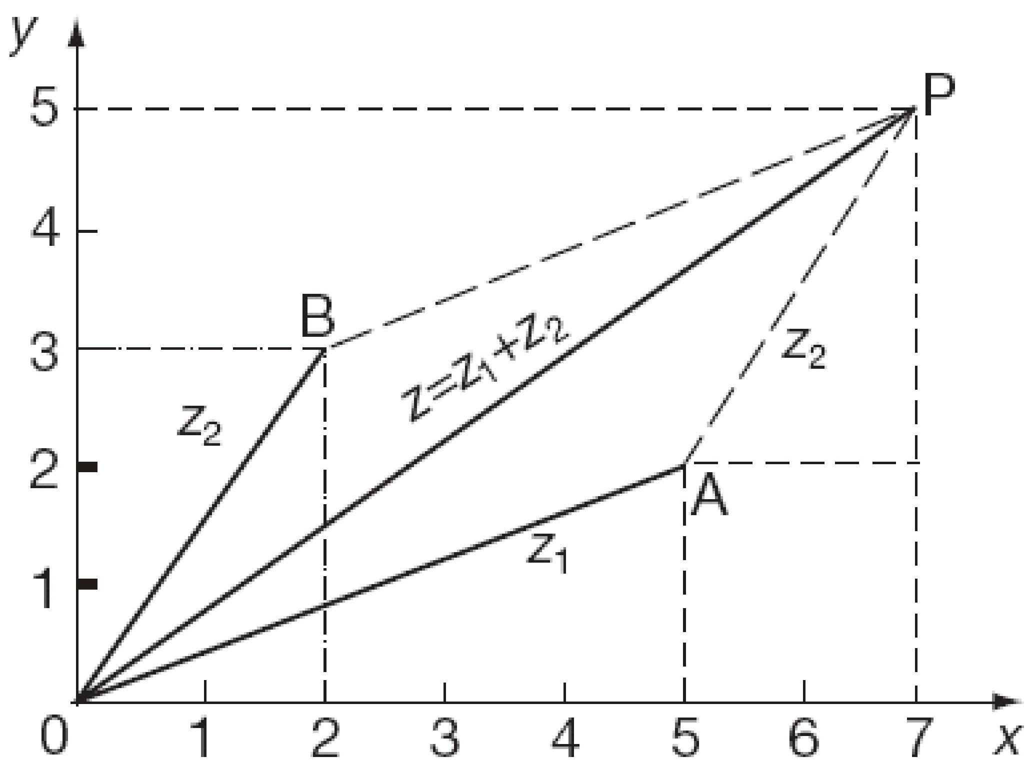
Complex Plane: Addition Example



$$(5 + j2) + (2 + j3) = 7 + j5$$

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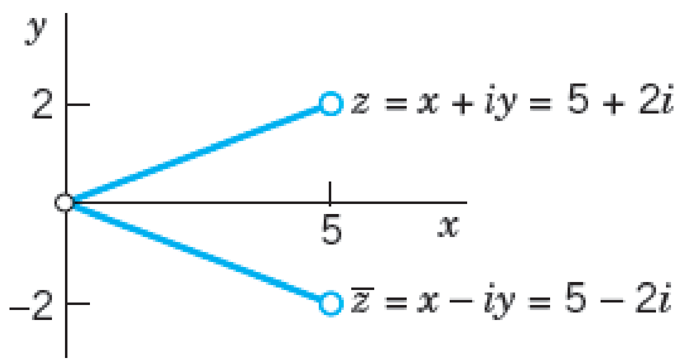
Complex Conjugate Numbers

- The complex conjugate $\bar{z}$ of a complex number $z = x + iy$ is defined by:

$$\bar{z} = x - iy$$

- Mathematically, replace i with $-i$.
- Graphically, flip z around the x-axis (real axis):



- Some references use the notation z^* for complex conjugate.

Complex Conjugate Numbers

- Prove the following:

$$z = x + iy,\ \bar{z} = x - iy$$

$$z\bar{z} = x^2 + y^2$$

$$z + \bar{z} = 2x$$

$$z - \bar{z} = 2iy$$

Conjugation Properties

- $\text{Re}\, z = x = \frac{1}{2}(z + \bar{z})$
- $Im\, z = y = \frac{1}{2i}(z - \bar{z})$
- If z is real $\rightarrow z = x \rightarrow z = \bar{z}$
- If z is imaginary $\rightarrow z = iy \rightarrow z = -\bar{z}$
- Working with conjugates is easy, since we have:

$$\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2, \qquad \overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2,$$

$$\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2, \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}.$$

Coordinate Systems

- The Cartesian coordinate system is commonly used to determine the location of a point in two or three dimensional space.
- The cylindrical and spherical coordinate systems – that will be addressed later – are also used to determine the location of a point in two or three dimensional space.
- The polar coordinate system is used to determine the location of a point in two dimensional space.
- The polar coordinate system is a special case of the cylindrical and the spherical coordinate systems.

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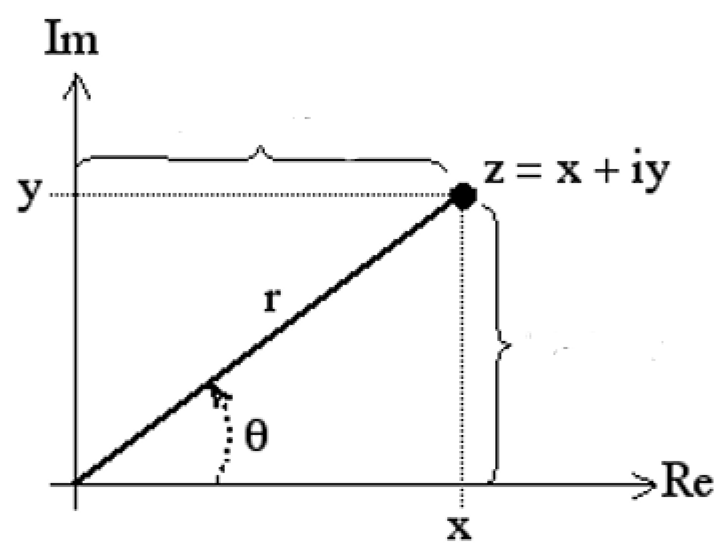
Polar Form: Absolute Value

- To find z in the polar form, we need to find r and θ
- The absolute value (aka: r, modulus, magnitude or amplitude) of a complex number in polar form are:

$$r = |z| = \sqrt{x^2 + y^2}$$



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Polar Form: Absolute Value

- $|z|$ is the distance between point z and the origin. The letter “r” stands for radius.
- In class: explain relationship to $|-5| = 5$

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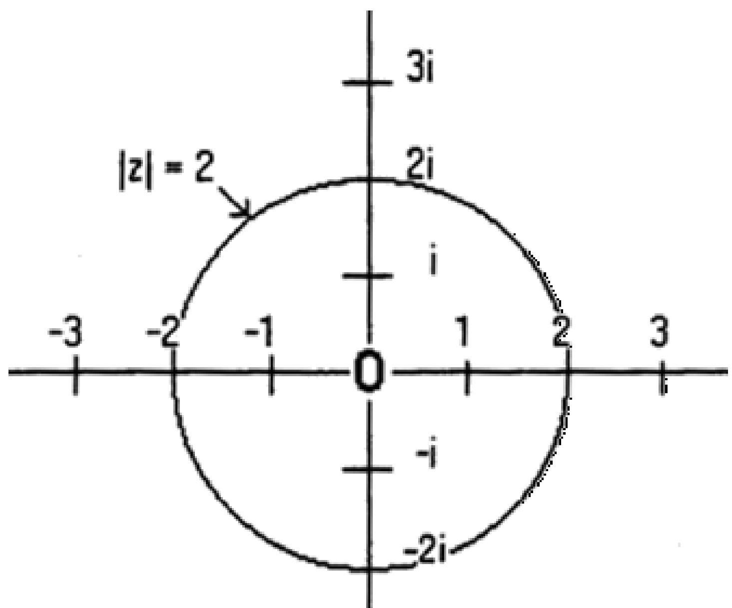
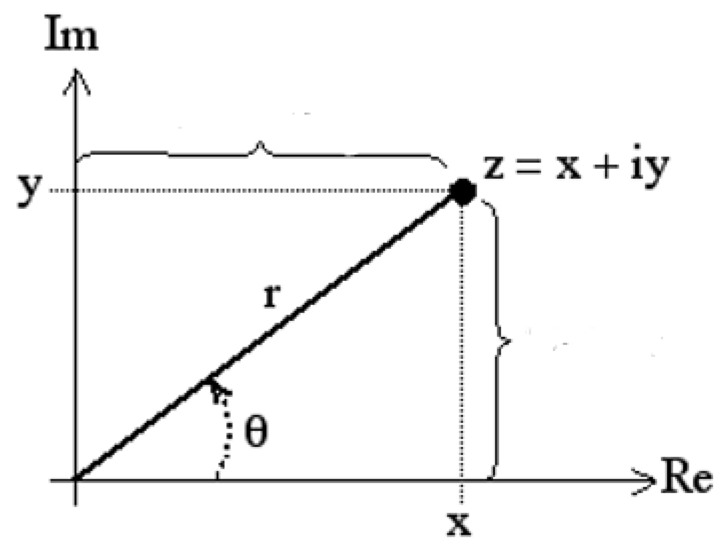
- $|z|$ = constant → circle

$$|z-(x+iy)| = C$$



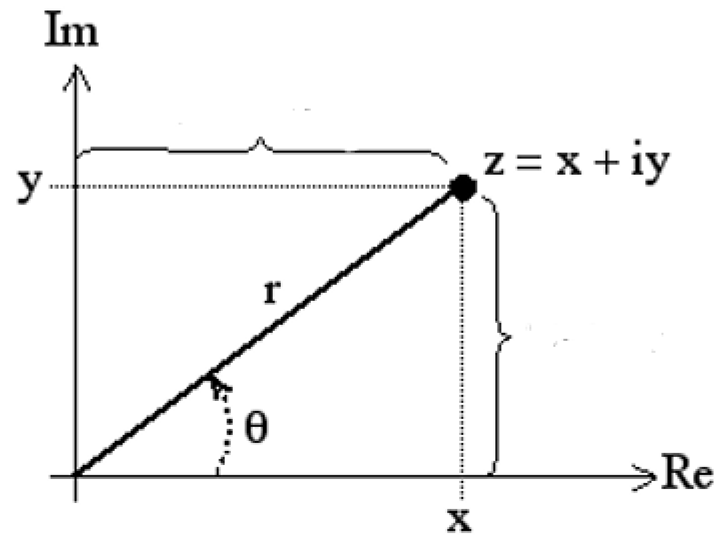


Polar Form: Argument

- The argument (or angle) of a complex number in polar form is:

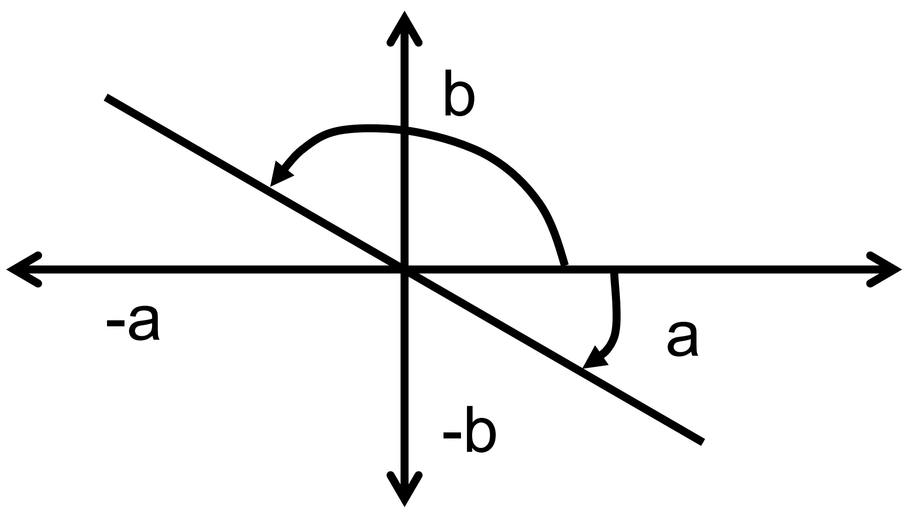
$$\arg z = \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

- Here, all angles are measured in radians and positive in the counter clockwise sense.
- For z=0, arg z is undefined (why?).



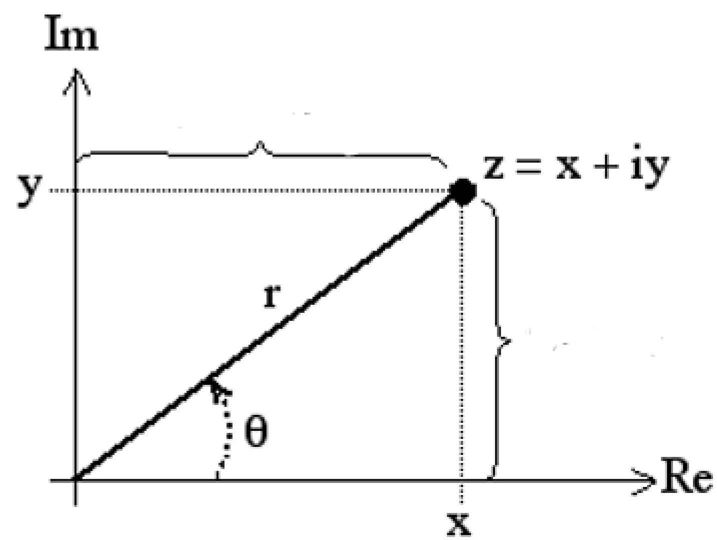
Warning!!!

$$\tan^{-1}\left(\frac{-b}{a}\right) \neq \tan^{-1}\left(\frac{b}{-a}\right) \qquad \tan^{-1}\left(\frac{b}{a}\right) \neq \tan^{-1}\left(\frac{-b}{-a}\right)$$



Polar Form: Argument

- For $z \neq 0$, arg z corresponds to the same value every 2π.
- Principal value Arg z: $-\pi < Arg\ z \leq \pi$.
- $\arg z = \text{Arg}\, z \pm n2\pi \ (n = 0, 1, 2, \ldots)$



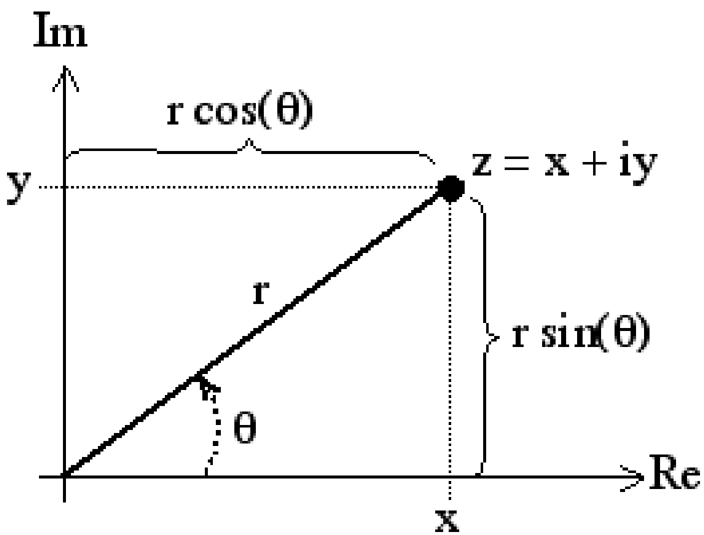
Polar Form

- If the polar form of z was given, i.e. we know r and θ, we can get the Cartesian form of z by doing projections on x & y axis.

$$x = r\cos\theta \quad and \ \ y = r\sin\theta$$

$$\rightarrow \quad z = r\cos\theta + i\, r\sin\theta = r\, e^{i\theta}$$

Euler's formula



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Euler's formula

Using Taylor series expansions we need to prove that [redacted] :

[redacted]

[redacted]

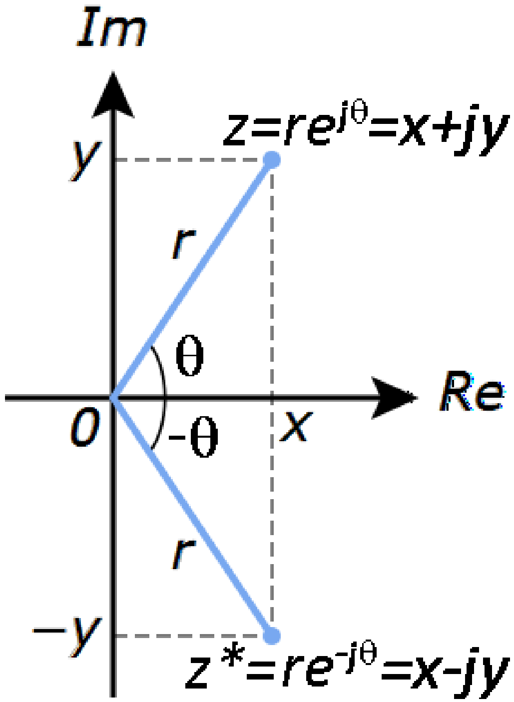
Polar Form: Conjugation

- For complex conjugates:

$z = re^{i\theta}$

$\bar{z} = re^{-i\theta}$

- $|\bar{z}| = |z|$
- $\arg \bar{z} = -\arg z$



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Example: Polar Form

- Find the polar form of $z = 1 + i \; and \; z = 3 + i\,3\sqrt{3}$:

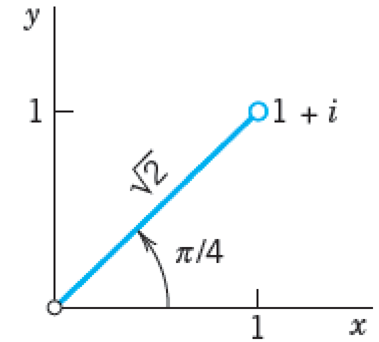
Polar Form of Complex Numbers. Principal Value Arg z

$z = 1 + i$ (Fig. 325) has the polar form $z = \sqrt{2}\,(\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi)$. Hence we obtain

$$|z| = \sqrt{2}, \quad \arg z = \tfrac{1}{4}\pi \pm 2n\pi \; (n = 0, 1, \cdots), \qquad \text{and} \qquad \operatorname{Arg} z = \tfrac{1}{4}\pi \quad \text{(the principal value).}$$

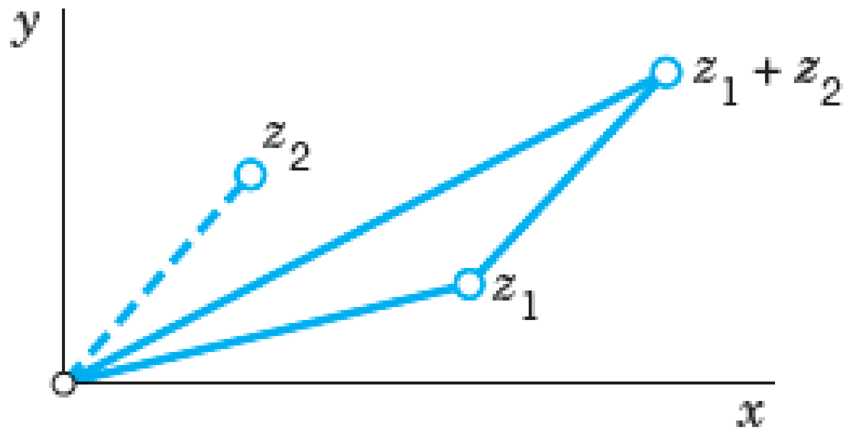
Similarly, $z = 3 + 3\sqrt{3}i = 6\,(\cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi)$, $|z| = 6$, and $\operatorname{Arg} z = \frac{1}{3}\pi$.

CAUTION! In using (4), we must pay attention to the quadrant in which z lies, since $\tan\theta$ has period π, so that the arguments of z and $-z$ have the same tangent. *Example:* for $\theta_1 = \arg(1 + i)$ and $\theta_2 = \arg(-1 - i)$ we have $\tan\theta_1 = \tan\theta_2 = 1$.



Triangle Inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$



- $|z_1 + z_2| = |z_1| + |z_2|$ when z_1 and z_2 lie on the same straight line through the origin.

Polar Form: Multiplication

- Multiplying two complex numbers gives a complex number whose modulus is the product of the two moduli and whose argument is the sum of the two arguments.

If $z_1 = r_1 e^{i\theta_1}\ \ and\ \ z_2 = r_2 e^{i\theta_2}$

Then $z_1 z_2 = r_1 e^{i\theta_1}\, r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$

Polar Form: Division

- Dividing two complex numbers gives a complex number whose modulus is the quotient of the two moduli and whose argument is the difference of the two arguments.

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$

Then $z_1/z_2 = r_1 e^{i\theta_1} / (r_2 e^{i\theta_2}) = (\frac{r_1}{r_2}) e^{i(\theta_1 - \theta_2)}$

Example: Polar Multiplication & Division

- Multiply and divide z_1 and z_2 in polar form:

Let $z_1 = -2 + 2i$ and $z_2 = 3i$. Then $z_1 z_2 = -6 - 6i$, $z_1/z_2 = \frac{2}{3} + (\frac{2}{3})i$. Hence (make a sketch)

$$|z_1 z_2| = 6\sqrt{2} = 3\sqrt{8} = |z_1||z_2|, \qquad |z_1/z_2| = 2\sqrt{2}/3 = |z_1|/|z_2|,$$

and for the arguments we obtain $\operatorname{Arg} z_1 = 3\pi/4$, $\operatorname{Arg} z_2 = \pi/2$,

$$\operatorname{Arg}(z_1 z_2) = -\frac{3\pi}{4} = \boxed{\operatorname{Arg} z_1 + \operatorname{Arg} z_2 - 2\pi,} \qquad \operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \frac{\pi}{4} = \operatorname{Arg} z_1 - \operatorname{Arg} z_2.$$

Integer Powers of Complex Numbers

- De Moivre's Formula: If a complex number is raised to the power n the result is a complex number whose modulus is the original modulus raised to the power n and whose argument is the original argument multiplied by n.

If $z = re^{i\theta}$

Then $z^n = (re^{i\theta})^n = r^n e^{in\theta}$

Example: Integer Powers of Complex Numbers

- Find $\left(\frac{1}{2}+i\,\frac{1}{2}\right)^{10}$

$|z| = 1/\sqrt{2}$

$Arg\ z = \pi/4$

$z^{10} = (1/32)e^{i\pi/2}$

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Integer Roots of Complex Numbers

- Need to calculate $w = \sqrt[k]{z}$ → $w^k = z$

- Let $z = re^{i\theta}$ and

  $w = Re^{i\emptyset}$

→ $w^k = R^k e^{ik\emptyset} = re^{i\theta} = z$

- Then, $R = \sqrt[k]{r}$.

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Integer Roots of Complex Numbers

- $w^k = R^k e^{ik\emptyset} = re^{i\theta} = z$

- Does $\emptyset = \frac{\theta}{k}$? The answer is NO!

- Since θ is determined only up to integer multiples of 2π (i.e. $\theta \equiv \theta \pm n2\pi$), then $\emptyset = \frac{\theta + n2\pi}{k} = \frac{\theta}{k} + n\frac{2\pi}{k}$.

- For $n = 0, 1, 2, k-1$ we get k distinct values of w. Further integers of n would give values already obtained. $\boxed{\mathrm{n} = \mathrm{k} \rightarrow n2\pi/\mathrm{k} = 2\pi}$ $\equiv$ $\boxed{\mathrm{n} = 0 \rightarrow n2\pi/\mathrm{k} = 0.}$

Integer Roots of Complex Numbers

$$\rightarrow\ w = \sqrt[k]{z} = \sqrt[k]{r}\ \exp i\left(\frac{\theta}{k} + \frac{n2\pi}{k}\right),$$

$$where\ n = 0, 1, 2, k - 1$$

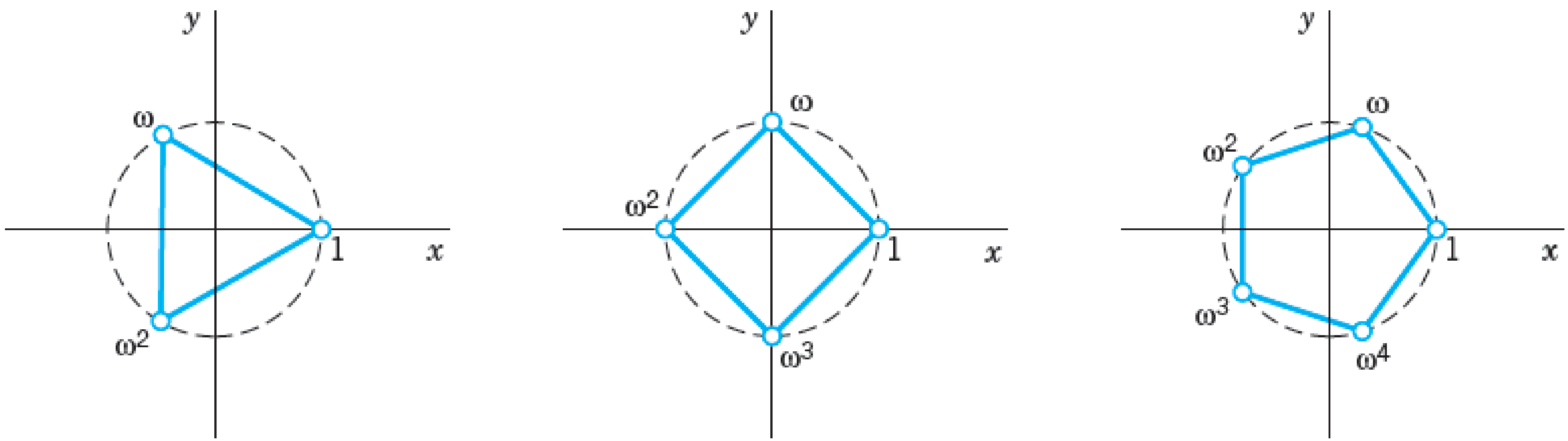
- Hence, there are k distinct roots of a complex number z.

- These k roots lie on a circle of radius $\sqrt[k]{r}$ and are separated by $2\pi/\mathrm{k}$ from their neighbouring root.

Examples: Integer Roots of Complex Numbers

- Find the roots of: $\sqrt[3]{1}$, $\sqrt[4]{1}$ $\sqrt[5]{1}$:

$\sqrt[3]{1} = 1, -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}i$, $\sqrt[4]{1} = \pm 1, \pm i$, and $\sqrt[5]{1}$.



Examples: Integer Roots of Complex Numbers

- Find the roots of:

21. $\sqrt[3]{1+i}$

23. $\sqrt[3]{216}$

25. $\sqrt[4]{i}$

21. $\sqrt[6]{2}\,(\cos \frac{1}{12}k\pi + i \sin \frac{1}{12}\pi), \quad k = 1, 9, 17$

23. $6, \quad -3 \pm 3\sqrt{3}i$

25. $\cos(\frac{1}{8}\pi + \frac{1}{2}k\pi) + i \sin(\frac{1}{8}\pi + \frac{1}{2}k\pi), \quad k = 0, 1, 2, 3$

Complex Numbers – Part 2

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13.5 Exponential Function e^z

- $z = x + iy$
- $e^z = e^{x+iy} = e^x \boxed{e^{iy}}$
- $e^z = e^x \boxed{(\cos y + i \sin y)}$

F.Y.I.

Euler's formula

Using Taylor series expansions we need to prove that [redacted] :

[redacted]

[redacted]

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Exponential Function e^z

- The complex exponential function e^z can be expressed as:

$$e^z = e^x \, e^{iy}$$

$$e^z = e^x(\cos y + i \sin y)$$

- Don’t confuse the previous definitions with the polar form of the complex number z:

$$z = re^{i\theta}$$

$$z = r(\cos\theta + i \sin\theta)$$

Properties of e^z

- $(e^z)' = e^z$ $\quad (e^z)' = \frac{d}{dx}(e^z)$

$$(e^z)' = (e^x \cos y)_x + i(e^x \sin y)_x = e^x \cos y + ie^x \sin y = e^z.$$

- $e^{z_1}e^{z_2} = e^{z_1+z_2}$
- $e^{i2\pi} = 1$, but why?
- Also, $e^{i\pi/2} = i,\ \ e^{i\pi} = -1, e^{-i\pi/2} = -i$
- How can you plot these on the complex plane?

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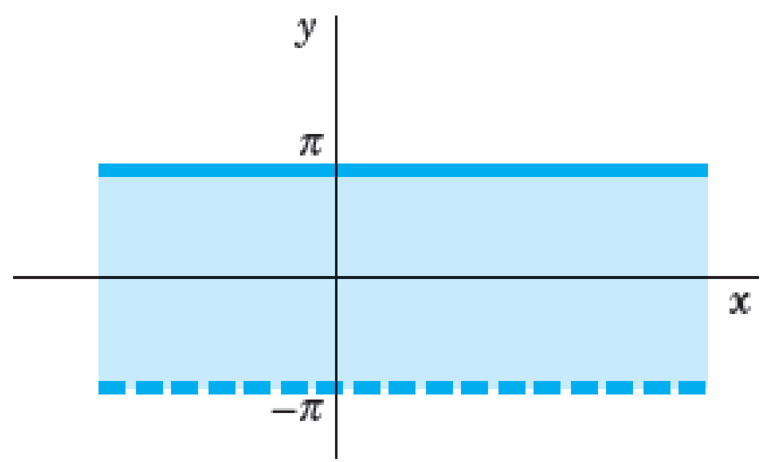
Properties of e^z

$$e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$$

- $|e^{iy}| = |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1$→ pure imaginary part amplitude=1.
- Hence, $|e^z| = e^x$ and $\arg e^z = y \pm n2\pi$
- e^z is periodic with period = $i2\pi$
  - $e^{z \pm in2\pi} = e^z$.
  - All values for $w = e^z$ are within a region = 2π.
  - The fundamental region of e^z is: $-\pi < y \leq \pi$.



Examples: Exponential Function

- Find the Cartesian & polar form of $e^{1.4-0.6i}$

$$e^{1.4-0.6i} = e^{1.4}(\cos 0.6 - i\sin 0.6) = 4.055(0.8253 - 0.5646i) = 3.347 - 2.289i$$

$$|e^{1.4-0.6i}| = e^{1.4} = 4.055, \qquad \operatorname{Arg} e^{1.4-0.6i} = -0.6.$$

- To prove that $e^{z_1}e^{z_2} = e^{z_1+z_2}$ for:

$$e^{2+i} = e^2(\cos 1 + i\sin 1) \qquad \text{and} \qquad e^{4-i} = e^4(\cos 1 - i\sin 1)$$

and verify that it equals $e^2e^4(\cos^2 1 + \sin^2 1) = e^6 = e^{(2+i)+(4-i)}$.

- Solve $e^z = 3 + 4i$:

To solve the equation $e^z = 3 + 4i$, note first that $|e^z| = e^x = 5$, $x = \ln 5 = 1.609$ is the real part of all solutions. Now, since $e^x = 5$,

$$e^x\cos y = 3, \qquad e^x \sin y = 4, \qquad \cos y = 0.6, \qquad \sin y = 0.8, \qquad y = 0.927.$$

Ans. $z = 1.609 + 0.927i \pm 2n\pi i$ $(n = 0, 1, 2, \cdots)$. These are infinitely many solutions (due to the periodicity of e^z). They lie on the vertical line $x = 1.609$ at a distance 2π from their neighbors.

Trigonometric Functions

- Euler's formula:

$$e^{ix} = \cos x + i\, sinx \ldots (eq1) \quad e^{-ix} = \cos x - i\, sinx \ldots (eq2)$$

- Add the previous two eq's (eq1+eq2) to get:

$$e^{ix} + e^{-ix} = 2\cos x \rightarrow \cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

- Subtract those eq's (eq1 – eq2) to get:

$$e^{ix} - e^{-ix} = 2i\sin x \rightarrow \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

- Similarly, for a complex value $z = x + iy$:

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \qquad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

Trigonometric Functions

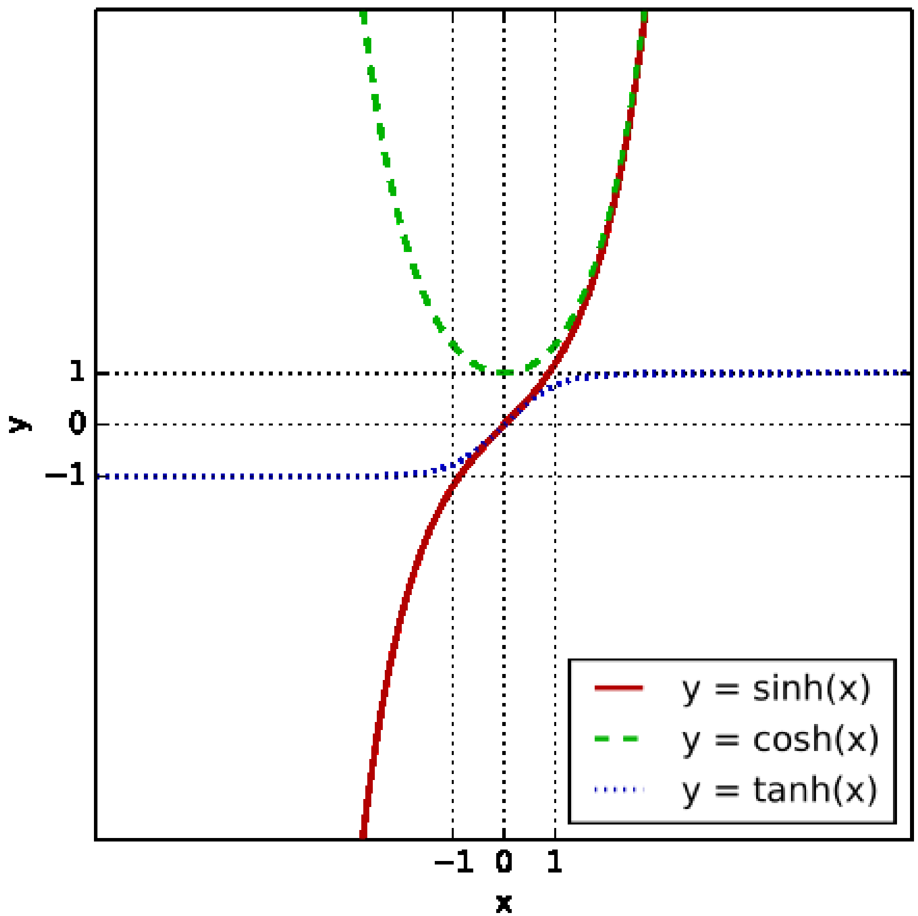
- The other trigonometric functions are defined as:

$$\tan z = \frac{\sin z}{\cos z} \qquad \cot z = \frac{\cos z}{\sin z}$$

$$\sec z = \frac{1}{\cos z} \qquad \csc z = \frac{1}{\sin z}$$

Hyperbolic Functions

$$\sinh x = \frac{1}{2}(e^{x} - e^{-x}) \qquad \cosh x = \frac{1}{2}(e^{x} + e^{-x})$$



Hyperbolic Functions

- The complex hyperbolic cosine and sine are defined by the formulas:

$$\sinh z = \tfrac{1}{2}(e^{z} - e^{-z}) \qquad \cosh z = \tfrac{1}{2}(e^{z} + e^{-z})$$

- Complex Trigonometric and Hyperbolic Functions Are Related:

$$\cosh iz = \tfrac{1}{2}\left(e^{iz} + e^{-iz}\right) = \cos z$$

$$\sinh iz = \tfrac{1}{2}\left(e^{iz} - e^{-iz}\right) = i \sin z$$

- Conversely:

$$\cos iz = \cosh z \qquad \sin iz = i \sinh z$$

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Hyperbolic Functions

- The other trigonometric functions are defined as:

$$\tanh z = \frac{\sinh z}{\cosh z} \qquad \coth z = \frac{\cosh z}{\sinh z}$$

$$\operatorname{sech} z = \frac{1}{\cosh z} \qquad \operatorname{csch} z = \frac{1}{\sinh z}$$

Examples: Trigonometric & Hyperbolic Functions

Prove the following equation 6a

$$\text{(6)} \quad \begin{aligned} &\text{(a)} \quad \cos z = \cos x \cosh y - i \sin x \sinh y \\ &\text{(b)} \quad \sin z = \sin x \cosh y + i \cos x \sinh y \end{aligned}$$

$$\begin{aligned} \cos z &= \tfrac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)}) \\ &= \tfrac{1}{2}e^{-y}(\cos x + i \sin x) + \tfrac{1}{2}e^{y}(\cos x - i \sin x) \\ &= \tfrac{1}{2}(e^{y} + e^{-y}) \cos x - \tfrac{1}{2}i(e^{y} - e^{-y}) \sin x. \end{aligned}$$

This yields (6a) since, as is known from calculus,

$$\text{(8)} \qquad \cosh y = \tfrac{1}{2}(e^{y} + e^{-y}), \qquad \sinh y = \tfrac{1}{2}(e^{y} - e^{-y});$$

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Examples: Trigonometric & Hyperbolic Functions

Prove the following equation 7a

(7)

$$\text{(a)} \qquad |\cos z|^2 = \cos^2 x + \sinh^2 y$$

$$\text{(b)} \qquad |\sin z|^2 = \sin^2 x + \sinh^2 y$$

From (6a) and $\cosh^2 y = 1 + \sinh^2 y$ we obtain

$$|\cos z|^2 = (\cos^2 x)(1 + \sinh^2 y) + \sin^2 x \sinh^2 y.$$

Since $\sin^2 x + \cos^2 x = 1$, this gives (7a), and (7b) is obtained similarly.

Example: Trigonometric & Hyperbolic Functions

- $\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \quad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$…(1)

Solve (a) $\cos z = 5$ (which has no real solution!), (b) $\cos z = 0$, (c) $\sin z = 0$.

Solution. (a) $e^{2iz} - 10e^{iz} + 1 = 0$ from (1) by multiplication by e^{iz}. This is a quadratic equation in e^{iz}, with solutions (rounded off to 3 decimals)

$$e^{iz} = e^{-y+ix} = 5 \pm \sqrt{25-1} = 9.899 \quad \text{and} \quad 0.101.$$

Thus $e^{-y} = 9.899$ or 0.101, $e^{ix} = 1$, $y = \pm 2.292$, $x = 2n\pi$. *Ans.* $z = \pm 2n\pi \pm 2.292i$ $(n = 0, 1, 2, \cdots)$.

(b) $\cos x = 0$, $\sinh y = 0$ by (7a), $y = 0$. *Ans.* $z = \pm\frac{1}{2}(2n+1)\pi$ $(n = 0, 1, 2, \cdots)$.

(c) $\sin x = 0$, $\sinh y = 0$ by (7b), *Ans.* $z = \pm n\pi$ $(n = 0, 1, 2, \cdots)$.

Hence the only zeros of $\cos z$ and $\sin z$ are those of the real cosine and sine functions.

Natural Logarithm Function

- The natural logarithm of z is denoted by:

$$\boldsymbol{w = \ln z} \rightarrow e^w = z \;\; for \;\; z \neq 0$$

- Let $z = re^{i\theta}$:

$$\rightarrow \ln z = \ln r + i\,\theta \qquad OR \qquad \ln z = \ln r + i \arg z$$

- Since arg z corresponds to the same value every 2π, $\ln z$ has infinite values (multivalued).

- The value of ln z corresponding to the principal value Arg z is denoted by $\mathrm{Ln}\, z$:

$$\mathrm{Ln}\, z = \ln r + i\,\mathrm{Arg}\,\mathrm{z}$$

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Natural Logarithm Function

- $\mathrm{Ln}\, z$ is called the principal value of $\ln z$:

$$\mathrm{Ln}\, z = \ln|z| + i\, \mathrm{Arg}\, \mathrm{z}$$

$$\ln z = \mathrm{Ln}\, z \pm i\, n 2\pi$$

- If z is positive real, then Arg z = 0 and Ln z becomes regular ln(x) function from calculus.

- If z is negative real, (remember ln(-x) is not defined in calculus):

$$\mathrm{Ln}\, z = \ln|z| + i\, \mathrm{Arg}\, \mathrm{z} = \ln|z| + i\, \pi$$

Natural Logarithm Function

- From: $\ln z = \ln r + i\theta$, it follows that:

$$e^{\ln z} = e^{\ln r} e^{i\theta} = z \text{ (single-valued)}$$

- Since $\arg(e^z) = y \pm n2\pi$, it follows that:

$$\ln e^z = z \pm i\, n2\pi \text{ (multi-valued)}$$

Remember:

e^z is periodic → $e^z = e^{z \pm \boldsymbol{i}\, \boldsymbol{n} 2\pi}$

$$\ln e^z = \ln e^{z \pm i\, n2\pi} = \ln e^{x + iy \pm i\, n2\pi} = x + iy \pm i\, n2\pi$$
$$= z \pm i\, n2\pi$$

Examples: Natural Logarithm

Natural Logarithm. Principal Value

$$\ln 1 = 0, \pm 2\pi i, \pm 4\pi i, \cdots \qquad \mathrm{Ln}\, 1 = 0$$

$$\ln 4 = 1.386294 \pm 2n\pi i \qquad \mathrm{Ln}\, 4 = 1.386294$$

$$\ln(-1) = \pm\pi i, \pm 3\pi i, \pm 5\pi i, \cdots \qquad \mathrm{Ln}\,(-1) = \pi i$$

$$\ln(-4) = 1.386294 \pm (2n+1)\pi i \qquad \mathrm{Ln}\,(-4) = 1.386294 + \pi i$$

$$\ln i = \pi i/2, -3\pi/2, 5\pi i/2, \cdots \qquad \mathrm{Ln}\, i = \pi i/2$$

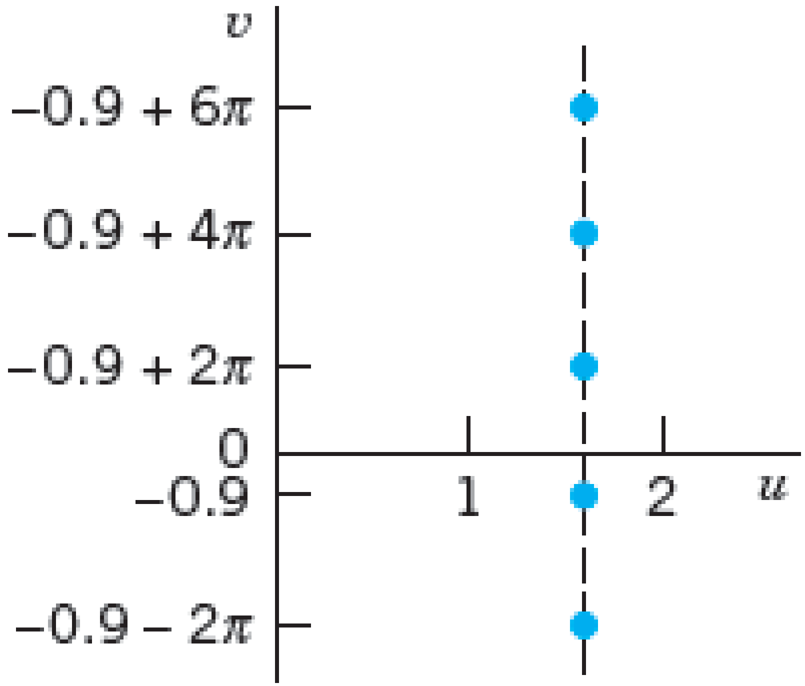
$$\ln 4i = 1.386294 + \pi i/2 \pm 2n\pi i \qquad \mathrm{Ln}\, 4i = 1.386294 + \pi i/2$$

$$\ln(-4i) = 1.386294 - \pi i/2 \pm 2n\pi i \qquad \mathrm{Ln}\,(-4i) = 1.386294 - \pi i/2$$

$$\ln(3-4i) = \ln 5 + i \arg(3-4i) \qquad \mathrm{Ln}\,(3-4i) = 1.609438 - 0.927295i$$

$$= 1.609438 - 0.927295i \pm 2n\pi i \qquad \text{(Fig. 337)}$$

Examples: Natural Logarithm



Some values of $\ln(3 - 4i)$

Natural Logarithm Function

- The familiar relations for the natural logarithm continue to hold for complex values:

$$\ln(z_1 z_2) = \ln(z_1) + \ln(z_2)$$

$$\ln(z_1/z_2) = \ln(z_1) - \ln(z_2)$$

Example: Natural Logarithm

- Prove that $\ln(z_1 z_2) = \ln(z_1) + \ln(z_2)$ and that $\mathrm{Ln}(z_1 z_2) \neq \mathrm{Ln}(z_1) + \mathrm{Ln}(z_2)$ for $z_1 = z_2 = -1$

Let

$$z_1 = z_2 = e^{\pi i} = -1.$$

If we take the principal values

$$\mathrm{Ln}\, z_1 = \mathrm{Ln}\, z_2 = \pi i,$$

then (5a) holds provided we write $\ln(z_1 z_2) = \ln 1 = 2\pi i$; however, it is not true for the principal value, $\mathrm{Ln}(z_1 z_2) = \mathrm{Ln}\, 1 = 0$.

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General Powers

- $z = x + iy$
- $z^c = e^{\ln z^c} = e^{c \ln z} \quad (c\ complex, z \neq 0).$
- $\ln z$ is multivalued → z^c is multivalued.
- Hence, there is a principal value of z^c which is:

$$principal(z^c) = e^{c \operatorname{Ln} z}$$

General Powers

- $z^c = e^{c \ln z} \quad (c\ complex, z \neq 0).$

- If **c = n** = 1, 2, … then z^n is **single-valued** and identical with the usual nth power of z.

- If **c = n** = -1, -2, … then z^n is also **single-valued** .

- If **c = 1/n**, where n = 2, 3, … , then:

$$z^c = z^{1/n} = \sqrt[n]{z}$$

→ Same **finite** n roots explained previously.

General Powers

- $z^c = e^{c \ln z} \quad (c\ complex, z \neq 0).$
- If c=m/n, the quotient of two positive integers, the situation is similar, and z^c has only **finite** n distinct values (n roots).
- However, if c is real irrational or complex, then z^c is **infinitely** many-valued.
- Remember: an irrational number is any real number that cannot be expressed as a ratio of integers. Example: $\pi, \sqrt{2}$.

Example: General Powers

- Find the values of: i^i, $(1+i)^{2-i}$:

$$i^i = e^{i \ln i} = \exp(i \ln i) = \exp\left[i\left(\frac{\pi}{2} i \pm 2n\pi i\right)\right] = e^{-(\pi/2) \mp 2n\pi}.$$

All these values are real, and the principal value ($n = 0$) is $e^{-\pi/2}$.

Similarly, by direct calculation and multiplying out in the exponent,

$$\begin{aligned}(1+i)^{2-i} &= \exp\left[(2-i)\ln(1+i)\right] = \exp\left[(2-i)\{\ln\sqrt{2} + \tfrac{1}{4}\pi i \pm 2n\pi i\}\right] \\ &= 2e^{\pi/4 \pm 2n\pi}\left[\sin\left(\tfrac{1}{2}\ln 2\right) + i\cos\left(\tfrac{1}{2}\ln 2\right)\right].\end{aligned}$$

Linear Algebra: Matrices and Vectors – Part 1

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Matrices

- A matrix is a rectangular array of numbers or functions which we will enclose in brackets.
- The numbers (or functions) are called entries or, less commonly, elements of the matrix.

$$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}, \quad \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}$$

Linear Systems

- Linear systems is a major application of matrices, example:

$$\begin{aligned} 4x_1 + 6x_2 + 9x_3 &= 6 \\ 6x_1 \qquad\quad - 2x_3 &= 20 \\ 5x_1 - 8x_2 + \ x_3 &= 10 \end{aligned}$$

- where x_1, x_2, x_3 are the unknowns. We form the coefficient matrix "A" (a_{22} = zero):

$$\mathbf{A} = \begin{bmatrix} 4 & 6 & 9 \\ 6 & 0 & -2 \\ 5 & -8 & 1 \end{bmatrix}$$

Linear Systems

- We form another matrix, the augmented matrix of the system "$\tilde{A}$":

$$\tilde{\mathbf{A}} = \begin{bmatrix} 4 & 6 & 9 & 6 \\ 6 & 0 & -2 & 20 \\ 5 & -8 & 1 & 10 \end{bmatrix}$$, remember: $$\begin{aligned} 4x_1 + 6x_2 + 9x_3 &= 6 \\ 6x_1 \qquad\quad - 2x_3 &= 20 \\ 5x_1 - 8x_2 + x_3 &= 10 \end{aligned}$$

- Matrix operations will be used LATER to find the values for x_1, x_2, x_3 (the solution for the linear system).

- For the previous system: $x_1 = 3,\ x_2 = 0.5,\ x_3 = -1$

General Concepts and Notations

- We shall denote matrices by capital boldface letters $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$
- or by writing the general entry in brackets $\boldsymbol{A} = [ajk]$.
- m x n matrix is a matrix with m rows and n columns .
- m x n is called the size of the matrix.

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

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General Concepts and Notations

- The following matrices have the sizes 2 x 3, 3 x 3, 2 x 2, 1 x 3, and 2 x 1 respectively.

$$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}, \quad [a_1 \quad a_2 \quad a_3], \quad \begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}$$

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General Concepts and Notations

- If m=n, we call $\boldsymbol{A}$ an n x n square matrix.
- Then its diagonal containing the entries a_{11}, a_{22}, …, a_{nn} is called the main diagonal of $\boldsymbol{A}$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- If m≠n, we call $\boldsymbol{A}$ an n x m rectangular matrix.
- A vector is a matrix with only one row or column.
- Its entries are called the components of the vector.

General Concepts and Notations

- We shall denote vectors by lowercase boldface letters $\boldsymbol{a}, \boldsymbol{b}, \ldots$
- or by its general component in brackets $\boldsymbol{a} = [aj]$.
- row vector

$$\mathbf{a} = [a_1 \quad a_2 \quad \cdots \quad a_n]. \qquad \text{For instance,} \qquad \mathbf{a} = [-2 \quad 5 \quad 0.8 \quad 0 \quad 1].$$

- column vector:

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \qquad \text{For instance,} \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}.$$

Matrix Addition and Scalar Multiplication

Equality of Matrices

Two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are **equal**, written $\mathbf{A} = \mathbf{B}$, if and only if they have the same size and the corresponding entries are equal, that is, $a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on. Matrices that are not equal are called **different**. Thus, matrices of different sizes are always different.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}.$$

$$\mathbf{A} = \mathbf{B} \quad \text{if and only if} \quad \begin{array}{ll} a_{11} = 4, & a_{12} = 0, \\ a_{21} = 3, & a_{22} = -1. \end{array}$$

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Matrix Addition and Scalar Multiplication

Addition of Matrices

The **sum** of two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ ***of the same size*** is written $\mathbf{A} + \mathbf{B}$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$. Matrices of different sizes cannot be added.

If $\quad \mathbf{A} = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$, then $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}$.

Matrix Addition and Scalar Multiplication

Scalar Multiplication (Multiplication by a Number)

The **product** of any $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ and any **scalar** c (number c) is written $c\mathbf{A}$ and is the $m \times n$ matrix $c\mathbf{A} = [ca_{jk}]$ obtained by multiplying each entry of $\mathbf{A}$ by c.

$$\text{If} \quad \mathbf{A} = \begin{bmatrix} 2.7 & -1.8 \\ 0 & 0.9 \\ 9.0 & -4.5 \end{bmatrix}, \quad \text{then} \quad -\mathbf{A} = \begin{bmatrix} -2.7 & 1.8 \\ 0 & -0.9 \\ -9.0 & 4.5 \end{bmatrix}, \quad \frac{10}{9}\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 0 & 1 \\ 10 & -5 \end{bmatrix}, \quad 0\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

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Rules for Matrix Addition

(a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

(b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ (written $\mathbf{A} + \mathbf{B} + \mathbf{C}$)

(c) $\mathbf{A} + \mathbf{0} = \mathbf{A}$

(d) $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}.$

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Rules for Matrix Scalar Multiplication

(a) $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$

(b) $(c + k)\mathbf{A} = c\mathbf{A} + k\mathbf{A}$

(c) $c(k\mathbf{A}) = (ck)\mathbf{A}$ (written $ck\mathbf{A}$)

(d) $1\mathbf{A} = \mathbf{A}.$

Matrix Multiplication

- The entry c_{jk} is obtained by multiplying each entry in the jth row of A by the corresponding entry in the kth column of B and then adding these n products.

$$\begin{array}{ccccc} \mathbf{A} & \mathbf{B} & = & \mathbf{C} \\ [m \times \boxed{n] \, [n} \times p] & & = & [m \times p]. \end{array}$$

$$m = 4 \left\{ \overbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}}^{n=3} \right. \overbrace{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}}^{p=2} = \left. \overbrace{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \\ c_{41} & c_{42} \end{bmatrix}}^{p=2} \right\} m = 4$$

Notations in a product $\mathbf{AB} = \mathbf{C}$

Matrix Multiplication

- Matrix multiplication means multiplication of matrices by matrices.

Multiplication of a Matrix by a Matrix

The **product** $\mathbf{C} = \mathbf{AB}$ (in this order) of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ times an $r \times p$ matrix $\mathbf{B} = [b_{jk}]$ is defined if and only if $r = n$ and is then the $m \times p$ matrix $\mathbf{C} = [c_{jk}]$ with entries

$$c_{jk} = \sum_{l=1}^{n} a_{jl}b_{lk} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk} \qquad \begin{array}{l} j = 1, \cdots, m \\ k = 1, \cdots, p. \end{array} \tag{1}$$

$$\begin{array}{ccccc} \mathbf{A} & \mathbf{B} & = & \mathbf{C} \\ [m \times \boxed{n] \ [n} \times p] & & = & [m \times p]. \end{array}$$

Matrix Multiplication

- Matrix multiplication means multiplication of matrices by matrices.

Multiplication of a Matrix by a Matrix

The product $\mathbf{C} = \mathbf{AB}$ (in this order) of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ times an $r \times p$ matrix $\mathbf{B} = [b_{jk}]$ is defined if and only if $r = n$ and is then the $m \times p$ matrix $\mathbf{C} = [c_{jk}]$ with entries

$$c_{jk} = \sum_{l=1}^{n} a_{jl}b_{lk} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk} \qquad \begin{array}{l} j = 1, \cdots, m \\ k = 1, \cdots, p. \end{array} \tag{1}$$

Example1: Matrix Multiplication

$$\mathbf{AB} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & \boxed{14} & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

Here $c_{11} = 3 \cdot 2 + 5 \cdot 5 + (-1) \cdot 9 = 22$, and so on. The entry in the box is $c_{23} = 4 \cdot 3 + 0 \cdot 7 + 2 \cdot 1 = 14$. The product **BA** is not defined.

Example2,3: Matrix Multiplication

- Matrix multiplication is not commutative
- **AB ≠ BA**
- Examples: rectangular matrices:

Multiplication of a Matrix and a Vector

$$\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \cdot 3 + 2 \cdot 5 \\ 1 \cdot 3 + 8 \cdot 5 \end{bmatrix} = \begin{bmatrix} 22 \\ 43 \end{bmatrix} \quad \text{whereas} \quad \begin{bmatrix} 3 \\ 5 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} \quad \text{is undefined.}$$

Products of Row and Column Vectors

$$[3 \quad 6 \quad 1]\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = [19], \qquad \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}[3 \quad 6 \quad 1] = \begin{bmatrix} 3 & 6 & 1 \\ 6 & 12 & 2 \\ 12 & 24 & 4 \end{bmatrix}.$$

Example4: Matrix Multiplication

- Matrix multiplication is not commutative
- **AB ≠ BA**
- Examples: square matrices:

$$\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{but} \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} = \begin{bmatrix} 99 & 99 \\ -99 & -99 \end{bmatrix}.$$

- Note that this also shows that **AB=0** does not necessarily imply **BA=0** or **A=0** or **B=0**.

Matrix Multiplication Rules

(a) $(k\mathbf{A})\mathbf{B} = k(\mathbf{AB}) = \mathbf{A}(k\mathbf{B})$ *written* $k\mathbf{AB}$ *or* $\mathbf{A}k\mathbf{B}$

(b) $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ *written* $\mathbf{ABC}$

(c) $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$

(d) $\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB}$

(e) Integer powers of a mat: $\mathbf{A}^2 = \mathbf{AA},\ \mathbf{A3} = \mathbf{AAA}$ …etc

(b) Associative law.

(c) and (d) Distributive laws.

F.Y.I.

Matrix Multiplication in Matlab

```
>> a=[1 2; 3 4]
a =
    1     2
    3     4

>> b=[1 0; 1 0]
b =
    1     0
    1     0

>> a*b
ans =
    3     0
    7     0
```

F.Y.I.

Matrix Multiplication in Matlab

```
>> a*b
ans =
    3     0
    7     0

>> b*a
ans =
    1     2
    1     2
```

Matrix Multiplication in Matlab

```
>> c=[a,b]
c =
    1    2    1    0
    3    4    1    0

>> a*c
ans =
    7   10    3    0
   15   22    7    0

>> c*a
??? Error using ==> mtimes
Inner matrix dimensions must agree.
```

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F.Y.I.

Matrix Multiplication: Method #2

- Since matrix multiplication is a multiplication of rows into columns, we can write the defining formula more compactly as:

$$\boxed{c_{jk} = \mathbf{a}_j \mathbf{b}_k,} \qquad j = 1, \cdots, m; \quad k = 1, \cdots, p,$$

where $\mathbf{a}_j$ is the jth row vector of **A** and $\mathbf{b}_k$ is the kth column vector of **B**:

$$\mathbf{a}_j \mathbf{b}_k = [a_{j1} \quad a_{j2} \quad \cdots \quad a_{jn}] \begin{bmatrix} b_{1k} \\ \vdots \\ b_{nk} \end{bmatrix} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk}.$$

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Example: Matrix Multiplication: Method #2

If $\mathbf{A} = [a_{jk}]$ is of size 3×3 and $\mathbf{B} = [b_{jk}]$ is of size 3×4, then

$$\text{(4)} \qquad \mathbf{AB} = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_1\mathbf{b}_2 & \mathbf{a}_1\mathbf{b}_3 & \mathbf{a}_1\mathbf{b}_4 \\ \mathbf{a}_2\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2 & \mathbf{a}_2\mathbf{b}_3 & \mathbf{a}_2\mathbf{b}_4 \\ \mathbf{a}_3\mathbf{b}_1 & \mathbf{a}_3\mathbf{b}_2 & \mathbf{a}_3\mathbf{b}_3 & \mathbf{a}_3\mathbf{b}_4 \end{bmatrix}.$$

Taking $\mathbf{a}_1 = [3 \quad 5 \quad -1]$, $\mathbf{a}_2 = [4 \quad 0 \quad 2]$, etc., verify (4) for the product in Example 1.

F.Y.I.

Matrix Multiplication: Method #3

- Parallel processing of products on the computer is facilitated by a variant method #1 for computing **C=AB**.

- In this method, **A** is used as given, **B** is taken in terms of its column vectors, and the product is computed columnwise:

$$\mathbf{AB} = \mathbf{A}[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_p] = [\mathbf{Ab}_1 \quad \mathbf{Ab}_2 \quad \cdots \quad \mathbf{Ab}_p].$$

- Columns of **B** are then assigned to different processors, which simultaneously compute the columns of the product matrix $\mathbf{Ab}_1$, $\mathbf{Ab}_2$, etc.

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F.Y.I.

Example: Matrix Multiplication: Method #3

To obtain

$$\mathbf{AB} = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 7 \\ -1 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 11 & 4 & 34 \\ -17 & 8 & -23 \end{bmatrix}$$

from (5), calculate the columns

$$\begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ -17 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 34 \\ -23 \end{bmatrix}$$

of $\mathbf{AB}$ and then write them as a single matrix, as shown in the first formula on the right.

F.Y.I.

Motivation of Multiplication by Linear Transformations

- Let us now motivate the matrix multiplication by its use in linear transformations. For $n = 2$ variables these transformations are of the form:

$$y_1 = a_{11}x_1 + a_{12}x_2$$

$$y_2 = a_{21}x_1 + a_{22}x_2$$

- In these equations, we may relate an x_1x_2-coordinate system to a y_1y_2-coordinate system in the plane:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{A}\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}.$$

F.Y.I.

Motivation of Multiplication by Linear Transformations

- Now suppose further that the x_1x_2-system is related to a w_1w_2-system by another linear transformation:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{Bw} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} b_{11}w_1 + b_{12}w_2 \\ b_{21}w_1 + b_{22}w_2 \end{bmatrix}.$$

- Then the y_1y_2-system is related to the w_1w_2-system indirectly via the x_1x_2-system, which is a linear transformation:

$$\mathbf{y} = \mathbf{Cw} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} c_{11}w_1 + c_{12}w_2 \\ c_{21}w_1 + c_{22}w_2 \end{bmatrix}.$$

F.Y.I.

Motivation of Multiplication by Linear Transformations

- Substituting x into y:

$$
\begin{aligned}
y_1 &= a_{11}(b_{11}w_1 + b_{12}w_2) + a_{12}(b_{21}w_1 + b_{22}w_2) \\
&= (a_{11}b_{11} + a_{12}b_{21})w_1 + (a_{11}b_{12} + a_{12}b_{22})w_2 \\
y_2 &= a_{21}(b_{11}w_1 + b_{12}w_2) + a_{22}(b_{21}w_1 + b_{22}w_2) \\
&= (a_{21}b_{11} + a_{22}b_{21})w_1 + (a_{21}b_{12} + a_{22}b_{22})w_2.
\end{aligned}
$$

- Comparing this with **y=Cw**:

$$
\begin{aligned}
c_{11} &= a_{11}b_{11} + a_{12}b_{21} & c_{12} &= a_{11}b_{12} + a_{12}b_{22} \\
c_{21} &= a_{21}b_{11} + a_{22}b_{21} & c_{22} &= a_{21}b_{12} + a_{22}b_{22}.
\end{aligned}
$$

→ **C=AB**

Transposition

- It provides a transition from row vectors to column vectors and vice versa.

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

$$\mathbf{A}^\mathsf{T} = [a_{kj}] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdot & \cdot & \cdots & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

Examples: Transposition

$$\begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 7 \\ 8 & -1 & 5 \\ 1 & -9 & 4 \end{bmatrix}^T = \begin{bmatrix} 3 & 8 & 1 \\ 0 & -1 & -9 \\ 7 & 5 & 4 \end{bmatrix},$$

$$\begin{bmatrix} 6 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix} \cdot \qquad \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}^T = \begin{bmatrix} 6 & 2 & 3 \end{bmatrix}.$$

Rules of Transposition

(a) $(\mathbf{A}^T)^T = \mathbf{A}$

(b) $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$

(c) $(c\mathbf{A})^T = c\mathbf{A}^T$

(d) $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T.$

- Note that in (d) the transposed matrices are in reversed order.

Symmetric and Skew-Symmetric Matrices

- Symmetric and Skew-Symmetric Matrices are square matrices whose transpose equals the matrix itself or minus the matrix.

$$\mathbf{A}^{\mathsf{T}} = \mathbf{A} \qquad (\text{thus } a_{kj} = a_{jk}), \qquad \mathbf{A}^{\mathsf{T}} = -\mathbf{A} \qquad (\text{thus } a_{kj} = -a_{jk}, \text{ hence } a_{jj} = 0).$$

Symmetric Matrix

Skew-Symmetric Matrix

- Example:

$$\mathbf{A} = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix} \quad \text{is symmetric, and} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \quad \text{is skew-symmetric.}$$

Triangular Matrices

- **Upper triangular matrices:** square matrices with nonzero entries only on and above the main diagonal.
- **Lower triangular matrices:** square matrices with nonzero entries only on and below the main diagonal.
- Any entry on the main diagonal of a triangular matrix may be zero or not.
- Example:

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}}_{\text{Upper triangular}}, \quad \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 9 & 3 & 6 \end{bmatrix}}_{\text{Lower triangular}}.$$

Example: Diagonal, Scalar, & Unit Matrices

Diagonal

Scalar

Identity

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Diagonal, Scalar, & Unit Matrices

- Diagonal matrix: square matrix that can have nonzero entries only on the main diagonal.

- Scalar matrix $\boldsymbol{S}$: diagonal matrix with all diagonal entries having the same value $=\ c$. Why call is scalar matrix?

$$\boldsymbol{AS} = \boldsymbol{SA} = c\boldsymbol{A}$$

- Unit or identity matrix $\boldsymbol{I}$: scalar matrix with $c = 1$. Why call is unit matrix?

$$\boldsymbol{AI} = \boldsymbol{IA} = \boldsymbol{A}$$

Linear Algebra: Matrices and Vectors – Part 2

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Linear Systems of Equations

- A linear system of m equations in n unknowns $x_1, \ldots, x_n$ is a set of equations of the form:

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

- The system is called linear because each variable appears in the first power only.

Linear Systems of Equations

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots&\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

- a_{11}, …, a_{mn} are given numbers, called the coefficients of the system.
- b_{11}, …, b_{mn} on the right are also given numbers.
- If all $b_j=0$ → homogeneous system.

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Matrix Form of the Linear System

- From the definition of matrix multiplication the “m” equations may be written as:

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

coefficient matrix $\mathbf{A} = [a_{jk}]$ is the $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Matrix Form of the Linear System

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}
\longrightarrow \boxed{\mathbf{A}\mathbf{x} = \mathbf{b}}
$$

$$
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}
$$

Matrix Form of the Linear System

- The matrix below is called the augmented matrix of the system. The dashed vertical line could be omitted:

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc:c} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

Example: Existence and Uniqueness of Solutions – 2 unknowns

If $m = n = 2$, we have two equations in two unknowns x_1, x_2

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2.$$

Example: Existence and Uniqueness of Solutions – 2 unknowns

- Interpret x_1, x_2 on the x_1x_2-plane.
- The two equations represent straight lines.
- Solutions are points on both lines at the same time.

(*a*) Precisely one solution if the lines intersect

(*b*) Infinitely many solutions if the lines coincide

(*c*) No solution if the lines are parallel

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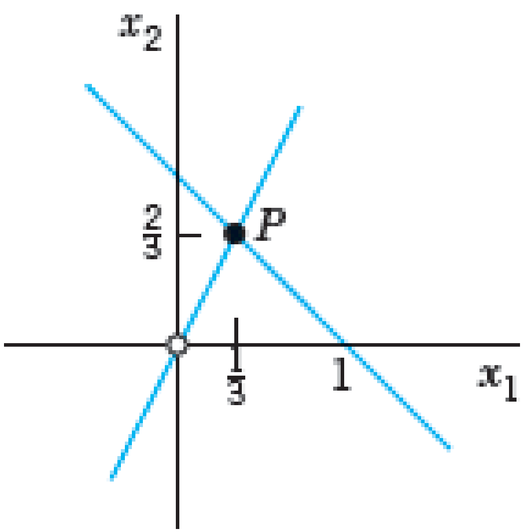
Example: Existence and Uniqueness of Solutions – 2 unknowns

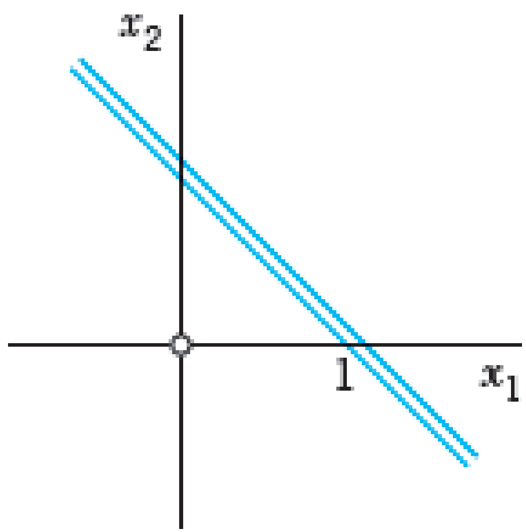
$$x_1 + x_2 = 1$$
$$2x_1 - x_2 = 0$$

Case (a)



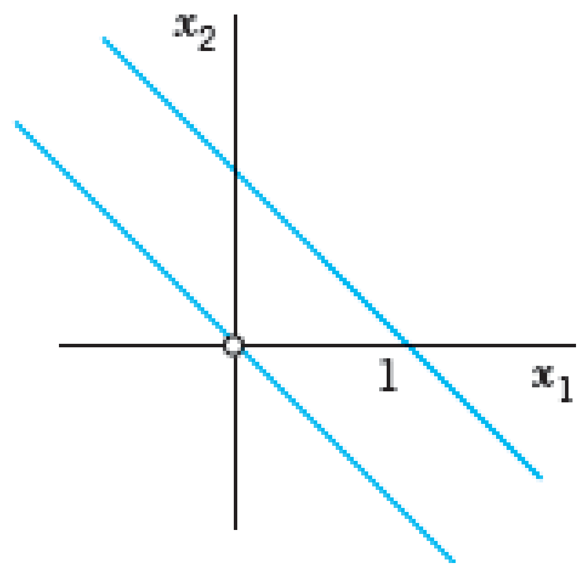
$$x_1 + x_2 = 1$$
$$2x_1 + 2x_2 = 2$$

Case (b)



$$x_1 + x_2 = 1$$
$$x_1 + x_2 = 0$$

Case (c)



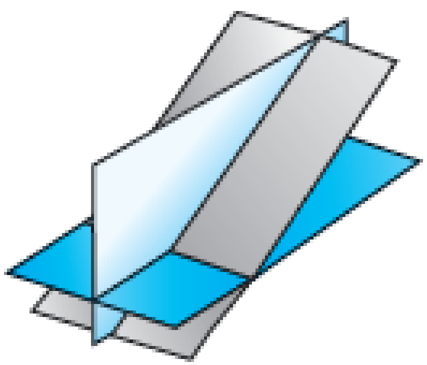
Example: Existence and Uniqueness of Solutions – 2 unknowns

- If the system is homogenous, Case (c) cannot happen, because then those two straight lines pass through the origin, whose coordinates constitute the trivial solution.

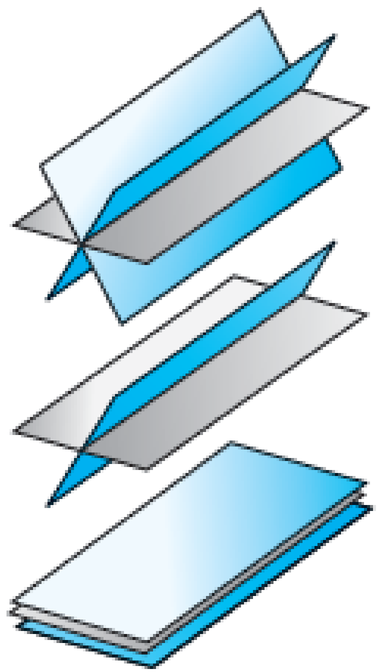
- Similarly, our present discussion can be extended from two equations in two unknowns to three equations in three unknowns.

- Instead of straight lines we have planes and the solution depends on the positioning of these planes in space relative to each other.

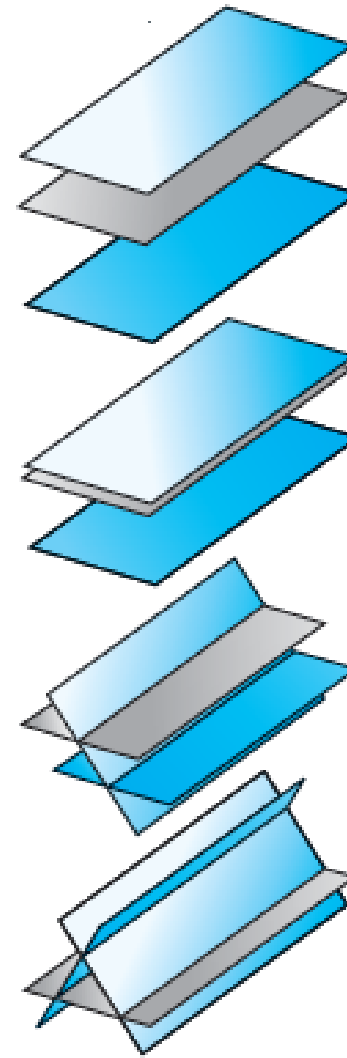
Example: Existence and Uniqueness of Solutions – 3 unknowns



Unique solution



Infinitely many solutions



No solution

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Gauss Elimination and Back Substitution

- The Gauss elimination method can be motivated as follows. Consider a linear system that is in triangular form (in full, upper triangular form) such as:

$$\begin{aligned} 2x_1 + 5x_2 &= 2 \\ 13x_2 &= -26 \end{aligned}$$

- Rearrange eq. 2:

$$x_2 = -26/13 = -2$$

- Back substitution: substitute x_2 into eq. 1:

$$x_1 = \frac{1}{2}(2 - 5x2) = \frac{1}{2}(2 - 5.(-2)) = 6$$

Elementary Row Operations. Row-Equivalent Systems

- Elementary Row Operations for Matrices:
  - Interchange of two rows.
  - Multiplication of a row by a nonzero constant c.
  - Addition of a constant multiple of one row to another row.
- CAUTION! Row operation not for columns!
- System $\mathbf{S_1}$ → row operations → system $\mathbf{S_2}$
- $\mathbf{S_1}$ and $\mathbf{S_2}$ are called row equivalent systems
- Row-equivalent linear systems have the same set of solutions.

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Example 1: Gauss Elimination and Back Substitution

- This gives us the idea of first reducing a general system to triangular form. For instance, let the given system be:

$$\begin{aligned} 2x_1 + 5x_2 &= 2 \\ -4x_1 + 3x_2 &= -30. \end{aligned}$$

Its augmented matrix is $\begin{bmatrix} 2 & 5 & 2 \\ -4 & 3 & -30 \end{bmatrix}$.

- We eliminate x_1 from the second equation, to get a triangular system.

$$\begin{bmatrix} 2 & 5 & 2 \\ -4 & 3 & -30 \end{bmatrix} \longrightarrow \begin{matrix} \\ \text{Row 2} + 2\,\text{Row 1} \end{matrix} \begin{bmatrix} 2 & 5 & 2 \\ 0 & 13 & -26 \end{bmatrix}$$

Example 1: Gauss Elimination and Back Substitution

$$\begin{array}{l} \\ \text{Row 2} + 2\,\text{Row 1} \end{array} \begin{bmatrix} 2 & 5 & 2 \\ 0 & 13 & -26 \end{bmatrix} \longrightarrow \begin{aligned} 2x_1 + 5x_2 &= 2 \\ 13x_2 &= -26 \end{aligned}$$

- This is the Gauss elimination for 2 unknowns giving the triangular form, from which back substitution now yields $x_2 = -2 \; and \; x_1 = 6$.

Example 2: Gauss Elimination: Electrical Network

- Solve the linear system:

$$
\begin{aligned}
x_1 - x_2 + x_3 &= 0 \\
-x_1 + x_2 - x_3 &= 0 \\
10x_2 + 25x_3 &= 90 \\
20x_1 + 10x_2 &= 80.
\end{aligned}
$$

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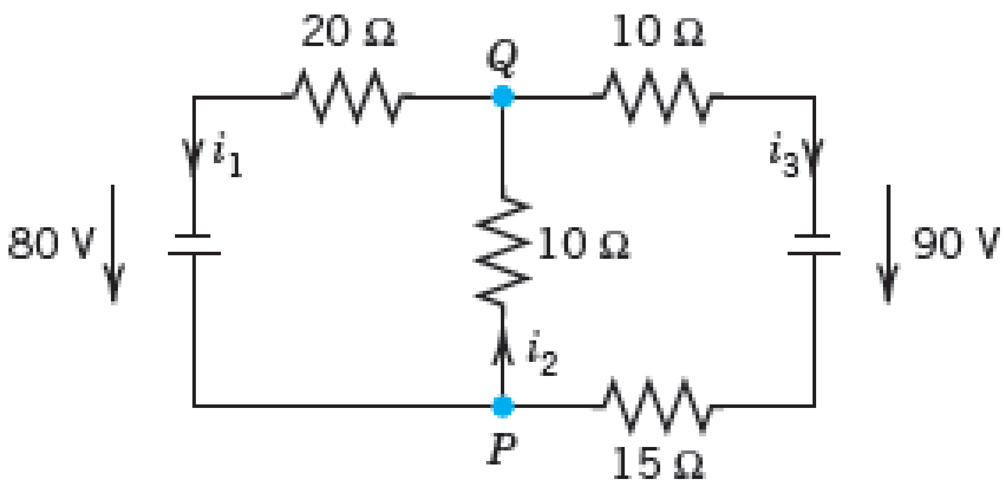
Example 2: Gauss Elimination: Electrical Network

- Derivation from the circuit below



$$
\begin{array}{lrrrl}
\text{Node } P: & i_1 - & i_2 + & i_3 = & 0 \\
\text{Node } Q: & -i_1 + & i_2 - & i_3 = & 0 \\
\text{Right loop:} & & 10i_2 + & 25i_3 = & 90 \\
\text{Left loop:} & 20i_1 + & 10i_2 & = & 80
\end{array}
$$

$$x_1 = i_1, x_2 = i_2, x_3 = i_3$$

Example 2: Gauss Elimination: Electrical Network

- Creation of augmented matrix $\tilde{A}$ and pivot 1:

Augmented Matrix $\tilde{\mathbf{A}}$

$$\begin{array}{l} \text{Pivot 1} \longrightarrow \\ \\ \text{Eliminate} \longrightarrow \\ \\ \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

- Elimination of x_1

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right] \begin{array}{l} \\ \text{Row 2 + Row 1} \\ \\ \text{Row 4 − 20 Row 1} \end{array}$$

Example 2: Gauss Elimination: Electrical Network

- Creation of pivot 10:

$$
\begin{array}{r}
\\
\text{Pivot 10} \longrightarrow \\
\text{Eliminate 30} \longrightarrow \\
\\
\end{array}
\left[\begin{array}{rrr|r}
1 & -1 & 1 & 0 \\
0 & 10 & 25 & 90 \\
0 & \boxed{30} & -20 & 80 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

- Elimination of x_2

$$
\left[\begin{array}{rrr|r}
1 & -1 & 1 & 0 \\
0 & 10 & 25 & 90 \\
0 & 0 & -95 & -190 \\
0 & 0 & 0 & 0
\end{array}\right]
\begin{array}{l}
\\
\\
\text{Row 3} - 3\,\text{Row 2} \\
\\
\end{array}
$$

Example 2: Gauss Elimination: Electrical Network

- Elimination of x_2:

- New system equations:

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ -95x_3 &= -190 \\ 0 &= 0. \end{aligned}$$

Example 2: Gauss Elimination: Electrical Network

- Back Substitution. Determination of x_3, x_2, x_1 (in this order):

$$
\begin{aligned}
-95x_3 &= -190 & \qquad x_3 &= i_3 = 2\ [\mathrm{A}] \\
10x_2 + 25x_3 &= 90 & \qquad x_2 &= \tfrac{1}{10}(90 - 25x_3) = i_2 = 4\ [\mathrm{A}] \\
x_1 - x_2 + x_3 &= 0 & \qquad x_1 &= x_2 - x_3 = i_1 = 2\ [\mathrm{A}]
\end{aligned}
$$

where A stands for "amperes." This is the answer to our problem. The solution is unique.

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Elementary Row Operations. Row-Equivalent Systems

- A linear system is called determined if number of equations m = number of unknowns n, as in example 1.
- overdetermined if it has more equations than unknowns, as in example 2.
- underdetermined if it has fewer equations than unknowns.

Elementary Row Operations. Row-Equivalent Systems

- A system is called consistent if it has at least one solution (thus, one solution or infinitely many solutions).
- A system is called inconsistent if it has no solutions at all, as:

$$x_1 + x_2 = 1$$

$$x_1 + x_2 = 0$$

Gauss Elimination: The Three Possible Cases of Systems

- Systems that have a unique solution: Examples 1 & 2 solved above.
- Systems that have infinitely many solutions: Example 3 solved below.
- Systems that don’t have any solution: Example 4 solved below.

Example 3: Gauss Elimination if Infinitely Many Solutions Exist

- Three equations and four unknowns.

$$\left[\begin{array}{cccc|c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right].$$

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Example 3: Gauss Elimination if Infinitely Many Solutions Exist

- Circle pivot 1 and box terms of equations:

$$
\begin{aligned}
3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\
\boxed{0.6x_1} + 1.5x_2 + 1.5x_3 - 5.4x_4 &= 2.7 \\
\boxed{1.2x_1} - 0.3x_2 - 0.3x_3 + 2.4x_4 &= 2.1.
\end{aligned}
$$

- Eliminate x_1:

$$
\left[\begin{array}{cccc|c}
3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\
0 & 1.1 & 1.1 & -4.4 & 1.1 \\
0 & -1.1 & -1.1 & 4.4 & -1.1
\end{array}\right]
\begin{array}{l}
\\
\text{Row 2} - 0.2 \text{ Row 1} \\
\text{Row 3} - 0.4 \text{ Row 1}
\end{array}
$$

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Example 3: Gauss Elimination if Infinitely Many Solutions Exist

- Circle pivot 2 and box terms of equations:

$$
\begin{aligned}
3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 &= 8.0 \\
1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \\
\boxed{-1.1x_2} - 1.1x_3 + 4.4x_4 &= -1.1.
\end{aligned}
$$

- Eliminate x_2:

$$
\left[\begin{array}{cccc|c}
3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\
0 & 1.1 & 1.1 & -4.4 & 1.1 \\
0 & 0 & 0 & 0 & 0
\end{array}\right] \quad \text{Row 3 + Row 2}
$$

Example 3: Gauss Elimination if Infinitely Many Solutions Exist

- Previous elimination gives:

$$3.0x_1 + 2.0x_2 + 2.0x_3 - 5.0x_4 = 8.0$$

$$1.1x_2 + 1.1x_3 - 4.4x_4 = 1.1$$

$$0 = 0.$$

- From the 2nd eq: $x_2 = 1 - x_3 + 4x_4.$
- From this and the 1st eq: $x_1 = 2 - x_4.$
- x_1 and x_2 depend on x_3 and x_4 → x_3 and x_4 remain arbitrary and we have infinitely many solutions.
- If we choose a value of x_3 and x_4, then the corresponding values of x_1 and x_2 and are uniquely determined.

Example 4: Gauss Elimination if no Solution Exists

- This case happens when gauss elimination produces a contradiction.
- Three equations and three unknowns.

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

Example 4: Gauss Elimination if no Solution Exists

- Circle pivot 1 and box terms of equations:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 3 \\ \boxed{2x_1} + x_2 + x_3 &= 0 \\ \boxed{6x_1} + 2x_2 + 4x_3 &= 6. \end{aligned}$$

- Eliminate x_1:

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & -2 & 2 & 0 \end{array}\right] \quad \begin{array}{l} \\ \text{Row 2} - \frac{2}{3}\,\text{Row 1} \\ \text{Row 3} - 2\,\text{Row 1} \end{array}$$

Example 4: Gauss Elimination if no Solution Exists

- Circle pivot 2 and box terms of equations:

$$
\begin{aligned}
3x_1 + 2x_2 + x_3 &= 3 \\
-\tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 &= -2 \\
\boxed{-2x_2} + 2x_3 &= 0.
\end{aligned}
$$

- Eliminate x_2:

$$
\left[\begin{array}{ccc|c}
3 & 2 & 1 & 3 \\
0 & -\tfrac{1}{3} & \tfrac{1}{3} & -2 \\
0 & 0 & 0 & 12
\end{array}\right]
\quad \text{Row 3} - 6\ \text{Row 2}
$$

Example 4: Gauss Elimination if no Solution Exists

- Previous elimination yields:

$$\begin{aligned} 3x_1 + 2x_2 + \phantom{\tfrac{1}{3}}x_3 &= 3 \\ -\tfrac{1}{3}x_2 + \tfrac{1}{3}x_3 &= -2 \\ 0 &= 12. \end{aligned}$$

- The false statement 0=12 shows that the system has no solution.

Row Echelon Form and Information From it

- At the end of the Gauss elimination the form of the coefficient matrix, the augmented matrix, and the system itself are called the row echelon form.

- In it, rows of zeros, if present, are the last rows, example 4:

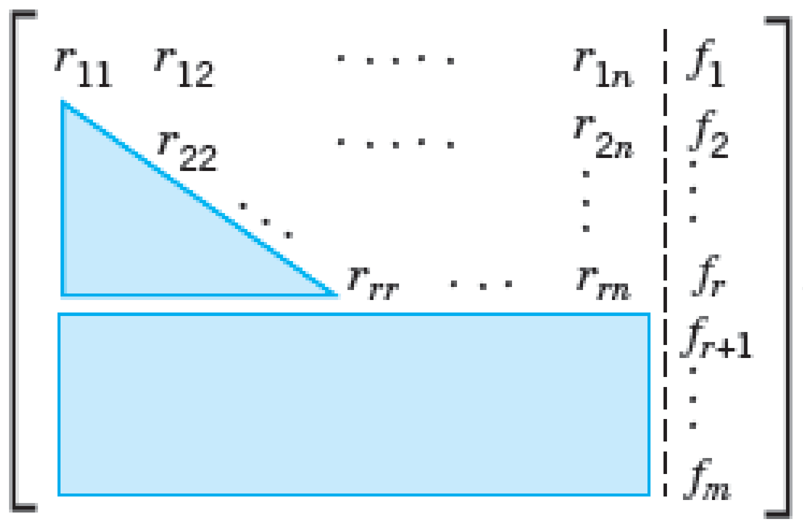
$$\left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right].$$

- The original system of m equations in n unknowns has augmented matrix **[A|b]**. This is to be row reduced to matrix **[R|f]**.

F.Y.I.

Row Echelon Form and Information From It

- At the end of the Gauss elimination the row echelon form of the augmented matrix [**R**|**f**] will be:



- Here $r \le m$, $r_{11} \neq 0$, and all entries in blue are zero.

- The number of nonzero rows, r, in the row-reduced coefficient matrix R is called the rank of R and also the rank of A.

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F.Y.I.

Row Echelon Form and Information From It

- Possible solution cases:
- **Unique solution**: if $r = n$. f_{r+1} to f_m if present are zero. In example 2, $r = n = 3$, and $m=4$.
- **Infinitely many solutions**: if $r<n$ and f_{r+1} to f_m if present are zero. See example 3.
- **No solution**: if $r<m$ and at least one of the numbers f_{r+1} to f_m is non-zero. See example 4.

Linear Systems in Matlab

- Eg 2: unique solution

linsolve([1 -1 1 ; 0 10 25 ; 20 10 0] , [0 ; 90 ; 80])
ans =
  2.0000
  4.0000
  2.0000

- Eg 3: infinitely many solutions

linsolve([3 2 2 -5 ; 0.6 1.5 1.5 -5.4 ; 1.2 -0.3 -0.3 2.4] , [8 ; 2.7 ; 2.1])
Warning: Rank deficient, **rank = 2**, tol = 6.8752e-015.
ans =
  2.2500
    0
    0
  -0.2500

$$x_2 = 1 - x_3 + 4x_4.$$

$$x_1 = 2 - x_4.$$

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Linear Systems in Matlab

- Eg 4: no solution

```
linsolve( [ 3 2 1 ; 2 1 1 ; 6 2 4 ] , [ 3 ; 0 ; 6 ] )
Warning: Matrix is close to singular or badly scaled.
        Results may be inaccurate. RCOND = 3.364312e-018.
ans =
  1.0e+016 *
    1.8014
   -1.8014
   -1.8014
```

Linear Algebra: Matrices and Vectors – Part 3

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7.4 Linear Independence of Vectors

- A linear combination of m vectors $\mathbf{a}_{(1)}, \ldots, \mathbf{a}_{(m)}$ is:

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)}$$

where $c_1, c_2, \ldots, c_m$ are any scalars. Rearrange:

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)} = \mathbf{0}.$$

- This can be satisfied if all c's are zero, because then it becomes $\mathbf{0} = \mathbf{0}$.

Linear Independence of Vectors

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)} = \mathbf{0}.$$

- If this equation also holds with scalars not all zero, we call these vectors linearly dependent. Otherwise, they are linearly independent.
- Linear dependence means that we can express at least one of the vectors as a linear combination of the other vectors. For example, if $c_1 \neq 0$:

$$\mathbf{a}_{(1)} = k_2\mathbf{a}_{(2)} + \cdots + k_m\mathbf{a}_{(m)} \qquad \text{where } k_j = -c_j/c_1.$$

Example0: Linear Independence of Vectors

- The independence of TWO vectors is easy to identify
- Example:

$a_1 = [2 \;\; 3 \;\; 15]$

$a_2 = [1 \;\; 1.5 \;\; 7.5]$

$a_1 = 2a_2 \rightarrow dependent$

Example1: Linear Independence of Vectors

The three vectors

$$\begin{aligned} \mathbf{a}_{(1)} &= [\;3 \quad 0 \quad 2 \quad 2] \\ \mathbf{a}_{(2)} &= [-6 \quad 42 \quad 24 \quad 54] \\ \mathbf{a}_{(3)} &= [\;21 \quad -21 \quad 0 \quad -15] \end{aligned}$$

are linearly dependent because

$$6\mathbf{a}_{(1)} - \tfrac{1}{2}\mathbf{a}_{(2)} - \mathbf{a}_{(3)} = \mathbf{0}.$$

- Easily checked but not so easy to discover.
- A systematic method can be done by calculating the rank of a matrix explained next.

Rank of a Matrix

- Definition: The rank of a matrix **A** is the maximum number of linearly independent row vectors of **A**. It is denoted by rank **A**.

- Matrix $\mathbf{A_1}$ → row operations → matrix $\mathbf{A_2}$
- Definition: $\mathbf{A_1}$ and $\mathbf{A_2}$ are called row equivalent matricies.

→ Matrices in Gauss elimination steps are row-equivalent.

- **Theorem:** Row-equivalent matrices have the same rank.

Example1: Rank of a Matrix

- The following three vectors (given previously):

$$\mathbf{a}_{(1)} = [\,3 \quad 0 \quad 2 \quad 2\,]$$

$$\mathbf{a}_{(2)} = [\,-6 \quad 42 \quad 24 \quad 54\,]$$

$$\mathbf{a}_{(3)} = [\,21 \quad -21 \quad 0 \quad -15\,]$$

can be represented by the matrix:

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ \boxed{\begin{matrix}-6 \\ 21\end{matrix}} & \begin{matrix}42 \\ -21\end{matrix} & \begin{matrix}24 \\ 0\end{matrix} & \begin{matrix}54 \\ -15\end{matrix} \end{bmatrix} \quad \text{(given)}$$

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix} \begin{matrix} \\ \text{Row 2} + 2\text{ Row 1} \\ \text{Row 3} - 7\text{ Row 1} \end{matrix}$$

Example1: Rank of a Matrix

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & \boxed{-21} & -14 & -29 \end{bmatrix} \begin{array}{l} \\ \text{Row 2} + 2\,\text{Row 1} \\ \text{Row 3} - 7\,\text{Row 1} \end{array}$$

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \\ \text{Row 3} + \frac{1}{2}\,\text{Row 2.} \end{array}$$

- The rank of a matrix in row-echelon form is the number of non-zero rows.
- Hence rank A=2

Previous Examples: Rank of a Matrix

$$\begin{bmatrix} 2 & 5 & 2 \\ -4 & 3 & -30 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 5 & 2 \\ 0 & 13 & -26 \end{bmatrix}$$

Hence, rank = 2

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ \boxed{-1} & 1 & -1 & 0 \\ \boxed{0} & 10 & 25 & 90 \\ \boxed{20} & 10 & 0 & 80 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & \boxed{30} & -20 & 80 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, rank = 3

$$\begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{bmatrix}. \rightarrow \begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{bmatrix} \rightarrow \begin{bmatrix} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, rank = 2

$$\begin{bmatrix} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & -2 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

Hence, rank = 3

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Linear Independence of Vectors

- **Theorem 2: Linear Independence and Dependence of Vectors:** p number of vectors are linearly independent if the matrix formed with these vectors has rank p. Otherwise linearly dependent.

If Rank(A) = # of vectors → vectors are independent

- **Theorem 3: Rank in Terms of Column Vectors:** The rank r of a matrix **A** equals the maximum number of linearly independent column vectors of **A**.

- Hence **A** and its transpose $\mathbf{A}^T$ have the same rank.
  Proof: see textbook.

F.Y.I.

Example1: Rank in Terms of Column Vectors

Recall the matrix from example1 above:

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

Performing the following column operations concludes - as before - that rank = 2:

$$\text{Column } 3 = \tfrac{2}{3}\text{Column } 1 + \tfrac{2}{3}\text{Column } 2 \qquad \text{and} \qquad \text{Column } 4 = \tfrac{2}{3}\text{Column } 1 + \tfrac{29}{21}\text{Column } 2.$$

Linear Independence of Vectors

- **<u>Theorem 4:</u> Linear Dependence of Vectors:** Consider p vectors each having n components. If n<p then these vectors are linearly dependent.

- Eg:

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 10 & 25 \\ 0 & 0 & -95 \\ 0 & 0 & 0 \end{bmatrix}$$

F.Y.I.

Linear Independence of Vectors

rref : Reduced row echelon form

```
>> rref ( [ 1 -1  ; 0 10  ; 20 10  ] )
ans =
     1     0
     0     1
     0     0

>> rref ( [ 5 -1  ; 0 10  ; 20 10  ] )
ans =
     1     0
     0     1
     0     0

>> rref ( [ 5 -1  ; 0 10  ; 20 0  ] )
ans =
     1     0
     0     1
     0     0
```

7.5 Solutions of Linear Systems: Existence, Uniqueness

- Rank gives complete information about existence, uniqueness, and general structure of the solution set of linear systems as follows.

- A linear system of equations in n unknowns has:
  - a **unique** solution if rank($\boldsymbol{A}$) = rank($\widetilde{\boldsymbol{A}}$) = n.
  - **infinitely** many solutions if if rank($\boldsymbol{A}$) = rank($\widetilde{\boldsymbol{A}}$) < n.
  - **no solution** if rank($\boldsymbol{A}$) ≠ rank($\widetilde{\boldsymbol{A}}$).

- If solutions exist, they can all be obtained by the **Gauss elimination**. (This method will automatically reveal whether or not solutions exist

Previous Examples: Rank of a Matrix

$$\begin{bmatrix} 2 & 5 & 2 \\ 0 & 13 & -26 \end{bmatrix}$$

rank($\boldsymbol{A}$) = rank($\widetilde{\boldsymbol{A}}$) = n = 2 → unique soln

$$\left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

rank($\boldsymbol{A}$) = rank($\widetilde{\boldsymbol{A}}$) = n = 3 → unique soln

$$\left[\begin{array}{cccc:c} 3.0 & 2.0 & 2.0 & -5.0 & 8.0 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

rank($\boldsymbol{A}$) = rank($\widetilde{\boldsymbol{A}}$) = 2 < (n=4) → infinite soln's

$$\left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

rank($\boldsymbol{A}$) ≠ rank($\widetilde{\boldsymbol{A}}$) → no soln

Determinants

- Determinants were originally introduced for solving linear systems. They have important engineering applications in eigenvalue problems (Sec. 8.1), differential equations, vector algebra (Sec. 9.3), and in other areas.

- A determinant of order n is a scalar associated with an n × n matrix:

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Second-Order Determinants

- A determinant of second order can be defined by:

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Third-Order Determinants

- A determinant of third order can be defined by:

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

$$D = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{21}a_{13}a_{32} - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22}.$$

Third-Order Determinants

- A determinant of third order can be defined by:

$$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \boxed{a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

$$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - \boxed{a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

$$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + \boxed{a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}}.$$

Third-Order Determinants

$$D = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = \boxed{1\begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix}}$$

$$= 1(12 - 0) - 3(4 + 4) + 0(0 + 6) = -12.$$

This is the expansion by the first row. The expansion by the third column is

$$\boxed{D = 0\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix} - 4\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0 - 12 + 0 = -12.}$$

Verify that the other four expansions also give the value -12.

F.Y.I.

n-order Determinants

- A determinant is defined as follows:

For $n = 1$, this determinant is defined by

(2) $$D = a_{11}.$$

For $n \geqq 2$ by

(3a) $$D = a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn} \qquad (j = 1, 2, \cdots, \text{or } n)$$

or "Column-wise" expansion

(3b) $$D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} \qquad (k = 1, 2, \cdots, \text{or } n).$$

Here, "Row-wise" expansion (shown in previous slide)

$$C_{jk} = (-1)^{j+k}M_{jk}$$

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Minors and Cofactors of Third-Order Determinants

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \boxed{a_{21} \quad a_{22} \quad a_{23}} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \qquad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \qquad M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

Minors and Cofactors

$$C_{jk} = (-1)^{j+k} M_{jk}$$

- C_{jk} is the cofactor of a_{jk} in D
- M_{jk} is the minor of a_{jk} in D (determinant of order n-1): the determinant of the submatrix of $\boldsymbol{A}$ obtained from $\boldsymbol{A}$ by omitting the row and column of the entry.

- Equation (3) above can be rewritten as :

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \qquad (j = 1, 2, \cdots, \text{or } n) \tag{4a}$$

$$D = \sum_{j=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \qquad (k = 1, 2, \cdots, \text{or } n). \tag{4b}$$

Minors and Cofactors of Third-Order Determinants

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \boxed{a_{21} \quad a_{22} \quad a_{23}} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \qquad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \qquad M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{21} = -M_{21}, C_{22} = +M_{22}, \text{ and } C_{23} = -M_{23}.$$

Checkerboard pattern for C_{jk}

$$\begin{matrix} + & - & + \\ \boxed{- \quad + \quad -} \\ + & - & + \end{matrix}$$

Third-Order Determinants

- The determinants can be expanded using any row or column. The following D using 1st column:

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \boxed{\begin{matrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{matrix}} \\ a_{31} & & \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

- Minor of a_{11} is $M_{11} = \boxed{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}$

- Cofactor of a_{11} is $C_{11} = (-1)^{j+k} M_{11} = (-1)^2 M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Third-Order Determinants

- Use the row or column with the most zeros.

$$D = \begin{vmatrix} \boxed{1 \quad 3 \quad 0} \\ 2 \quad 6 \quad 4 \\ -1 \quad 0 \quad 2 \end{vmatrix} = \boxed{1\begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix}}$$

Cofactor Matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7, \quad C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13, \quad C_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8,$$

$$C_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2, \quad C_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2, \quad C_{23} = -\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2,$$

$$C_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3, \quad C_{32} = -\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = 7, \quad C_{33} = \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = -2,$$

$$cof(A) = \begin{bmatrix} -7 & -13 & 8 \\ 2 & -2 & 2 \\ 3 & 7 & -2 \end{bmatrix}$$

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Cofactor Matrix

```
>> A=[-1 1 6;-3 7 0;0 -2 5]

A =

    -1     1     6
    -3     7     0
     0    -2     5

>> cof(A)

ans =

   35.0000   15.0000    6.0000
  -17.0000   -5.0000   -2.0000
  -42.0000  -18.0000   -4.0000
```

Google cofactor matlab

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cof=cof(a) - File Exchange - MATLAB Central
www.mathworks.com/matlabcentral/...cofactor.../cof.m The MathWorks
Feb 8, 2012 - function cof=cof(a); % % COF=COF(A) generates matrix of **cofactor** values for an M-by-N matrix % A : an M-by-N matrix % % Example: Find the ...

Adjoint (or Adjugate) Matrix

- $(adj\ \boldsymbol{A}) = (cof\ \boldsymbol{A})^T$

- $\boldsymbol{A}\,(adj\ \boldsymbol{A}) = (adj\ \boldsymbol{A})\,\boldsymbol{A} = (\det \boldsymbol{A})\,\boldsymbol{I}$

- Using MATLAB:

- Cofactor: cof(A)
- Transpose: transpose(A) or A’
- Adjoint: transpose(cof(A)) or cof(A)’

Example: Determinant of a Triangular Matrix

$$\begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix} = -3 \begin{vmatrix} 4 & 0 \\ 2 & 5 \end{vmatrix} = -3 \cdot 4 \cdot 5 = -60.$$

- Hence, the determinant of a **diagonal or triangular matrix** is just the product of its diagonal entries.

General Properties of Determinants

Theorem 1:

Behavior of an nth-Order Determinant under Elementary Row Operations

(a) *Interchange of two rows multiplies the value of the determinant by* -1.

(b) *Addition of a multiple of a row to another row does not alter the value of the determinant.*

(c) *Multiplication of a row by a nonzero constant c multiplies the value of the determinant by c.* (This holds also when $c = 0$, but no longer gives an elementary row operation.)

- (a) can be realized just by looking at the checkerboard mentioned above.

A=[-3 0 0 ; 6 4 0 ; -1 2 5]; det(A) = - 60

A=[-1 2 5 ; 6 4 0 ; -3 0 0]; det(A) = 60

A=[-1 2 5 ; -3 0 0 ; 6 4 0]; det(A) = - 60

- (b) points us to an attractive way of finding determinants: by reduction to triangular form.
- from (c): $\det(c\mathbf{A}) = c^n \det(\mathbf{A})$

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General Properties of Determinants

Theorem 2:

Further Properties of nth-Order Determinants

(a)–(c) *in Theorem 1 hold also for columns.*

(d) ***Transposition*** *leaves the value of a determinant unaltered.*

(e) ***A zero row or column*** *renders the value of a determinant zero.*

(f) ***Proportional rows or columns*** *render the value of a determinant zero. In particular, a determinant with two identical rows or columns has the value zero.*

- (f) can be proven from Theorem 1 and (e).
- from (f) & (b): a matrix with rank < n has det = zero.

eg:
$$\begin{matrix} 3 & 2 & 1 \\ 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{matrix}$$

Example: Finding Determinants by Reduction to Triangular Form

$$D = \begin{vmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{vmatrix}$$

Example: Finding Determinants by Reduction to Triangular Form

$$D = \begin{vmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 2 & 6 & -1 \\ 0 & 8 & 3 & 10 \end{vmatrix} \begin{array}{l} \\ \text{Row 2} - 2\ \text{Row 1} \\ \\ \text{Row 4} + 1.5\ \text{Row 1} \end{array}$$

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Example: Finding Determinants by Reduction to Triangular Form

$$=\begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & -11.4 & 29.2 \end{vmatrix} \begin{array}{l} \\ \\ \text{Row 3} - 0.4 \text{ Row 2} \\ \text{Row 4} - 1.6 \text{ Row 2} \end{array}$$

Example: Finding Determinants by Reduction to Triangular Form

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & 0 & 47.25 \end{vmatrix} \quad \text{Row 4 + 4.75 Row 3}$$

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Example: Finding Determinants by Reduction to Triangular Form

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & 0 & 47.25 \end{vmatrix} \quad \text{Row 4} + 4.75 \text{ Row 3}$$

$$= 2 \cdot 5 \cdot 2.4 \cdot 47.25 = 1134.$$

Linear Algebra: Matrices and Vectors – Part 4

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Cramer's Rule for Linear Systems of Two Equations

(2)

$$\text{(a)}\quad a_{11}x_1 + a_{12}x_2 = b_1$$

$$\text{(b)}\quad a_{21}x_1 + a_{22}x_2 = b_2$$

(3)

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} = \frac{b_1a_{22} - a_{12}b_2}{D},$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D} = \frac{a_{11}b_2 - b_1a_{21}}{D}$$

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

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Cramer's Rule for Linear Systems of Two Equations

$$(2) \qquad \begin{aligned} &\text{(a)} \quad a_{11}x_1 + a_{12}x_2 = b_1 \\ &\text{(b)} \quad a_{21}x_1 + a_{22}x_2 = b_2 \end{aligned}$$

We prove (3). To eliminate x_2 multiply $\boxed{\text{(2a) by } a_{22}}$ and $\boxed{\text{(2b) by } -a_{12}}$ and add,

$$\underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{D}x_1 = \underbrace{b_1a_{22} - a_{12}b_2}_{D_1}.$$

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Cramer's Rule for Linear Systems of Two Equations

$$(2) \qquad \begin{array}{ll} \text{(a)} & a_{11}x_1 + a_{12}x_2 = b_1 \\ \text{(b)} & a_{21}x_1 + a_{22}x_2 = b_2 \end{array}$$

We prove (3). To eliminate x_2 multiply $\boxed{\text{(2a) by } a_{22}}$ and $\boxed{\text{(2b) by } -a_{12}}$ and add,

$$\underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{D}x_1 = \underbrace{b_1a_{22} - a_{12}b_2}_{D_1}.$$

Similarly, to eliminate x_1 multiply $\boxed{\text{(2a) by } -a_{21}}$ and $\boxed{\text{(2b) by } a_{11}}$ and add,

$$\underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{D}x_2 = \underbrace{a_{11}b_2 - b_1a_{21}}_{D_2}.$$

Assuming that $D = a_{11}a_{22} - a_{12}a_{21} \neq 0$, dividing, and writing the right sides of these two equations as determinants, we obtain (3).

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Cramer's Rule for Linear Systems of Two Equations

We prove (3). To eliminate x_2 multiply $\boxed{\text{(2a) by } a_{22}}$ and $\boxed{\text{(2b) by } -a_{12}}$ and add,

D $\qquad\qquad$ D_1

$$(a_{11}a_{22} - a_{12}a_{21})x_1 = b_1a_{22} - a_{12}b_2.$$

Similarly, to eliminate x_1 multiply $\boxed{\text{(2a) by } -a_{21}}$ and $\boxed{\text{(2b) by } a_{11}}$ and add,

D $\qquad\qquad$ D_2

$$(a_{11}a_{22} - a_{12}a_{21})x_2 = a_{11}b_2 - b_1a_{21}.$$

Assuming that $D = a_{11}a_{22} - a_{12}a_{21} \neq 0$, dividing, and writing the right sides of these two equations as determinants, we obtain (3).

D_1

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} = \frac{b_1a_{22} - a_{12}b_2}{D},$$

(3)

D_2

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D} = \frac{a_{11}b_2 - b_1a_{21}}{D}$$

Example 1: Cramer's Rule for Linear Systems of Two Equations

If
$$\begin{aligned} 4x_1 + 3x_2 &= 12 \\ 2x_1 + 5x_2 &= -8 \end{aligned}$$
then

$$x_1 = \frac{\begin{vmatrix} 12 & 3 \\ -8 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \frac{84}{14} = 6, \qquad x_2 = \frac{\begin{vmatrix} 4 & 12 \\ 2 & -8 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \frac{-56}{14} = -4.$$

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Cramer's Rule for Linear Systems of Three Equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D} \qquad (D \neq 0)$$

$$D_1 = \begin{vmatrix} \boxed{\begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix}} & \begin{matrix} a_{12} \\ a_{22} \\ a_{32} \end{matrix} & \begin{matrix} a_{13} \\ a_{23} \\ a_{33} \end{matrix} \end{vmatrix}, \quad D_2 = \begin{vmatrix} \begin{matrix} a_{11} \\ a_{21} \\ a_{31} \end{matrix} & \boxed{\begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix}} & \begin{matrix} a_{13} \\ a_{23} \\ a_{33} \end{matrix} \end{vmatrix}, \quad D_3 = \begin{vmatrix} \begin{matrix} a_{11} \\ a_{21} \\ a_{31} \end{matrix} & \begin{matrix} a_{12} \\ a_{22} \\ a_{32} \end{matrix} & \boxed{\begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix}} \end{vmatrix}.$$

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Cramer's Rule for Linear Systems of n Equations

(a) *If a linear system of n equations in the same number of unknowns* $x_1, \cdots, x_n$

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
\tag{6}
$$

has a nonzero coefficient determinant $D = \det \mathbf{A}$, *the system has precisely one solution. This solution is given by the formulas*

$$
x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \cdots, \quad x_n = \frac{D_n}{D} \qquad \textbf{(Cramer's rule)}
\tag{7}
$$

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Cramer's Rule for Linear Systems of n Equations

(7) $$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \cdots, \quad x_n = \frac{D_n}{D}$$ **(Cramer's rule)**

where D_k is the determinant obtained from D by replacing in D the kth column by the column with the entries $b_1, \cdots, b_n$.

(b) *Hence if the system (6) is* ***homogeneous*** *and $D \neq 0$, it has only the trivial solution $x_1 = 0, x_2 = 0, \cdots, x_n = 0$. If $D = 0$, the homogeneous system also has nontrivial solutions.*

- This last rule will be used in Eigenvalues problems

Inverse of a Matrix. Gauss–Jordan Elimination

- For a matrix to have an inverse, it has to be a square matrix.
- The inverse of an n x n matrix, is denoted by **A^{-1}** which is also an n x n matrix:

$$\boldsymbol{AA^{-1}} = \boldsymbol{A^{-1}A} = \boldsymbol{I} \; (n \times n \; identity \; matrix)$$

- If **A** has an inverse, then **A** is called a non-singular (or invertible) matrix. Otherwise it is called singular.
- If **A** has an inverse then the inverse is unique.
- **A** has an inverse iff rank(**A**) = n.
- **A** has an inverse iff det(**A**) ≠ 0.

Example: Finding Inverse of a Matrix by Gauss–Jordan Elimination

Determine the inverse $\mathbf{A}^{-1}$ of

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

- $\boldsymbol{A}$ → $[\boldsymbol{A}|\boldsymbol{I}]$ → Gauss-Jordan → $[\boldsymbol{I}|\boldsymbol{A}^{-1}]$

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Example: Finding Inverse of a Matrix by Gauss–Jordan Elimination

$$[\mathbf{A} \quad \mathbf{I}] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \text{Row 2} + 3\,\text{Row 1} \\ \text{Row 3} - \text{Row 1} \end{array}$$

Example: Finding Inverse of a Matrix by Gauss–Jordan Elimination

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & \boxed{2} & 2 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \text{Row 2} + 3\,\text{Row 1} \\ \text{Row 3} - \text{Row 1} \end{array}$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] \begin{array}{l} \\ \\ \text{Row 3} - \text{Row 2} \end{array}$$

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Example: Finding Inverse of a Matrix by Gauss–Jordan Elimination

- Gauss-Jordan extra steps: reducing $\boldsymbol{U\ to\ I}$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

$$\begin{array}{cl} \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 3.5 & 1.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] & \begin{array}{l} -\text{Row } 1 \\ 0.5 \text{ Row } 2 \\ -0.2 \text{ Row } 3 \end{array} \end{array}$$

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Example: Finding Inverse of a Matrix by Gauss–Jordan Elimination

- Gauss-Jordan extra steps: reducing $\boldsymbol{U\ to\ I}$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & \boxed{-2} & -1 & 0 & 0 \\ 0 & 1 & \boxed{3.5} & 1.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0.6 & 0.4 & -0.4 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \begin{array}{l} \text{Row 1} + 2\ \text{Row 3} \\ \text{Row 2} - 3.5\ \text{Row 3} \\ \end{array}$$

Example: Finding Inverse of a Matrix by Gauss–Jordan Elimination

- Gauss-Jordan extra steps: reducing $\boldsymbol{U\ to\ I}$

$$\left[\begin{array}{ccc|ccc} 1 & \boxed{-1} & 0 & 0.6 & 0.4 & -0.4 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \quad \text{Row 1 + Row 2}$$

Example: Finding Inverse of a Matrix by Gauss–Jordan Elimination

The last three columns constitute $\mathbf{A}^{-1}$. Check:

$$\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. Similarly, $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

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Inverse of a Matrix. The cofactor method

The inverse of a nonsingular $n \times n$ *matrix* $\mathbf{A} = [a_{jk}]$ *is given by*

$$(4) \qquad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}[C_{jk}]^{\mathsf{T}} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \cdot & \cdot & \cdots & \cdot \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix},$$

where C_{jk} *is the cofactor of* a_{jk} *in* $\det \mathbf{A}$

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Inverse of a Matrix.
The cofactor method

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad is \quad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

Here $cof(A) = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$ checkerboard: $\begin{matrix} + & - \\ - & + \end{matrix}$

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}, \quad \mathbf{A}^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$$

The cofactor method

- Find the inverse of: $$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

Solution. We obtain $\det \mathbf{A} = -1(-7) - 1 \cdot 13 + 2 \cdot 8 = 10$, and in (4),

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7, \quad C_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2, \quad C_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3,$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13, \quad C_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2, \quad C_{32} = -\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = 7,$$

$$C_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8, \quad C_{23} = -\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2, \quad C_{33} = \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = -2,$$

$$\mathbf{A}^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}.$$

Inverse of Diagonal Matrices

- A diagonal matrix has an inverse if $a_{jj} \neq 0$
- A^{-1} is also diagonal with entries $\frac{1}{a_{11}}, \frac{1}{a_{22}}, \dots, \frac{1}{a_{nn}}$

$$\mathbf{A} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \mathbf{A}^{-1} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Properties of Matrix Operations

- $(\mathbf{AC})^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}$
- $(\mathbf{AC} \ldots \mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1} \ldots \mathbf{C}^{-1}\mathbf{A}^{-1}$
- Matrix multiplication is **NOT** commutative: $\mathbf{AB} \neq \mathbf{BA}$
- $\mathbf{AB} = \mathbf{0}$, does not generally imply that $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$ or $\mathbf{BA} = \mathbf{0}$
  - $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
  - $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$
- $\mathbf{AC} = \mathbf{AD}$, does not imply that $\mathbf{C} = \mathbf{D}$ even when $\mathbf{A} \neq \mathbf{0}$

Properties of Matrix Operations

- $\mathbf{AB} = \mathbf{0}$, does not generally imply that $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$ or $\mathbf{BA} = \mathbf{0}$

Let **A**, **B**, **C** be n x n matrices:

- If rank($\mathbf{A}$) = n and $\mathbf{AB} = \mathbf{0}$ implies that $\mathbf{B} = \mathbf{0}$.

Proof: $\mathbf{A}^{-1}$ exists, $\mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{0} \rightarrow \mathbf{B} = \mathbf{0}$

- Hence, if $\mathbf{AB} = \mathbf{0}$, but $\mathbf{A} \neq \mathbf{0}$ & $\mathbf{B} \neq \mathbf{0}$ then rank($\mathbf{A}$) < n and rank($\mathbf{B}$) < n

Proof: if rank($\mathbf{A}$) = n $\rightarrow$ $\mathbf{A}^{-1}$ exists $\rightarrow$ $\mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{0} \rightarrow \mathbf{B} = \mathbf{0}$

also: if rank($\mathbf{B}$) = n $\rightarrow$ $\mathbf{B}^{-1}$ exists $\rightarrow$ $\mathbf{ABB}^{-1} = \mathbf{0B}^{-1} \rightarrow \mathbf{A} = \mathbf{0}$

Properties of Matrix Operations

- **AB** = **AC**, does not imply that **B** = **C** even when **A** ≠ **0**

Let **A**, **B**, **C** be n x n matrices:

- If rank(**A**) = n and **AB = AC** then **B** = **C**

Proof: $\mathbf{A}^{-1}$ exists**,** $\mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{AC} \rightarrow \mathbf{B} = \mathbf{C}$

Properties of Matrix Operations

Let **A**, **B**, **C** be n x n matrices then:

- If **A** is singular → **BA** and **AB** are singular

  Proof: see textbook

For any n x n matrices **A** and **B**

- det (**AB**) = det (**BA**) = det **A** det **B**.

  Proof: see textbook

Example: Solving Systems of Linear Equations Using Matrix Inverse

$$\begin{aligned} x_1 + x_2 - x_3 &= 3 \\ -x_1 + x_2 + x_3 &= -1 \\ x_1 - x_2 + x_3 &= 5 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

A x = b

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Example: Solving Systems of Linear Equations Using Matrix Inverse

A x = b

$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

Multiply by inverse:

$$A^{-1} A x = A^{-1} b \rightarrow x = A^{-1} b$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

Linear Algebra: Matrix Eigenvalue Problems – Part 1

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Eigenvalue Problems

- The determination of the eigenvalues and eigenvectors of a system is extremely important in physics and engineering.
- Solving eigenvalue problems is equivalent to matrix diagonalization and has several applications:
    - Stability analysis
    - The physics of rotating bodies
    - Small oscillations of vibrating systems
- You might encounter these and/or other applications of eigenvalue problems in other courses.

Matrix Eigenvalue Problems

- A matrix eigenvalue problem considers the vector equation:

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$$

Known

Here, **A** is a given square matrix,

– λ is an unknown scalar

– $\boldsymbol{x}$ is an unknown vector

- In a matrix eigenvalue problem, the task is to determine:

– λ's that satisfy the eq. above (called eigenvalues).

– $\boldsymbol{x}$'s that satisfy the eq. above (called eigenvectors) excluding $\boldsymbol{x} = \boldsymbol{0}$ which is always a solution.

Matrix Eigenvalue Problems

- The set of all the eigenvalues of **A** is called the spectrum of **A**.
- The spectrum consists of at least one eigenvalue and at most of n numerically different eigenvalues (where n is the matrix size).

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Example 1: Finding Eigenvalues and Eigenvectors

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}.$$

- For the matrix above, **eigenvalues** must be determined first:

$$\mathbf{Ax} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \qquad \text{in components,} \qquad \begin{aligned} -5x_1 + 2x_2 &= \lambda x_1 \\ 2x_1 - 2x_2 &= \lambda x_2. \end{aligned}$$

- By rearranging these equations we get:

$$\begin{aligned} (-5 - \lambda)x_1 + \quad 2x_2 \quad &= 0 \\ 2x_1 \quad + (-2 - \lambda)x_2 &= 0. \end{aligned}$$

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Example 1: Finding Eigenvalues and Eigenvectors

$$\begin{aligned} (-5-\lambda)x_1 + \quad 2x_2 \quad &= 0 \\ 2x_1 \quad + (-2-\lambda)x_2 &= 0. \end{aligned}$$

- This can be written in matrix notation:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

Proof: $\boldsymbol{Ax} = \lambda\boldsymbol{x} \rightarrow$ $\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{Ax} - \lambda\mathbf{Ix} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0},$

- So we've transferred the original eigenvalue equation to a **homogeneous linear system**.
- By Cramer's rule, $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \mathbf{0}$ has a nontrivial solution $\boldsymbol{x} \neq \mathbf{0}$ iff the determinant of coefficient matrix $(\boldsymbol{A} - \lambda\boldsymbol{I})$ is zero.

Example 1: Finding Eigenvalues and Eigenvectors

$$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix}$$

$$= (-5-\lambda)(-2-\lambda) - 4 = \lambda^2 + 7\lambda + 6 = 0.$$

- $D(\lambda)$ is the characteristic determinant (or characteristic polynomial) and $D(\lambda) = 0$ is the characteristic equation of **A**.
- Solving the characteristic equations gives the two **eigenvalues**: $\lambda = -1, \lambda = -6,$
- Solution of quadratic equation:

$$ax^2 + bx + c = 0 \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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Example 1: Finding Eigenvalues and Eigenvectors

- **Eigenvector x** of **A** corresponding to $\lambda = -1$ can be obtained from:

$$\begin{aligned} (-5-\lambda)x_1 + \quad 2x_2 \quad &= 0 \\ 2x_1 \quad + (-2-\lambda)x_2 &= 0. \end{aligned}$$

- by substituting $\lambda = -1$:

$$\begin{aligned} -4x_1 + 2x_2 &= 0 \\ 2x_1 - \quad x_2 &= 0. \end{aligned}$$

- Gauss elimination will zero row 2 which means we have infinite solutions. Rearranging row 1 or row 2 gives the solution: $x_2 = 2x_1$

Example 1: Finding Eigenvalues and Eigenvectors

- solution: $x_2 = 2x_1$
- Hence for $\lambda = -1,\ \mathbf{x} = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix}$
- If we choose $x_1 = 1$ we obtain the eigenvector $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
- Check:

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \lambda \mathbf{x}$$

Example 1: Finding Eigenvalues and Eigenvectors

- **Eigenvector x** of **A** corresponding to $\lambda = -6$ can be obtained from:

$$\begin{aligned} (-5-\lambda)x_1 + \quad 2x_2 \quad &= 0 \\ 2x_1 \quad + (-2-\lambda)x_2 &= 0. \end{aligned}$$

- by substituting $\lambda = -6$:

$$x_1 + 2x_2 = 0$$

$$2x_1 + 4x_2 = 0.$$

- Gauss elimination will also zero row 2 which means we have infinite solutions. Rearranging row 1 or row 2 gives the solution: $x_2 = -x_1/2$

Example 1: Finding Eigenvalues and Eigenvectors

- solution: $x_2 = -x_1/2$
- Hence for $\lambda = -6$, $\mathbf{x} = \begin{bmatrix} x_1 \\ -x_1/2 \end{bmatrix}$

- If we choose $x_1 = 2$ we obtain the eigenvector $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$
- Check:

$$\mathbf{Ax} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -12 \\ 6 \end{bmatrix} = \lambda \mathbf{x}$$

Finding Eigenvalues and Eigenvectors: General Case

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= \lambda x_1 \\
a_{21}x_1 + \cdots + a_{2n}x_n &= \lambda x_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{n1}x_1 + \cdots + a_{nn}x_n &= \lambda x_n.
\end{aligned}
$$

- Transferring the terms on the right side to the left side:

$$
\begin{array}{ccccccccc}
(a_{11}-\lambda)x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = 0 \\
a_{21}x_1 & + & (a_{22}-\lambda)x_2 & + & \cdots & + & a_{2n}x_n & = 0 \\
\cdots\cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\
a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & (a_{nn}-\lambda)x_n & = 0.
\end{array}
$$

- Which is equivalent to: $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$

Finding Eigenvalues and Eigenvectors: General Case

- By Cramer's theorem this homogeneous linear system of equations has a nontrivial solution if and only if the corresponding determinant of the coefficients is zero:

$$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0.$$

- Which is equivalent to: $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \mathbf{0}$

Eigenvalue Problems Steps

- Steps for solving Eigenvalue Problems:

– Solve the c/s equation to get the eigenvalues:

$$D(\lambda) = \det\,(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$$

– Substitute each λ into $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$

– Solve the system of linear equations $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$ (i.e. find $\boldsymbol{x}$ for each λ). These vectors $\boldsymbol{x}$ are the eigenvectors).

– You can always check your solution by substituting your λ and the corresponding $\boldsymbol{x}$ into:

$$\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$$

Definitions

- $\boldsymbol{A} - \lambda\boldsymbol{I}$ is called the characteristic matrix.
- $\mathrm{D}(\lambda) = det(\boldsymbol{A} - \lambda\boldsymbol{I})$ is called the characteristic determinant of **A.**
- $\mathrm{D}(\lambda) = 0$ is called the characteristic equation of **A**.
- By developing $\mathrm{D}(\lambda)$ we obtain a polynomial of nth degree in λ. This is called the characteristic polynomial of **A**.
- **Theorem 1:** The eigenvalues of a square matrix **A** are the roots of the characteristic equation of **A**.
- Hence an n x n matrix has at least one eigenvalue and at most n different eigenvalues.

Definitions

- **Theorem 2: Eigenvectors, Eigenspace:** If **w** and **x** are eigenvectors of a matrix **A** corresponding to the same eigenvalue λ, so are **w + x** (provided **x** $\neq$ –**w**) and k**x** for any k≠0.
- Hence the eigenvectors corresponding to one and the same eigenvalue λ of **A**, together with **0**, form a vector space, called the eigenspace of **A** corresponding to that λ.

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Example 2: Multiple Eigenvalues

- Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

- For our matrix, the characteristic determinant gives the characteristic equation:

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0.$$

- The roots (**eigenvalues** of **A**) are:

$$\lambda_1 = 5, \lambda_2 = \lambda_3 = -3.$$

Example 2: Multiple Eigenvalues

- For $\lambda = 5$ the characteristic matrix is:

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \rightarrow \mathbf{A} - \lambda\mathbf{I} = \mathbf{A} - 5\mathbf{I} = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}.$$

- After two steps of gauss elimination:

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & -\frac{24}{7} & -\frac{48}{7} \\ 0 & 0 & 0 \end{bmatrix}.$$

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Example 2: Multiple Eigenvalues

- For $\lambda = 5$:

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & -\frac{24}{7} & -\frac{48}{7} \\ 0 & 0 & 0 \end{bmatrix}.$$

- From row 2 → $x_2 = -2x_3$
- From row 1 & $x_2 = -2x_3$ → $x_1 = -x_3$
- Hence for $\lambda = 5,\ \mathbf{x} = \begin{bmatrix} -x_3 \\ -2x_3 \\ x_3 \end{bmatrix}$
- If we choose $x_3 = -1$ we obtain $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

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Example 2: Multiple Eigenvalues

- For $\lambda = -3$ the characteristic matrix is:

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \rightarrow \quad \mathbf{A} - \lambda\mathbf{I} = \mathbf{A} + 3\mathbf{I} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$$

- After two steps of gauss elimination:

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 2: Multiple Eigenvalues

- For $\lambda = -3$: $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
- From row 1 $x_1 = -2x_2 + 3x_3$,
- Hence for $\lambda = -3$, $\mathbf{x} = \begin{bmatrix} -2x_2 + 3x_3 \\ x_2 \\ x_3 \end{bmatrix}$
- If we choose $x_2 = 1$, $x_3 = 0$ we obtain $\mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

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Example 5: Real Matrices with Complex Eigenvalues & Eigenvectors

- Find the eigenvalues and eigenvectors of the following skew-symmetric matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0.$$

- Solving the characteristic equations gives the two **eigenvalues**:$\lambda = i, \lambda = -i$.

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Example 5: Real Matrices with Complex Eigenvalues & Eigenvectors

- Eigenvector of **A** corresponding to $\lambda = i$ can be obtained from:

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \quad \text{Gauss elimination} \rightarrow \quad \begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix}$$

- From row 1 → $x_2 = ix_1$
- Hence for $\lambda = i, \; \mathbf{x} = \begin{bmatrix} x_1 \\ ix_1 \end{bmatrix}$
- If we choose $x_1 = 1$ we obtain the eigenvector $\mathbf{x} = \begin{bmatrix} 1 \\ i \end{bmatrix}$

F.Y.I.

Example 5: Real Matrices with Complex Eigenvalues & Eigenvectors

- Eigenvector of **A** corresponding to $\lambda = -i$ can be obtained from:

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \quad \text{Gauss elimination} \rightarrow \quad \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}$$

- From row 1 → $x_2 = -ix_1$
- Hence for $\lambda = -i,\ \mathbf{x} = \begin{bmatrix} x_1 \\ -ix_1 \end{bmatrix}$
- If we choose $x_1 = 1$ we obtain the eigenvector $\mathbf{x} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

Eigenvalues of The Transpose of a Matrix

- **Theorem3: Eigenvalues of the Transpose:** The transpose $\mathbf{A}^T$ of a square matrix $\mathbf{A}$ has the same eigenvalues as $\mathbf{A}$.

Linear Algebra: Matrix Eigenvalue Problems – Part 2

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8.3 Symmetric, Skew-symmetric, and Orthogonal Matrices

- Symmetric matrix: $\boldsymbol{A}^T = \boldsymbol{A}$
- Skew-symmetric matrix: $\boldsymbol{A}^T = -\boldsymbol{A}$
- Orthogonal matrix: $\boldsymbol{A}^T = \boldsymbol{A}^{-1}$

- Examples:

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

Symmetric Skew-symmetric Orthogonal

8.3 Symmetric, Skew-symmetric, and Orthogonal Matrices

- Any real square matrix A may be written as the sum of a symmetric matrix R and a skew-symmetric matrix S, where:

$$\boldsymbol{R} = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T) \quad and \quad \boldsymbol{S} = \frac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^T)$$

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Example 6: Finding R and S of a Square Matrices

$$\boldsymbol{R} = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T) \qquad and \qquad \boldsymbol{S} = \frac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^T)$$

$$\mathbf{A} = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix} = \mathbf{R} + \mathbf{S} = \begin{bmatrix} 9.0 & 3.5 & 3.5 \\ 3.5 & 3.0 & -2.0 \\ 3.5 & -2.0 & 3.0 \end{bmatrix} + \begin{bmatrix} 0 & 1.5 & -1.5 \\ -1.5 & 0 & -6.0 \\ 1.5 & 6.0 & 0 \end{bmatrix}$$

- Remember: Symmetric (R): $\boldsymbol{A}^T = \boldsymbol{A}$, Skew-symmetric (S): $\boldsymbol{A}^T = -\boldsymbol{A}$

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Symmetric & Skew-symmetric Matrices

- **Theorem 1: Eigenvalues of Symmetric and Skew-Symmetric Matrices:**

  (a) The eigenvalues of a symmetric matrix are real.

  (b) The eigenvalues of a skew-symmetric matrix are pure imaginary or zero. **Example 7.**

- Remember: Symmetric: $\boldsymbol{A}^T = \boldsymbol{A}$, Skew-symmetric: $\boldsymbol{A}^T = -\boldsymbol{A}$

Orthogonal Matrices

- **Theorem 3: Orthogonal matrices:** Orthonormality of Column and Row Vectors: A real square matrix is orthogonal iff its column vectors (and also its row vectors) form an orthonormal system:

$$\mathbf{a}_j \bullet \mathbf{a}_k = \mathbf{a}_j^{\mathsf{T}}\mathbf{a}_k = \begin{cases} 0 & \text{if} \quad j \neq k \\ 1 & \text{if} \quad j = k. \end{cases}$$

Proof: $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I} \;\rightarrow\; \boldsymbol{A}^{T}\boldsymbol{A} = \boldsymbol{I}$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

- Remember: Orthogonal matrix: $\boldsymbol{A}^{T} = \boldsymbol{A}^{-1}$

Orthogonal Matrices

- **Theorem 4: Determinant of an Orthogonal Matrix:** The determinant of an orthogonal matrix has the value +1 or –1.
- **Theorem 5: Eigenvalues of an Orthogonal Matrix:** The eigenvalues of an orthogonal matrix A are real or complex conjugates in pairs and have absolute value 1.

- Remember: Orthogonal matrix: $\boldsymbol{A}^T = \boldsymbol{A}^{-1}$

8.4 Diagonalization of Matrices: Similar Matrices

- **Definition: Similar Matrices:** An n x n $\hat{\boldsymbol{A}}$ matrix is called similar to an n x n matrix $\boldsymbol{A}$ if

$$\hat{\boldsymbol{A}} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$$

where **P** is a non-singular n x n matrix.

- **Theorem 3: Eigenvalues and Eigenvectors of Similar Matrices:** If $\hat{\boldsymbol{A}}$ is similar to $\boldsymbol{A}$, then $\hat{\boldsymbol{A}}$ has the same eigenvalues as $\boldsymbol{A}$. Furthermore, if **x** is an eigenvector of $\boldsymbol{A}$, then $\boldsymbol{y} = \boldsymbol{P}^{-1}\boldsymbol{x}$ is an eigenvector of $\hat{\boldsymbol{A}}$ corresponding to the same eigenvalue.

Example 9: Similar Matrices

- Let **A** and **P** be the following:

Given: $$\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{P} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}.$$

$$\mathbf{P}^{-1} = \frac{1}{\det \mathbf{P}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \rightarrow \quad \mathbf{P}^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

- $\hat{A} = P^{-1}AP$

$$\hat{\mathbf{A}} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

Example 9: Similar Matrices

- Eigenvalues of $\boldsymbol{A}$:

$$\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}$$

$$det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \begin{vmatrix} 6-\lambda & -3 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

→ $\lambda^2 - 5\lambda + 6 = 0$

→ $(\lambda - 3)(\lambda - 2) = 0$

$\lambda = 3$

$\lambda = 2$

Example 9: Similar Matrices

- Eigenvalues of $\hat{\boldsymbol{A}}$: $= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

$$det(\hat{\boldsymbol{A}} - \lambda \boldsymbol{I}) = \begin{vmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

→ $(\lambda - 3)(\lambda - 2) = 0$

$\lambda = 3$

$\lambda = 2$

Example 9: Similar Matrices

- Eigenvectors of $\boldsymbol{A}$: $\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}$
- For λ = 3

$$\rightarrow \mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 3 & -3 \\ 4 & -4 \end{bmatrix} \rightarrow \quad x_2 = x_1$$

- Hence for $\lambda = 3,\ \mathbf{x} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}$
- If we choose $x_1 = 1$ we obtain the eigenvector $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Example 9: Similar Matrices

- Eigenvectors of $\boldsymbol{A}$: $\mathbf{A} = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}$

- For λ = 2

$\rightarrow \mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 4 & -3 \\ 4 & -3 \end{bmatrix} \rightarrow \quad x_2 = \frac{4}{3} x_1$

- Hence for $\lambda = 2,\ \mathbf{x} = \begin{bmatrix} x_1 \\ 4/3\ x_1 \end{bmatrix}$

- If we choose $x_1 = 3$ we obtain the eigenvector $\mathbf{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Example 9: Similar Matrices

- Eigenvectors of $\widehat{\boldsymbol{A}}$: $= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.
- For λ = 2, $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- For λ = 3, $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

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Diagonalization of a Matrix

Diagonalization: converting a square matrix into a diagonal matrix.

Diagonalization of a Matrix

If an $n \times n$ matrix $\mathbf{A}$ has a basis of eigenvectors, then

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X} \tag{5}$$

is diagonal, with the eigenvalues of $\mathbf{A}$ as the entries on the main diagonal. Here $\mathbf{X}$ is the matrix with these eigenvectors as column vectors.

Also $\mathbf{A} = \mathbf{X}\mathbf{D}\mathbf{X}^{-1}$. Can you prove that?

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Example 10: Diagonalization of a Matrix

$$\mathbf{A} = \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1.0 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}$$

- The characteristic determinant gives the characteristic equation: $-\lambda^3 - \lambda^2 + 12\lambda = 0.$

- The eigenvalues (roots): $\lambda_1 = 3, \lambda_2 = -4, \lambda_3 = 0.$

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Example 10: Diagonalization of a Matrix

- Eigenvectors of A can be: $\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix},$

$$\mathbf{X} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}, \quad \mathbf{X}^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}.$$

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X} = \underbrace{\begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}}_{X^{-1}} \underbrace{\begin{bmatrix} -3 & -4 & 0 \\ 9 & 4 & 0 \\ -3 & -12 & 0 \end{bmatrix}}_{AX} = \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D}.$$

Example 11: Diagonalization of a Matrix

- From eigenvalues example 2

the matrix A is:

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- The eigenvalues/eigenvectors are:

$\lambda = 5 \quad \rightarrow \quad \mathbf{x} = [1 \quad 2 \quad -1]^T$

$\lambda = -3 \quad \rightarrow \mathbf{x} = [-2 \quad 1 \quad 0]^T \quad and \quad \mathbf{x} = [3 \quad 0 \quad 1]^T$

Note that $\mathbf{x}$'s for $\lambda = -3$ are independent → **X** is invertable

$$\rightarrow \mathbf{X} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \mathbf{X}^{-1}\ \mathbf{A}\ \mathbf{X} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Vector Calculus – Part 1

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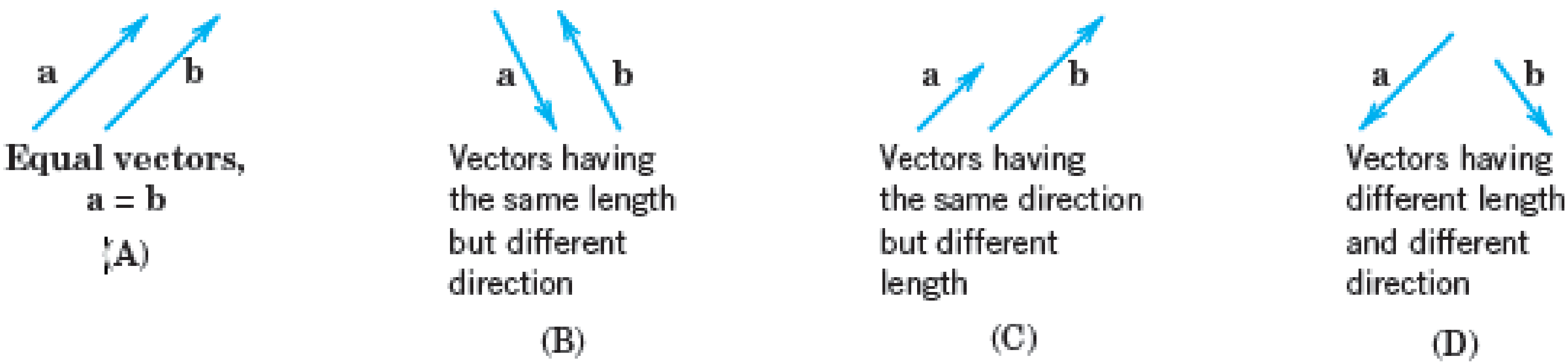
Vectors

- Two kinds of quantities:
  - Scalar: determined by its magnitude: voltage, temperature, length, speed.
  - Vector: has both a magnitude and a direction: force, displacement, velocity.
- A vector is denoted in drawing by an arrow that has length ≡ magnitude (aka norm or Euclidean norm). Magnitude of vector **a** is denoted by |**a**|.
- Tail: initial point, tip: terminal point, and direction of the arrow: direction of the vector.

Vectors

- In writing, a vector is denoted by lower case boldface $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{v})$, or by using an arrow $(\vec{a}, \vec{b}, \vec{v})$.
- A vector of length 1 is called a unit vector.

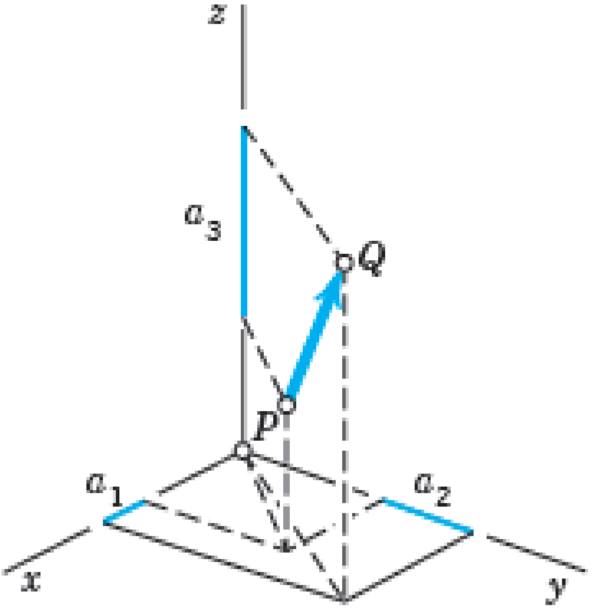


Components of a Vector

- Using Cartesian coordinate system, let **a** be a given vector with initial point P: (x_1, y_1, z_1) and terminal point Q: (x_2, y_2, z_2), then:

$$a_1 = x_2 - x_1 \qquad a_2 = y_2 - y_1 \qquad a_3 = z_2 - z_1$$

are called the components of the vector a: $\boldsymbol{a} = [a_1, a_2, a_3]$



- By the Pythagorean theorem:

$$|\boldsymbol{a}| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$$

Example: Components of a Vector

The vector **a** with initial point P: (4, 0, 2) and terminal point Q: (6, −1, 2) has the components

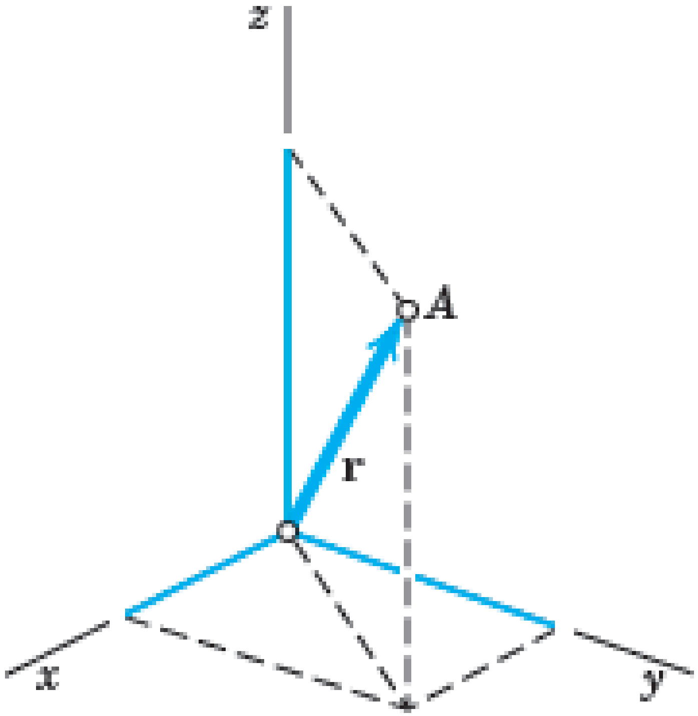
$$a_1 = 6 - 4 = 2, \qquad a_2 = -1 - 0 = -1, \qquad a_3 = 2 - 2 = 0.$$

Hence $\mathbf{a} = [2, -1, 0]$. (Can you sketch **a**, as in Fig. 168?) Equation (2) gives the length

$$|\mathbf{a}| = \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5}.$$

Position Vector

- The position vector **r** of a point A: (x, y, z) is the vector with the origin (0, 0, 0) as the initial point and A as the terminal point.



Vector Addition

Addition of Vectors

The **sum** $\mathbf{a} + \mathbf{b}$ of two vectors $\mathbf{a} = [a_1, a_2, a_3]$ and $\mathbf{b} = [b_1, b_2, b_3]$ is obtained by adding the corresponding components,

$$\mathbf{a} + \mathbf{b} = [a_1 + b_1, \quad a_2 + b_2, \quad a_3 + b_3]. \tag{3}$$

Geometrically, place the vectors as in Fig. 170 (the initial point of $\mathbf{b}$ at the terminal point of $\mathbf{a}$); then $\mathbf{a} + \mathbf{b}$ is the vector drawn from the initial point of $\mathbf{a}$ to the terminal point of $\mathbf{b}$.

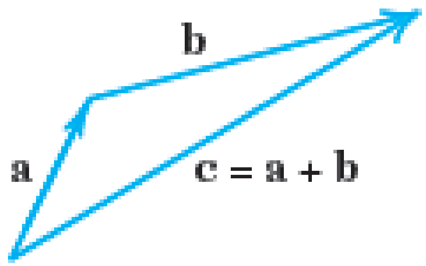


Fig. 170. Vector addition

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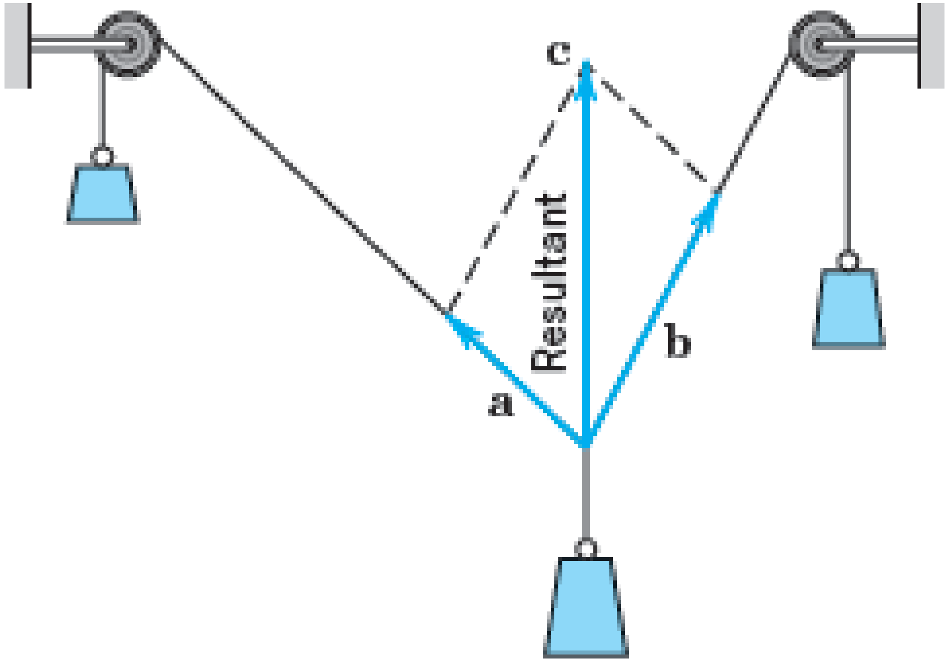
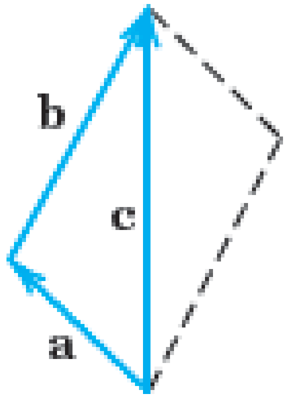
Vector Addition

- For forces, this addition is the parallelogram law by which we obtain the resultant of two forces in mechanics.

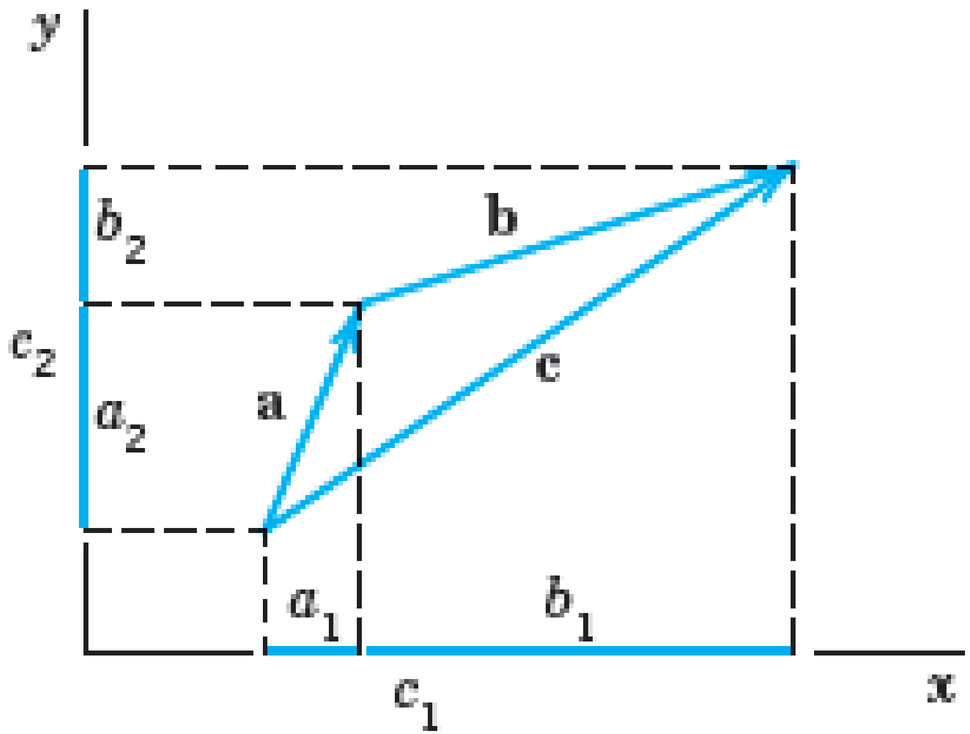




Vector Addition

- The "algebraic way" and the "geometric way" of vector addition give the same vector:

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$$



Basic Properties of Vector Addition

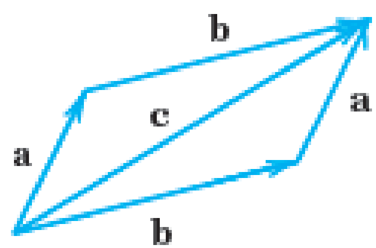
(a) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ *(Commutativity)*

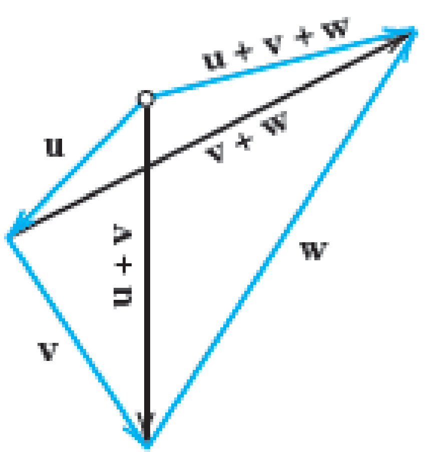
(b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ *(Associativity)*

(c) $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$

(d) $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}.$

- $-\mathbf{a}$ denotes the vector having the length $|\mathbf{a}|$ and the direction opposite to that of $\mathbf{a}$.
- (a) and (b) are verified in the following figures:

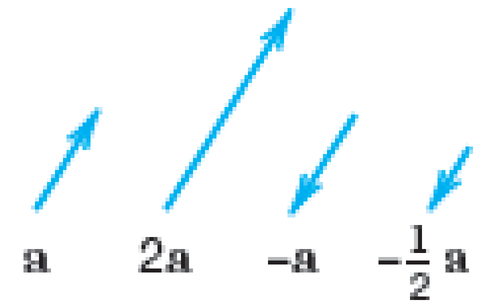




Scalar Multiplication

The product $c\mathbf{a}$ of any vector $\mathbf{a} = [a_1, a_2, a_3]$ and any scalar c (real number c) is the vector obtained by multiplying each component of $\mathbf{a}$ by c,

$$c\mathbf{a} = [ca_1, ca_2, ca_3]. \tag{5}$$



Basic Properties of Scalar Multiplication:

(a) $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$

(b) $(c + k)\mathbf{a} = c\mathbf{a} + k\mathbf{a}$

(c) $c(k\mathbf{a}) = (ck)\mathbf{a}$ (written $ck\mathbf{a}$)

(d) $1\mathbf{a} = \mathbf{a}.$

Example: Vector Addition and Scalar Multiplication

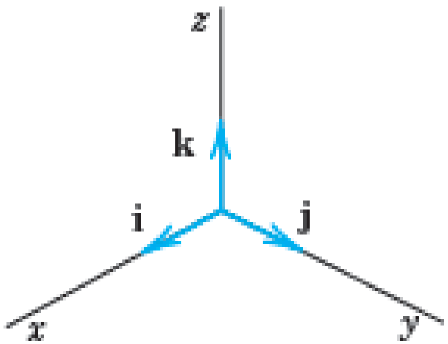
$$\mathbf{a} = [4, 0, 1] \quad \text{and} \quad \mathbf{b} = [2, -5, \tfrac{1}{3}].$$

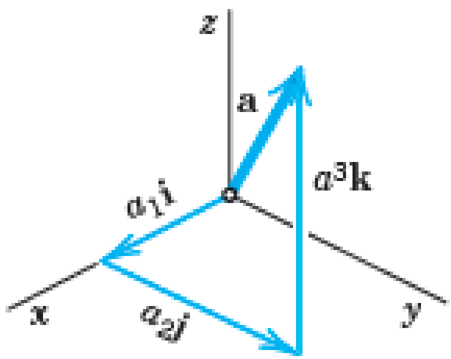
Then $-\mathbf{a} = [-4, 0, -1]$, $7\mathbf{a} = [28, 0, 7]$, $\mathbf{a} + \mathbf{b} = [6, -5, \frac{4}{3}]$, and

$$2(\mathbf{a} - \mathbf{b}) = 2[2, 5, \tfrac{2}{3}] = [4, 10, \tfrac{4}{3}] = 2\mathbf{a} - 2\mathbf{b}.$$

Unit Vectors i, j, k

- Besides $\boldsymbol{a} = [a_1, a_2, a_3]$ another popular way of writing vectors is: $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$





Example:

$$\mathbf{a} = 4\mathbf{i} + \mathbf{k},\ \mathbf{b} = 2\mathbf{i} - 5\mathbf{j} + \tfrac{1}{3}\mathbf{k},$$

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Unit Vectors i, j, k

- $\mathbf{i} = [1,0,0]$
- $\mathbf{j} = [0,1,0]$
- $\mathbf{k} = [0,0,1]$

9.2 Inner Product (Dot Product)

The **inner product** or **dot product** $\mathbf{a} \cdot \mathbf{b}$ (read "**a** dot **b**") of two vectors **a** and **b** is the product of their lengths times the cosine of their angle (see Fig. 178),

$$(1) \qquad \begin{aligned} \mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}||\mathbf{b}| \cos \gamma & \text{if} \quad & \mathbf{a} \neq \mathbf{0}, \mathbf{b} \neq \mathbf{0} \\ \mathbf{a} \cdot \mathbf{b} &= 0 & \text{if} \quad & \mathbf{a} = \mathbf{0} \text{ or } \mathbf{b} = \mathbf{0}. \end{aligned}$$

The angle γ, $0 \leqq \gamma \leqq \pi$, between **a** and **b** is measured when the initial points of the vectors coincide, as in Fig. 178. In components, $\mathbf{a} = [a_1, a_2, a_3]$, $\mathbf{b} = [b_1, b_2, b_3]$, and

$$(2) \qquad \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

→ The result of an inner product is always a scalar.

9.2 Inner Product (Dot Product)

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

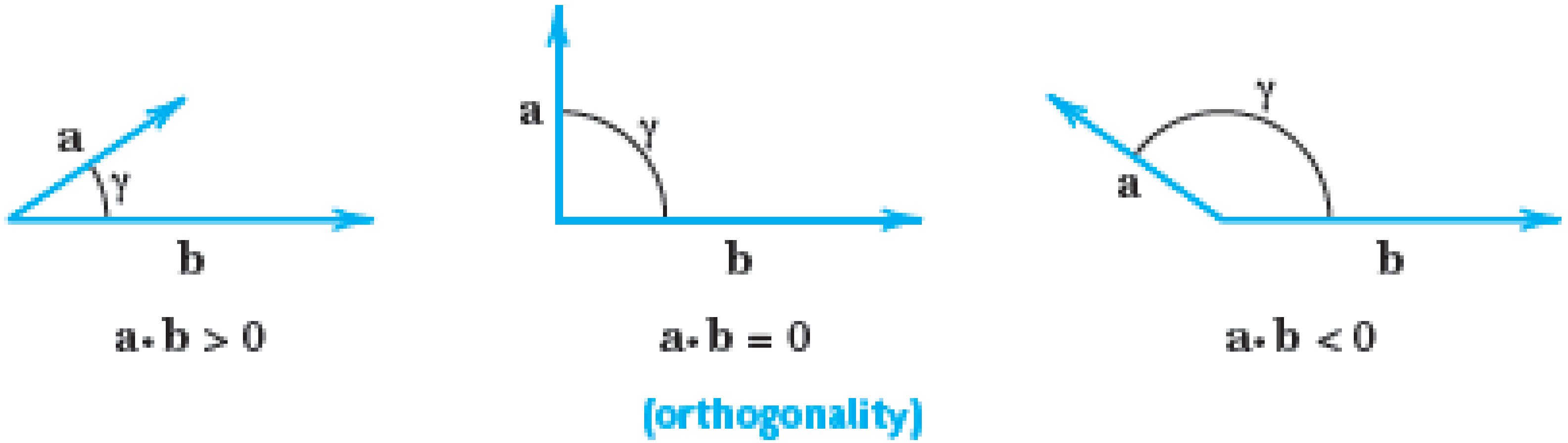

Fig. 178. Angle between vectors and value of inner product

- Vector **a** is orthogonal to vector **b** if **a.b** = 0, **a** ≠ **0** and **b** ≠ **0**.
- "Orthogonal" is a term used for more general objects, like planes and functions, whereas "perpendicular" is used <u>only</u> with lines.

Length and Angle

$$|\mathbf{a}| = \sqrt{\mathbf{a} \bullet \mathbf{a}}.$$

$$\cos \gamma = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Find the inner product and the lengths of $\mathbf{a} = [1, 2, 0]$ and $\mathbf{b} = [3, -2, 1]$ as well as the angle between these vectors.

Solution. $\mathbf{a} \bullet \mathbf{b} = 1 \cdot 3 + 2 \cdot (-2) + 0 \cdot 1 = -1$, $|\mathbf{a}| = \sqrt{\mathbf{a} \bullet \mathbf{a}} = \sqrt{5}$, $|\mathbf{b}| = \sqrt{\mathbf{b} \bullet \mathbf{b}} = \sqrt{14}$, and (4) gives the angle

$$\gamma = \arccos \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \arccos(-0.11952) = 1.69061 = 96.865°.$$

Properties of Inner Product

(a) $(q_1\mathbf{a} + q_2\mathbf{b}) \cdot \mathbf{c} = q_1\mathbf{a} \cdot \mathbf{c} + q_1\mathbf{b} \cdot \mathbf{c}$ *(Linearity)*

(b) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ *(Symmetry)*

(c) $\left.\begin{array}{l} \mathbf{a} \cdot \mathbf{a} \geqq 0 \\ \mathbf{a} \cdot \mathbf{a} = 0 \quad \text{if and only if} \quad \mathbf{a} = \mathbf{0} \end{array}\right\}$ *(Positive-definiteness).*

Since $|\cos\gamma| \leq 1$:

$|\mathbf{a} \cdot \mathbf{b}| \leqq |\mathbf{a}|\,|\mathbf{b}|$ *(Cauchy–Schwarz inequality).*

| a + b | $\leq$ **| a | + | b |** (Triangular inequality)

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Properties of Inner Product

$$|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2) \quad (\textit{Parallelogram equality}).$$

Prove that $\boxed{\mathbf{a} \bullet \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.}$

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \text{ and } \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$

$$\mathbf{a} \bullet \mathbf{b} = a_1b_1\mathbf{i} \bullet \mathbf{i} + a_1b_2\mathbf{i} \bullet \mathbf{j} + \cdots + a_3b_3\mathbf{k} \bullet \mathbf{k}.$$

where $\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1$

$$\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{j} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{i} = \boldsymbol{i} \cdot \boldsymbol{k} = 0$$

$\rightarrow$ $\boxed{\mathbf{a} \bullet \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.}$

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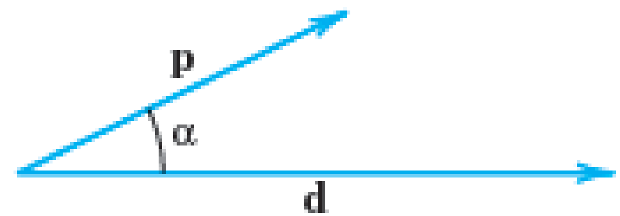
Applications of Inner Products

Work Done by a Force Expressed as an Inner Product

- **p**: constant force
- **d**: displacement
- α : angle between p and d



The work done by **p** in the displacement is:

$$W = |\mathbf{p}||\mathbf{d}|\cos\alpha = \mathbf{p}\cdot\mathbf{d}$$

- If $\alpha < 90$ → $W > 0$
- If $\mathbf{p}$ & $\mathbf{d}$ are orthogonal then $W = 0$
- If $\alpha > 90$ then $W < 0$ → work against the force

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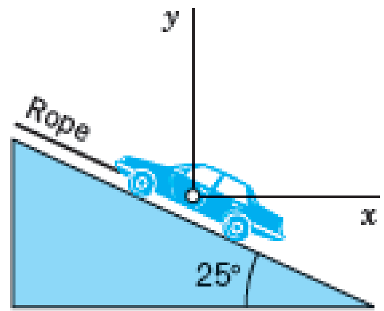
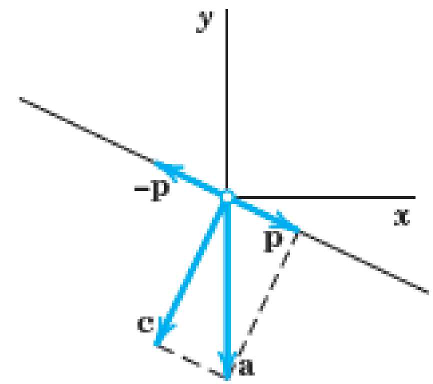
Example: Applications of Inner Products

What force in the rope in will hold a car of 5000 lb in equilibrium if the ramp makes an angle of 25° with the horizontal?





- weight: $\boldsymbol{a} = [0, -5000]$
- **c** is the force the car exerts on the ramp
- **p** is the force parallel to the ramp → $\boldsymbol{a} = \boldsymbol{p} + \boldsymbol{c}$
- $\gamma = 90 - 25 = 65^o$

→ $|\mathbf{p}| = |\boldsymbol{a}| \cos\gamma = 5000 \cos 65 = 2113 \text{ lb}$

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9.3 Vector Product (Cross Product)

The **vector product** or **cross product** $\mathbf{a} \times \mathbf{b}$ (read "**a** cross **b**") of two vectors **a** and **b** is the vector **v** denoted by

$$\mathbf{v} = \mathbf{a} \times \mathbf{b}$$

I. If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, then we define $\mathbf{v} = \mathbf{a} \times \mathbf{b} = \mathbf{0}$.

II. If both vectors are nonzero vectors, then vector **v** has the length

$$|\mathbf{v}| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin \gamma, \tag{1}$$

III. If **a** and **b** lie in the same straight line, i.e., **a** and **b** have the same or opposite directions, then γ is $0°$ or $180°$ so that $\sin \gamma = 0$. In that case $|\mathbf{v}| = 0$ so that $\mathbf{v} = \mathbf{a} \times \mathbf{b} = \mathbf{0}$.

IV. If cases I and III do not occur, then **v** is a nonzero vector. The direction of $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ is perpendicular to both **a** and **b** such that **a**, **b**, **v**—precisely in this order (!)—form a right-handed triple as shown in Figs. 185–187 and explained below.

Cross Product

- The direction of $\mathbf{v}$ is perpendicular both **a** & **b**
- If $\mathbf{a}, \mathbf{b}, \mathbf{v}$ are in the order of: $\mathbf{v} = \mathbf{a} \times \mathbf{b}$, the cross product follows the right-handed triple

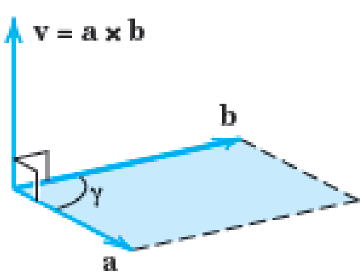


Fig. 185. Vector product

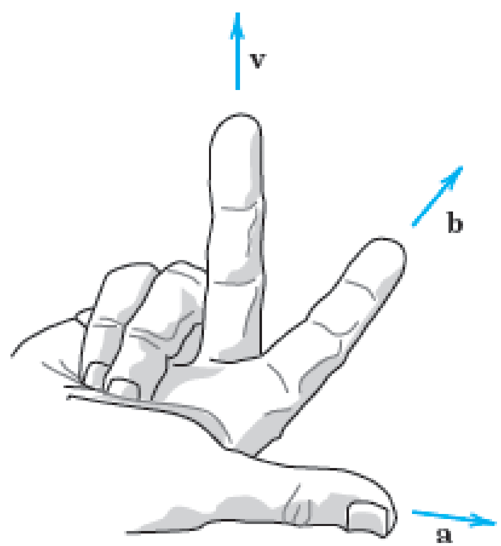


Fig. 186. Right-handed triple of vectors **a**, **b**, **v**

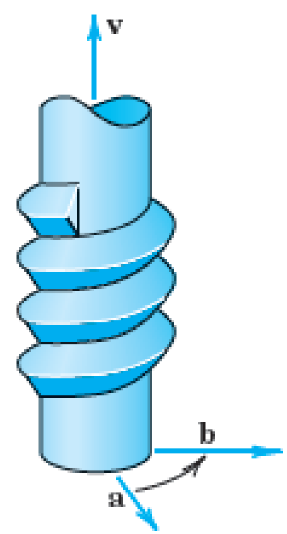


Fig. 187. Right-handed screw

Cross Product

- The direction of $\mathbf{v}$ is perpendicular to both **a** & **b**
- If $\mathbf{a}, \mathbf{b}, \mathbf{v}$ are in the order of: $\mathbf{v} = \mathbf{a} \times \mathbf{b}$, the cross product follows the right-handed triple

$$\mathbf{v} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}.$$

Cross Product

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \qquad \mathbf{i} \times \mathbf{i} = 0$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{j} \times \mathbf{j} = 0$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j} \qquad \mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{k} = 0$$

Example: Cross Product

For the vector product $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ of $\mathbf{a} = [1, 1, 0]$ and $\mathbf{b} = [3, 0, 0]$ in right-handed coordinates we obtain from (2)

$$v_1 = 0, \qquad v_2 = 0, \qquad v_3 = 1 \cdot 0 - 1 \cdot 3 = -3.$$

We confirm this by (2**):

$$\mathbf{v} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} \mathbf{k} = -3\mathbf{k} = [0, 0, -3].$$

Sketch in class **a**, **b**, **a** x **b**, and **b** x **a.**

Properties of Cross Product

(a) *For every scalar l,*

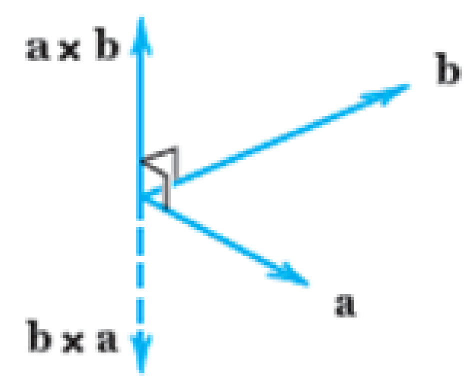
(4) $$(l\mathbf{a}) \times \mathbf{b} = l(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (l\mathbf{b}).$$

(b) *Cross multiplication is distributive with respect to vector addition; that is,*

(5)
$$(\alpha) \quad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}),$$
$$(\beta) \quad (\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}).$$

(c) *Cross multiplication is **not commutative** but **anticommutative**; that is,*

(6) $$\boxed{\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})}$$



(d) *Cross multiplication is **not associative**; that is,*

(7) $$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

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Applications of Cross Product

Moment of a Force:

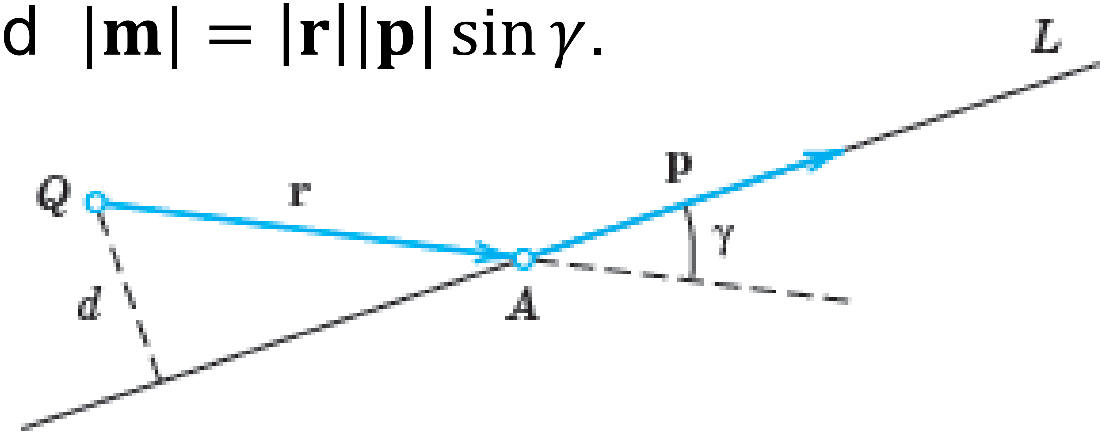
- Moment $\mathbf{m}$ of a force p about point Q is defined as

$$|\mathbf{m}| = |\mathbf{p}|d$$

- d is the perpendicular distance between Q and the line of action L of $\mathbf{p}$.

- If $\mathbf{r}$ is the vector from Q to any point A on L, then $d = |\mathbf{r}|\sin\gamma$ and $|\mathbf{m}| = |\mathbf{r}||\mathbf{p}|\sin\gamma$.

- Since γ is the angle between $\mathbf{r}$ and $\mathbf{p}$ then: $\mathbf{m} = \mathbf{r} \times \mathbf{p}$



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Example: Applications of Cross Product

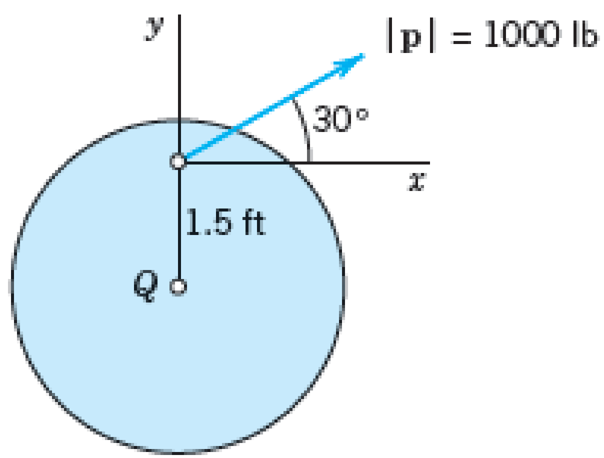
Find the moment of the force $\mathbf{p}$ about the center Q of a wheel, as given in Fig. 191.

Solution. Introducing coordinates as shown in Fig. 191, we have

$$\mathbf{p} = [1000\cos 30°,\quad 1000\sin 30°,\quad 0] = [866,\quad 500,\quad 0],\qquad \mathbf{r} = [0,\quad 1.5,\quad 0].$$

(Note that the center of the wheel is at $y = -1.5$ on the y-axis.) Hence (8) and (2**) give

$$\mathbf{m} = \mathbf{r}\times\mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1.5 & 0 \\ 866 & 500 & 0 \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + \begin{vmatrix} 0 & 1.5 \\ 866 & 500 \end{vmatrix}\mathbf{k} = [0, 0, -1299].$$



Scalar Triple Product

- Scalar triple product or mixed product of three vectors

$$(\boldsymbol{a}\ \ \boldsymbol{b}\ \ \boldsymbol{c}) = \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$$

$$\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \bullet \mathbf{v} = a_1 v_1 + a_2 v_2 + a_3 v_3$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

(10) $$(\mathbf{a}\quad \mathbf{b}\quad \mathbf{c}) = \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Properties of Scalar Triple Product

(a) *In* (10) *the dot and cross can be interchanged:*

$$(\mathbf{a}\quad \mathbf{b}\quad \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}. \tag{11}$$

(b) Geometric interpretation. *The absolute value* $|(\mathbf{a}\quad \mathbf{b}\quad \mathbf{c})|$ *of* (10) *is the volume of the parallelepiped* (oblique box) *with* **a**, **b**, **c** *as edge vectors* (Fig. 193).

(c) Linear independence. *Three vectors in* R^3 *are linearly independent if and only if their scalar triple product is not zero.*

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Application of Scalar Triple Product

Volume of a box:

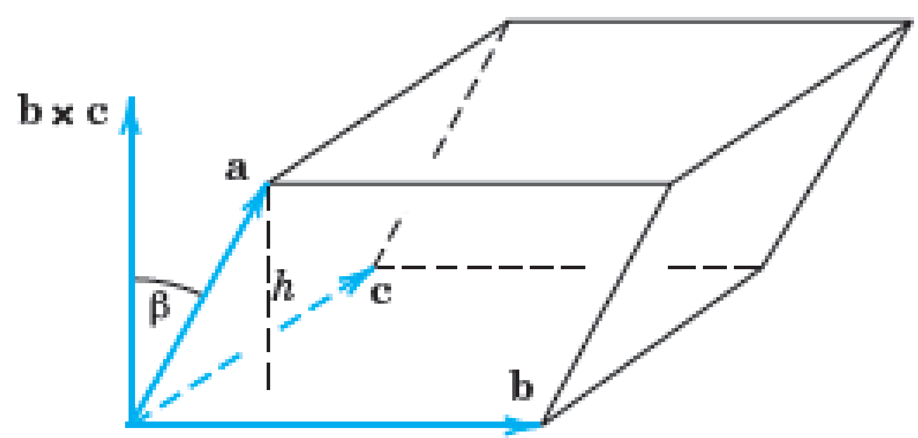
$$\text{Volume} = (\text{Height})\,(\text{Area of the base})$$

$$\text{Height} = |\mathbf{a}||\cos\beta|$$

$$\text{Area of base} = |\mathbf{b}\times\mathbf{c}|$$

$$\text{Volume} = |\mathbf{a}||\mathbf{b}\times\mathbf{c}||\cos\beta|$$

$$\text{Volume} = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$$



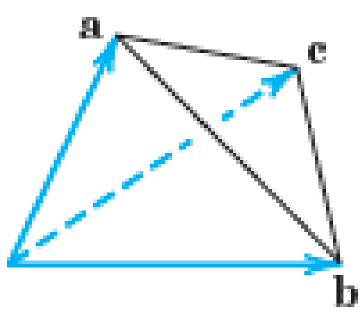
Example: Scalar Triple Product

- A tetrahedron is determined by three edge vectors **a, b, c**. Find the volume when **a**=[2, 0, 3] **b**=[0, 4, 1], **c**=[5, 6, 0].
- Volume of the parallelepiped with these vectors as edge vectors is the absolute value of the scalar triple product:

$$(\mathbf{a}\quad \mathbf{b}\quad \mathbf{c}) = \begin{vmatrix} 2 & 0 & 3 \\ 0 & 4 & 1 \\ 5 & 6 & 0 \end{vmatrix} = 2\begin{vmatrix} 4 & 1 \\ 6 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 4 \\ 5 & 6 \end{vmatrix} = -12 - 60 = -72.$$

- The volume of the tetrahedron is $\frac{1}{6}$ of that of the parallelepiped.

→ $\text{volume} = 72/6 = 12$



Vector Calculus – Part 2

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9.4 Vector and Scalar Functions and Their Fields

- Vector functions (whose values are vectors) In Cartesian coordinate system:

$$\mathbf{v}(x, y, z) = [v_1(x, y, z), \quad v_2(x, y, z), \quad v_3(x, y, z)].$$

$$\mathbf{v}(t) = [v_1(t), v_2(t), v_3(t)]$$

- Scalar function (whose values are scalars). In Cartesian coordinate system:

$$f(P) = f(x, y, z).$$

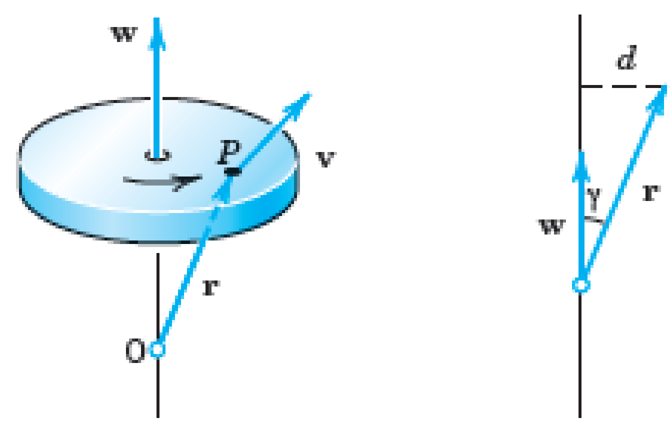
F.Y.I.

9.4 Vector Field: Velocity of a Rotating Body

- Rotation of a rigid body is described by a vector $\boldsymbol{w}$.
- Direction of $\boldsymbol{w}$ is that of the axis of rotation
- $|\boldsymbol{w}|$ = angular speed.
- Let P be any point on the body and d its distance from the axis. Then P has the speed ω_d.

$$d = |\boldsymbol{r}| \sin\gamma$$

$$\rightarrow \omega_d = |\boldsymbol{w}||\boldsymbol{r}| \sin\gamma = |\boldsymbol{w} \times \boldsymbol{r}|$$



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9.4 Vector Field: Velocity of a Rotating Body

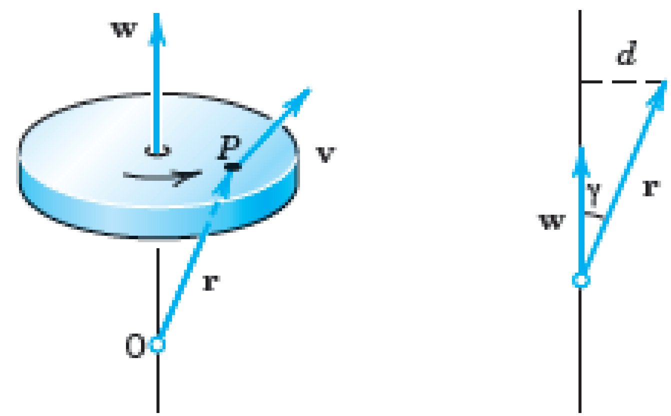
- Let P be any point of *B* and d its distance from the axis. Then P has the speed ω_d.

$$d = |\boldsymbol{r}| \sin\gamma$$

$$\rightarrow \omega_d = |\boldsymbol{w}||\boldsymbol{r}| \sin\gamma = |\boldsymbol{w} \times \boldsymbol{r}|$$

→ the velocity vector **v** of P:

$$\mathbf{v} = \boldsymbol{w} \times \boldsymbol{r}$$



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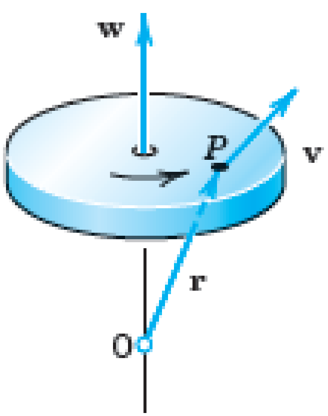
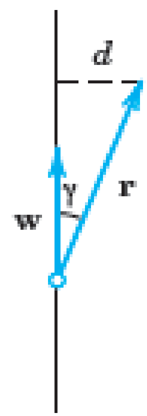
F.Y.I.

9.4 Vector Field: Velocity of a Rotating Body

- If the z-axis is the axis of rotation and $\boldsymbol{w}$ is in the +ve z-direction, then $\boldsymbol{w} = w\,\boldsymbol{k}$

$$\mathbf{v} = \boldsymbol{w} \times \boldsymbol{r} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & w \\ x & y & z \end{vmatrix} = w(-\mathrm{y}\boldsymbol{i} + x\boldsymbol{j})$$





F.Y.I.

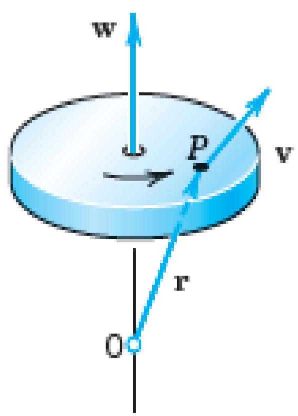
9.4 Vector Field: Velocity of a Rotating Body: Example

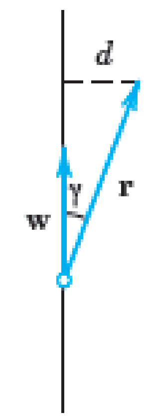
- A wheel is rotating about the y-axis with angular speed $w = 10$ rounds/sec.
- The rotation appears clock-wise if one looks from origin, in the +ve y-direction.
- Find the velocity and speed at P = [4, 3, 0].

$$\boldsymbol{r} = [(4-0),(3-0),(0-0)] = [4,3,0]$$

$$\mathbf{v} = \boldsymbol{w} \times \boldsymbol{r} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 10 & 0 \\ 4 & 3 & 0 \end{vmatrix} = -40\boldsymbol{k}$$

$$\text{speed} = |\mathbf{v}| = 40$$





9.4 Derivative of a Vector Function

- For a vector function $\mathbf{v}(\mathrm{t})$:

$$\mathbf{v}(t) = [v_1(t), v_2(t), v_3(t)] = v_1(t)\mathbf{i} + v_2(t)\mathbf{j} + v_3(t)\mathbf{k}.$$

- $\mathbf{v}'(\mathrm{t})$ is defined as the time derivative of $\mathbf{v}(\mathrm{t})$:

$$\mathbf{v}'(t) = [v_1'(t), \quad v_2'(t), \quad v_3'(t)].$$

- $\mathbf{v}'(\mathrm{t})$ is obtained by differentiating each component separately.

- Example: $\mathbf{v}(\mathrm{t}) = [\, t, t^2, 0 \,]$

$$\mathbf{v}'(\mathrm{t}) = [\, 1, 2t, 0 \,]$$

9.4 Derivative of a Vector Function: Rules

$$(c\mathbf{v})' = c\mathbf{v}' \qquad (c \text{ constant}),$$

$$(\mathbf{u} + \mathbf{v})' = \mathbf{u}' + \mathbf{v}'$$

$$(\mathbf{u} \bullet \mathbf{v})' = \mathbf{u}' \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{v}'$$

$$(\mathbf{u} \times \mathbf{v})' = \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}'$$

$$(\mathbf{u} \quad \mathbf{v} \quad \mathbf{w})' = (\mathbf{u}' \quad \mathbf{v} \quad \mathbf{w}) + (\mathbf{u} \quad \mathbf{v}' \quad \mathbf{w}) + (\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}').$$

9.4 Partial Derivatives of a Vector Function

- Suppose that the components of a vector function:

$$\mathbf{v} = [v_1, \quad v_2, \quad v_3] = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

are differentiable functions of n variables t_1, t_2, … t_m.

- Then the partial derivative of v with respect to t_m is denoted by $\partial\mathbf{v}/\partial t_m$ and is defined as the vector function:

$$\frac{\partial \mathbf{v}}{\partial t_m} = \frac{\partial v_1}{\partial t_m}\mathbf{i} + \frac{\partial v_2}{\partial t_m}\mathbf{j} + \frac{\partial v_3}{\partial t_m}\mathbf{k}.$$

9.4 Partial Derivatives of a Vector Function

- Then the partial derivative of v with respect to t_m is denoted by $\partial\mathbf{v}/\partial t_m$ and is defined as the vector function:

$$\frac{\partial \mathbf{v}}{\partial t_m} = \frac{\partial v_1}{\partial t_m}\mathbf{i} + \frac{\partial v_2}{\partial t_m}\mathbf{j} + \frac{\partial v_3}{\partial t_m}\mathbf{k}.$$

- Similarly, second partial derivatives are

$$\frac{\partial^2 \mathbf{v}}{\partial t_l \partial t_m} = \frac{\partial^2 v_1}{\partial t_l \partial t_m}\mathbf{i} + \frac{\partial^2 v_2}{\partial t_l \partial t_m}\mathbf{j} + \frac{\partial^2 v_3}{\partial t_l \partial t_m}\mathbf{k},$$

9.4 Partial Derivatives of a Vector Function

- Similarly, second partial derivatives are

$$\frac{\partial^2 \mathbf{v}}{\partial t_l \partial t_m} = \frac{\partial^2 v_1}{\partial t_l \partial t_m}\mathbf{i} + \frac{\partial^2 v_2}{\partial t_l \partial t_m}\mathbf{j} + \frac{\partial^2 v_3}{\partial t_l \partial t_m}\mathbf{k},$$

- Note d^2v / (dt dr) = d/dt (dv/dr)

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9.4 Partial Derivatives of a Vector Function: Example

Let $\mathbf{r}(t_1, t_2) = a \cos t_1\, \mathbf{i} + a \sin t_1\, \mathbf{j} + t_2\, \mathbf{k}.$

Then $$\frac{\partial \mathbf{r}}{\partial t_1} = -a \sin t_1\, \mathbf{i} + a \cos t_1\, \mathbf{j}$$

and $$\frac{\partial \mathbf{r}}{\partial t_2} = \mathbf{k}.$$

- Various physical and geometric applications of derivatives of vector functions are discussed in the next sections as well as in Chap. 10.

9.7 Gradient of a Scalar Field

- Gradient of a given scalar function $f(P) = f(x, y, z)$. is denoted by (grad f) or (∇ f).
- ∇ is pronounced Nabla.
- The differential operator ∇ is defined as:

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}$$

9.7 Gradient of a Scalar Field

$$grad\ f(x,y,z) = \nabla f(x,y,z) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

$$= \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}$$

- A gradient gives the rate of change of $f\ (x,y,z)$ in any direction in space.
- This is done by obtaining by deriving a vector field from a scalar field.

9.7 Gradient of a Scalar Field: Example

- Find the gradient of a scalar function described by:

$$f(x,y,z) = 2y^3 + 4xz + 3x$$

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

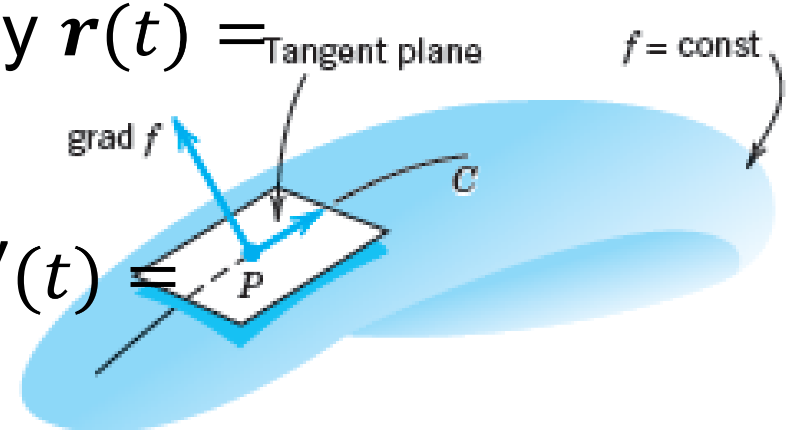
$$\nabla f = [4z + 3, \quad 6y^2, \quad 4x]$$

$$\nabla f = (4z + 3)\boldsymbol{i} + (6y^2)\boldsymbol{j} + (4x)\boldsymbol{k}$$

F.Y.I.

.7 Gradient as a Surface Normal Vector

- Gradients have an important application in connection with surfaces as surface normal vectors.
- Let *S* be a surface represented by $f\,(x, y, z) = c = const$, where *f* is differentiable.
- Now let *C* be a curve (a line) on *S* through a point *P* of the surface *S*.
- The curve *C* is represented by $\boldsymbol{r}(t) = [x(t), y(t), z(t)]$
- A tangent vector of *C* is $\boldsymbol{r}'(t) = [x'(t), y'(t), z'(t)]$



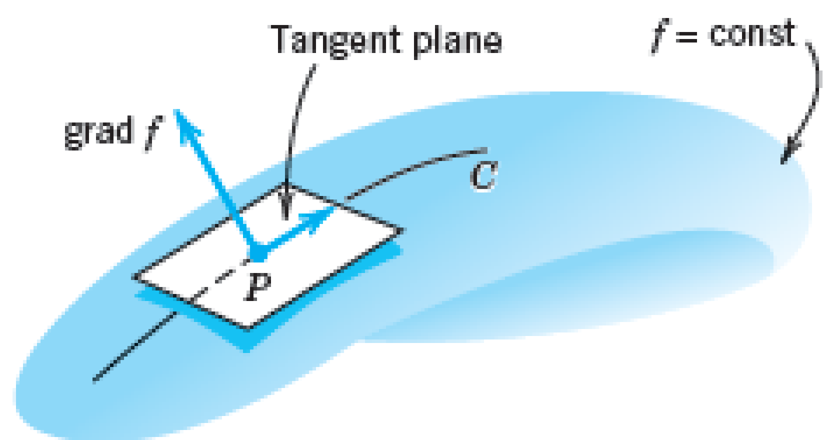
F.Y.I.

.7 Gradient as a Surface Normal Vector

- A tangent vector of *C* → tangent plane of *S* at *P.*
- A vector in the direction of the surface normal is called a surface normal vector of *S* at *P*.
- Surface normal vector of *S* ($f\,(x, y, z)$) at point *P* equals $\nabla \mathrm{f(P)} = \mathrm{grad\ f\ (P)}$
- Since $\mathrm{grad\ f\ (P)}$ is perpendicular to tangent $\boldsymbol{r}'(t)$:

$$\frac{\partial f}{\partial x}x' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial z}z' =$$

$$= (\mathrm{grad}\, f) \bullet \mathbf{r}' = 0.$$



9.7 Gradient as a Surface Normal Vector: Example

- Find a unit normal vector **n** of the cone of revolution $z^2 = 4(x^2 + y^2)$ $at\ point\ P(1,0,2)$

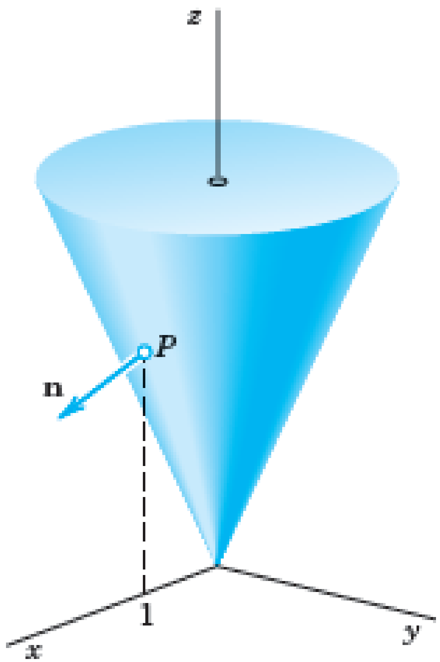
$$\rightarrow f(x, y, z) = 4x^2 + 4y^2 - z^2 = 0$$

$$\nabla \mathrm{f}(x, y, z) = [8\mathrm{x}, 8\mathrm{y}, -2\mathrm{z}]$$

normal vector: $\nabla \mathrm{f}(1,0,2) = [8, 0, -4]$

unit normal vector: $= \dfrac{\nabla \mathrm{f}(1,0,2)}{|\nabla \mathrm{f}(1,0,2)|} =$

$$\frac{[8,0,-4]}{\sqrt{80}} = \left[\frac{2}{\sqrt{5}},\ 0,\ \frac{-1}{\sqrt{5}}\right]$$



9.7 Laplacian of a Scalar Field

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- ∇^2 : nabla squared: laplacian.
- Remember that f is a scalar function and ∇^2 gives also a scalar function.
- Differentiate f twice with respect to x, y, and z and add the derivatives.

9.7 Laplacian of a Scalar Field

$\nabla^2 f$ is also denoted by Δf. The differential operator

(11)
$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

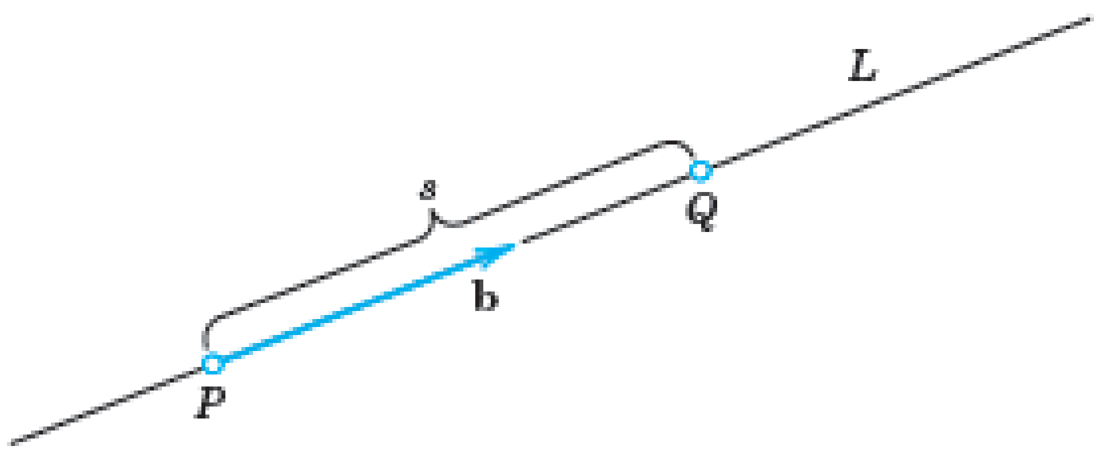
(read "nabla squared" or "delta") is called the **Laplace operator**.

9.7 Directional Derivative of a Scalar Field

- The directional derivative $D_b f$ or df/ds of a function f(x, y, z) at a point P in the direction of a unit vector **b** is defined by:

$$D_{\mathbf{b}} f = \frac{df}{ds} = \lim_{s \to 0} \frac{f(Q) - f(P)}{s}.$$

where Q is a variable point on the straight line L in the direction of **b**



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9.7 Directional Derivative of a Scalar Field

- Assuming that **a** is an arbitrary vector of any length ($\neq 0$), then the directional derivative of f $D_a f$ in the direction of vector **a** is:

$$D_{\mathbf{a}} f = \frac{df}{ds} = \frac{1}{|\mathbf{a}|} \mathbf{a} \cdot \operatorname{grad} f.$$

Remember: $grad\ f = \nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}$

9.7 Directional Derivative of a Scalar Field: Example

- Find the directional derivative of:

$$f(x, y, z) = 2x^2 + 3y^2 + z^2 \text{ at } P\text{: } (2, 1, 3)$$

in the direction of $\mathbf{a} = [1, 0, -2]$.

$$\text{grad f} = \nabla \text{f} = [4\text{x}, 6\text{y}, 2\text{z}]$$

$$\nabla \text{f(P)} = [8, 6,6]$$

$$|\mathbf{a}| = \sqrt{5}$$

$$\text{D}_\text{a}\text{f(P)} = \frac{\boldsymbol{a}\,.\nabla \text{f(P)}}{|\boldsymbol{a}|} = \frac{[1,0,-2]\,.\,[8,6,6]}{\sqrt{5}} = \frac{-4}{\sqrt{5}}$$

9.8 Divergence of a Vector Field

- Divergence measures the magnitude of a vector field's source at a given point, in terms of a scalar.
- Example: The velocity of the moving air at a point.
- If air is heated in a region it will expand in all directions such that the velocity field points outward from that region →positive divergence
- If the air cools and contracts →negative divergence, as the region is a sink

9.8 Divergence of a Vector Field

- Let $\mathbf{v}(x, y, z) =$ $\mathbf{v} = [v_1, \quad v_2, \quad v_3] = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

$$\operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

9.8 Divergence of a Vector Field: Example

- Let

$$\mathbf{v} = [3xz,\ 2xy,\ -yz^2] = 3xz\mathbf{i} + 2xy\,\mathbf{j} - yz^2\mathbf{k},$$

$$\text{then} \qquad \operatorname{div} \mathbf{v} = 3z + 2x - 2yz.$$

9.8 Divergence of a Vector Field

Another common notation for the divergence is

$$\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial \mathbf{y}}, \frac{\partial}{\partial \mathbf{z}}\right] \cdot [v_1, v_2, v_3]$$

$$= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z},$$

9.8 Divergence of a Vector Field

- Let $f\ (x, y, z)$ be a twice differentiable scalar function. Then its gradient exists,

$$\mathbf{v} = \operatorname{grad} f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

- Let's now find div **v:**

$$\operatorname{div} \mathbf{v} = \operatorname{div} (\operatorname{grad} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

- Hence, $div(grad\ f) = \nabla^2 f$

9.9 Curl of a Vector Field

- Curl describes the rotation of a 3-dimensional vector field.
- The direction of the curl is the axis of rotation, as determined by the right-hand rule, and the magnitude of the curl is the magnitude of rotation.
- Example: The flow velocity of a moving fluid, then the curl is the circulation density of the fluid.
- A vector field whose curl is zero is called irrotational.

9.9 Curl of a Vector Field

- Let $\mathbf{v}(x, y, z) = \mathbf{v} = [v_1, \quad v_2, \quad v_3] = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

$$\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}.$$

9.9 Curl of a Vector Field: Example

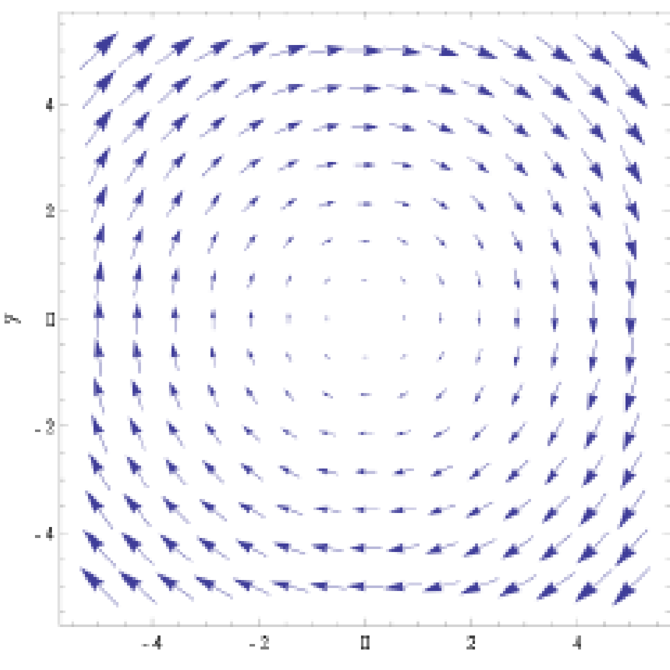
$$\text{Let } \mathbf{v} = [yz,\quad 3zx,\quad z] = yz\mathbf{i} + 3zx\mathbf{j} + z\mathbf{k}$$

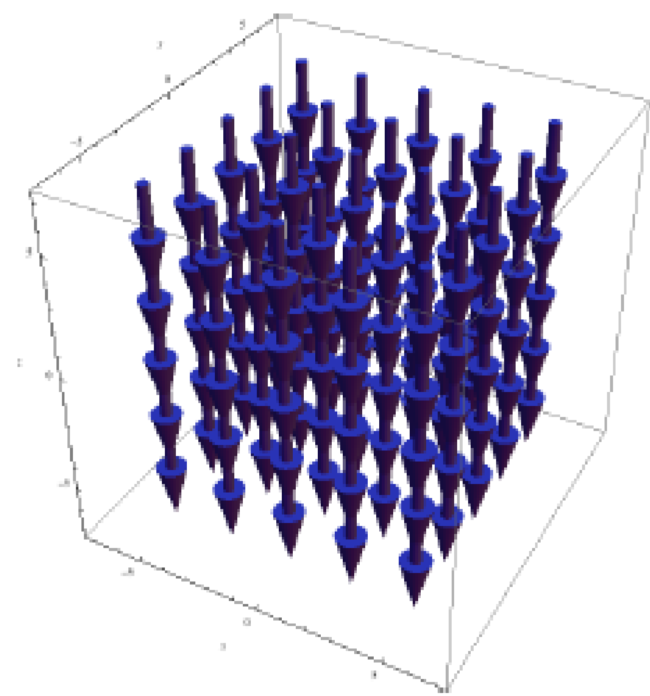
$$\operatorname{curl} \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix} = -3x\mathbf{i} + y\mathbf{j} + (3z - z)\mathbf{k} = -3x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}.$$

9.9 Curl of a Vector Field: Example

- Let $\mathbf{v}(x, y, z) = y\mathbf{i} - x\mathbf{j}$

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + \left[\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial x}(y)\right]\mathbf{k} = -2\mathbf{k}$$

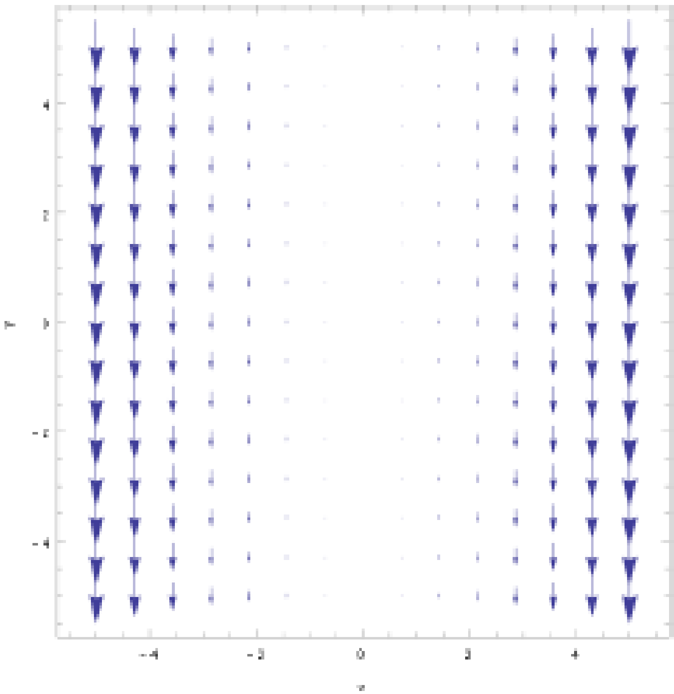


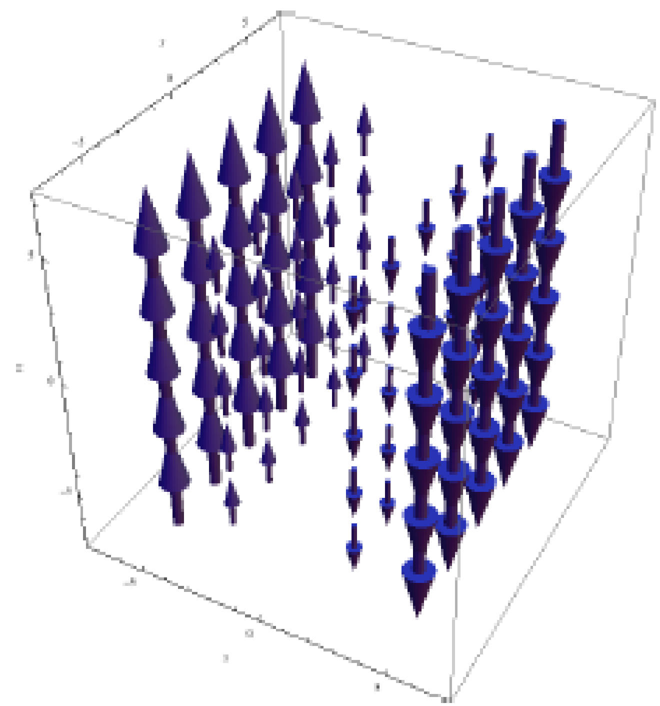


9.9 Curl of a Vector Field: Example

- Let $\mathbf{v}(x, y, z) = -x^2\mathbf{j}$

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -x^2 & 0 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + \left[\frac{\partial}{\partial x}(-x^2)\right]\mathbf{k} = -2x\mathbf{k}$$





Vector & Scalar Fields Operations

- Vector field $\xrightarrow{d/dt}$ vector field.
- Vector field $\xrightarrow{\partial/\partial t_m}$ vector field.
- Scalar field $\xrightarrow{\nabla f}$ vector field.
- Scalar field $\xrightarrow{\nabla^2 f}$ scalar field.
- Scalar field $\xrightarrow{D_a f}$ scalar field.
- Vector field $\xrightarrow{div\,\mathbf{v}}$ scalar field.
- Vector field $\xrightarrow{curl\,\mathbf{v}}$ vector field.

F.Y.I.

Other Vector & Scalar Field Operators

- Hessian is a square matrix of second-order partial derivatives of a scalar field.

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix}.$$

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Other Vector & Scalar Field Operators

- Jacobian is the matrix of all first-order partial derivatives of a vector field.

$$\mathbf{J} = \frac{d\mathbf{f}}{d\mathbf{x}} = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1} & \cdots & \frac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Basic Formulas for Grad, Div, Curl

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(f/g) = (1/g^2)(g\nabla f - f\nabla g)$$

$$\operatorname{div}(f\mathbf{v}) = f\operatorname{div}\mathbf{v} + \mathbf{v}\bullet\nabla f$$

$$\operatorname{div}(f\nabla g) = f\nabla^2 g + \nabla f\bullet\nabla g$$

$$\nabla^2 f = \operatorname{div}(\nabla f)$$

Basic Formulas for Grad, Div, Curl

$$\nabla^2(fg) = g\nabla^2 f + 2\nabla f \cdot \nabla g + f\nabla^2 g$$

$$\operatorname{curl}(f\mathbf{v}) = \nabla f \times \mathbf{v} + f \operatorname{curl} \mathbf{v}$$

$$\operatorname{div}(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \operatorname{curl} \mathbf{u} - \mathbf{u} \cdot \operatorname{curl} \mathbf{v}$$

$$\operatorname{curl}(\nabla f) = \mathbf{0}$$

$$\operatorname{div}(\operatorname{curl} \mathbf{v}) = 0.$$

- Important tip: Page 410 in textbook gives a useful summary of vector differential calculus: grad, div, curl.