Structural Mechanics

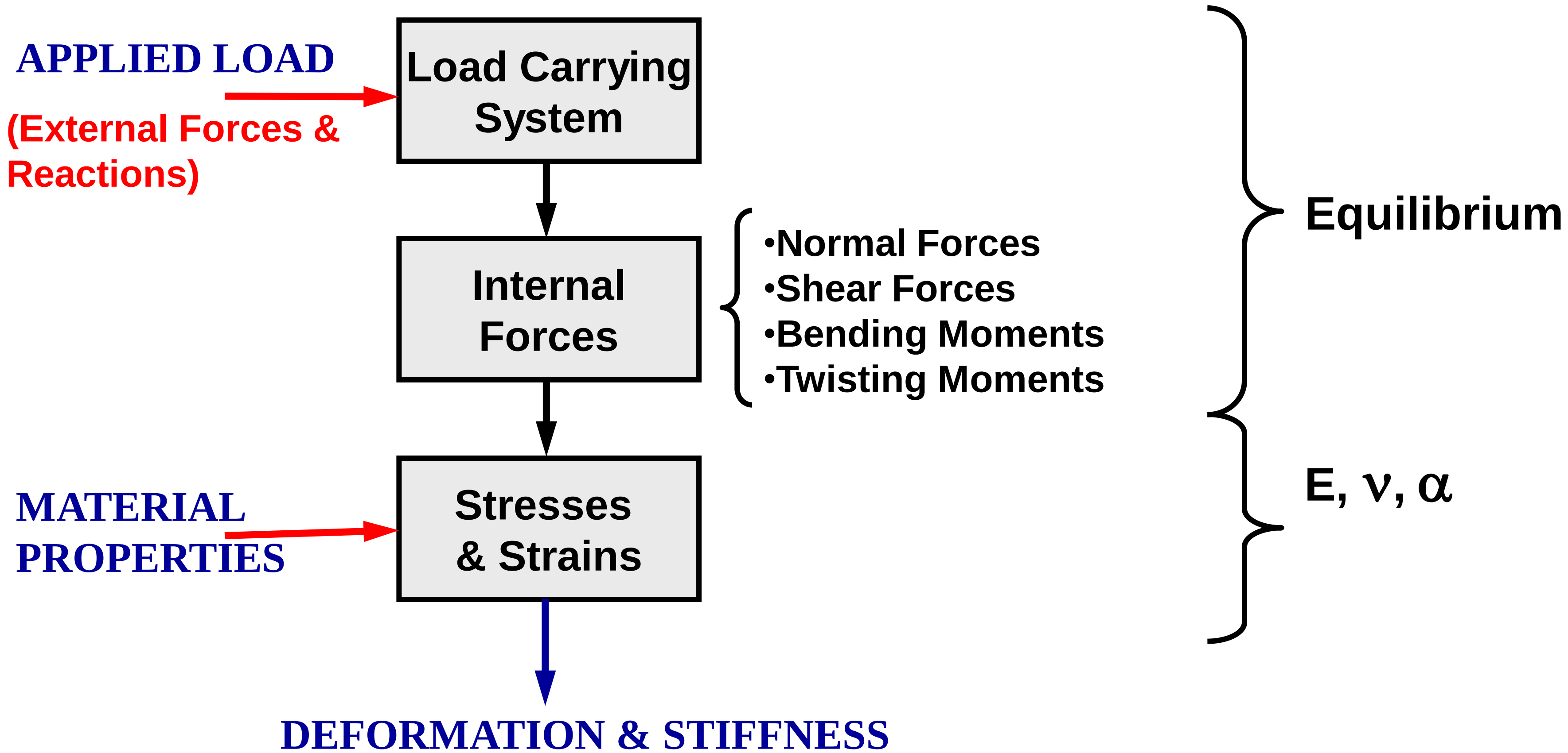
2104011212

Stress

Hibbeler, R.C. (2011). Mechanics of Materials. 8thEd. Pearson Prentice Hall.

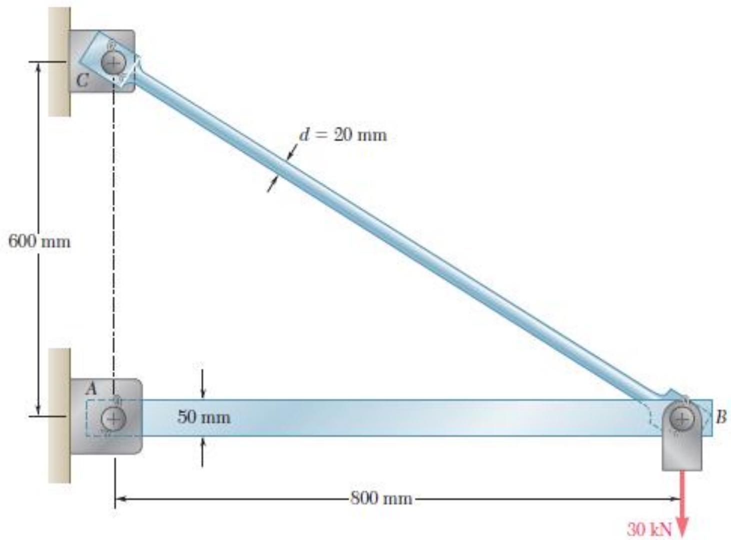
Mechanics of Materials, F. P. Beer et al, Mc Graw Hill.

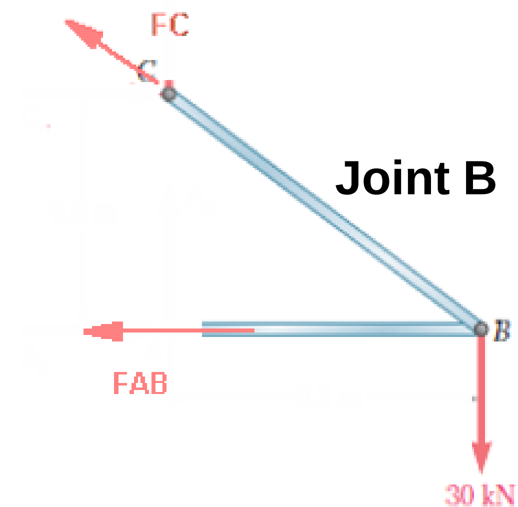
Iqbal Marie


APPLIED LOAD
(External Forces & Reactions)
Load Carrying System
Internal Forces
•Normal Forces
•Shear Forces
•Bending Moments
•Twisting Moments
Equilibrium
MATERIAL PROPERTIES
Stresses & Strains
E, ν, α
DEFORMATION & STIFFNESS

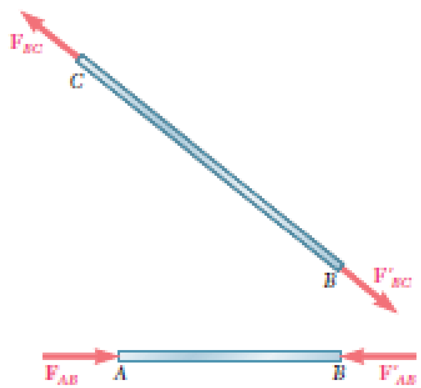
1.1- Equilibrium of rigid body and internal forces

1. draw a *free-body diagram* of the structure
2. Define the two force members
3. Apply equilibrium equations to find the support reactions
4. Be sure all the units are consistent
5. Find the forces in the two force members and specify if in tension or compression
6. make section and find the internal forces





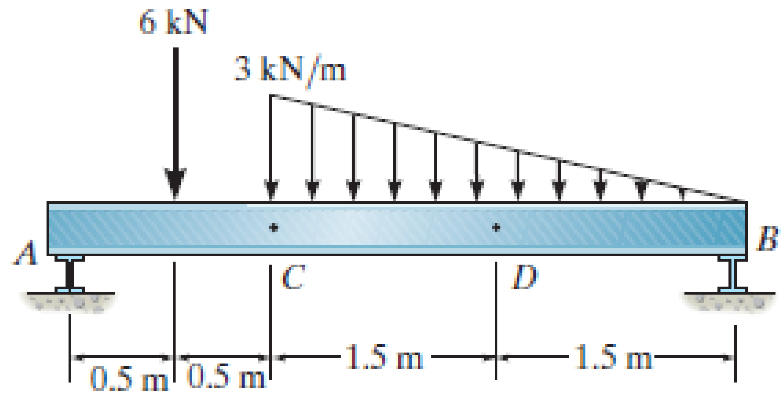
Joint B



Free-body diagrams of two-force members AB and BC.

FAB= -40 (C)
FAC= 50 (T)

Determine the resultant internal loadings on the cross section through point *C and D.*



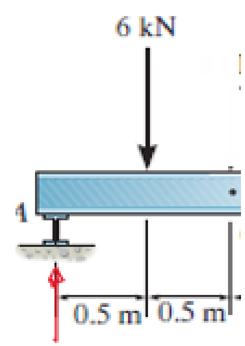
Support Reactions.

$(+\circlearrowleft)\ \Sigma M_A = 0 \Rightarrow -0.5\times6 - (4.5)(2) + 4B_y = 0$

$B_y = 3\ kN \uparrow$

$+\uparrow \Sigma F_y = 0 \Rightarrow A_y - 10.5 + 3 = 0$

$A_y = 7.5\ kN \uparrow$



$A_y = 7.5$

Internal forces at C

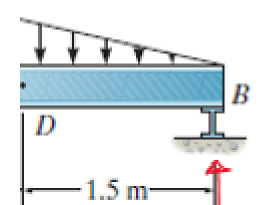
$\Sigma F_x = 0 \Rightarrow N_C = 0$

$+\uparrow \Sigma F_y = 0 \Rightarrow 7.5 - 6 - V_C = 0$

$V_C = 1.5\ kN \downarrow$

$(+\circlearrowleft)\ \Sigma M_C = 0 \Rightarrow -7.5\times1 + 6\times0.5 + M_C = 0$

$\Rightarrow M_C = +1.5\ kN.m$



$3 = B_y$

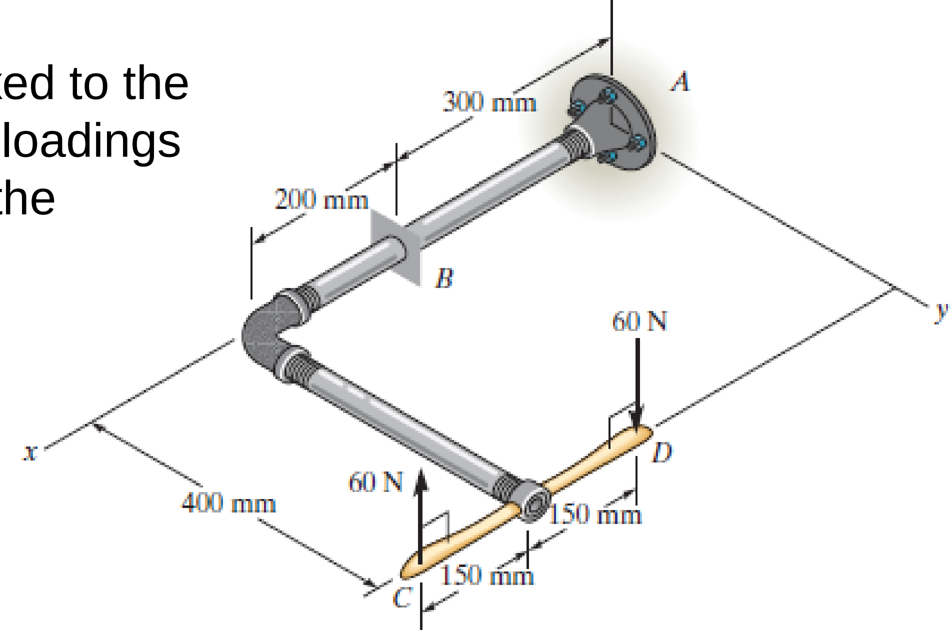
Internal forces at D.

$N_D = 0$

$\Sigma F_y = 0 \Rightarrow 3 - 1.125 + V_D = 0 \longrightarrow V_D = -1.875\ kN$

$(+\circlearrowleft)\ \Sigma M_D = 0 \Rightarrow -M_D - (1.125)(0.5) + (3)(1.5) = -3.94\ kNm$

The pipe has a mass of 12 kg/m. If it is fixed to the wall at *A*, determine the resultant internal loadings acting on the cross section at *B*. Neglect the weight of the wrench *CD*.



Section at B

$N_x = 0$

$V_y = 0$

$\Sigma F_z = 0 \Rightarrow -23.52 - 47.04 + V_z = 0$

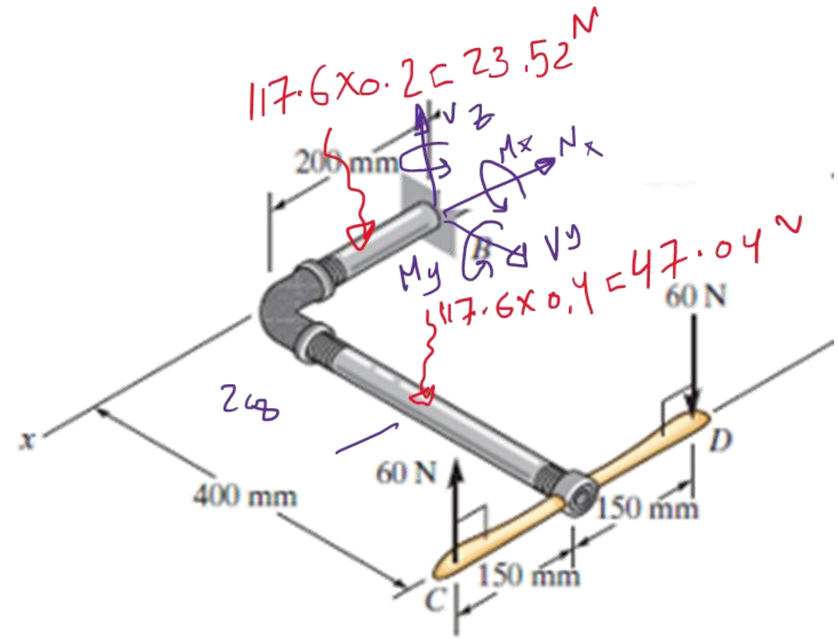
$V_z = 70.56$

$M_z = 0$

$\Sigma M_y = 0 \Rightarrow (-60 \times 0.3) - (23.52)(0.1) + 70.56 \times 0.2 + M_y = 0 \Rightarrow M_y = 6.24 \text{ N·m}$

$\Sigma M_x = 0 \Rightarrow (47.04)(0.2) + M_x = 0$

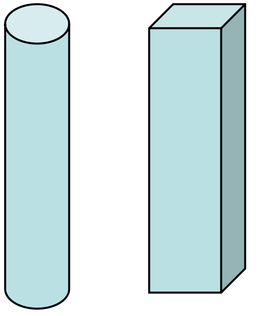
$M_x = -9.41 \text{ N·m}$



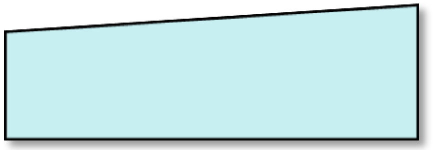
1.3 STRESSES

1.4 Average noirmal stress in an axially loaded bar Axial Stress

Prismatic bar: is a straight structural member having the same cross section throughout its length



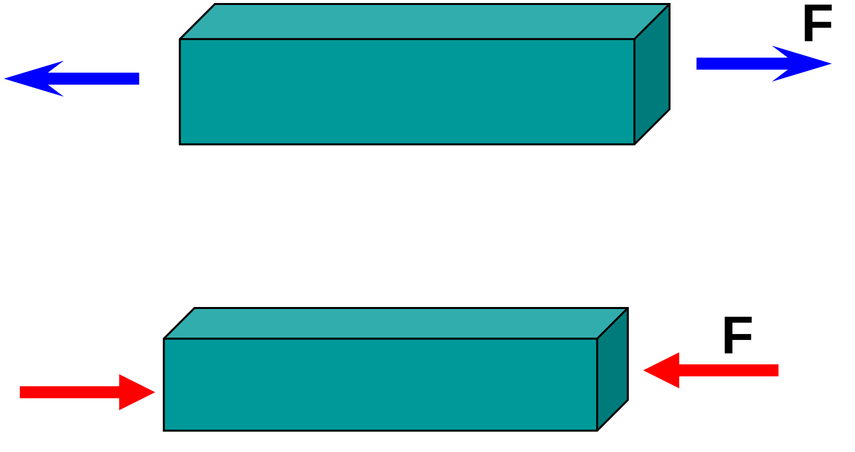
Non prismatic member with non uniform stresses



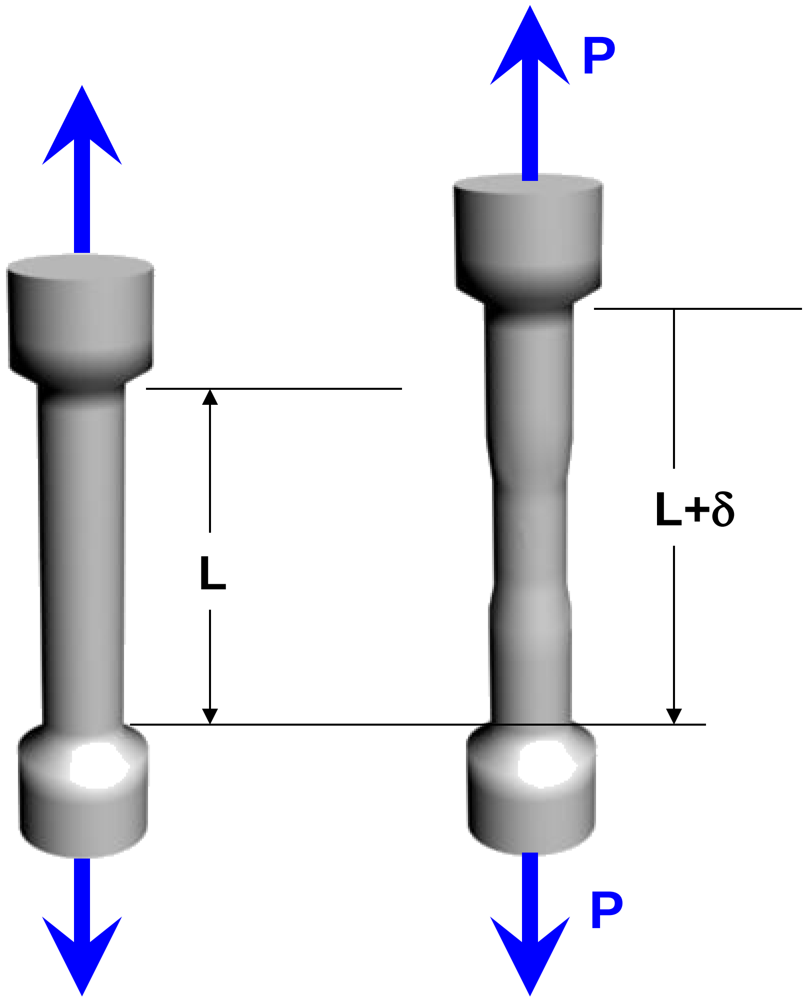
Homogeneous material: Have the same physical and mechanical properties through out its volume

isotropic material: Have the same properties in all directions

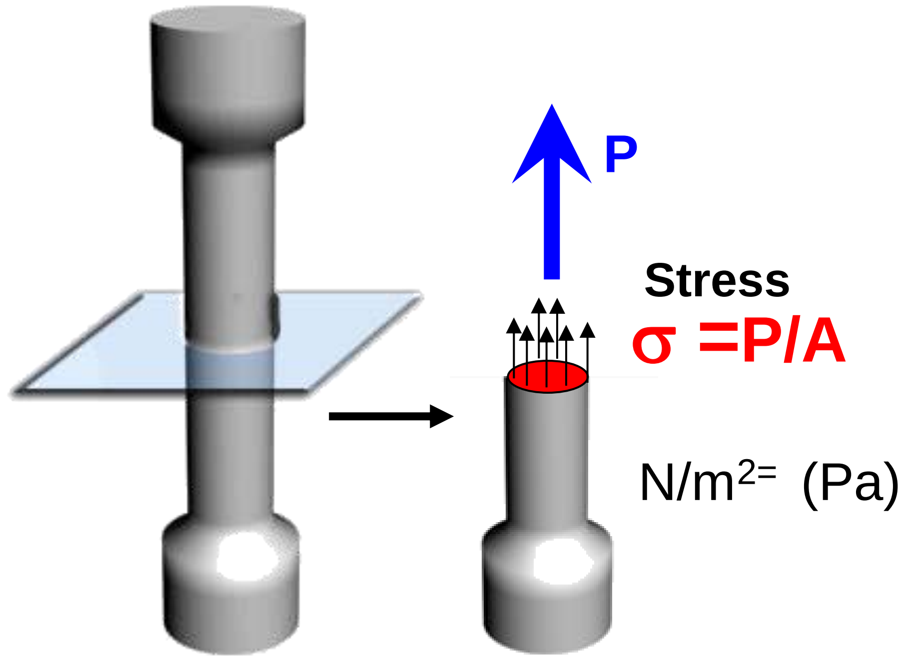
Axial force: A load directed along the axis of the member resulting either tension of compression in the bar.



Normal Stress



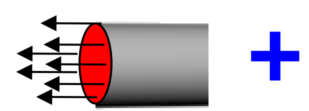
Prismatic Bar in tension



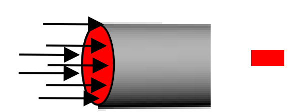
Normal Stress in the Bar

Sign convention:

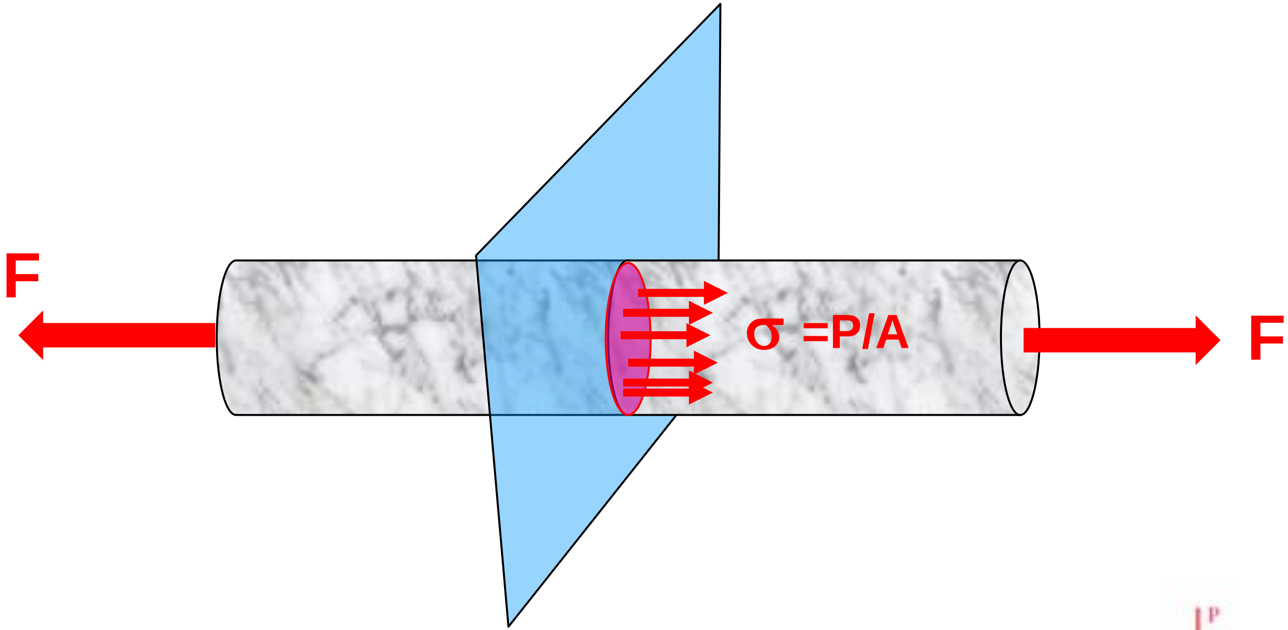
Tensile Stress: Positive

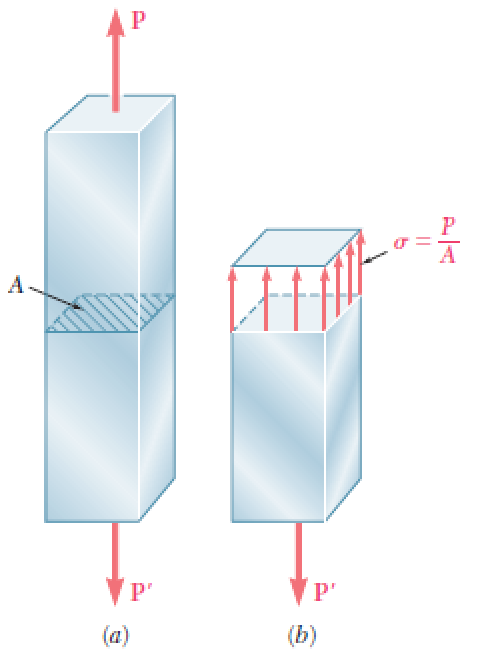


Compressive stress: Negative



The unit of stress is pascal (Pa), and multiples of the pascal i.e. kilopascal (kPa), megapascal (MPa= N/mm^2) the gigapascal (GPa = 10^3 MPa).

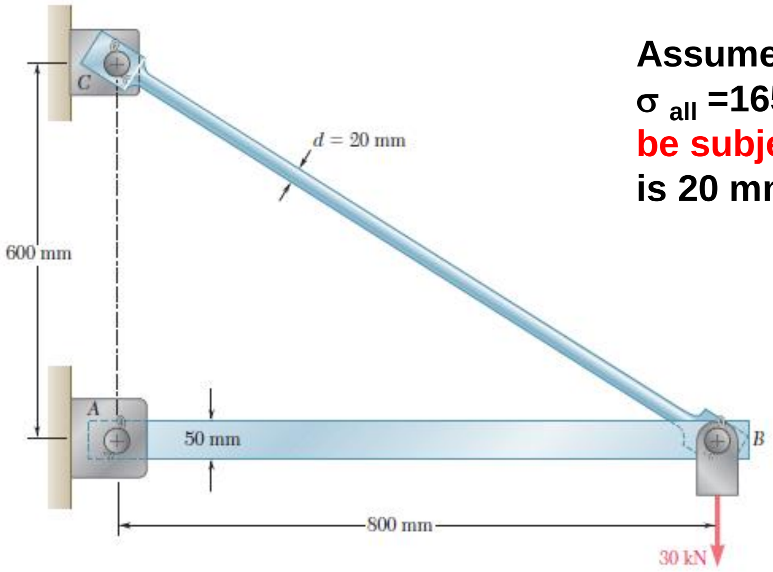




(a) Member with an axial load.
(b) Idealized uniform stress distribution at an arbitrary section.

A uniform distribution of stress is possible only if the line of action of the concentrated loads **P** *and* **P**’ *passes through the centroid of the section*. This type of loading is called ***centric loading*** and will take place in all straight two-force members found in trusses and pin-connected structures,

Centric loading having resultant forces passing through the centroid of the section.

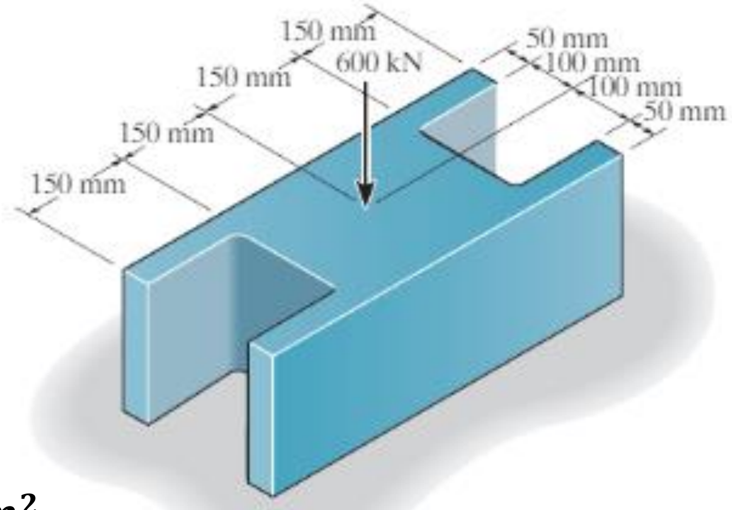


Assume rod BC is made of steel with maximum allowable stress σ_{all} =165 MPa. Can rod BC safely support the load which it will be subjected? Knowing that F_{BC} = 50 kN , the diameter of the rod is 20 mm

$$\boldsymbol{\sigma = \frac{p}{A} = \frac{50000}{\pi \times 10^2} = 159\ MPa < 165}$$

Then it is safe and can support the load

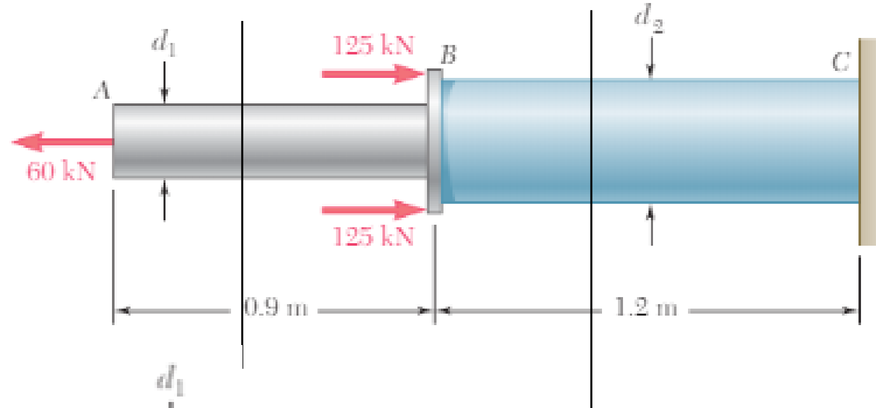
For the object shown determine the average normal stress in the material.



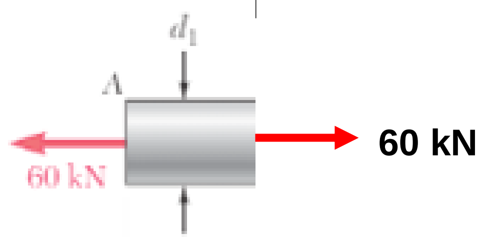
$$A = 600 \times 300 - \mathbf{150x200x2} = \mathbf{120000\ m}\boldsymbol{m^2}$$

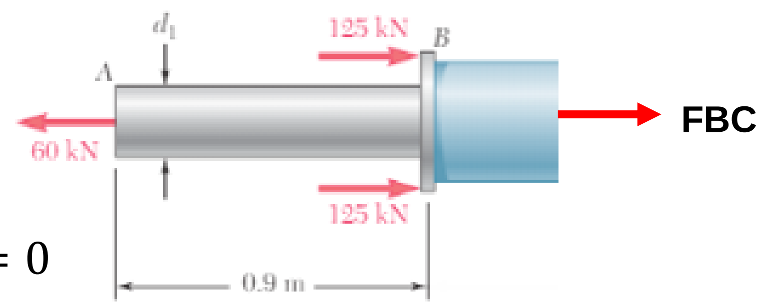
$$\boldsymbol{\sigma = \frac{p}{A} = \frac{600000}{120000} = 5\ MPa}$$

Two solid cylindrical rods AB and BC are welded together at B and loaded as shown. Knowing that $d_1 = 30$ mm and $d_2 = 50$ mm, find the average normal stress at the midsection of (*a*) rod AB, (*b*) rod BC.



$$\sigma_{AB} = \frac{p}{A} = \frac{60000}{\pi \times 15^2} = 84.9\ MPa$$



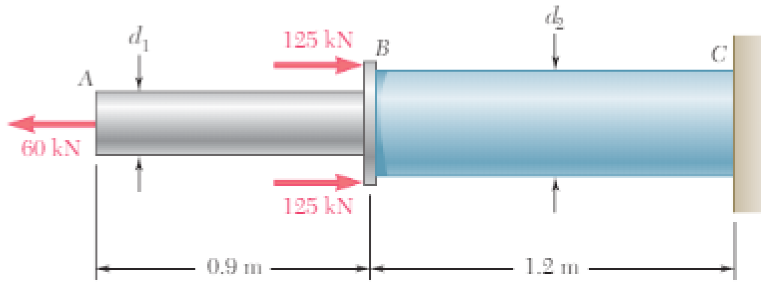


$$\sum F_x = 0 = -60 + 125 \times 2 + F_{BC} = 0$$

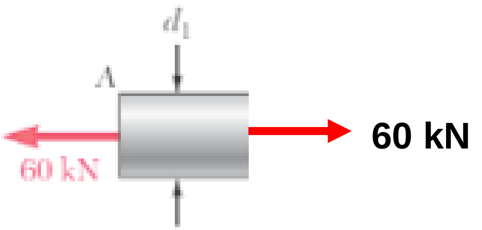
$$F_{BC} = -190\ (\mathrm{C})$$

$$\sigma_{AB} = \frac{p}{A} = \frac{-190000}{\pi \times 25^2} = 96.8MPa$$

Two solid cylindrical rods AB and BC are welded together at B and loaded as shown. Knowing that the average normal stress must not exceed 150 MPa in either rod, determine the smallest allowable values of the diameters d_1 and d_2.

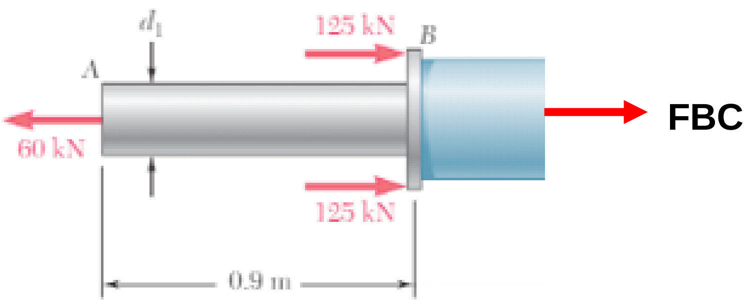


$$\sigma_{all} = \frac{p}{A} = \frac{60000}{(\pi \times d_1{}^2)/4} = 150MPa$$



$$d_1 = 22.57 \text{ mm}$$

$$\sum F_x = 0 = -60 + 125 \times 2 + F_{BC} = 0$$

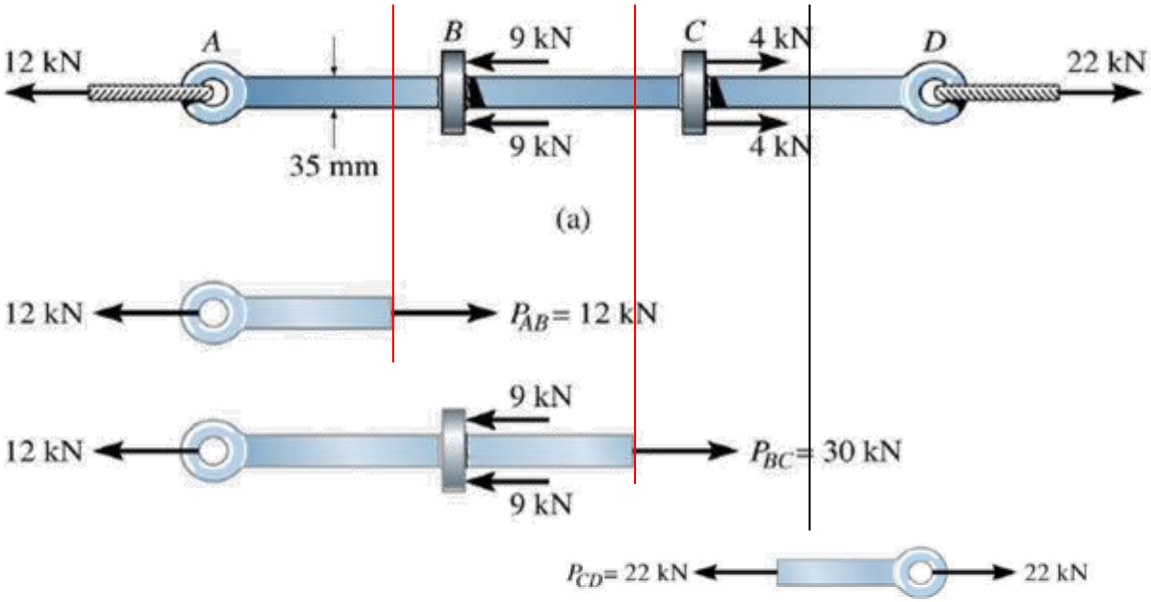


$$F_{BC} = -190 \text{ (C)}$$

$$\sigma_{all} = \frac{p}{A} = \frac{190000}{(\pi \times d_2{}^2)/4} = 150MPa$$

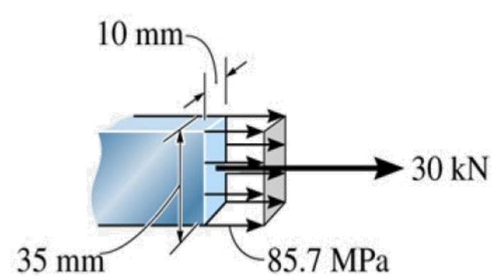
$$d_2 = 40.17 \text{ mm}$$

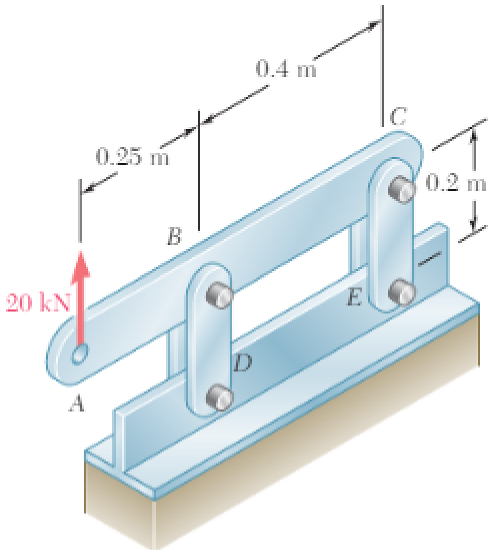
The bar has a constant width of 35 mm and a thickness of 10 mm. determine the maximum average normal stress in the bar when it is subjected to the loading shown



Since the bar has a constant cross section then the maximum internal force will cause the maximum stress

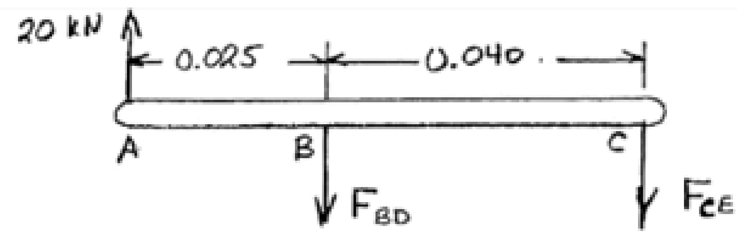
$$\sigma_{AB} = \frac{p}{A} = \frac{30000}{35\ x10} = 85.7\ MPa$$





Each of the four vertical links has an 8×36-mm uniform rectangular cross section and each of the four pins has a 16-mm diameter. Determine the maximum value of the average normal stress in the links connecting (*a*) points *B* and *D*, (*b*) points *C* and *E*.

Use bar *ABC* as a free body.



$$\Sigma M_C = 0: \quad (0.040) F_{BD} - (0.025 + 0.040)(20 \times 10^3) = 0$$

$$F_{BD} = 32.5 \times 10^3 \text{ N} \qquad \text{Link } BD \text{ is in tension.}$$

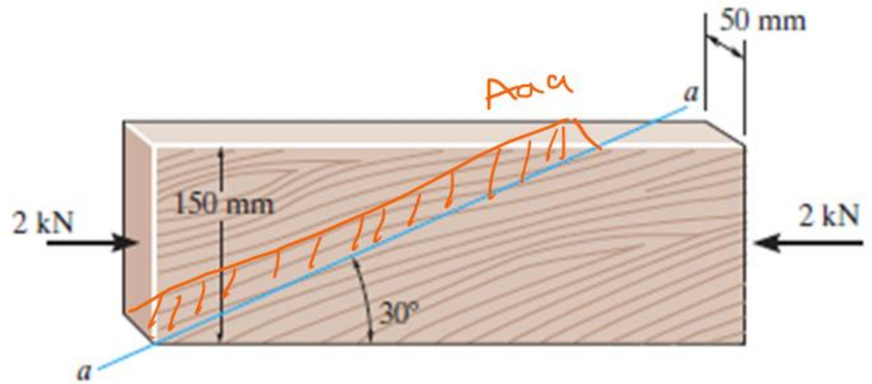
$$\Sigma M_B = 0: \quad -(0.040) F_{CE} - (0.025)(20 \times 10^3) = 0$$

$$F_{CE} = -12.5 \times 10^3 \text{ N} \qquad \text{Link } CE \text{ is in compression.}$$

Net area for link in tension (BD) =8x(36-16) =160 mm^2
Stress for one link of BD = (FBD /2) / 160 =32500/(2*160) = 101.6 MPa

Area for link in compression (CE) =8x(36) =288 mm^2
Stress for one link of BD = (FBD /2) / 288 = -12500/(2*288) = -21.7 MPa

The block is subjected to a compressive force of
2 kN Determine the average normal stress developed in the wood fibers that are oriented along section *a–a* at 30° with the axis of the block.



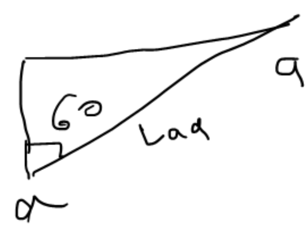
$$\uparrow \Sigma F_y = 0$$

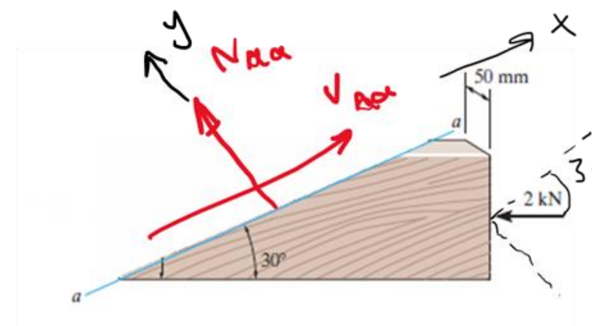
$$2 \sin 30 + N_{aa} = 0 \Rightarrow N_{aa} = 1 \text{ kN}$$

$$= 1000 \text{ N}$$

$$A_{aa} = (300)(50) = 1500 \text{ mm}^2$$

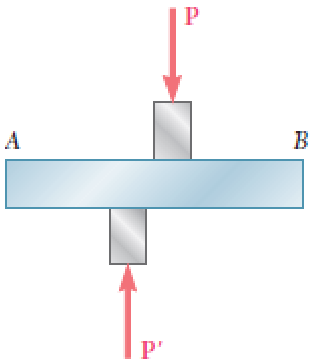
$$\sigma_{aa} = \frac{1000}{15000} = 0.067 \text{ MPa}$$

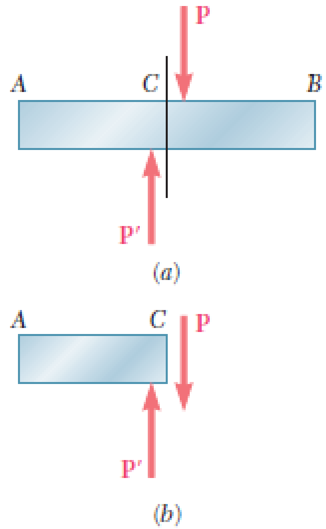


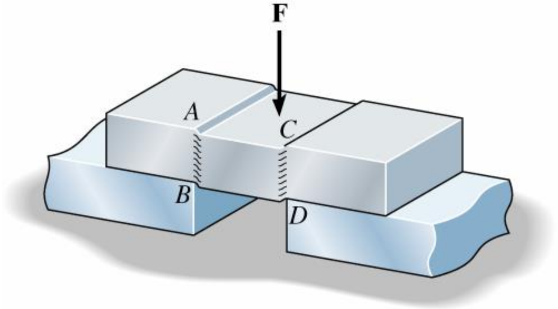


$$L_{aa} = \frac{150}{\cos 60} = 300 \text{ mm}$$

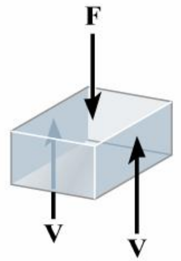
1.5 Average Shear Stress



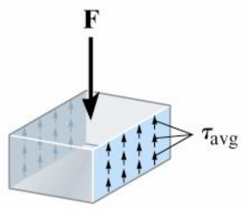




(a)



(b)



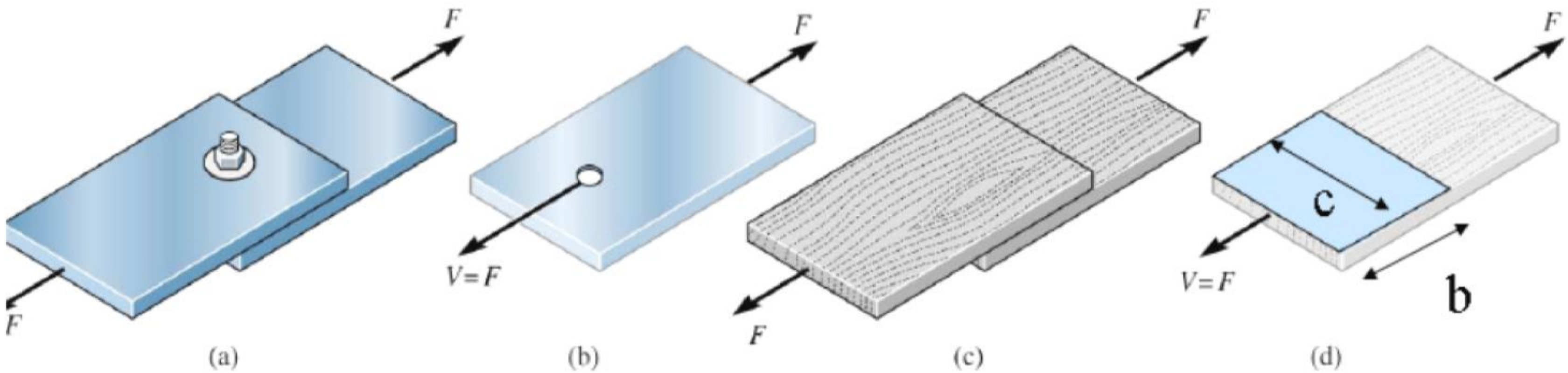
(c)

$\tau = V/A$ **average shear stress at the section, which is assumed to be the *same at each point located on the section***

V = internal resultant shear force at the section determined from the equations of equilibrium

A= area at the section (parallel to the shear force)

Single Shear

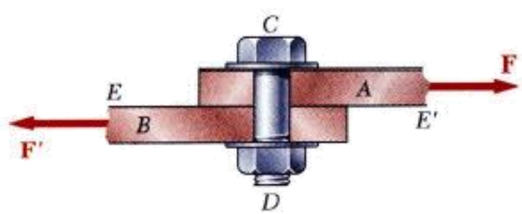


Shear stress on bonded area

Shear stress on bolt

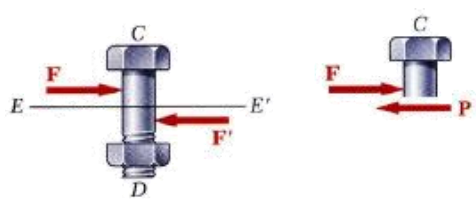
$\tau = F/A = F/\pi r^2$

Where r is the radius of the bolt

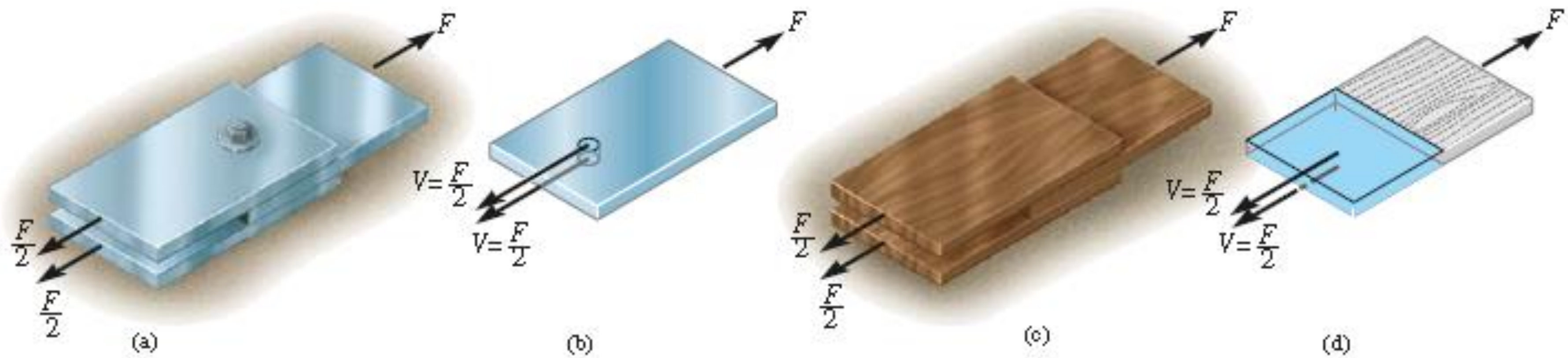


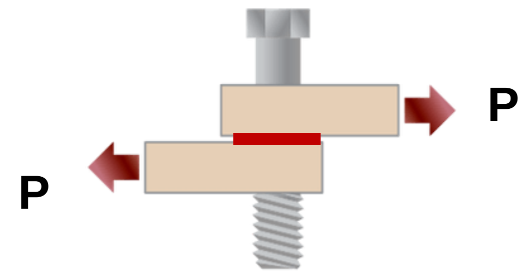
$\tau = F/bc$

Where (bc) is the area of contact subjected to the shear force

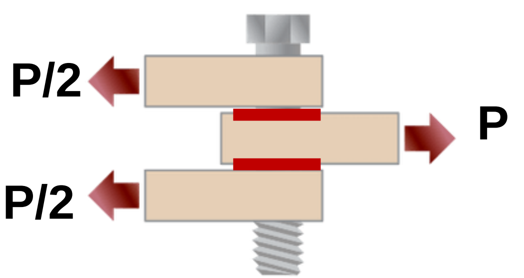


Double Shear





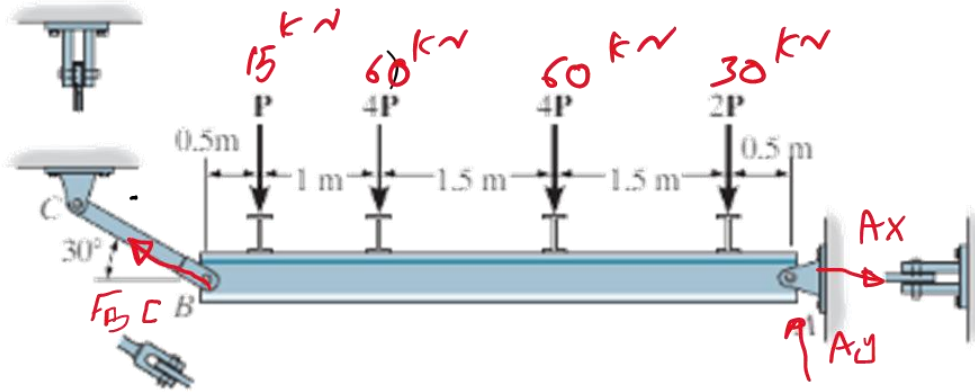
$$\textit{Single shear}\ \tau = \frac{p}{A}$$



$$\textit{Double shear}\ \tau = \frac{p}{2A}$$

The beam is supported by a pin at A and a short link BC. If P=15 kN, determine the average shear stress developed in the pins at A,B, and C. All pins are in double shear as shown, and each has a diameter of 18 mm

$$A_b = \frac{\pi\, 18^2}{4} = 254.47\, mm^2$$



Support reactions:

$$\circlearrowleft \Sigma M_B = 0$$

$$-15 \times 1.5 - 4\times 15 \times 1.5 - 4 \times 15 \times 3 + 2 \times 15 \times 4.5 + 5A_y = 0 \Rightarrow$$

$$A_y = 82.5 \text{ kN}$$

$$\uparrow \Sigma F_y = 0 + F_{BC}\sin 30 - 15 - 60 - 60 - 30 + 82.5 = 0 \rightarrow$$

$$F_{BC} = 165 \text{ kN}$$

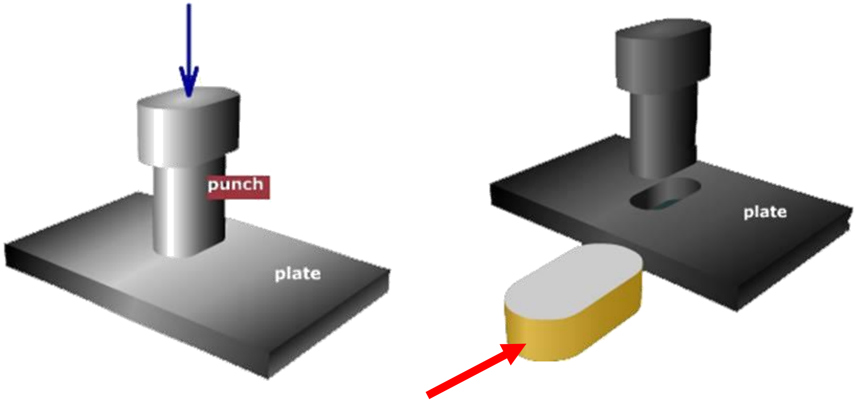
$$\Sigma F_x = 0 \Rightarrow A_x - 165\cos 30 = 0 \Rightarrow A_x = 142.8 \text{ kN}$$

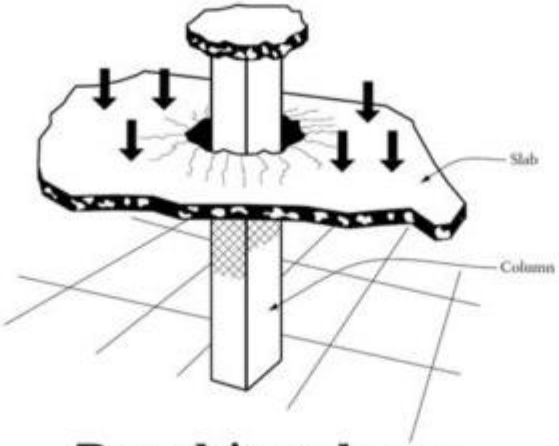
$$F_A = \sqrt{85.5^2 + 142.8^2} = 165 \text{ kN}$$

$$\tau_A = \frac{165000}{(2)(254.47)} = \underline{324 \text{ MPa}} \quad \Big| \quad \tau_B = \frac{165000}{(2)(254.47)} = \underline{324 \text{ MPa}}$$

Punch Shear

The punching shear is a failure mechanism in structural members like foundations by shear under the action of concentra



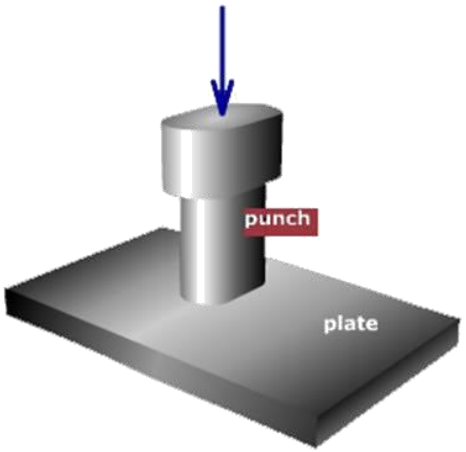


Punching shear

https://civilsnapshot.com/sliding-shear-punching-shear/

Stress acts on the perimeter surface of the slug. To compute the shear stress at failure, divide the applied load by the area of the slug perimeter

Determine the diameter of the largest circular hole that can be punched into a sheet of polystyrene 6mm thick, knowing that the force exerted by the punch is 45kN and that a 55 MPa average shearing stress is required to cause the material to fail.

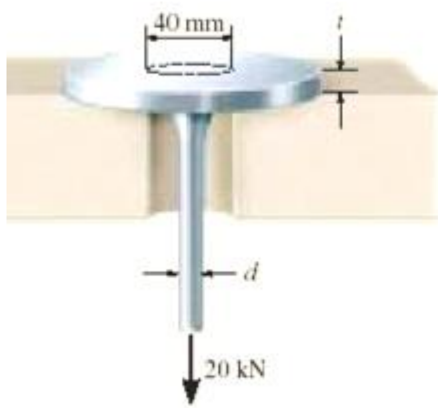


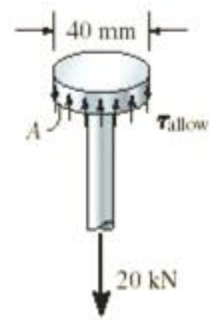
$$\tau = \frac{p}{A} = \frac{45000}{2*\pi*r*6} = 55\ MPa$$

$$r = \frac{45000}{2*\pi*55*6} = 21.7\text{mm}$$

d = 21.7*2= 43.4mm

If the rod passes through a 40mm diameter hole, determine the minimum required diameter of the rod and the minimum thickness of the disk needed to support the 20 kN load. The allowable normal stress for the rod is σ_{allow} = 60 MPa, and the allowable shear stress for the disk is $\tau_{\text{allow}} = 35\ \text{MPa}$





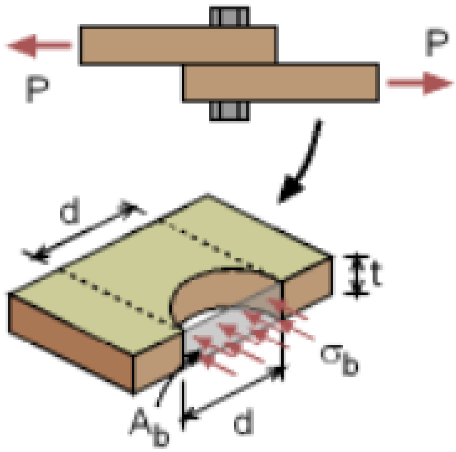
$$\boldsymbol{\sigma_{rod}} = \frac{\boldsymbol{p}}{\boldsymbol{A}} = \frac{\mathbf{20000}}{\pi * \boldsymbol{r}^{2}} = \mathbf{60}\ \boldsymbol{MPa}$$

r=10.3 mm **d=10.3x2 =20.6 mm**

$$\tau = \frac{\boldsymbol{p}}{\boldsymbol{A}} = \frac{\mathbf{20000}}{\mathbf{2} * \quad * \mathbf{20} * \boldsymbol{t}} = \mathbf{35}\ \boldsymbol{MPa}$$

t= 4.6 mm

Bearing Stress



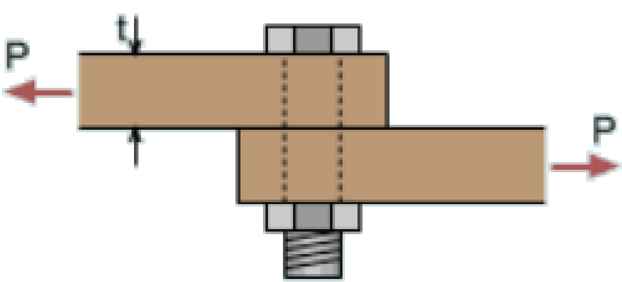
Bearing Stress Due to a Bolt

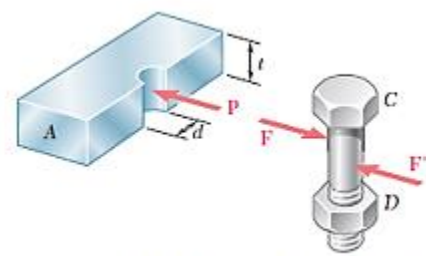
The average bearing stress is the force pushing against a structure divided by the area. Exact bearing stress is more complicated but for most applications, the following equation works well for the average,

$$\sigma_b = P/A_b$$

This relationship can be further refined by using the width and height of the bearing area as

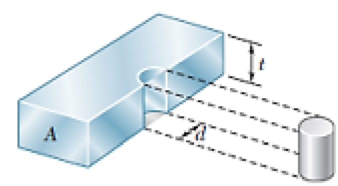
$$\sigma_b = \frac{P}{dt}$$





Equal and opposite forces between plate and bolt, exerted over bearing surfaces.

Bolts, pins, and rivets create stresses in the members they connect along the *bearing surface* or surface of contact

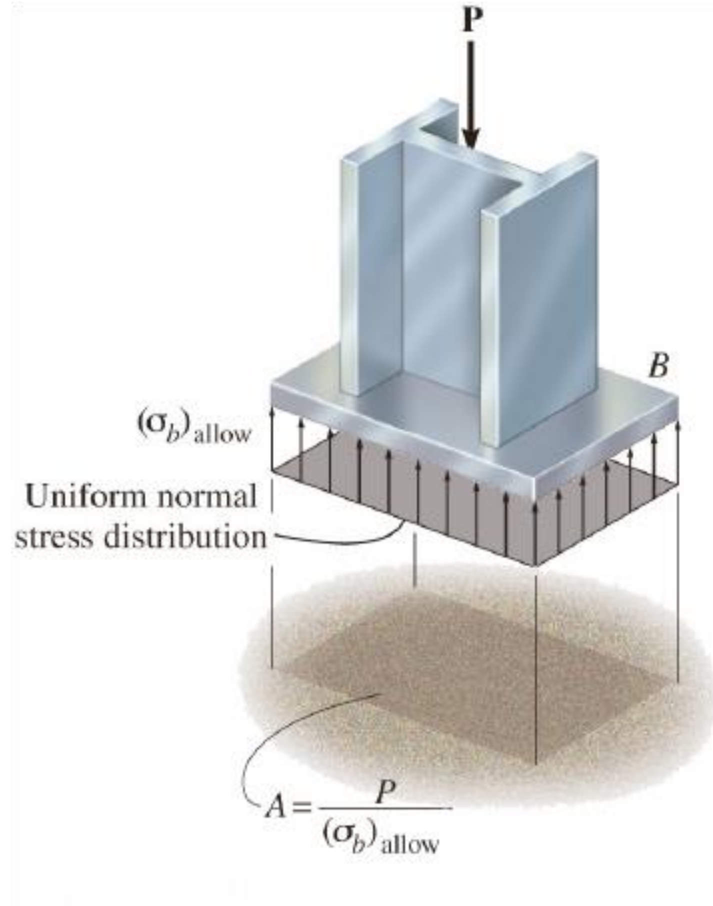


Dimensions for calculating bearing stress area.

Bearing stress

$$A = \frac{P}{(\sigma_b)_{allowable}}$$

where $(\sigma_b)_{allowable}$ is the allowable bearing stress of the weaker material



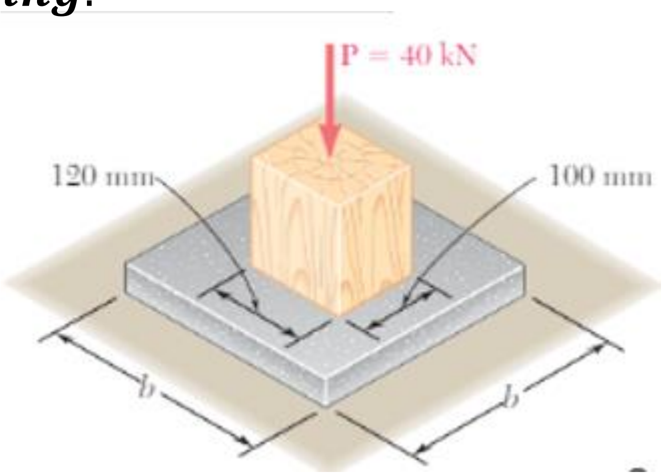
A 40kN axial load is applied to a short wooden post that is supported by a concrete footing resting on undisturbed soil. Determine:

a. The maximum bearing stress on the concrete footing

b. B. the size of the footing for which the average bearing stress in the soil is 145 kPa

***Maximum bearing stress on concrete footing*:**

$$\sigma_b = \frac{p}{A_b} = \frac{40000}{120 * 100} = 3.333\ MPa$$

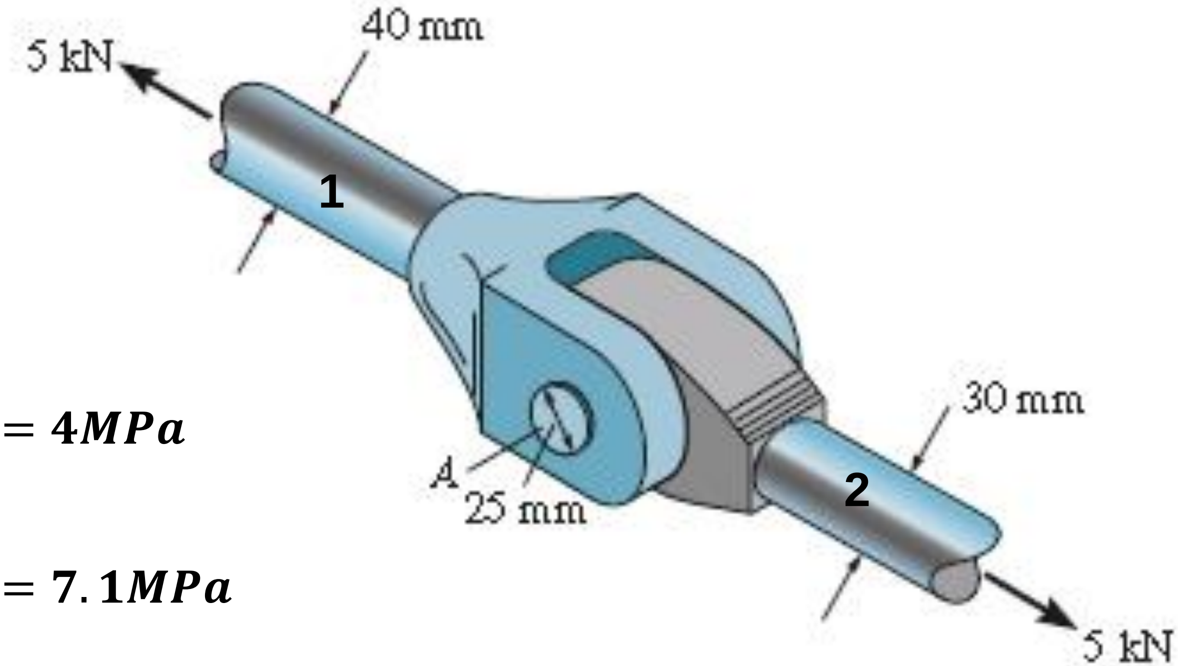


The size of the footing

$$\sigma_b = \frac{p}{A_b} = \frac{40000}{A} = \frac{145}{1000}\ MPa$$

$$A = 275.86\ x10^3\ mm^2 = 0.2786\ m^2$$

Determine the average normal stress in each rod and the average shear stress in the pin A



$$\boldsymbol{\sigma_{Rod1}} = \frac{p}{A} = \frac{5000}{3.14\ x20^{\ 2}} = 4MPa$$

$$\boldsymbol{\sigma_{Rod2}} = \frac{p}{A} = \frac{5000}{3.14\ x15^{\ 2}} = 7.1MPa$$

pin A (double shear)

$$\tau_A = \frac{p}{2A} = \frac{5000}{2\ (3.14\ x12.5^{\ 2})} = 5.1\ MPa$$

1.6 Allowable stress
1.7 Design of simple connections

The knowledge of stresses is used by engineers to assist in their most important task: the design of structures and machines that will safely and economically perform a specified function.

largest force is called the ***ultimate load*** and is denoted by P_U. Since the applied load is centric, the ultimate load is divided by the original cross-sectional area of the rod to obtain the ***ultimate normal stress*** of the material. This stress, also known as the *ultimate strength in tension*

$$\boldsymbol{P_u} \quad \rightarrow \sigma_U = \frac{\boldsymbol{P_u}}{A}$$

$$\boldsymbol{P_u} \quad \rightarrow \tau_U = \frac{\boldsymbol{P_u}}{A}$$

Allowable Stress:
Factor of Safety

The selection of the factor of safety to be used is one of the most important engineering tasks. If a factor of safety is too small, the possibility of failure becomes unacceptably large. On the other hand, if a factor of safety is unnecessarily large, the result is an uneconomical or nonfunctional design. The choice of the factor of safety for a given design application requires engineering judgment based on many considerations

Factors to be considered in design includes :

- **functionality,**
- **strength,**
- **appearance,**
- **economics and**
- **environmental protection.**

Factor of safety considerations:

- uncertainty in material properties
- uncertainty of loadings
- uncertainty of analyses
- number of loading cycles
- types of failure
- maintenance requirements and deterioration effects
- importance of member to integrity of whole structure
- risk to life and property
- influence on machine function

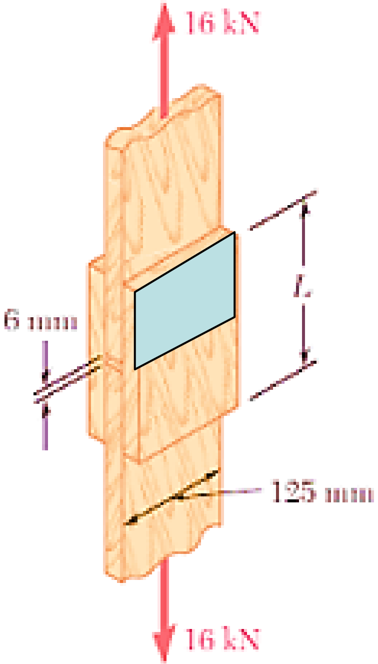
$$\text{Factor of safety} = F.S. = \frac{\text{ultimate load}}{\text{allowable load}}$$

$$\text{Factor of safety} = F.S. = \frac{\text{ultimate stress}}{\text{allowable stress}}$$

$$P_{\text{allow}} = \sigma_{\text{allow}} A$$

The factor of safety must be greater than one to avoid failure

Two wooden members are joined by plywood splice plates that are fully glued on the contact surfaces. Knowing that the clearance between the ends of the members is 6 mm and that the ultimate shearing stress in the glued joint is 2.5 MPa, determine the length L for which the factor of safety is 2.75 for the loading shown.



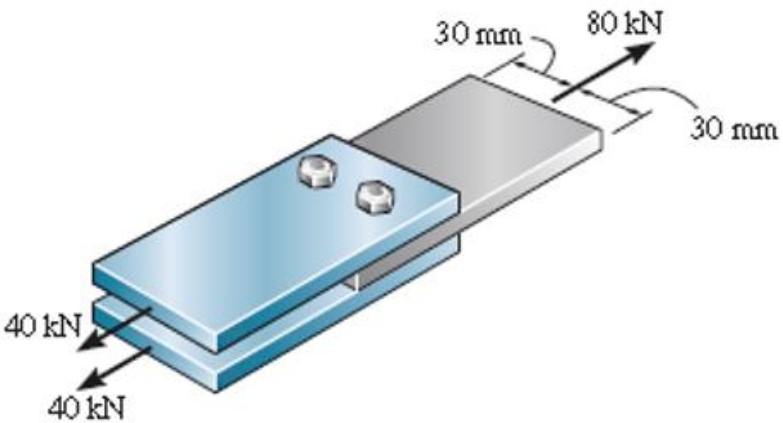
$$\tau_{all} = \frac{\tau_u}{FS} = \frac{2.5}{2.75} = 0.90909\ MPa$$

(double shear)

$$\tau = \frac{p}{2A} = \frac{16000}{2\ x125x((L-6)/2)} = 0.90909$$

L= 146.8 mm

The joint is fastened together using two bolts.
Determine the required diameter of allowable shear stress for the bolts is 110 MPa. Assume each bolt supports an equal portion of the load



Double shear with two bolts

$$\tau = \frac{p}{2nA_{bolt}}$$

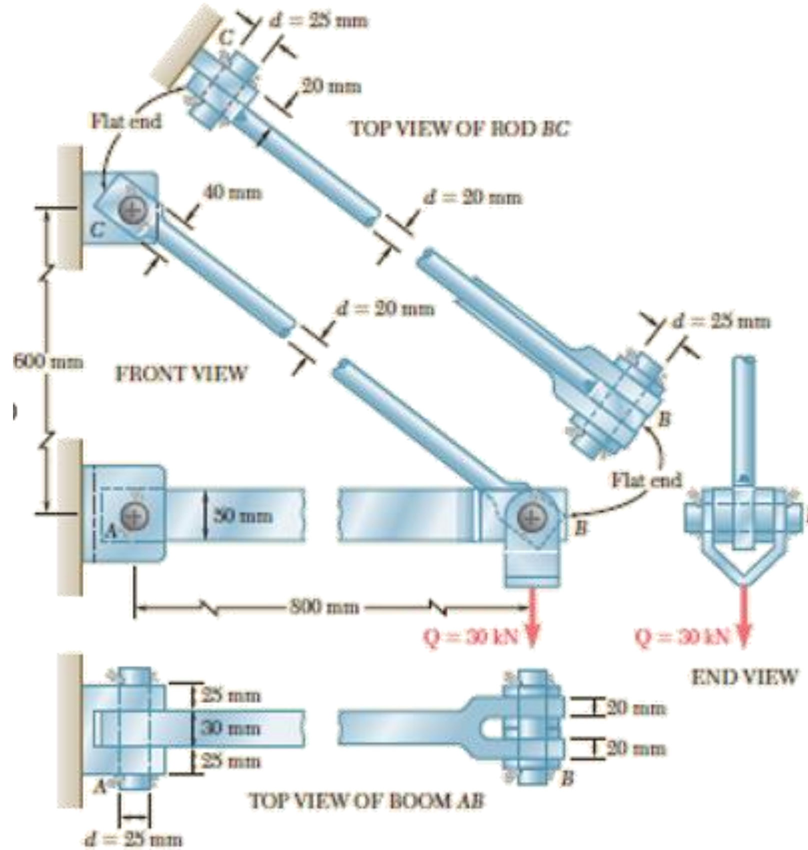
n= number of bolts at the section

$$\tau = \frac{80000}{2A * 2} = 110 \qquad A_{bolt} = 181.82\ mm^2$$

181.82=π d²/4 **d= 15.22 mm**

Application to the Analysis and Design of Simple Structures

Determine the normal stresses in the links and shearing, and bearing stresses in various connections
Force in BC = 50kN tension and in AB = -40kN (compression)



Normal stresses

$$\sigma_{AB} = \frac{p}{A_{AB}} = \frac{-40000}{50*30} = -26.7\ MPa$$

For members in tension : considers the critical section (net Area)

$$\sigma_{BC} = \frac{p}{A_{BC\,(Net)}} = \frac{50000}{(40-25)*20} = 166.7MPa$$

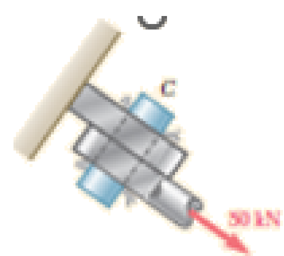
Shear and bearing stresses stresses

$$\tau_c = \frac{50000}{\pi(12.5)^2} = 102\ MPa \qquad \tau_A = \frac{-40000}{2\pi(12.5)^2} = -40.7\ MPa$$

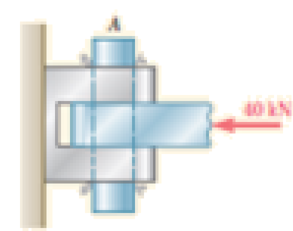
At A.

$$\sigma_{b(AB)} = \frac{p}{A_{AB}} = \frac{40000}{25*30} = 53.3\ MPa$$

$$\sigma_{b(bracket)} = \frac{p}{A_{AB}} = \frac{20000}{25*25} = 32\ MPa$$

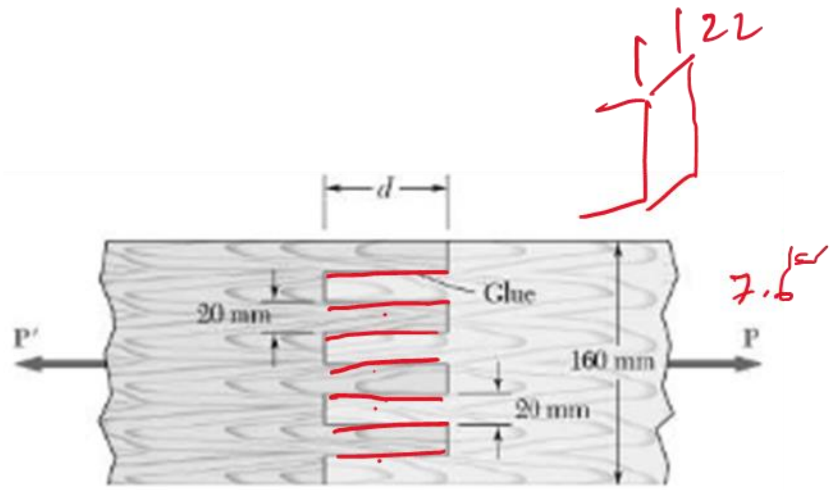


(a)



(a)

Two wooden planks, each 22 mm thick and 160 mm wide are joined by the glue. The joint will fail when the average shearing in the glue reaches 820 kPa, determine the smallest allowable length d of the cuts, if P= 7.6 kN.



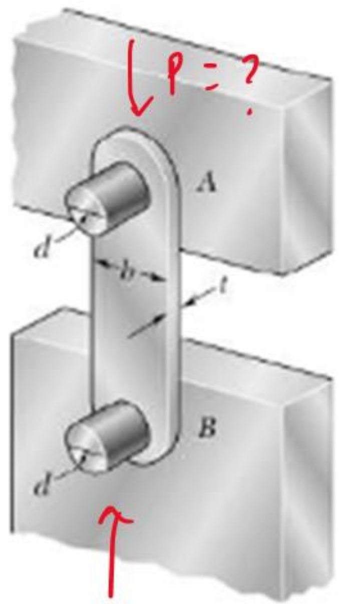
$$\tau = \frac{P}{A}$$

7 Glued Surfaces $\Rightarrow A = (7)(22\, d) = 154\, d \ mm^2$

$$\frac{7.6 \times 10^3}{154\, d} = \frac{820}{1000} \Rightarrow d = \underline{60.2\ mm}$$

For link AB : b=50 mm, t= 6mm. Knowing that the average normal stress in the link =-140 Mpa and that the average shearing stress in each pin is 80MPa. Determine:

a. Diameter d of the two pins
b. The average bearing stress in the link



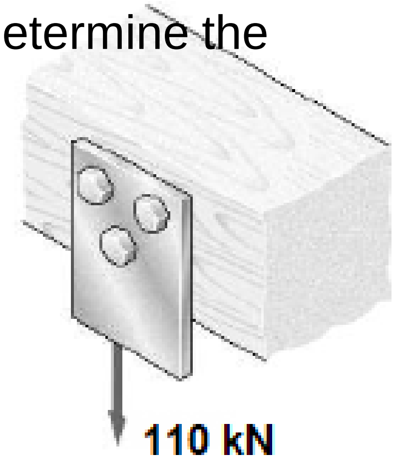
$$\sigma = \frac{P}{A} = 140 = \frac{P}{50 \times 6} \Rightarrow P = 42000\ N$$

$$\tau = \frac{P}{A_b} = \frac{42000}{A_b} = 80 \Rightarrow A_b = 525\ mm^2$$

$$A = \frac{\pi d^2}{4} = 525 \Rightarrow d = 25.85\ mm$$

$$\sigma_b = \frac{P}{A_b} = \frac{42000}{(25.85)(6)} = 270.8\ MPa$$

Three steel bolts are to be used to attach the steel plate shown to a wooden beam . . Ultimate shear stress = 360 MPa and factor of safety = 3.35 . Determine the diameter of the bolts



$$\tau_{all} = \frac{\tau_u}{F.S} = \frac{360}{3.35} = 107.46$$

$$\tau = \frac{P}{3A_b} = \frac{110000}{3A_b} = 107.46 \Rightarrow$$

$$A_b = 341.21\, m^2 = \frac{\pi d^2}{4} \Rightarrow$$

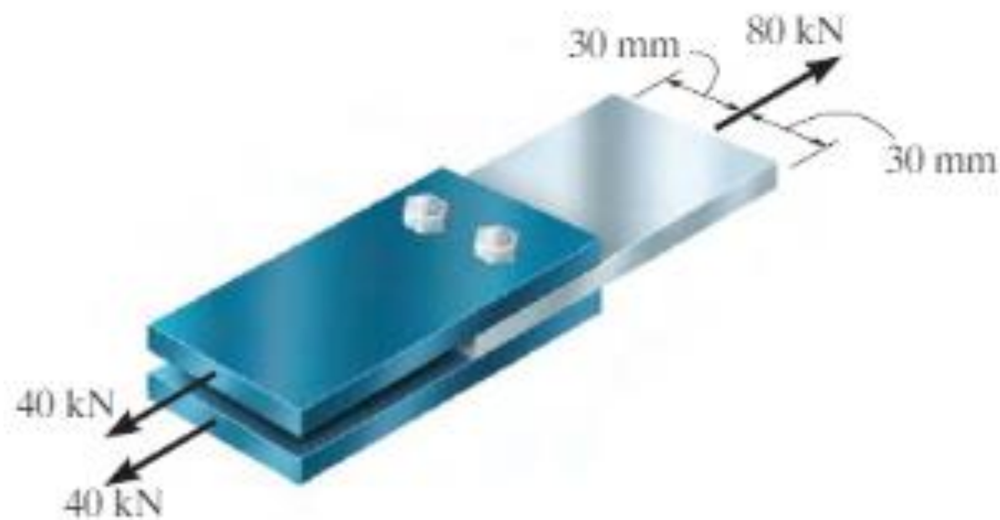
$$d = 20.8\ mm$$

The joint is fastened together using two bolts. Determine the required diameter of the bolts if the failure shear stress for the bolts is $\tau_{\text{fail}} = 350\ \text{MPa}$. Us $FS = 2.5$

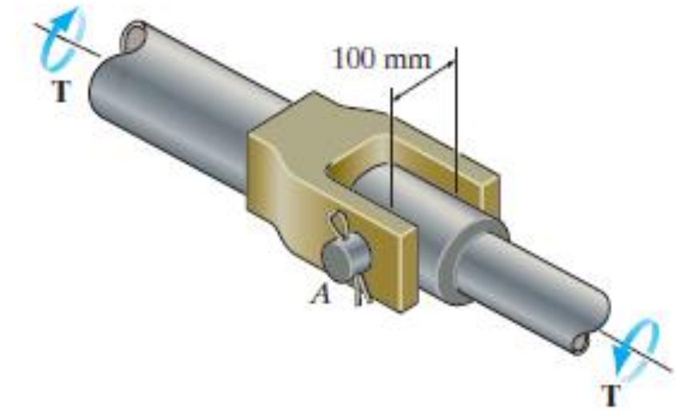
$$\tau_{\text{allow}} = 350/2.5 = 140\ \text{MPa}$$

$$\tau_{\text{allow}} = \frac{80000}{2x2x\pi r^2} = 140$$

r =6.75 mm **d = 2r =13.5 mm**



The joint is used to transmit a torque of T=3kN.m. Determine the required minimum diameter of the shear pin *A* if it is made from a material having a shear failure stress of 150 MPa. Apply a factor of safety of 3 against failure.

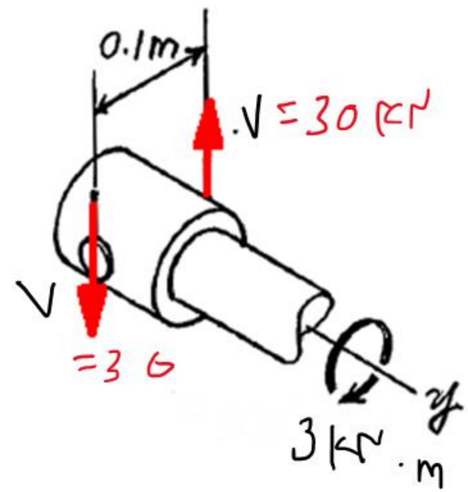


$$\Sigma M = 0 \Longrightarrow 0.1V = 3 \Longrightarrow V = 30\ \text{kN.m}$$

$$\tau_{all} = \frac{\tau_F}{F.S} = \frac{150}{3} = 50\ \text{MPa}$$

$$\tau_{all} = \frac{V}{A_{pin}} = \frac{30\,000}{\pi d^2/4} = 50$$

$$\underline{d = 27.65\ \text{mm}}$$



Design for Axial Loads and Direct Shear

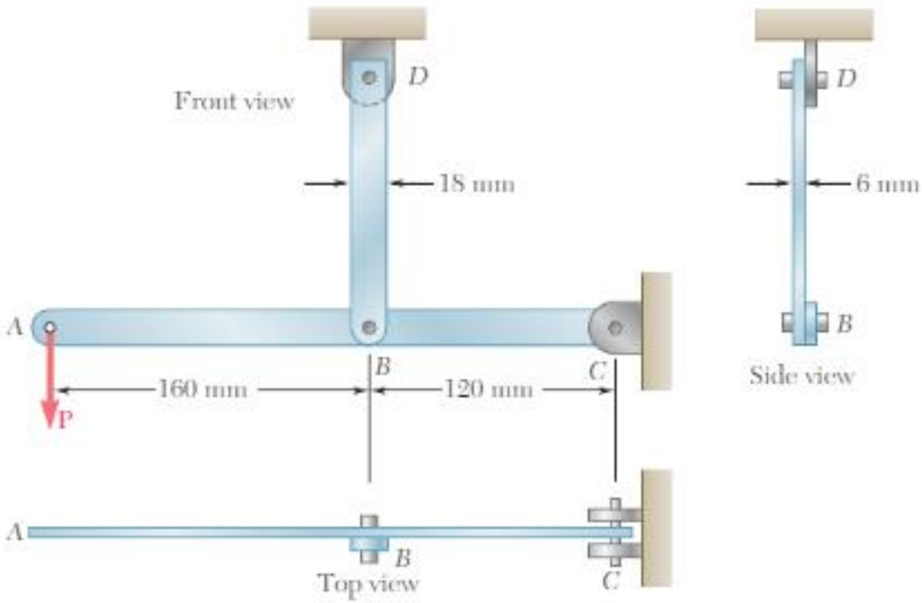
Analysis: Given the structure and loads, determine stresses and strains.
Design: Given the loads and allowable stresses, determine the properties of the structure.
Design for axial loads and direct shear entails finding the required area to carry the loads

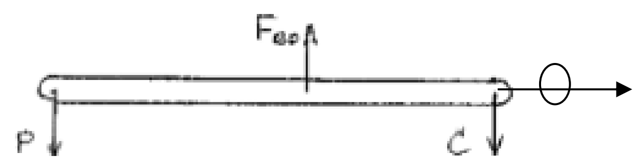
$$\text{Required area} = \frac{\text{Load to be transmitted}}{\text{Allowable stress}} \qquad \text{(i.e., Strength Consideration)}$$

Other design considerations include

- ***Stiffness***: Designing the structure to resist changes in shape.
- ***Stability***: Designing the structure to resist buckling under compressive loads.
- ***Optimization***: Designing the best structure to meet a particular goal.

In the steel structure shown, a **6 mm** diameter pin is used at C and a **10 mm diameter** pin is used at B and D. The ultimate shear stress = 150 MPa . The ultimate normal stress = 400 MPa in link BD. The factor of safety =3.
Determines the largest load P that can be applied at A.





$$\sum M_C = 0: \quad 0.280P - 0.120F_{BD} = 0$$

$$P = \frac{3}{7}F_{BD}$$

$$\sum M_B = 0: \quad 0.160P - 0.120C = 0$$

$$P = \frac{3}{4}C$$

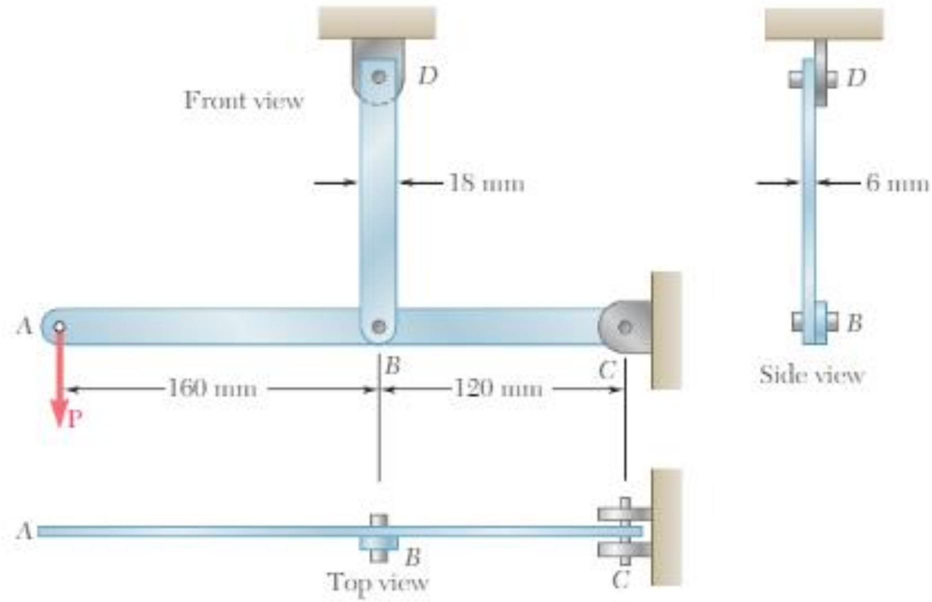
$$\sigma_{all} = \frac{\sigma_u}{FS} = \frac{400}{3} = 133.333MPa$$

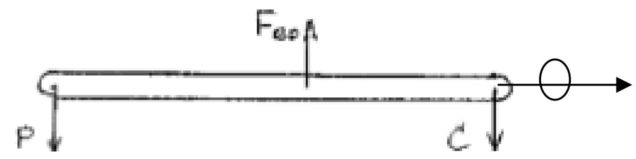
$$\frac{7P/3}{(18-10)*6} = 133.333MPa$$ **P= 2.74 N** **(member bd in tension – take net area)**

$$\tau_{Bor\ D} = \frac{7P/3}{\pi\,5^{\,2}} = \frac{150}{3}$$ $P = 1.683$ **N** **The smaller one controls**

$$\tau_C = \frac{4P/3}{2\pi\,3^{\,2}} = \frac{150}{3}$$ $P = 2.12$**N .... Double shear**

In the steel structure shown, a **6 mm** diameter pin is used at C and a **10 mm diameter** pin is used at B and D. The ultimate shear stress = 150 MPa . The ultimate normal stress = 400 MPa in link BD. The factor of safety =3. Determines the largest load P that can be applied at A.





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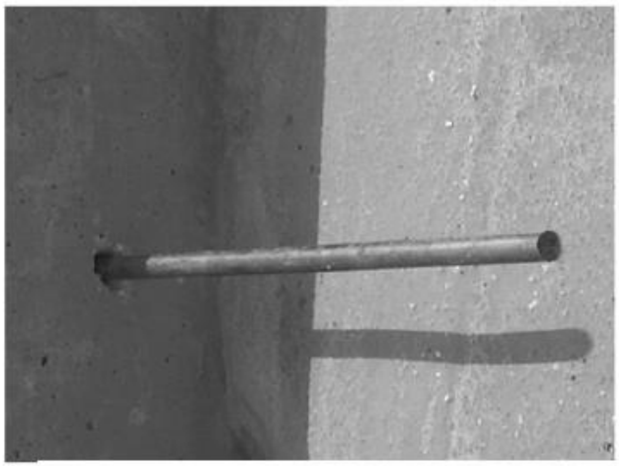
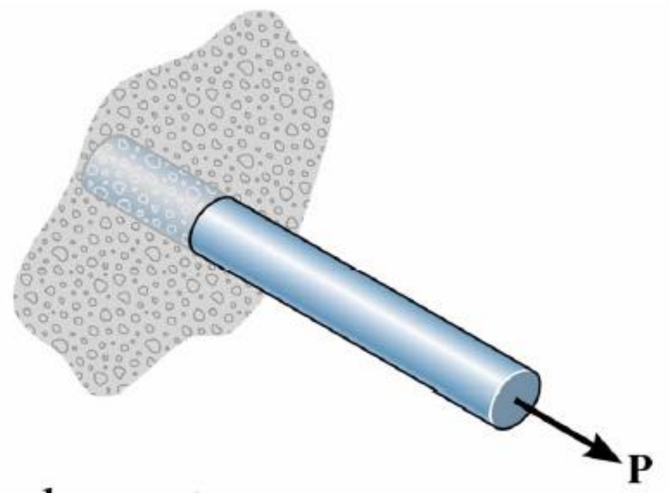
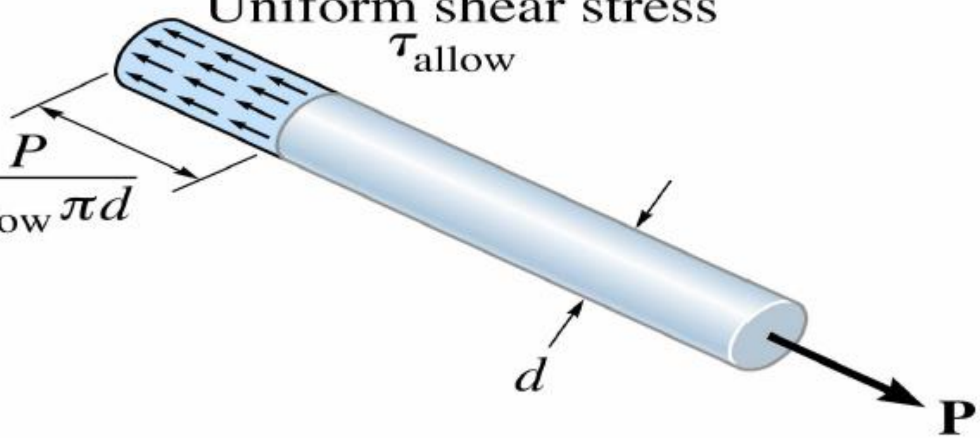
$$\tau_C = \frac{4P/3}{2\pi\, 3^2} = \frac{150}{3}$$ $P = 2.12$**N .... Double shear**

Required Area to resist shear caused by axial load







$$l = \frac{P}{\tau_{allow}\pi d}$$

Structural Mechanics

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Chapter 2 +Chapter 3

Strain

Mechanical properties of Materials

An important aspect of the analysis and design of structures relates to the *deformations* caused by the loads applied to a structure. It is important to avoid deformations so large that they may prevent the structure from fulfilling the purpose for which it was intended.

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Strain

2.1 Deformation

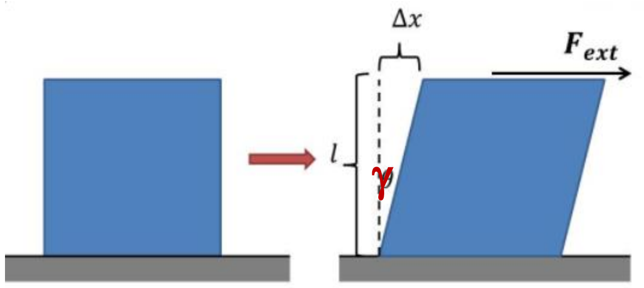
Whenever a force is applied to a body, it will tend to change the body's shape and size. These changes referred to as deformation

2.2 Strain

Strain is a *dimensionless quantity.*

Normal strain :$\varepsilon = \delta / L$

Shear strain :$\gamma = \Delta_x / l$ (radian)



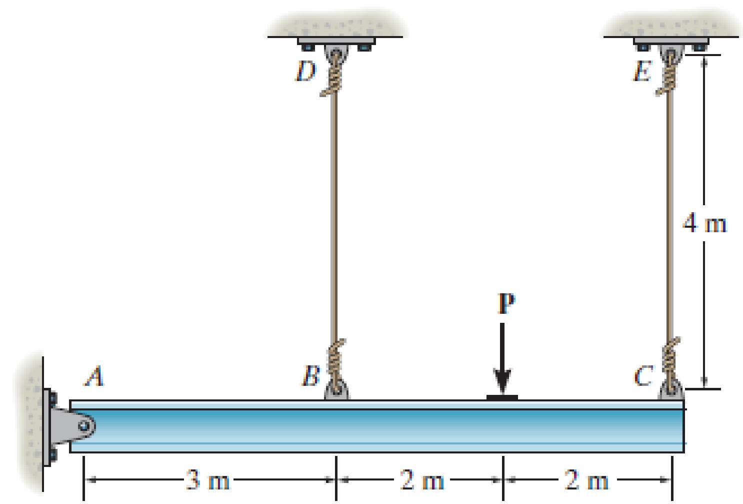
The rigid beam is supported by a pin at *A* and wires *BD* and *CE*. If the load **P** on the beam causes the end *C* to be displaced 10 mm downward, determine the normal strain developed in wires *CE* and *BD*.

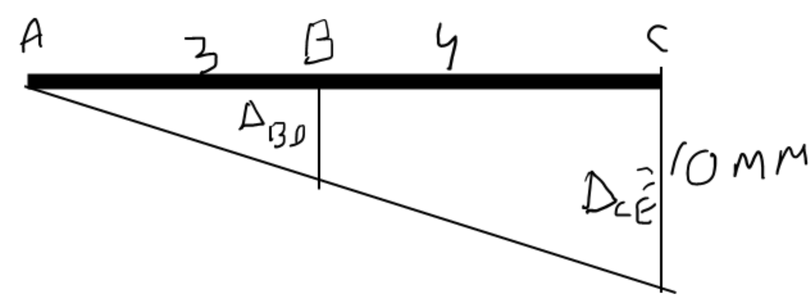
$$\frac{\Delta L_{BD}}{3} = \frac{\Delta L_{CE}}{7}$$

$$\Delta L_{BD} = \frac{3\,(10)}{7} = 4.286 \text{ mm}$$

$$\varepsilon_{CE} = \frac{\Delta L_{CE}}{L} = \frac{10}{4000} = 0.00250 \text{ mm/mm}$$

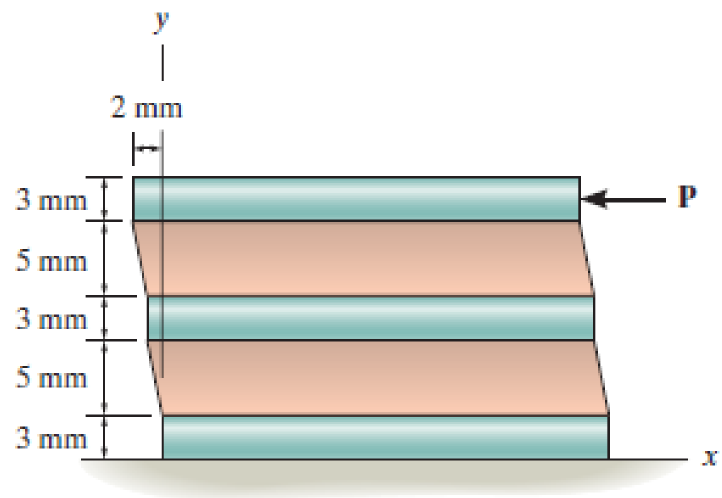
$$\varepsilon_{BD} = \frac{\Delta L_{BD}}{L} = \frac{4.286}{4000} = 0.00107 \text{ mm/mm}$$

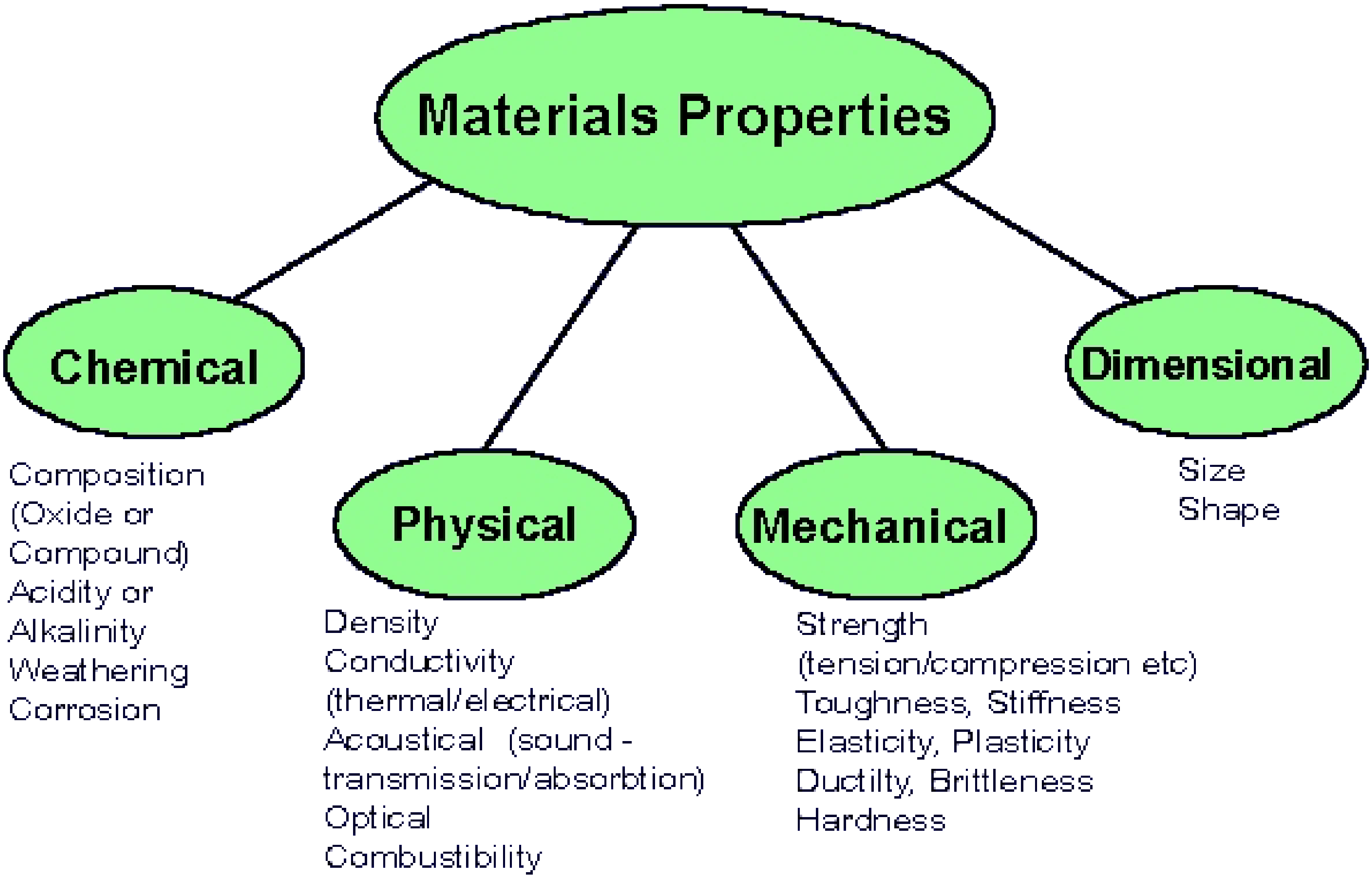




Nylon strips are fused to glass plates. When moderately heated the nylon will become soft while the glass stays approximately rigid. Determine the average shear strain in the nylon due to the load **P** when the assembly deforms as indicated.

$$\gamma = \tan^{-1}\left(\frac{2}{10}\right) = 11.31^\circ = 0.197 \text{ rad}$$

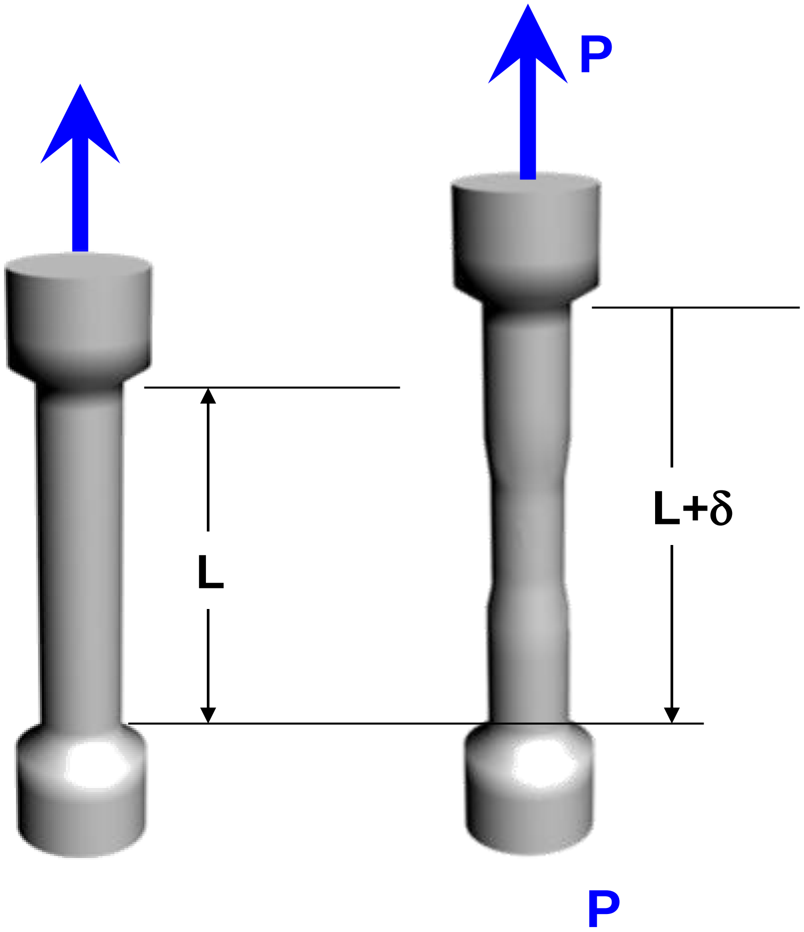




https://boeingconsult.com/tafe/structures/Stress-strain/stress-strain.html

3. Mechanical properties of materials

3.1 The tension and compression test



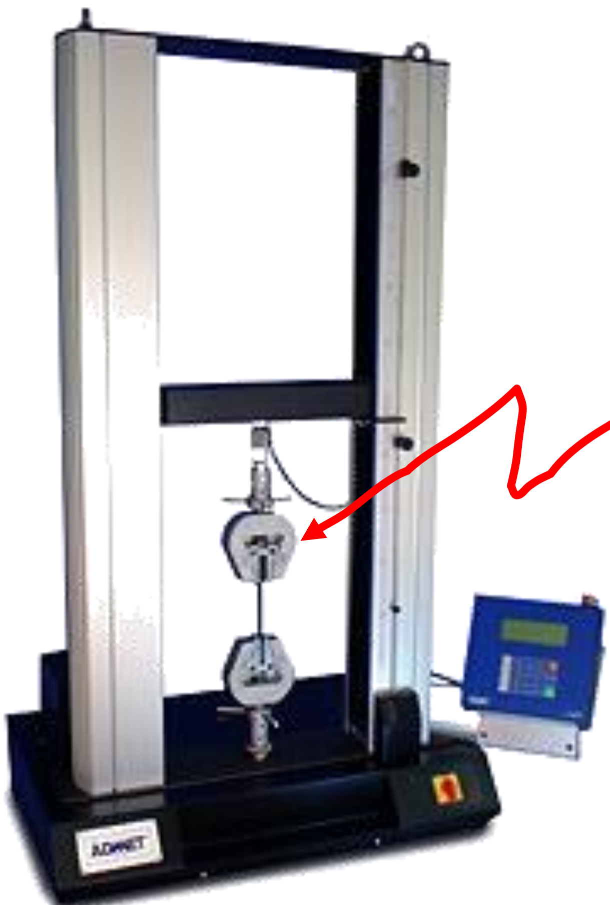
Strain: $\varepsilon = \delta / L$

Stress: $\sigma = P/A$

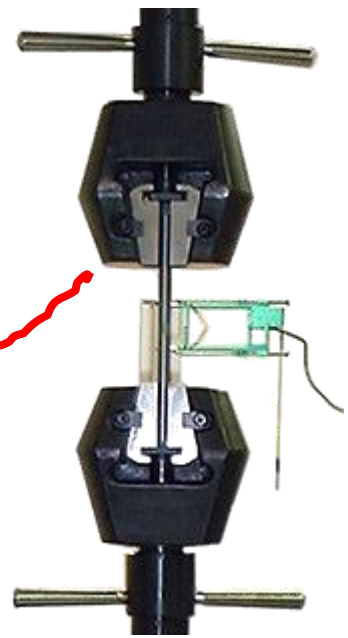
Plotting the stress against the strain results in a curve that is characteristic of the properties of the material but does not depend upon the dimensions of the specimen used. This curve is called ***a stress-strain diagram***.

Stress-Strain Diagram

The design of machines and structures so that they will function properly requires that we understand the *mechanical behavior of the materials* being used.



Tensile Testing Machine

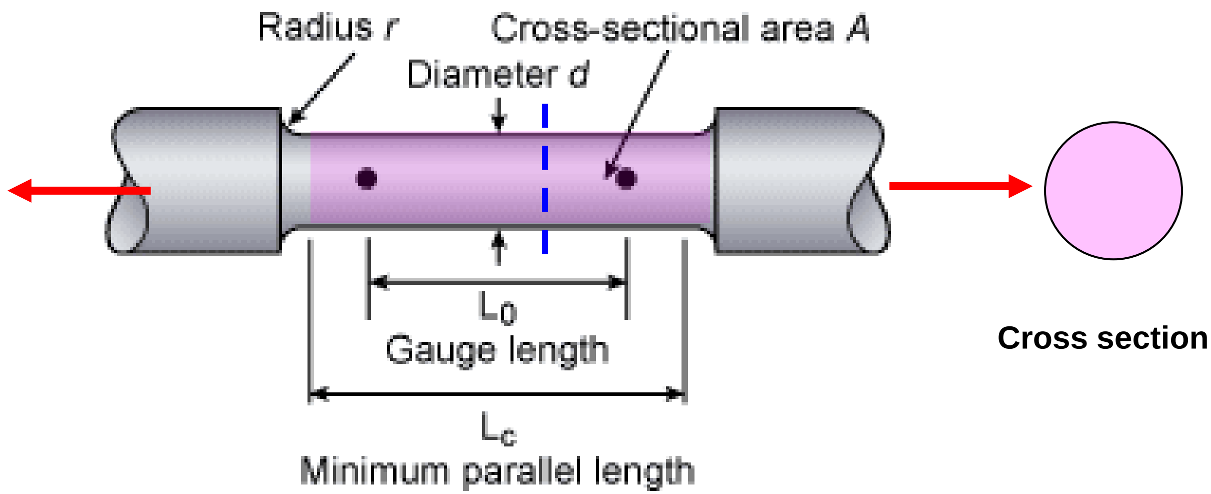


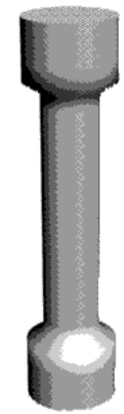
Gripping Devices

Tensile Testing

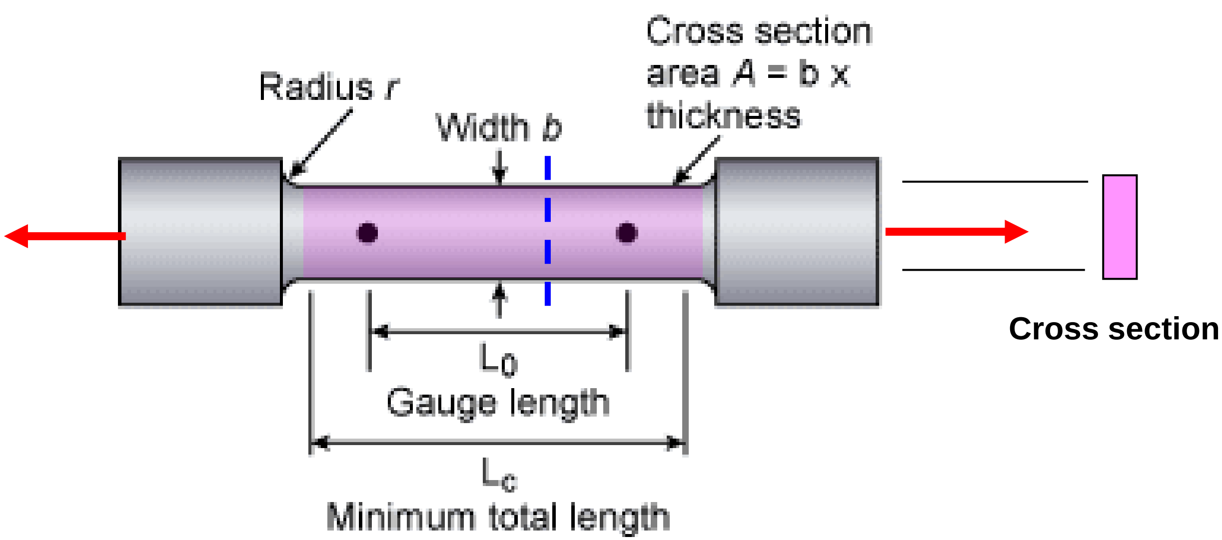
Tensile tests are carried out by gripping the ends of a suitably prepared standardised test piece in a tensile testing machine and then applying a continually increasing uni-axial load until such time as failure occurs

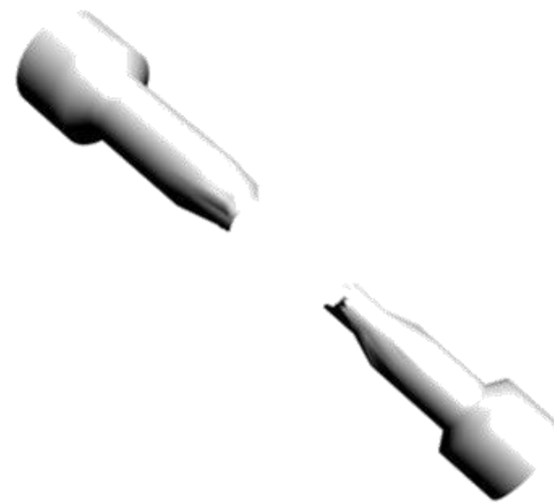
Tensile Test Specimen





(a) Round cross section





(b) Square cross section

3.2 Stress- Strain Diagram

σ = normal stress on a plane perpendicular to the longitudinal axis of the specimen

P = applied load

A = original cross sectional area

ε = normal strain in the longitudinal direction

δ = change in the specimen's gage length

L_o= original gage length

$$\sigma = \frac{P}{A} \qquad \varepsilon = \frac{\delta}{L}$$

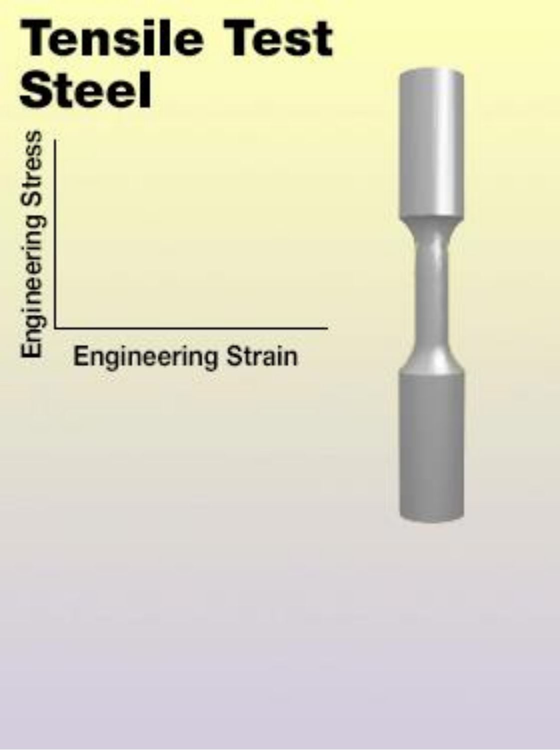
- **Engineering stress**

  $\sigma = P/A_0$
- **True stress**

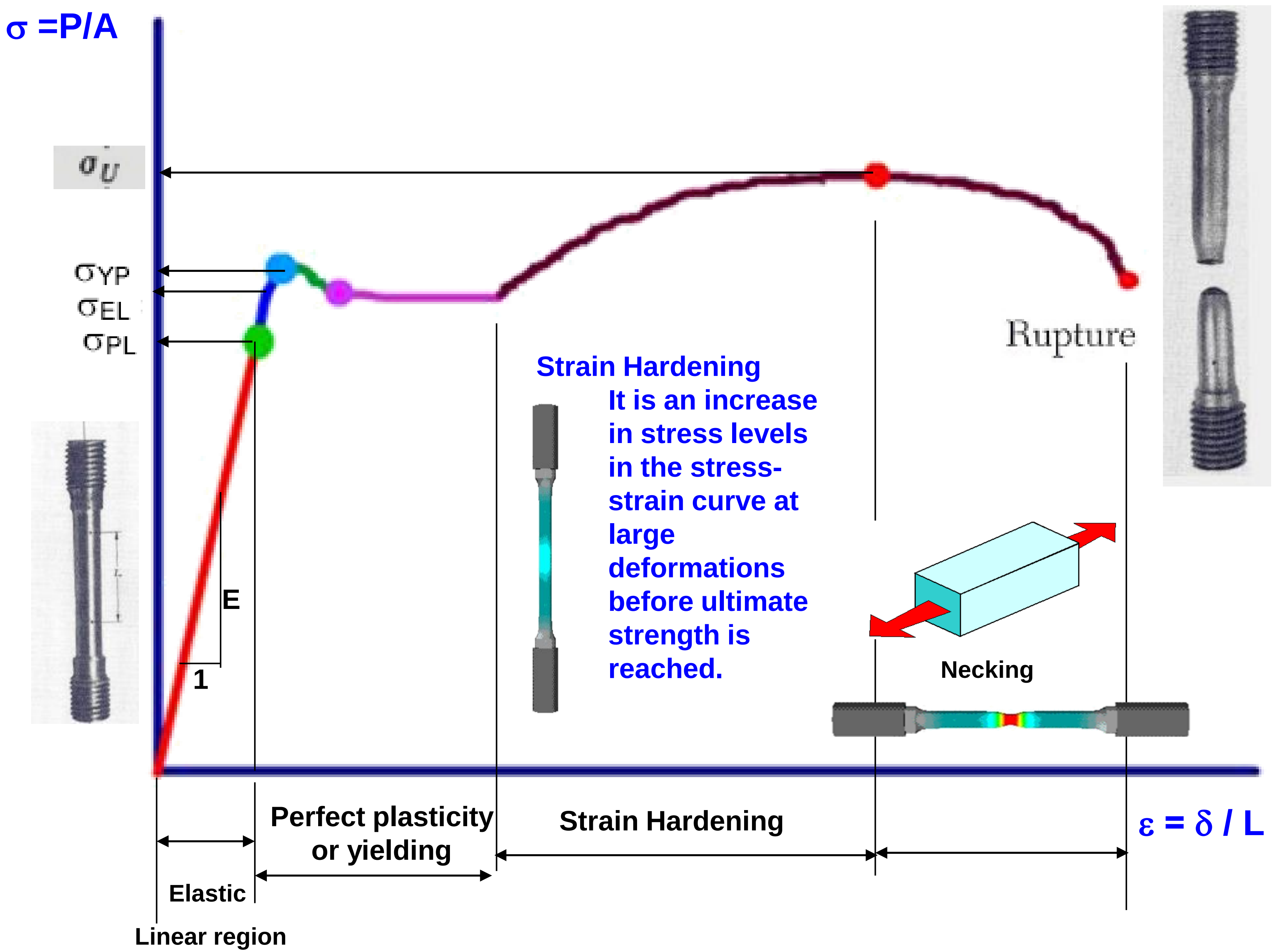
  $\sigma = P/A$
- **Engineering strain**

  $\varepsilon = (l - l_0)/l_0$
- **True strain (Logarithmic strain)**

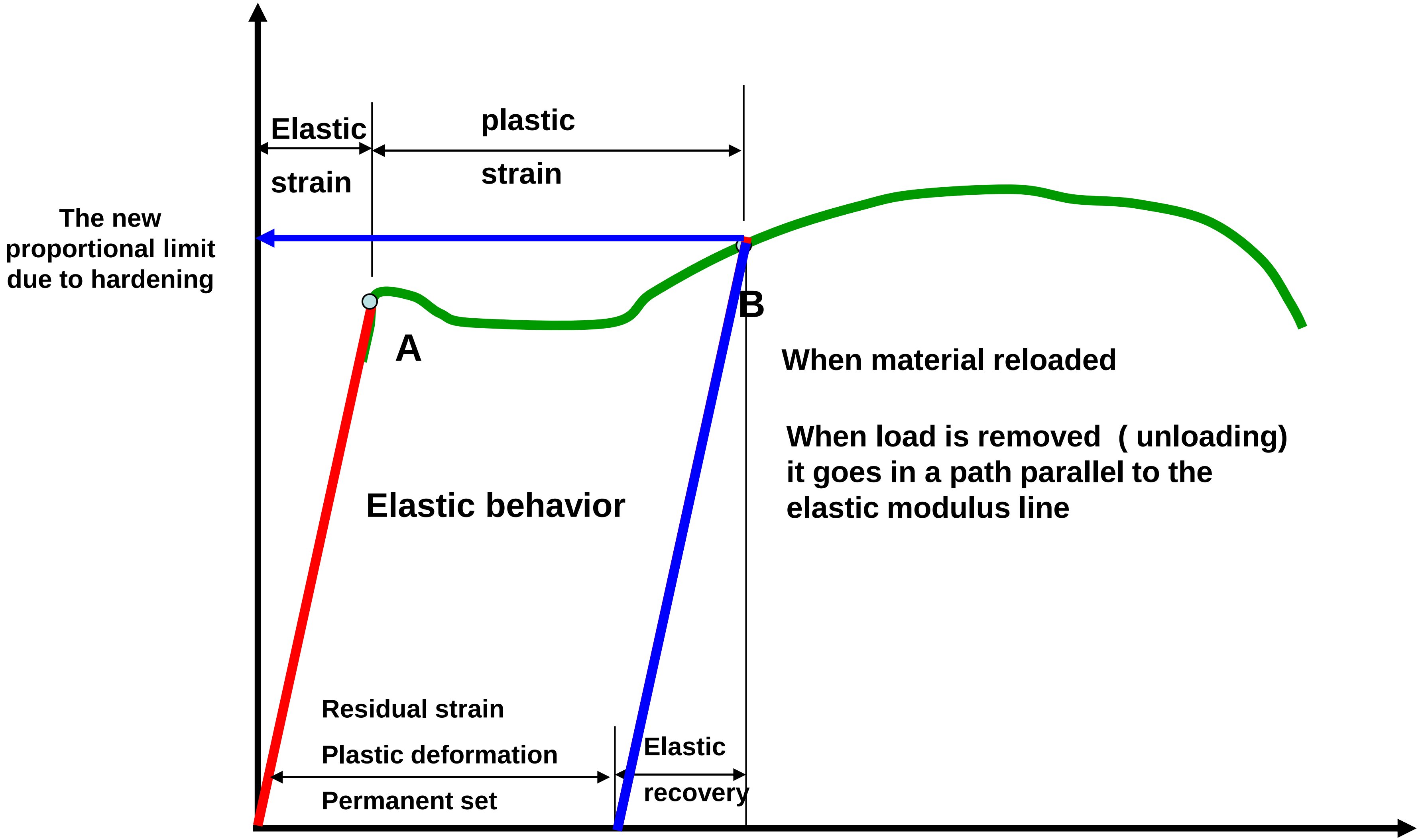
  $\varepsilon = \ln(l/l_0) = \ln(A/A_0)$
- **Volume must be the same** $Al = A_0 l_0$



stress-strain curve or diagram gives a direct indication of the material properties.



σ =P/A
σU
σYP
σEL
σPL
E
1
Strain Hardening
It is an increase in stress levels in the stress-strain curve at large deformations before ultimate strength is reached.
Rupture
Necking
Perfect plasticity or yielding
Strain Hardening
Elastic
Linear region
ε = δ / L



Linear relationship between stress and strain

Strain is temporary, meaning that all strain is fully recovered upon removal of the stress

The slope of this is called the elastic modulus

E . Modulus of Elasticity (Young's Modulus) - **Slope of the initial linear portion of the stress-strain diagram. The modulus of elasticity may also be characterized as the "stiffness" or ability of a material to resist deformation within the linear range.**

Proportional limit : **is the maximum value of the stress from the stress-strain diagram, where the stress and strain are proportional**

Elastic Limit **: is the maximum stress for a material to behave elastically, - the specimen will return to its original undeformed shape if the load is removed so long as the stress is below the elastic limit.**

Yield Point: **This defined as the maximum stress on stress-strain curve, where there is an appreciable increase in strain with no increase in stress. It is generally easier to determine than the proportional limit or elastic**

Some materials do not exhibit a distinct yield point

Yield Strength **:It is the stress which induces a specified permanent set. This is useful for materials which have no well defined yield point. The offset method is generally used to determine yield stress**

Tensile strength: **the maximum stress applied to the specimen.**

Failure stress: **the stress applied to the specimen at failure (usually less than the maximum tensile strength because necking reduces the cross-sectional area).**

Ductility

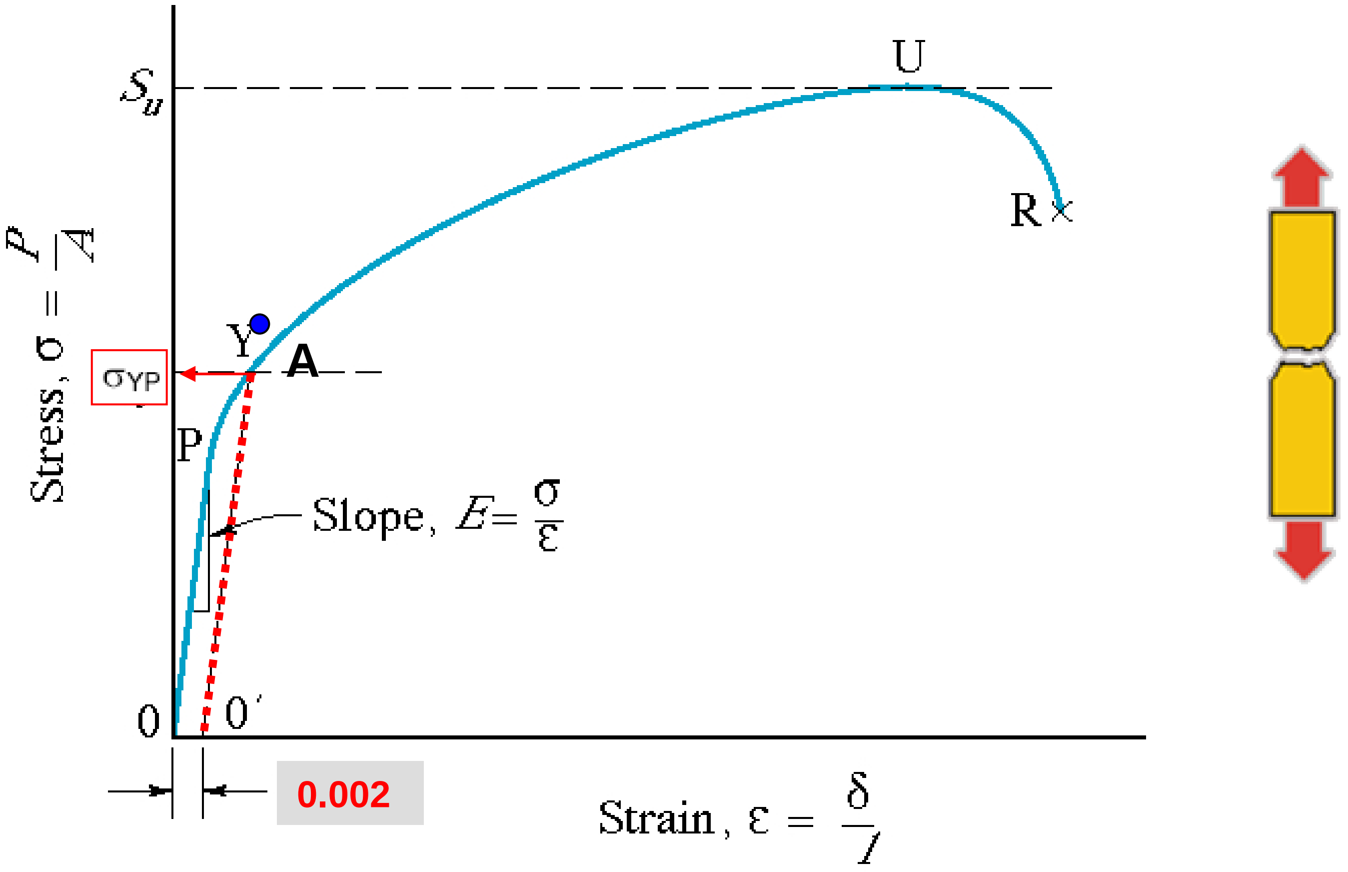
It is the ability of a material to deform plastically.
Two measurements of ductility:

Percent (%) elongation **of the member =** $(L_f - L_0) / (L_0) * 100.0$

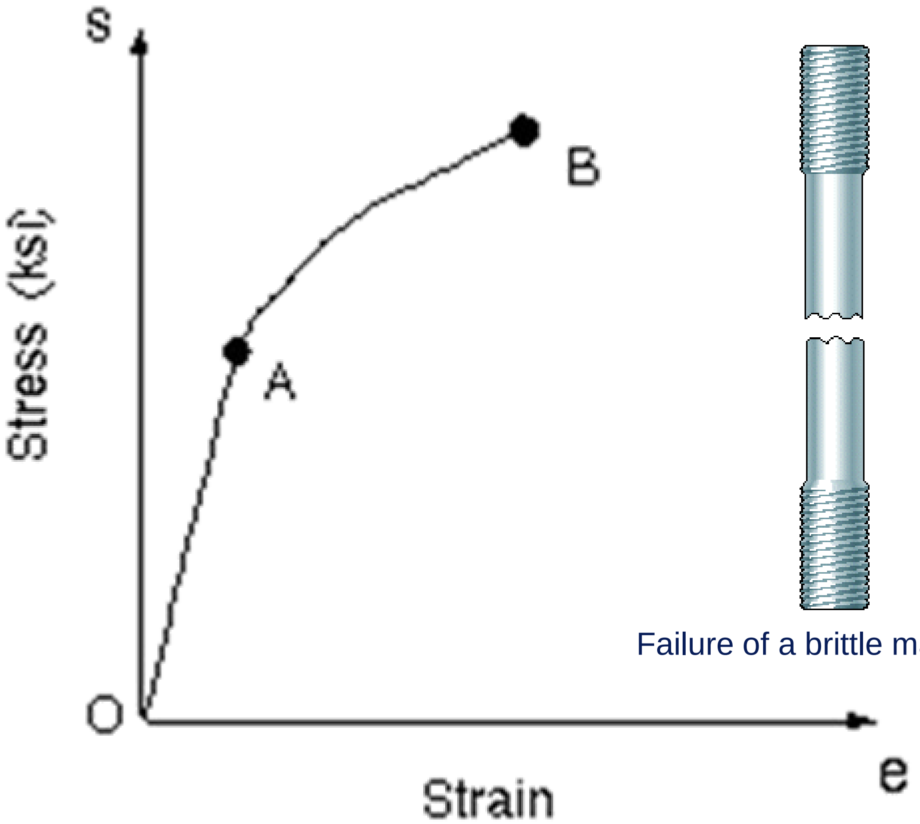
Percent (%) reduction in area at the location of fracture

% Area **=** $(A_0 - A_f) / (A_0) * 100.0$

3.3 the stress and strain diagram for ductile and brittle materials

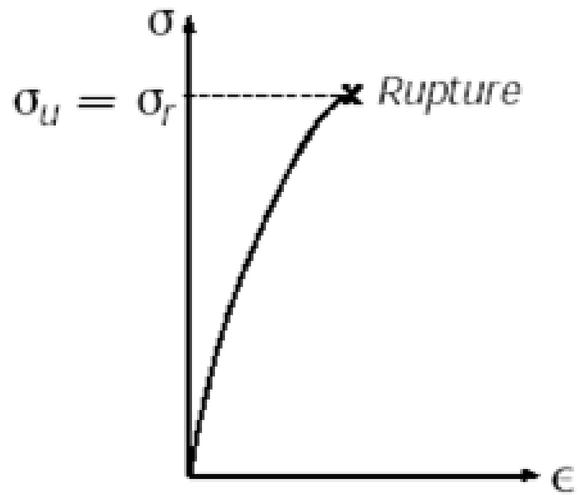


The yield point may be determined by the offset method . A line is drawn on the stress-strain diagram parallel to the initial linear part of the curve but is offset by some standard amount of strain, such as 0.002 or 0.2%. The intersection of the offset line and the stress-strain curve (point A in the figure) defines the yield stress.



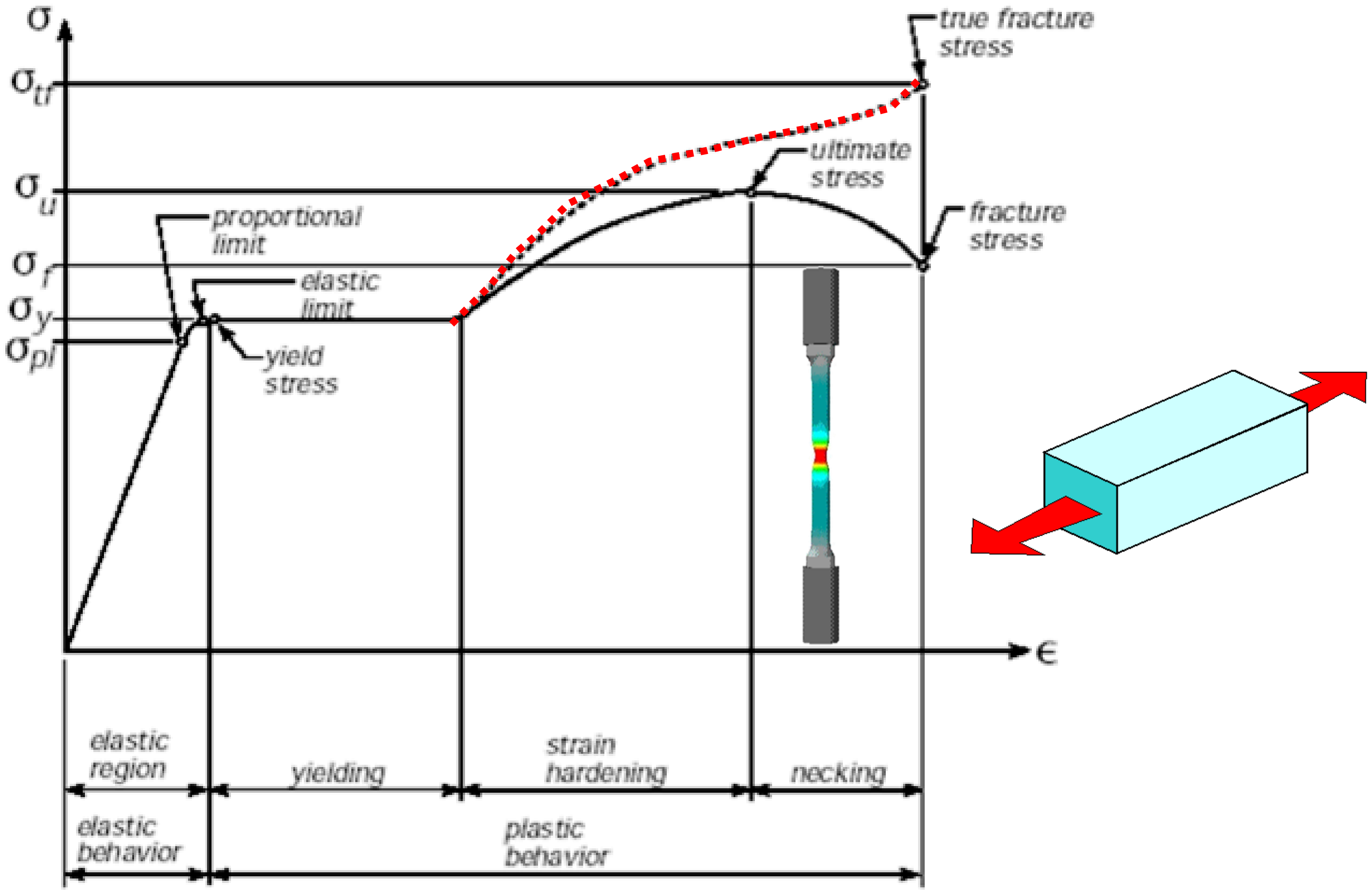
Failure of a brittle material

Ordinary glass is a nearly ideal brittle material, because it exhibits almost no ductility whatsoever



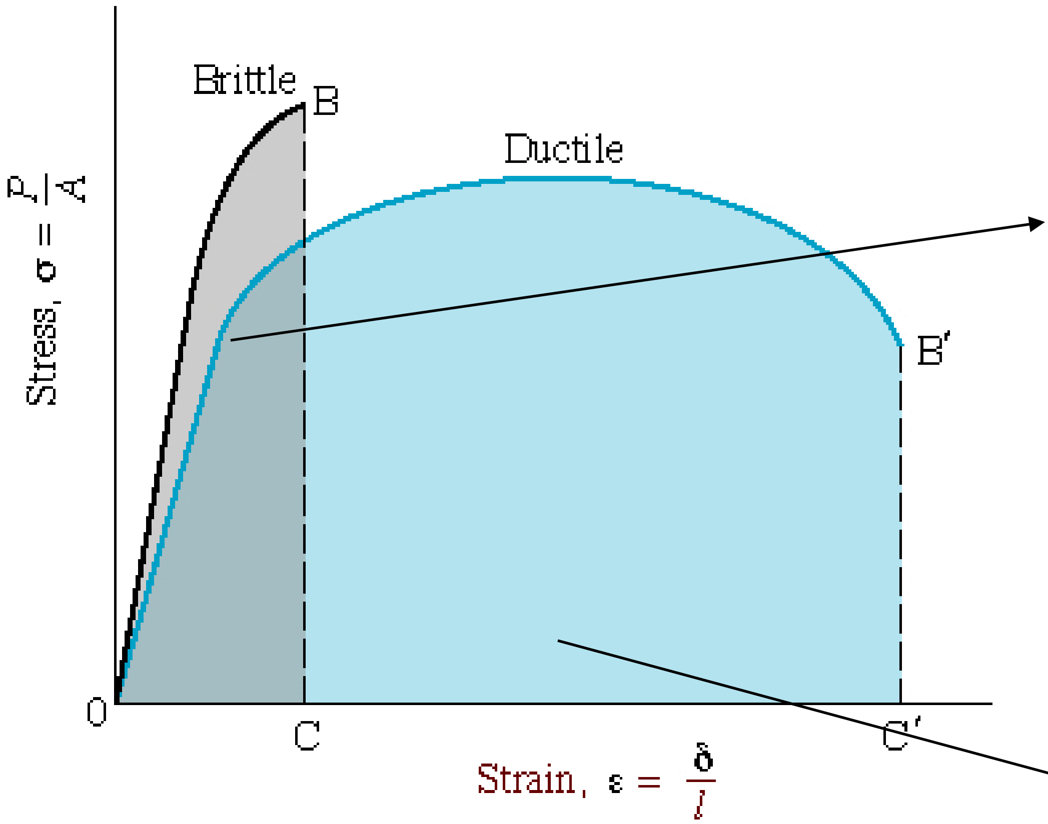
Stress –strain diagram for a brittle material

Materials that fail in tension at relatively low values of strain are classified as brittle materials. Examples are concrete, stone, cast iron, glass, ceramic materials, and many common metallic alloys. These materials fail with only little elongation after the proportional limit (point A) is exceeded, and the fracture stress (point B) is the same as the ultimate stress



- **The <u>true-stress vs. true-strain curve</u> is a plot of the stress in the sample at its minimum diameter, after necking has begun, vs the local elongation.**
- **This more accurately reflects the physical processes happening in the material, but is much more difficult to measure than the engineering stress and strain, which divide the applied load by the original cross-sectional area, and the total elongation by the original length.**

3.5 Strain energy

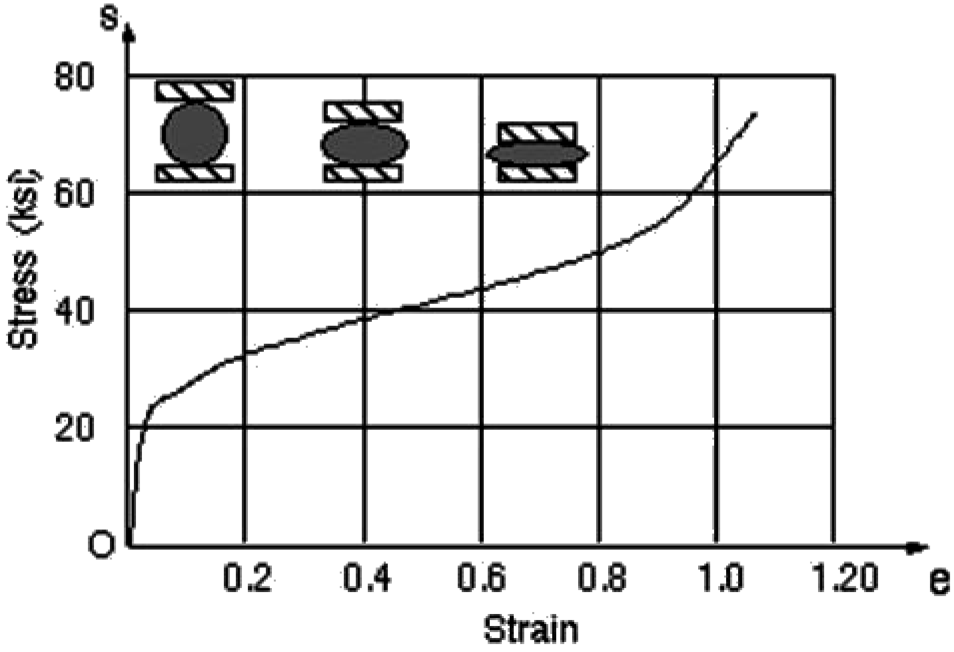


Modulus of resilience: **the area under the linear part of the curve, measuring the stored elastic energy**

Toughness: the total area under the curve, which measures the energy absorbed by the specimen in the process of breaking

tensile stress-strain diagrams for brittle and ductile metals loaded to fracture.

Compression Stress Strain Diagram



Compression stress-strain diagram for copper.

Stress-strain diagrams for compression have different shapes from those for tension. Ductile metals such as steel, aluminum, and copper have proportional limits in compression very close to those in tension, hence the initial regions of their compression stress-strain diagrams are very similar to the tension diagrams. When yielding begins, the behavior is quite different. In a tension test, the specimen is being stretched, necking may occur, and ultimately fracture takes place. When a small specimen of ductile material is compressed, it begins to bulge outward on the sides and become barrel shaped. With increasing load, the specimen is flattened out, thus offering increased resistance to further shortening (which means the stress-strain curve goes upward

Of brittle material (concrete) has different properties in tension and compression

Ductile Material – Materials that are **capable of undergoing large strains** (*at normal temperature*) before failure. An advantage of ductile materials is that visible distortions may occur if the loads before too large. Ductile materials are also **capable of absorbing large amounts of energy prior to failure**. Ductile materials include **mild steel, aluminum and some of its alloys, copper, magnesium, nickel, brass, bronze** and many others.

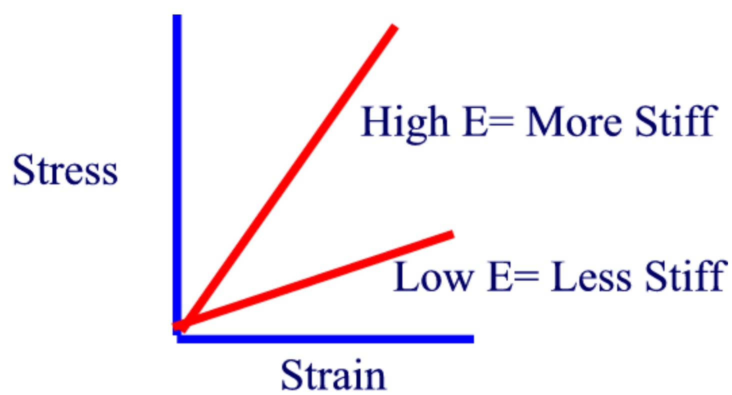
Brittle Material – Materials that exhibit **very little inelastic deformation**. In other words, materials that fail in tension at relatively low values of strain are considered brittle. Brittle materials include **concrete, stone, cast iron, glass and plaster.**

3.4 Hooke's Law; Modulus of Elasticity or Young's modulus (E)

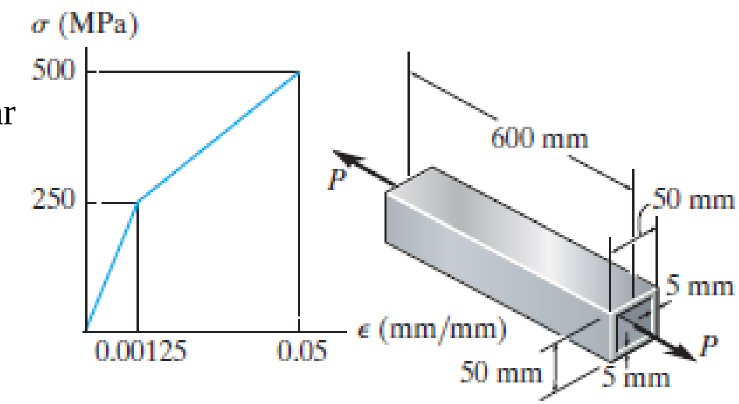
Most engineering structures are designed to undergo relatively small deformations, involving only the straight-line portion of the corresponding stress-strain diagram

Hooke's law $\sigma = E\varepsilon$

Units of E is pascals or one of its multiples



Determine the elongation of the square hollow bar when it is subjected to the axial force= 100 k. If this axial force is increased to 360 kN and released, find he permanent elongation of the bar. The bar is made of a metal alloy having a stress–strain diagram which can be approximated as shown.



Normal Stress and Strain: The cross-sectional area of the hollow bar is $A = 0.05^2 - 0.04^2 = 0.9(10^{-3})\text{m}^2$. When $P = 100\text{kN}$,

$$\sigma_1 = \frac{P}{A} = \frac{100(10^3)}{0.9(10^{-3})} = 111.11 \text{ MPa}$$

From the stress–strain diagram shown in Fig. *a*, the slope of the straight line *OA* which represents the modulus of elasticity of the metal alloy is

$$E = \frac{250(10^6) - 0}{0.00125 - 0} = 200 \text{ GPa}$$

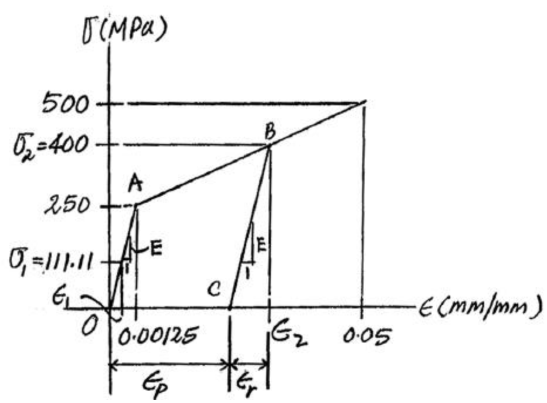
Since $\sigma_1 < 250$ MPa, Hooke's Law can be applied. Thus

$$\sigma_1 = E\varepsilon_1; \; 111.11(10^6) = 200(10^9)\varepsilon_1$$

$$\varepsilon_1 = 0.5556(10^{-3}) \text{ mm/mm}$$

Thus, the elongation of the bar is

$$\delta_1 = \varepsilon_1 L = 0.5556(10^{-3})(600) = 0.333 \text{ mm}$$



When $P = 360$ kN,

$$\sigma_2 = \frac{P}{A} = \frac{360(10^3)}{0.9(10^{-3})} = 400 \text{ MPa}$$

From the geometry of the stress–strain diagram

$$\frac{\varepsilon_2 - 0.00125}{400 - 250} = \frac{0.05 - 0.00125}{500 - 250} \qquad \varepsilon_2 = 0.0305 \text{ mm/mm}$$

When $P = 360$ kN is removed, the strain recovers linearly along line *BC*, Fig. *a*, parallel to *OA*. Thus, the elastic recovery of strain is given by

$$\sigma_2 = E\varepsilon_r; \qquad 400(10^6) = 200(10^9)\varepsilon_r$$

$$\varepsilon_r = 0.002 \text{ mm/mm}$$

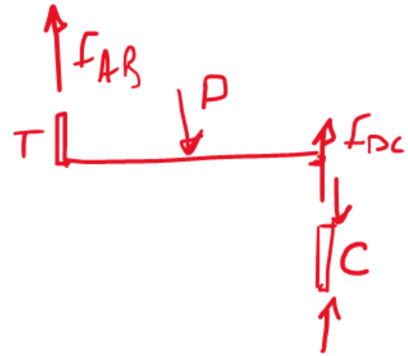
The permanent set is

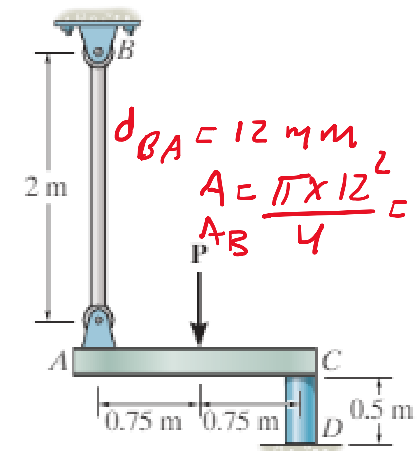
$$\varepsilon_P = \varepsilon_2 - \varepsilon_r = 0.0305 - 0.002 = 0.0285 \text{ mm/mm}$$

Thus, the permanent elongation of the bar is

$$\delta_P = \varepsilon_P L = 0.0285(600) = 17.1 \text{ mm}$$

The stress strain diagram for a polyester resin is shown. If the rigid beam Ac is supported by a strut AB and post CD made from the material, determine the largest load P that can be applied to the beam before it **ruptures.** The diameter of the strut is 12 mm and the diameter of the post is 40 mm.





$d_{BA} = 12\ mm$

$A_{AB} = \dfrac{\pi \times 12^2}{4} =$

$d_{CD} = 40mm$

$A_{CD} = \dfrac{40^2 \pi}{4}$

$F_{AB} = P/2$ Tension

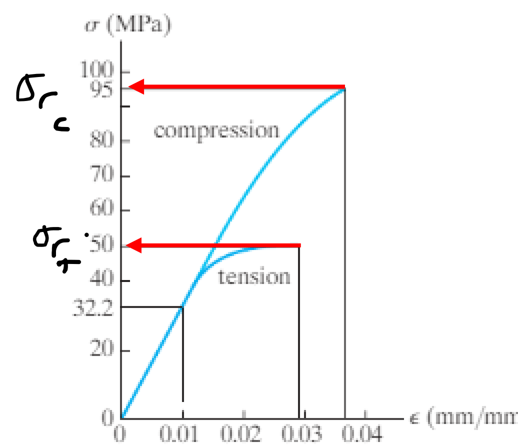
$F_{CD} = P/2$ Compression

$\sigma_{rup_{(T)}} = 50\ MPa \quad , \quad \sigma_{rup_{(C)}} = 95\ MPa$

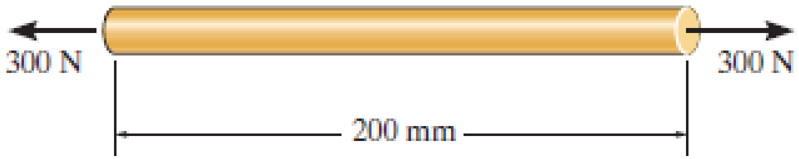
$$\sigma_{rup_{AB}} = \frac{P/2}{\frac{\pi}{4} \times 12^2} = 50 \Rightarrow \underline{\underline{P = 11300\ N}}$$

$$\sigma_{rup_{CD}} = \frac{P/2}{\frac{\pi \times 40^2}{4}} = 95 \Rightarrow P = 239000\ N$$

largest $\underline{\underline{P = 11.3\ kN}}$



The acrylic plastic rod is 200 mm long and 15 mm in diameter. If an axial load of 300 N is applied to it, determine the change in its length and the change in its diameter. $E_p = 2.70$ GPa, $\upsilon_p = 0.4$.



$$\sigma = \frac{P}{A} = \frac{300}{\frac{\pi}{4}(0.015)^2} = 1.697 \text{ MPa}$$

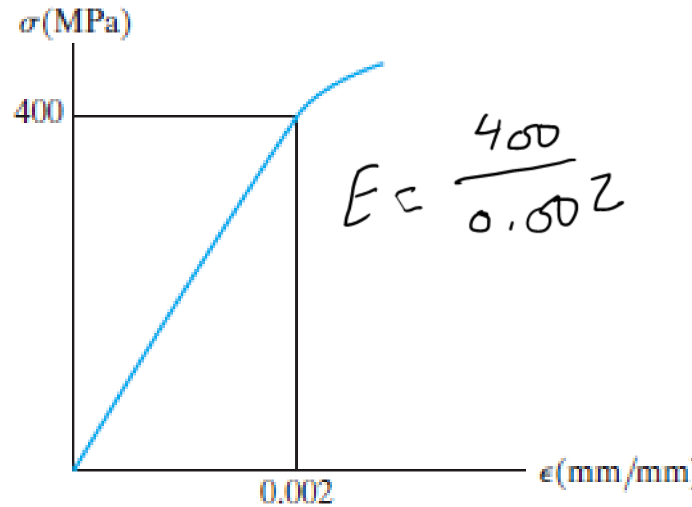
$$\varepsilon_{\text{long}} = \frac{\sigma}{E} = \frac{1.697(10^6)}{2.70(10^9)} = 0.0006288$$

$$\delta = \varepsilon_{\text{long}} L = 0.0006288\,(200) = 0.126 \text{ mm}$$

$$\varepsilon_{\text{lat}} = -V\varepsilon_{\text{long}} = -0.4(0.0006288) = -0.0002515$$

$$\Delta d = \varepsilon_{\text{lat}} d = -0.0002515\,(15) = -0.00377 \text{ mm}$$

The elastic portion of the stress–strain diagram for a steel alloy is shown. The specimen from which it was obtained had an original diameter of 13 mm and a gauge length of 50 mm. When the applied load on the specimen is 50 kN, the diameter is 12.99265 mm. Determine Poisson's ratio for the material.



$$E = \frac{400}{0.002}$$

$$\nu = \frac{\epsilon'}{\epsilon_a}, \quad \epsilon_a = \frac{\sigma}{E}, \quad \epsilon' = \frac{\Delta d}{d_o}$$

$$E = \frac{400}{0.002} = 200{,}000 \text{ MPa} = 200 \text{ GPa}$$

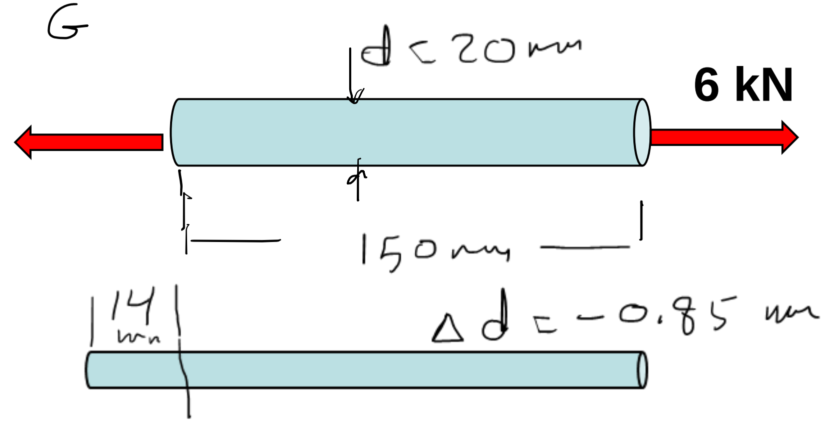
$$\sigma = \frac{P}{A_o} = \frac{50000 \times 4}{\pi \times 13^2} = 376.7 \text{ MPa}$$

$$\epsilon_a = \frac{\sigma}{E} = \frac{376.7}{200{,}000} = 1.8835 \times 10^{-3}$$

$$\epsilon' = \frac{\Delta d}{d} = \frac{13 - 12.99265}{13} = 0.56538 \times 10^{-3}$$

$$\nu = \frac{0.56538 \times 10^{-3}}{1.8835 \times 10^{-3}} = \underline{\underline{0.3}}$$

A 20 mm diameter rod made of plastic is subjected to tensile force = 6 kN. Knowing that an elongation of 14 mm and decrease in diameter of 0.85 mm are observed in a 150 mm length , determine the modulus of elasticity, the modulus of rigidity and poisons ratio for the material. ν E G



$$\sigma_a = E\epsilon_a \Rightarrow \frac{P}{A} = E\,\frac{\delta L}{L}$$

$$\frac{6000}{\pi\, 10^2} = E\,\frac{14}{150} \Rightarrow E = 204.6\ MPa$$

$$\nu = \frac{\epsilon'}{\epsilon_a} = \frac{\Delta d / d}{\delta L / L} = \frac{0.85/20}{14/150} = 0.455$$

$$G = \frac{E}{2(1+\nu)} = \frac{204.6}{2(1+0.455)} = 70.3\ MPa$$

A nylon thread is subjected to a <u>8.5 N tension force</u>. Knowing that <u>E= 3.3 GPa</u> and that the length of the thread <u>increases by 1.1%,</u> determine the **diameter of the thread (d)**

$$\sigma = E\epsilon = 3.3 \times 1000 \times \frac{1.1}{100} = \mathbf{36.3\,MPa}$$

$$\sigma = \frac{\mathbf{P}}{\mathbf{A}} = \mathbf{36.3} = \frac{\mathbf{8.5}}{\mathbf{A}} \qquad \mathbf{36.3} = \frac{\mathbf{8.5}}{\pi\frac{d^2}{4}} \qquad \mathbf{d = 0.546\ mm}$$

A control rod made of yellow brass must not <u>stretch more than 3 mm</u> when the tension in the wire <u>is 4 kN.</u> Knowing that <u>E = 105 GPa</u> and that the maximum <u>allowable normal stress is 180 MPa</u>,

Determine (*a*) the smallest diameter rod that should be used, (*b*) the corresponding maximum length of the rod.

$$\sigma = \frac{\mathbf{P}}{\mathbf{A}} = \mathbf{180} = \frac{\mathbf{4000}}{\pi\frac{\mathbf{d^2}}{\mathbf{4}}} \qquad \boldsymbol{d = 5.32\ mm}$$

$$\sigma = E\epsilon = 105 \times 1000\ \epsilon = 180 \qquad \epsilon = 1.714\text{x}10^{-3} = \delta/L$$

$3/L = 1.714\text{x}10^{-3}$ L= 1.75 m

The 4-mm-diameter cable *BC* is made of a steel with ***E* = 200 GPa.** Knowing that the maximum **stress in the cable must not exceed 190 MPa** and that the elongation of the cable must not exceed 6 mm, find the maximum load **P** that can be applied as shown.

$$\circlearrowleft + \sum M_A = 0 \Rightarrow 3.5P - \frac{2F_{BC}}{3.61} = 0 \Rightarrow$$

$$F_{BC} = 6.318\,P$$

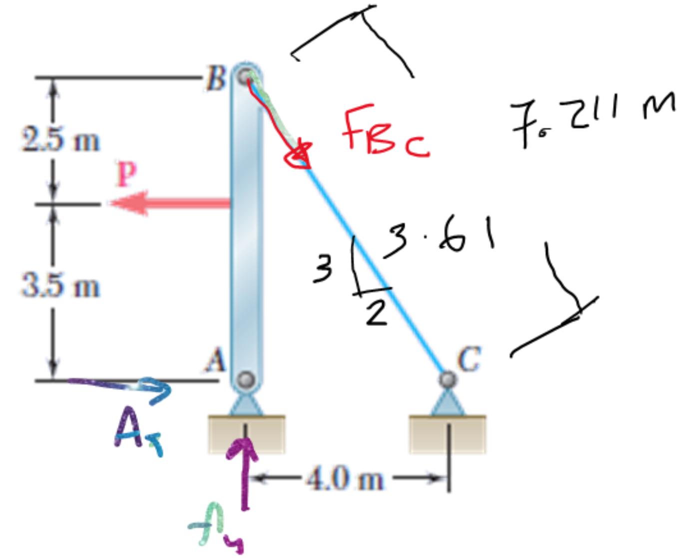
$$\sigma = \frac{P}{A} = 190 = \frac{6.318P}{\pi 4^2/4}$$

$$\Rightarrow P = 377.71\ \text{N}$$

$$\delta = \frac{PL}{EA} = 6\ \text{mm}$$

$$6 = \frac{(6.318P)(7.211\times10^3)}{(200\times10^3)\,\pi(2^2)} \Longrightarrow \underline{P = 330.8\ \text{N}}$$

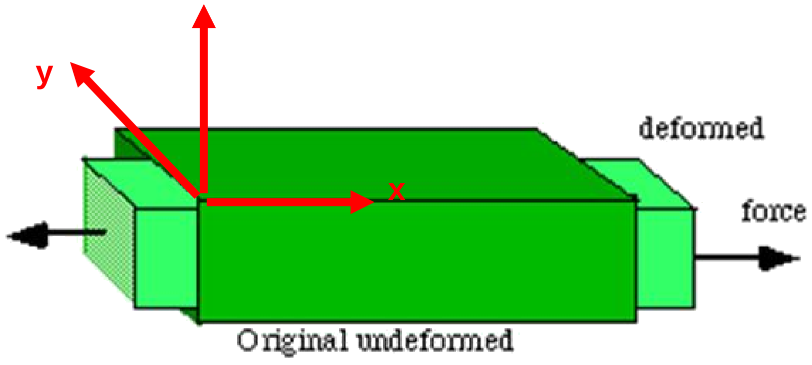
Take smaller

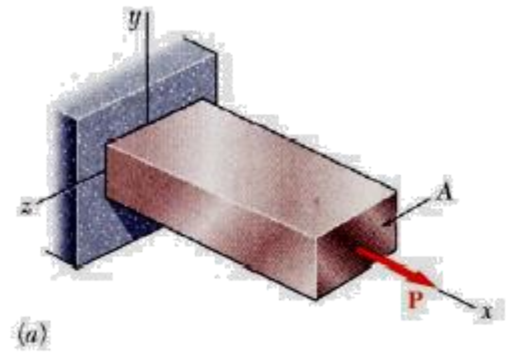


3.6 POISSON'S RATIO

Poisson's ratio is defined as

$$\nu = \left|\frac{\text{lateral strain}}{\text{axial strain}}\right| = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}$$





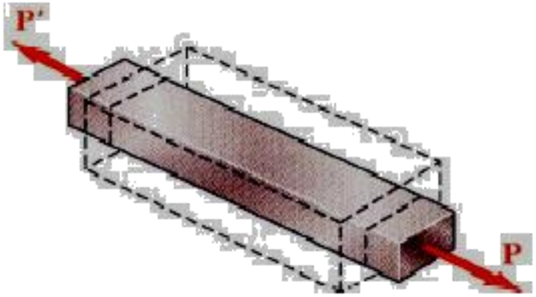
ε' (lateral strain) = - ν ε

$$\epsilon_x = \frac{\sigma_x}{E} \qquad \epsilon_y = \epsilon_z = -\frac{\nu\sigma_x}{E}$$

Isotropic – Isotropic materials have elastic properties that are independent of direction. Most common structural materials are isotropic.

Anisotropic – Materials whose properties depend upon direction. An important class of anisotropic materials is fiber-reinforced composites.

Homogeneous – A material is homogeneous if it has the same composition at every point in the body. A homogeneous material may or may not be isotropic.



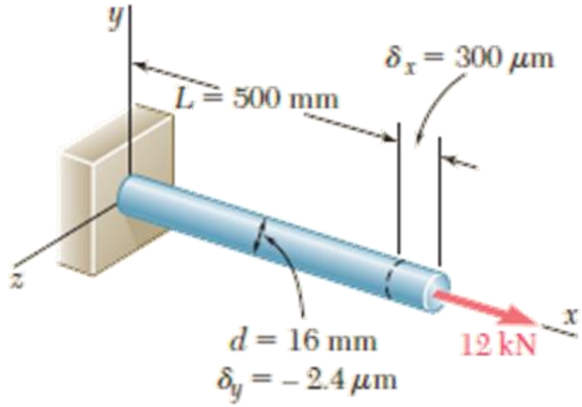


Fig. 2.31 Axially loaded rod.

A 500-mm-long, 16-mm-diameter rod made of a homogenous, isotropic material is observed to increase in length by 300 μm, and to decrease in diameter by 2.4 μm when subjected to an axial 12-kN load. Determine the modulus of elasticity and Poisson's ratio of the material.

The cross-sectional area of the rod is

$$A = \pi r^2 = \pi(8 \times 10^{-3}\,\text{m})^2 = 201 \times 10^{-6}\,\text{m}^2$$

Choosing the x axis along the axis of the rod

$$\sigma_x = \frac{P}{A} = \frac{12 \times 10^3\,\text{N}}{201 \times 10^{-6}\,\text{m}^2} = 59.7\ \text{MPa}$$

$$\epsilon_x = \frac{\delta_x}{L} = \frac{300\ \mu\text{m}}{500\ \text{mm}} = 600 \times 10^{-6}$$

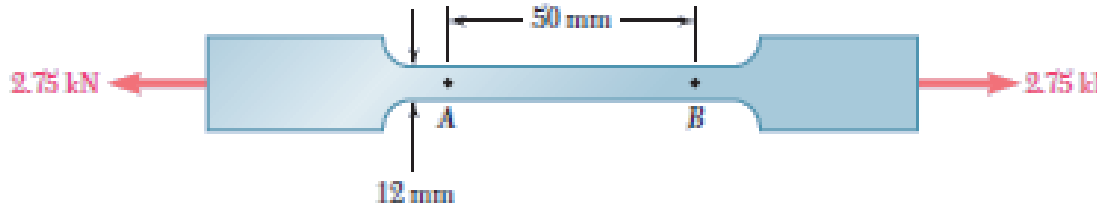
$$\epsilon_y = \frac{\delta_y}{d} = \frac{-2.4\ \mu\text{m}}{16\ \text{mm}} = -150 \times 10^{-6}$$

From Hooke's law, $\sigma_x = E\epsilon_x$,

$$E = \frac{\sigma_x}{\epsilon_x} = \frac{59.7\ \text{MPa}}{600 \times 10^{-6}} = 99.5\ \text{GPa}$$

$$\nu = -\frac{\epsilon_y}{\epsilon_x} = -\frac{-150 \times 10^{-6}}{600 \times 10^{-6}} = 0.25$$

A 2.75-kN tensile load is applied to a test coupon made from 1.6- mm flat steel plate **(E = 200 GPa, υ= 0.30).** Determine the resulting change (*a*) in the 50-mm gage length, (*b*) in the width of portion *AB* of the test coupon, (*c*) in the thickness of portion *AB*, (*d*) in the cross-sectional area of portion *AB.*



$\sigma_a = \frac{P}{A} = \frac{2750}{19.2} = 143.23\ MPa$

$A = (12)(1.6) = 19.2\ mm^2$

$\sigma_a = E\epsilon_a \Rightarrow 143.23 = 200\times10^3\ \epsilon_a$

$\epsilon_a = 7.1615\times10^{-4} = \delta_{AB}/50$

$\underline{\delta_{AB} = 0.036\ mm}$

$\epsilon_l = \nu\epsilon_a \Rightarrow \epsilon_L = (0.3)(7.1615\times10^{-4}) = 2.1485\times10^{-4} = \frac{\delta_w}{12} \Rightarrow$

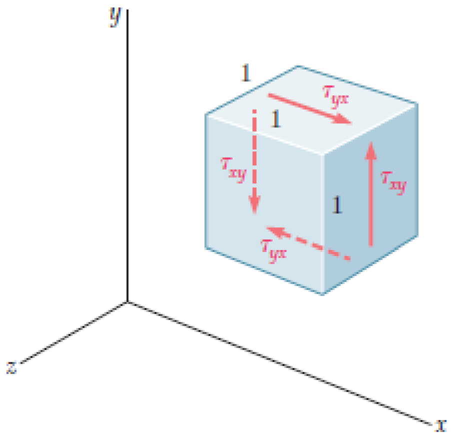
change in width $\delta_w = \underline{-2.578\times10^{-3}\ mm}$

change in thickness $\Rightarrow -2.1485\times10^{-4} = \frac{\delta_t}{1.6} \Rightarrow \delta_t = \underline{-3.4376\times10^{-4}}$

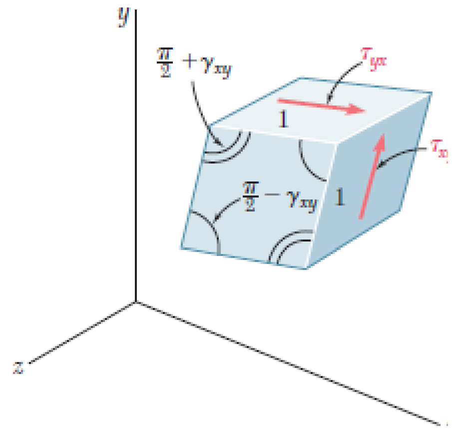
change in cross section $\Rightarrow \Delta A = A_0 - A'$

$= 19.2 - (12-2.578\times10^{-3})(1.6-3.4376\times10^{-4}) = \underline{0.00825\ mm^2}$

SHEARING STRAIN



Unit cubic element subjected to shearing stress.



Deformation of unit cubic element due to shearing stress.

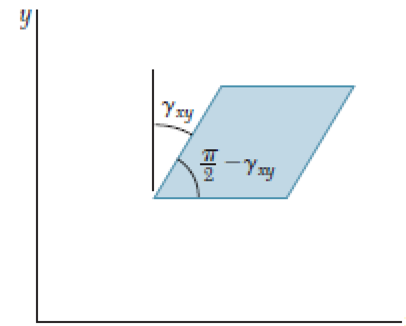


Fig. 2.38 Cubic element as viewed in *xy*-plane after rigid rotation.

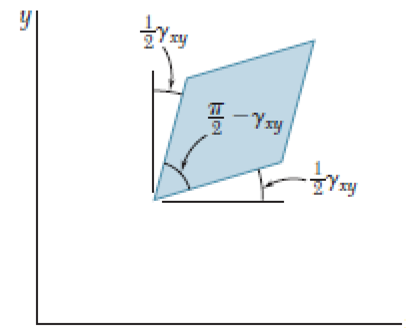


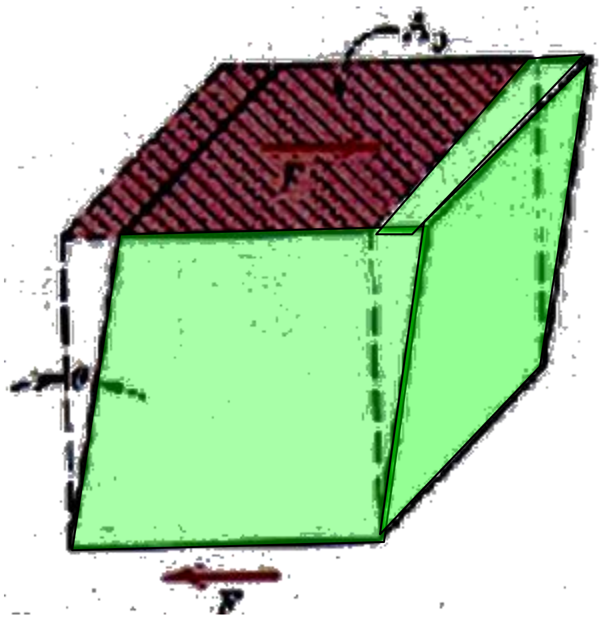
Fig. 2.39 Cubic element as viewed in *xy*-plane with equal rotation of *x* and *y* faces.

$$\tau_{xy} = G\gamma_{xy}$$

Hooke's law for shearing stress and strain, and the constant ***G*** is called the *modulus of rigidity* or *shear modulus* of the material

Hooke's Law in Shear

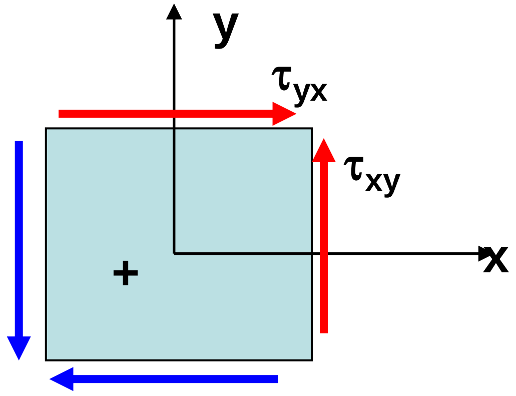
$$\tau = G\gamma$$



The constant *G* is called the shear modulus and relates the shear stress and strain in the elastic region .

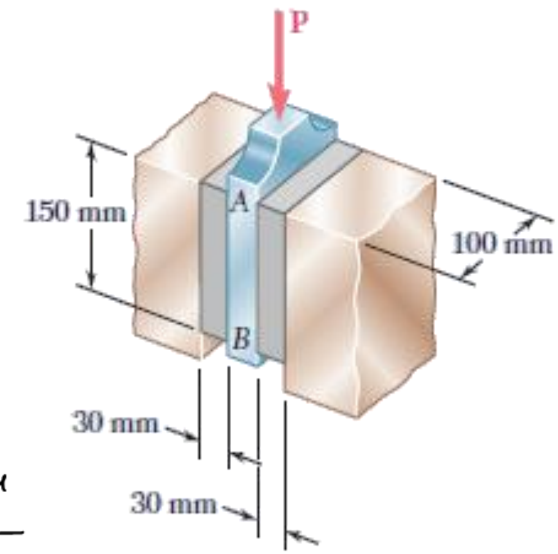
It is also used to relate shear and elastic moduli•

$$G = \frac{E}{2(1+\nu)}$$



A vibration isolation unit consists of two blocks of hard rubber bonded to a plate *AB* and to rigid supports as shown. Knowing that a force of magnitude $P = 25$ kN causes a deflection $\delta = 1.5$ mm of plate *AB*, determine the modulus of rigidity of the rubber used.

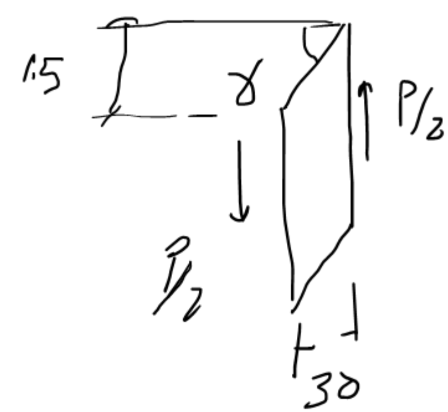
$$\tau = G\gamma \quad , \quad \tau = \frac{P}{A}$$



$$\tau = \frac{(25000/2)}{(100)(150)} = 0.833 \text{ MPa}$$

$$\tan\gamma = \gamma = \frac{1.5}{30} = 0.05 \text{ rad}$$

$$0.833 = G \times 0.05 \Rightarrow \underline{G = 16.66 \text{ MPa}}$$



Structural Mechanics

2104011212

Chapter 4

Axial Load

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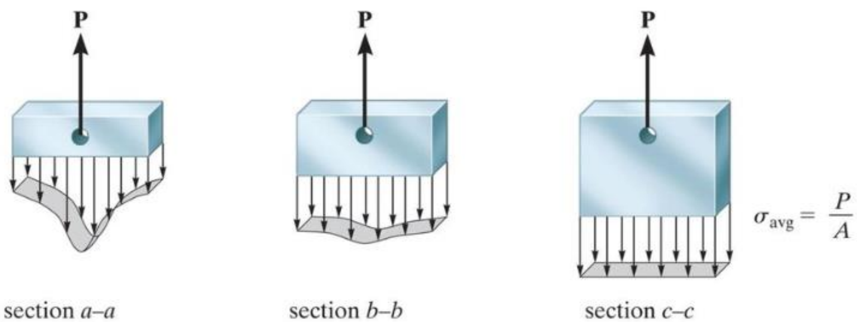
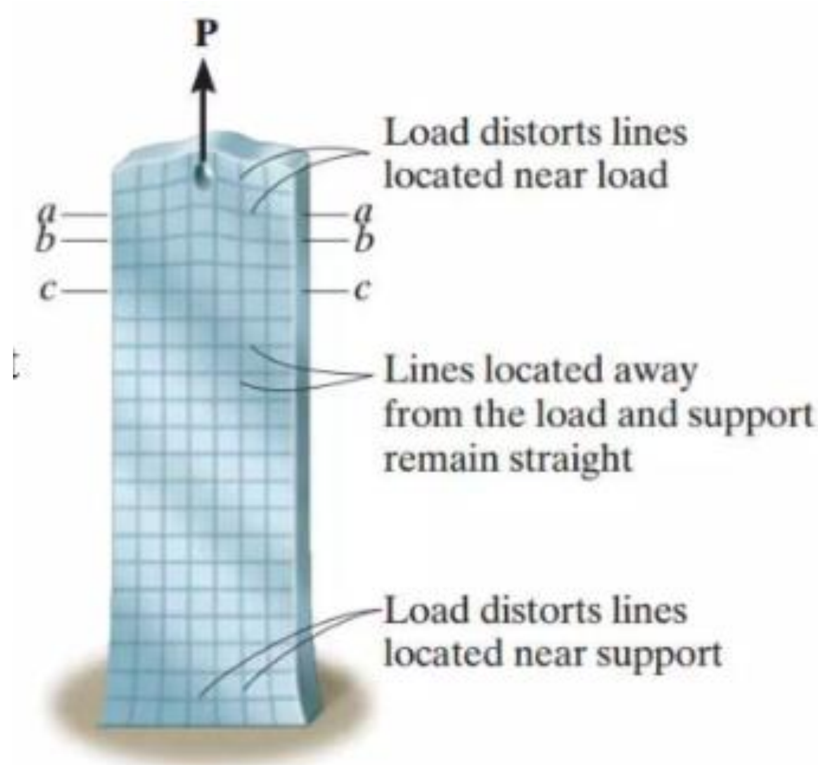
Hibbeler, R.C. (2011). Mechanics of Materials. 8thEd. Pearson Prentice Hall.

Mechanics of Materials, F. P. Beer et al, Mc Graw Hill.

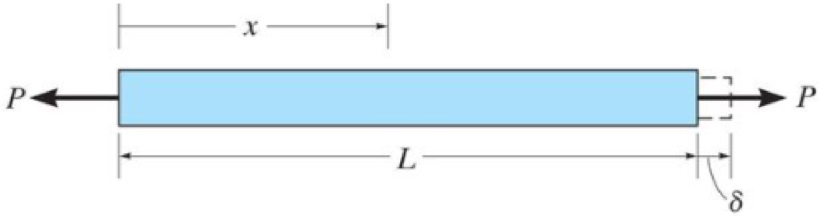
4.1 Saint-Venant’s principle - 1855





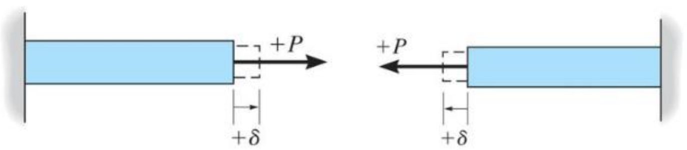
4.2 Elastic deformation of an axially loaded member

Constant Load and Cross-Section Area



For homogeneous prismatic bar

Sign Convention



ε = δ / L   **σ = P/A**   **σ = Eε**

$$\epsilon = \frac{\sigma}{E} = \frac{P}{EA}$$

$$\frac{\delta}{L} = \frac{p}{EA}$$

$$\delta = \frac{pL}{EA}$$

If the rod is loaded at other points, or consists of several portions of various cross sections and possibly of different materials, it must be divided into component parts

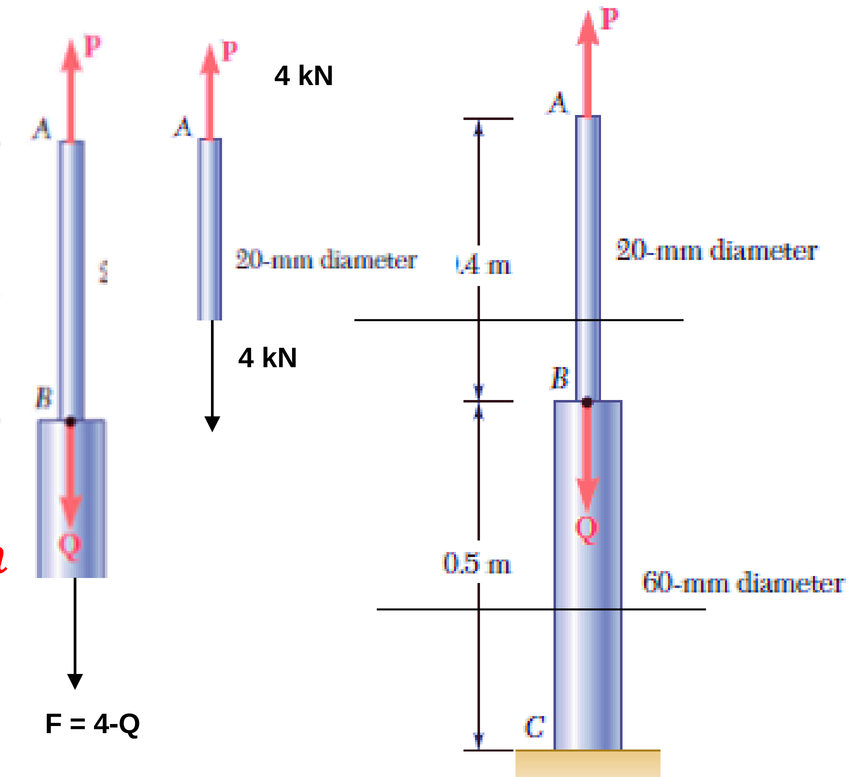
$$\delta = \sum \frac{p_i L_i}{E_i A_i}$$

Both portions of the rod *ABC* are made of an aluminum for which ***E* = 70 GPa.** Knowing that the magnitude of **P is 4 kN**, determine
(*a*) the value of **Q** so that the deflection at ***A* is zero,** (*b*) the corresponding deflection of *B.*

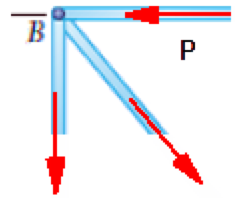
$$\delta = \sum \frac{p_i L_i}{E_i A_i} = \frac{4000*400}{70000*3.14*10^2} + \frac{(4000-Q)*500}{70000*3.14*30^2} = 0$$

Q= 32800 N = 32.8 kN

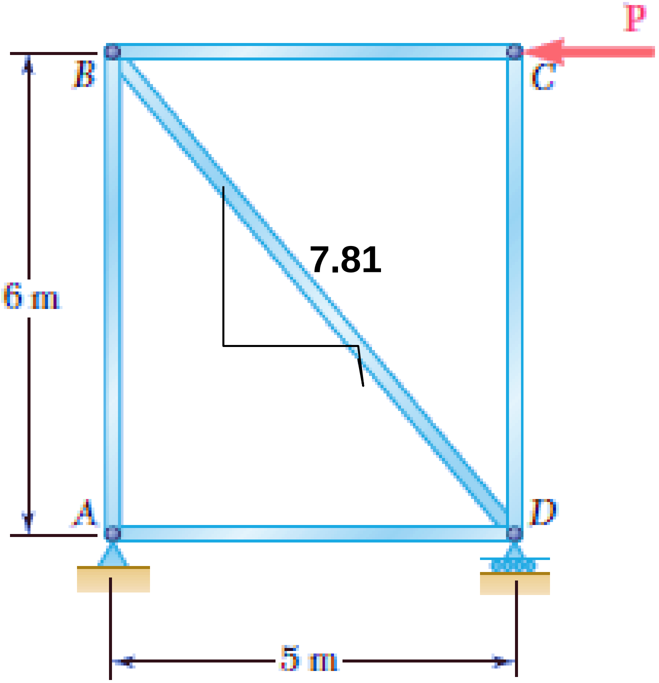
$$\delta_B = \frac{QL}{EA} = \frac{(4000-32800)*500}{70000*3.14*30^2} = \mathbf{-0.0728\ mm}$$



The steel frame (***E* = 200 GPa**) shown has a diagonal brace ***BD* with an area of 1920** mm². Determine the largest allowable load **P** if the **change in length of member *BD* is not to exceed 1.6 mm.**



FBD$_x$ =P
FBD = P*7.81/5 = 1.562 P

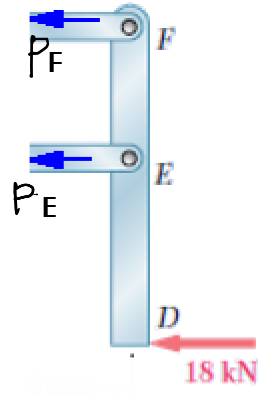


$$\delta = \frac{pL}{EA} = \frac{P*7.81*1000}{200000*1920} = 1.6$$

FBD = 78668.4 N = 78.668 kN

P= 50.36 kN

Members *ABC* and *DEF* are joined with steel links (***E* = 200 GPa)**. Each of the links is made of a **pair of 25x35-mm** plates. Determine the **change in length of member *BE***, and **member *CF.***



$\Sigma M_F=0 \qquad -18x440- \ 180P_E=0$

$P_E= \ -44$ kN (C)

$\Sigma Fx=0 \qquad -18+44 - P_F=0$

$P_F= \ 26$ kN (T)

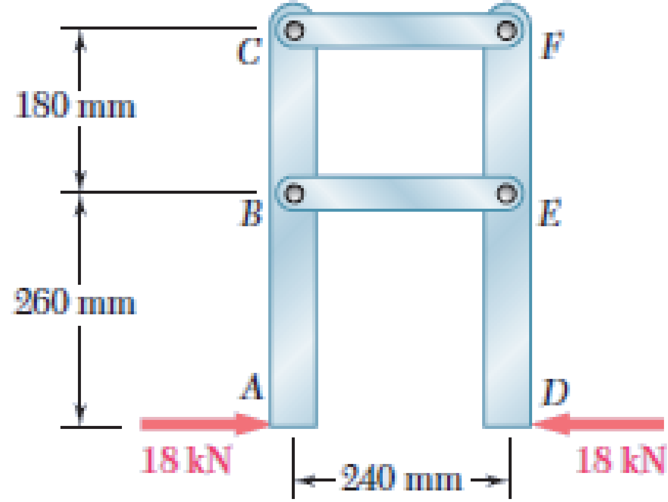
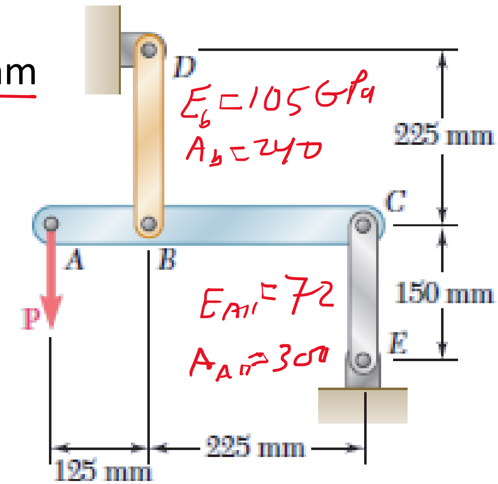


Fig. P2.26

$$\delta_{EB} = \frac{P_E L}{EA} = \frac{(-44000*240}{200000*2(\,25*35)} = \mathbf{-0.0302}\ \boldsymbol{mm}$$

$$\delta_{CF} = \frac{P_F L}{EA} = \frac{(26000*240}{200000*2(\,25*35)} = \mathbf{0.01783}\ \boldsymbol{mm}$$

Link *BD* is made of brass (E = 105 GPa) and has a cross-sectional area of 240 mm². Link *CE* is made of aluminum (E = 72 GPa) and has a cross-sectional area of 300 mm². Knowing that they support rigid member *ABC*, determine the maximum force **P** that can be applied vertically at point *A* if the deflection of *A* is not to exceed 0.35 mm

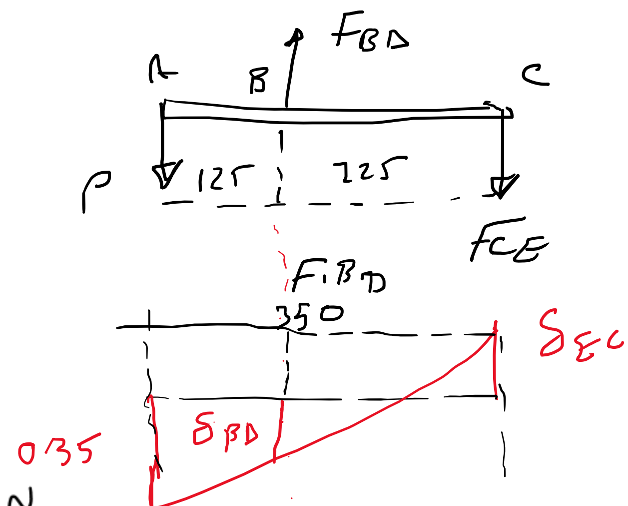


$\circlearrowleft^{+} \sum M_C = 0 \Rightarrow 350 - 225 F_{BD} = 0 \Rightarrow F_{BD} \overset{+}{=} 1.56P$ (T)

$+\uparrow \sum F_y = 0 \Rightarrow -P + 1.56P - F_{CE} = 0 \Rightarrow F_{CE} \overset{+}{=} 0.56P$

$$\delta_{BD} = \frac{(1.56P)(225)}{(105000)(240)} = 1.393 \times 10^{-5} P$$

$$\delta_{EC} = \frac{(0.56P)(150)}{(72000)(300)} = 3.89 \times 10^{-6} P$$



$$\frac{0.35 + \delta_{EC}}{350} = \frac{\delta_{BD} + \delta_{EC}}{225}$$

$$\frac{0.35 + 3.89 \times 10^{-6} P}{350} = \frac{1.393 \times 10^{-5} P + 3.89 \times 10^{-6} P}{225}$$

$1 \times 10^{-3} = 6.804 \times 10^{-8} P \Rightarrow \underline{P = 14.69 \text{ kN}}$

4.4 Statically indeterminate axially loaded members

There are many problems, however, where the internal forces cannot be determined from statics alone. In most of these problems, the reactions themselves—the external forces— cannot be determined by simply drawing a free-body diagram of the member and writing the corresponding equilibrium equations. The equilibrium equations must be complemented by relationships involving deformations obtained by considering the geometry of the problem. Because statics is not sufficient to determine either the reactions or the internal forces, problems of this type are called ***statically indeterminate***

$P_1 + P_2 = P$ statically indeterminate **1**

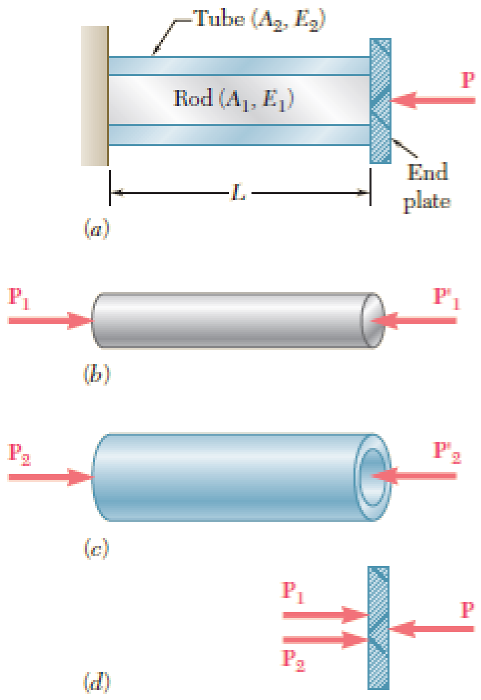
δ_1 and δ_2 of the rod and tube must be equal

$$\delta_1 = \frac{P_1 L}{A_1 E_1} = \delta_2 = \frac{P_2 L}{A_2 E_2}$$

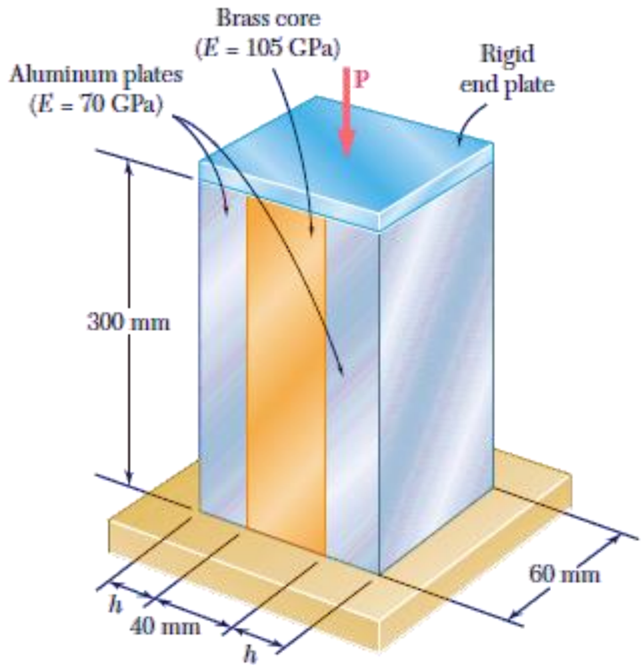
$$\frac{P_1}{A_1 E_1} = \frac{P_2}{A_2 E_2} \qquad \mathbf{2}$$

solve 1 & 2

$$P_1 = \frac{A_1 E_1 P}{A_1 E_1 + A_2 E_2} \qquad P_2 = \frac{A_2 E_2 P}{A_1 E_1 + A_2 E_2}$$



An axial centric force of magnitude *P*=450 kN is applied to the composite block shown by means of a rigid end plate. Knowing that *h* = 10 mm, determine the normal stress in (*a*) the brass core, (*b*) the aluminum plates.



$P_B + P_{Al} = P = 450000\ N$

$$\delta_1 = \frac{P_1 L}{A_1 E_1} = \delta_2 = \frac{P_2 L}{A_2 E_2}$$

$$P_1 = \frac{A_1 E_1 P}{A_1 E_1 + A_2 E_2} \qquad P_2 = \frac{A_2 E_2 P}{A_1 E_1 + A_2 E_2}$$

$$\delta_B = \frac{P_B L}{E_B A_B} = \frac{P_B * 300}{105000*(40*60)} = \mathbf{1.1905\ x10^{-6}\ P_B}$$

$$\delta_{All} = \frac{P_{All} L}{E_{All} A_{All}} = \frac{P_{All} * 300}{70000*2(10*60)} = \mathbf{3.5714x10^{-6}\ P_{All}}$$

$1.1905x10^{-6} P_B = 3.5714x10^{-6} P_{All}$

$P_B = 3P_{All}$

$3\ P_{all} + P_{all} = 450000\ N$

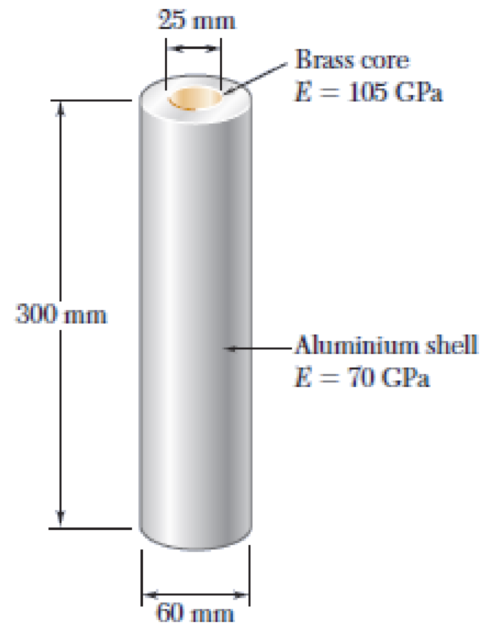
$P_{all} = 112500\ N$ $\qquad$ $P_B = 3x\ 112500 = 337500\ NN$

σ = P/A

σ_B = 337500/(40x60) = 140.625 MPA

σ_{ALL} = 112500/2(10x60) = 93.75 MPA

The length of the assembly shown decreases by 0.40 mm when an axial force is applied by means of rigid end plates. Determine (*a*) the magnitude of the applied force, (*b*) the corresponding stress in the brass core.



$$\delta = \frac{PL}{EA}$$

$$\delta_b = \delta_{Al} = \delta = 0.4$$

$$P_b + P_{Al} = P$$

$$\frac{300 P_b}{(105000)(\pi\, 12.5^2)} = \frac{300 P_{al}}{(70000)\,\pi(30^2 - 12.5^2)}$$

$$5.823 \times 10^{-6} P_b = 1.835 \times 10^{-6} P_{al}$$

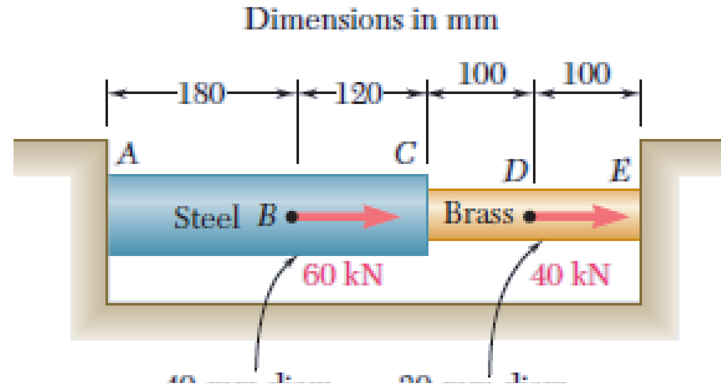
$$5.823 \times 10^{-6} P_b = 0.4 \Rightarrow P_b = 68.693 \text{ kN}$$

$$1.835 \times 10^{-6} P_{al} = 0.4 \Rightarrow P_{al} = 217.98 \text{ kN}$$

$$P = 68.693 + 217.98 = \underline{286.673 \text{ kN}}$$

$$\sigma_b = \frac{P_b}{A_b} = \frac{68.693 \times 10^3}{\pi\, 12.5^2} = \underline{140 \text{ MPa}}$$

Two cylindrical rods, one of steel and the other of brass, are joined at *C* and restrained by rigid supports at *A* and *E*. For the loading shown and knowing that $E_s = 200$ GPa and $E_b = 105$ GPa, determine (*a*) the reactions at *A* and *E*, (*b*) the deflection of point *C*.



SINCE E Can't move Relative to E

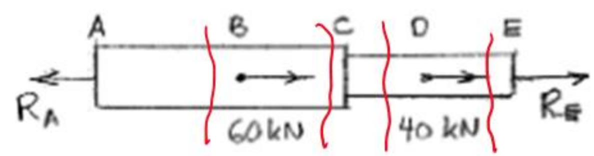
$\Sigma \delta = 0 \rightarrow \delta_{AB} + \delta_{BC} + \delta_{CD} + \delta_{DE} = 0$

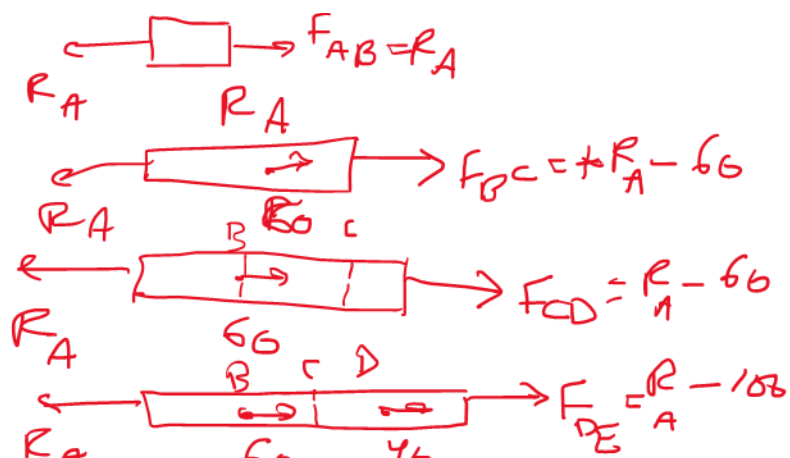
$\Rightarrow R_A$

$\Sigma f_x = 0 \Rightarrow R_E$

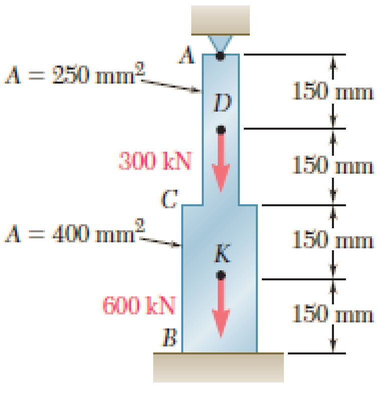
$\delta_C = \delta_{AB} + \delta_{BC}$

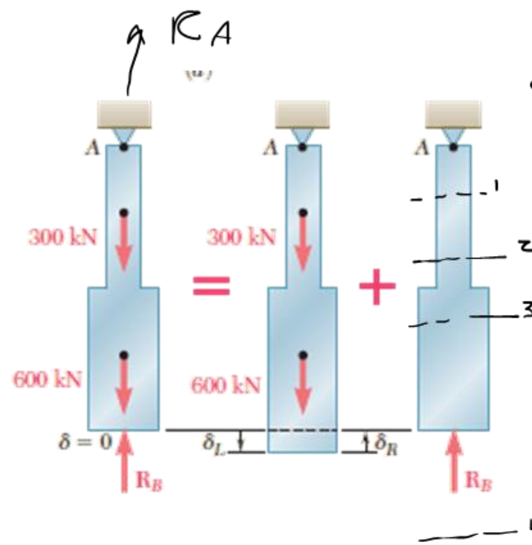
$\delta = \dfrac{PL}{EA}$





Determine the reactions at *A* and *B* for the steel bar and loading shown, assuming a close fit at both supports before the loads are applied.





$$\Sigma \delta_i = \Sigma \frac{PL}{EA}$$

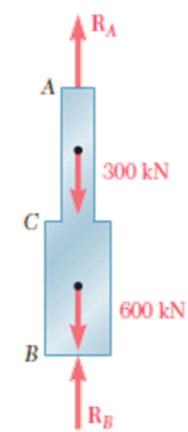
$$\delta_1 + \delta_2 + \delta_3 + \delta_4 = 0$$

$$\Sigma \delta = \frac{R_B\,(150)}{400E} + \frac{(R_B-600)(150)}{E\times 400} + \frac{(R_B-600)(150)}{E\times 250} + \frac{(R_B-900)(150)}{E\times 250} = 0$$

$$0.375R_B + 0.375R_B - 225 + R_B - 600 + 0.6R_B - 540 = 0$$

$$2.35R_B = 1365 \Rightarrow \underline{R_B = 580.85 \text{ kN}}$$

$$\uparrow \Sigma F_y = 0 \Rightarrow 580.85 - 600 - 300 + R_A = 0 \Rightarrow \underline{R_A = 319.15 \text{ kN}}$$



4.6 Thermal stresses

If the temperature of the rod is raised by $\Delta \boldsymbol{T}$, the rod elongates by an amount δ_T that is proportional to both the temperature change ΔT and the length L of the rod

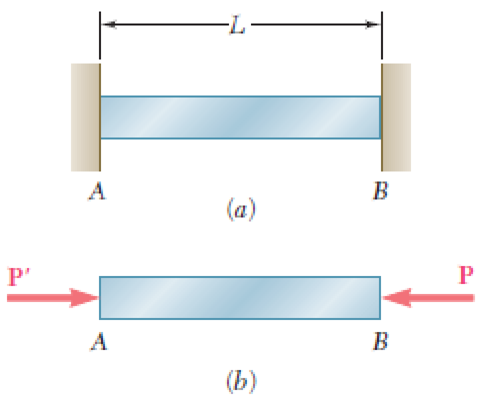
$$\delta_T = \alpha(\Delta T)L$$

$$\epsilon_T = \delta_T / L.$$

$$\epsilon_T = \alpha \Delta T$$

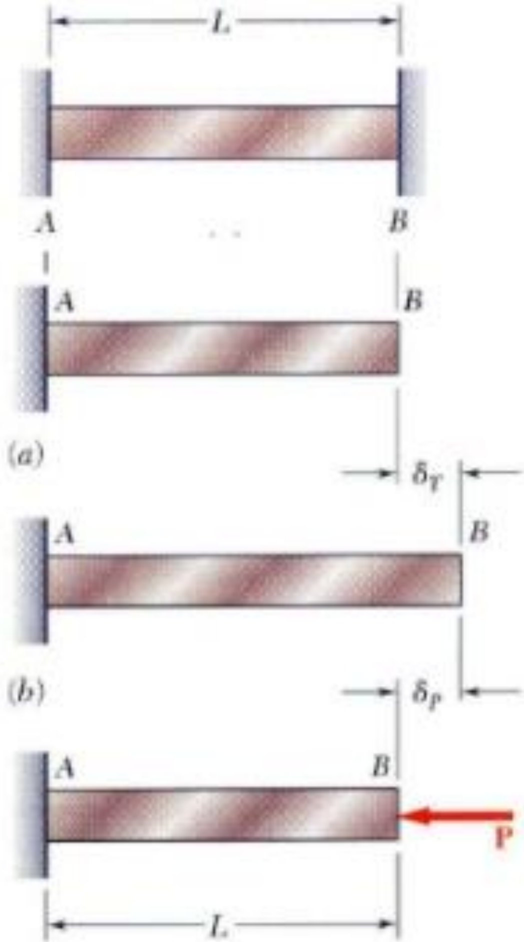
α : ***coefficient of thermal expansion***. (*per degree C)*

ε_T is called a ***thermal strain***



Force P develops when the temperature of the rod increases while ends A and B are restrained.

problem created by the temperature change D*T* is statically indeterminate. Therefore, the magnitude *P* of the reactions at the supports
is determined from the condition that the elongation of the rod is zero



- A temperature change results in a change in length or *thermal strain*. There is no stress associated with the thermal strain unless the elongation is restrained by the supports.

- Treat the additional support as redundant and apply the principle of superposition.

$$\delta_T = \alpha(\Delta T)L \qquad \delta_P = \frac{PL}{AE}$$

$\alpha =$ thermal expansion coef.

- The thermal deformation and the deformation from the redundant support must be compatible.

$$\delta = \delta_T + \delta_P = 0$$

$$\alpha(\Delta T)L + \frac{PL}{AE} = 0$$

$$\delta = \delta_T + \delta_P = 0$$

$$P = -AE\alpha(\Delta T)$$

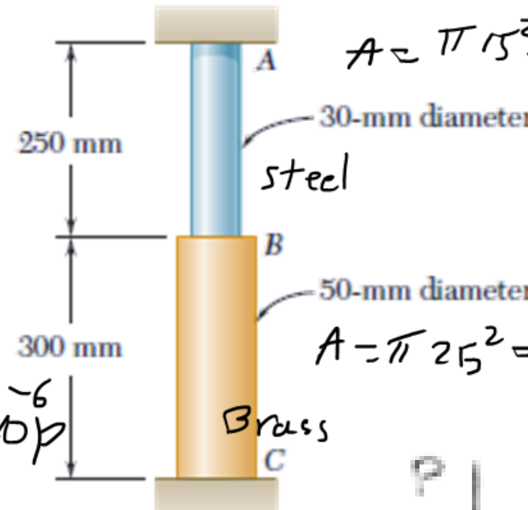
$$\sigma = \frac{P}{A} = -E\alpha(\Delta T)$$

A rod consisting of two cylindrical portions *AB* and *BC* is restrained at both ends. Portion *AB* is made of steel (E_s = 200 GPa, α_s = 11.7 3 10^{-6}/°C) and portion *BC* is made of brass (E_b = 105 GPa, α_b = 20.9 3 10^{-6}/°C). Knowing that the rod is initially unstressed, determine the compressive force induced in *ABC* when there is a **temperature rise of 50°C.**

$\Delta T = 50°C \quad : \delta = \Sigma \alpha \Delta T L$

$\Sigma \delta_T = 11.73\times10^{-6} \times 50 \times 250 + 20.93\times10^{-6} \times 50 \times 300$

$= 0.461\ m$ [expansion)



$A = \pi 15^2 = 706.5\ mm^2$

steel

$A = \pi 25^2 = 1962.5\ mm^2$

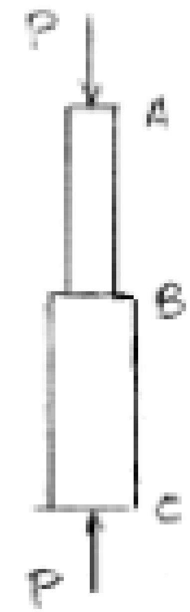
Brass

$\delta_P = \Sigma \frac{PL}{EA}$

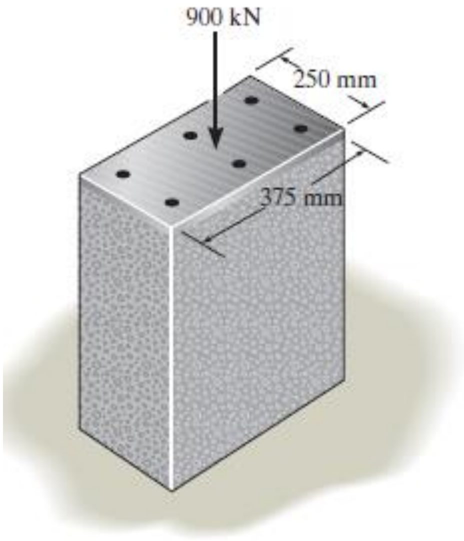
$= \frac{250 P}{200\times10^3 \times 706.5} + \frac{300 P}{105\times10^3 \times 1962.5} = 3.225\times10^{-6} P$

$\delta_P = \delta_T \Rightarrow 3.225\times10^{-6} P = 0.461$

$\underline{P = 142.95\ N}$



The concrete post is reinforced using six steel reinforcing rods, each having a diameter of 20 mm. Determine the stress in the concrete and the steel if the post is subjected to an axial load of 900 kN. E_c = 25 GPa. Est = 200 GPa,



$$+\uparrow \Sigma F_y = 0; \qquad P_{con} + 6P_{st} - 900 = 0$$

$$\partial_{con} = \delta_{st}$$

$$\frac{P_{con}\, L}{A_{con}\, E_{con}} = \frac{P_{st}\, L}{A_{st}\, E_{st}}$$

$$\frac{P_{con}\, \cancel{L}}{\left[0.25(0.375) - 6(\frac{\pi}{4})(0.02^2)\right]\left[25(10^9)\right]} = \frac{P_{st}\, \cancel{L}}{(\frac{\pi}{4})(0.02^2)\left[200(10^9)\right]}$$

$$P_{con} = 36.552\, P_{st}$$

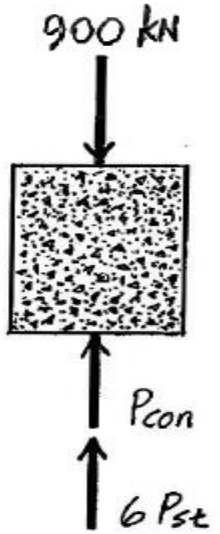
Solving Eqs (1) and (2) yields

$$P_{st} = 21.15 \text{ kN} \qquad P_{con} = 773.10 \text{ kN}$$

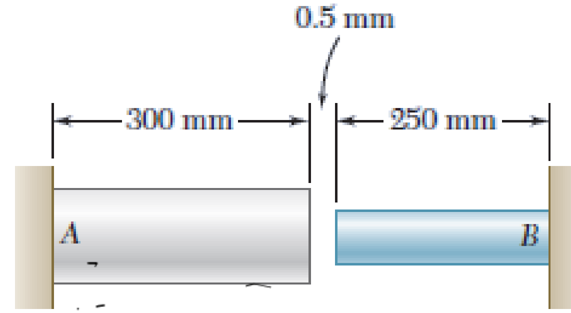
Thus,

$$\sigma_{con} = \frac{P_{con}}{A_{con}} = \frac{773.10(10^3)}{0.15(0.375) - 6(\frac{\pi}{4})(0.02^2)} = 8.42 \text{ MPa}$$

$$\sigma_{st} = \frac{P_{st}}{A_{st}} = \frac{21.15(10^3)}{\frac{\pi}{4}(0.02^2)} = 67.3 \text{ MPa}$$



At room temperature (20°C) a 0.5-mm gap exists between the ends of the rods shown. At a later time when the temperature has reached 140°C, determine (*a*) the normal stress in the aluminum rod, (*b*) the change in length of the aluminum rod.



Aluminum
$A = 2000\ \text{mm}^2$
$E = 75\ \text{GPa}$
$\alpha = 23 \times 10^{-6}/°\text{C}$

Stainless steel
$A = 800\ \text{mm}^2$
$E = 190\ \text{GPa}$
$\alpha = 17.3 \times 10^{-6}/°\text{C}$

$\Delta T = 140 - 20 = 120°C$

$\delta T_{Al} = \alpha \Delta T L = (23\times10^{-6})(120)(300) = 0.828$ mm

$\delta T_{st} = \alpha \Delta T L = (17.3\times10^{-6})(120)(250) = 0.519$

$\delta = 0.828 + 0.519 - 0.5 = 0.847$

$\delta_P = \Sigma \frac{PL}{EA} = \frac{300P}{75000\times2000} + \frac{250P}{190000\times800}$

$= 3.6447 P\times10^{-6}$

$\delta_P = \delta_T \Rightarrow 0.847 = 3.6447P\times10^{-6} \longrightarrow \underline{P = 232.44\ kN}$

$\sigma_{Al} = \frac{P}{A} = \frac{232.44\times10^3}{2000} = \underline{116.22\ MPa}$

$\delta_{Al} = \delta_T - \delta_P = (23\times10^{-6})(120)(300) - \frac{(232.44\times1000)(300)}{75\times10^3\times2000}$

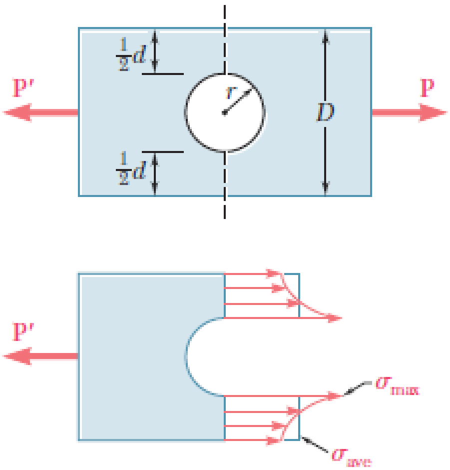
$= 0.828 - 0.465 = \underline{0.363\ mm}$

4.7 STRESS CONCENTRATIONS

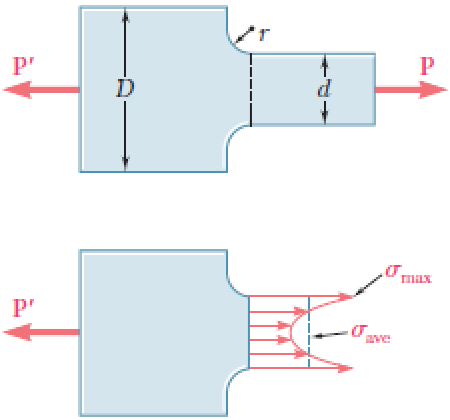
stress-concentration factor

$$K = \frac{\sigma_{max}}{\sigma_{ave}}$$

To determine the maximum stress occurring near a discontinuity in a given member subjected to a given axial load P, the designer needs to compute the average stress $\sigma = P/A$ in the critical section and multiply the result obtained by the appropriate value of the stress-concentration factor K.



Stress distribution near circular hole in flat bar under axial loading.

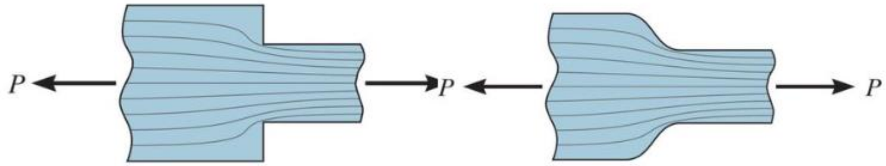


Stress distribution near fillets in flat bar under axial loading.

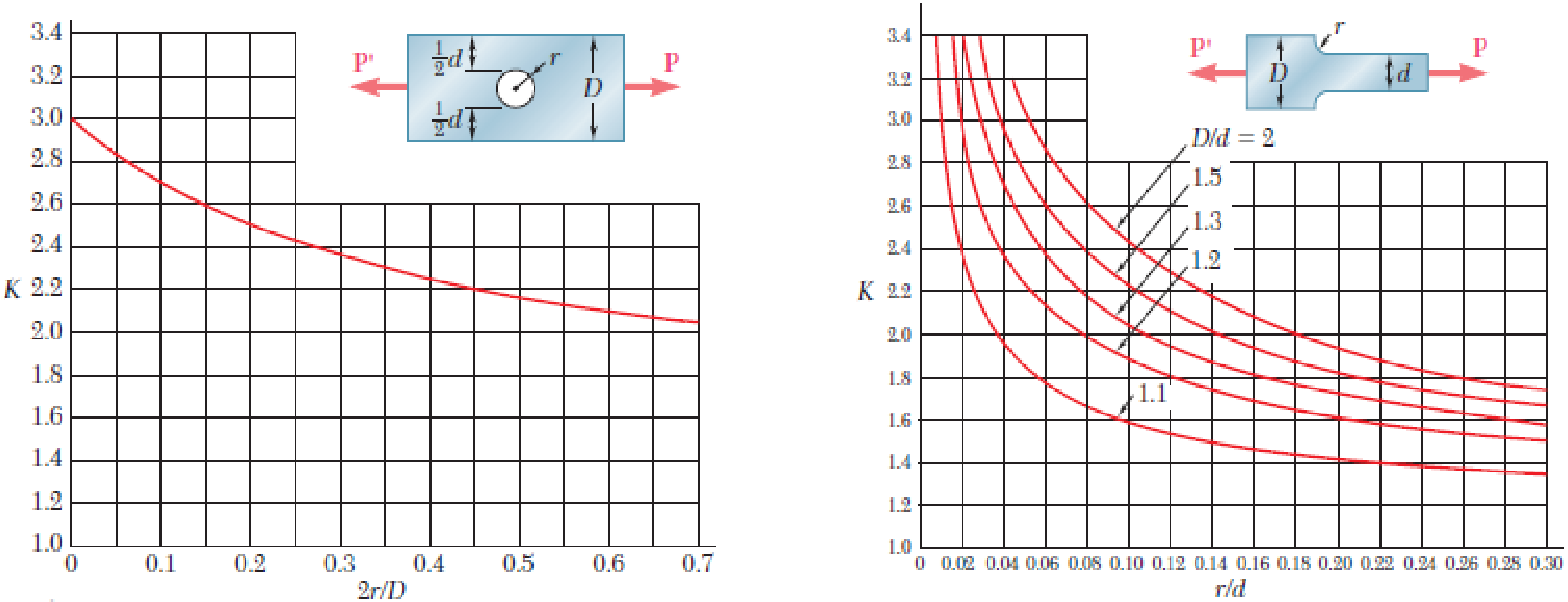
Reduction of Stress Concentration

* Avoid abrupt and/or large-magnitude changes in cross section.

* Avoid sharp corners using round and fillet.



(a) (b)

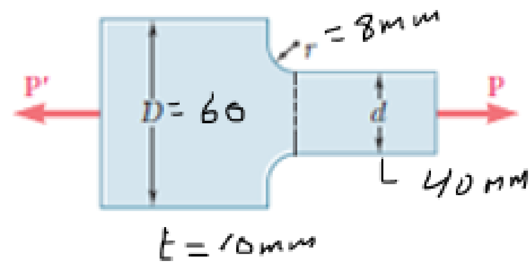


(*a*) Flat bars with holes

(*b*) Flat bars with fillets

Stress concentration factors for flat bars under axial loading. Note that the average stress must be computed across the narrowest section: $\sigma_{ave} = P/td$, where t is the thickness of the bar. (Source: W. D. Pilkey and D.F. Pilkey, *Peterson's Stress Concentration Factors*, 3rd ed., John Wiley & Sons, New York, 2008.)

Determine the largest axial load **P** that can be safely supported by a flat steel bar consisting of two portions, both 10 mm thick and, respectively, 40 and 60 mm wide, connected by fillets of radius *r* = 8 mm. Assume an allowable normal stress of 165 MPa.

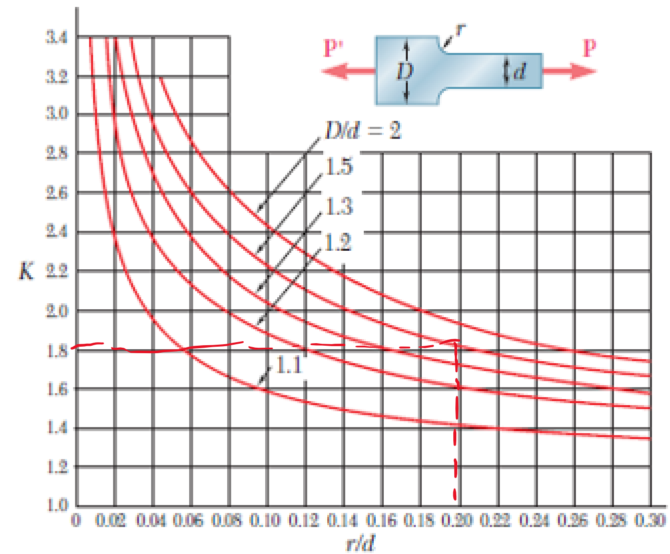


$$\frac{r}{d} = \frac{8}{40} = 0.2$$

$$\frac{D}{d} = \frac{60}{40} = 1.5$$

use chart to get K

$K = 1.82$

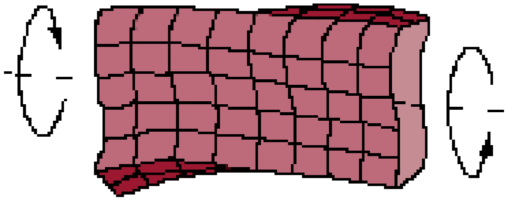


$$\sigma = K\frac{P}{A} = 165 = 1.82\frac{P}{(40)(10)}$$

$$\underline{P = 36263.7\ N} = \underline{36.3\ kN}$$

Chapter 5

Torsion

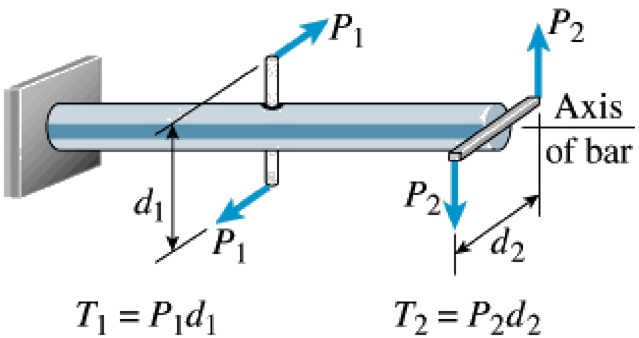


I. Marie

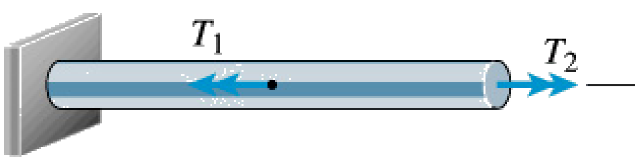
Lee, D.V., Beer, Johnston, DeWolf & Mazurek, “ Mechanics of Materials, McGraw Hill

Introduction

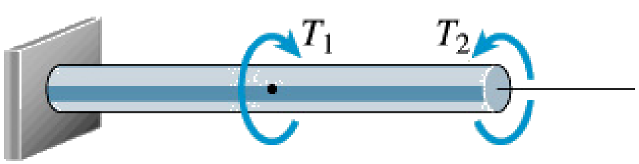
Torque is **a moment that twists a structure**



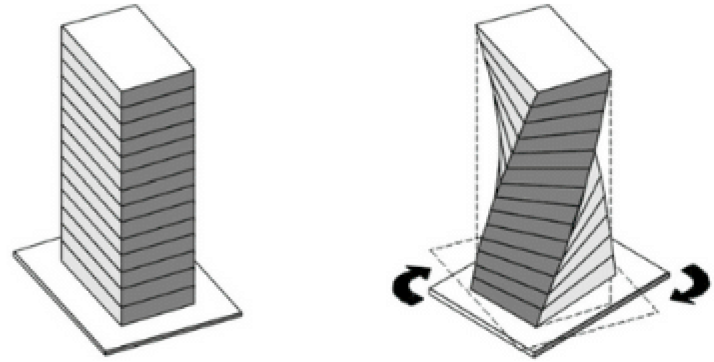
(a)

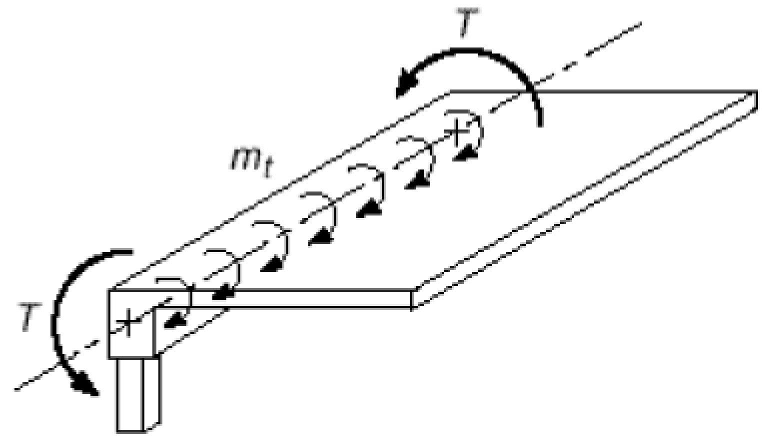


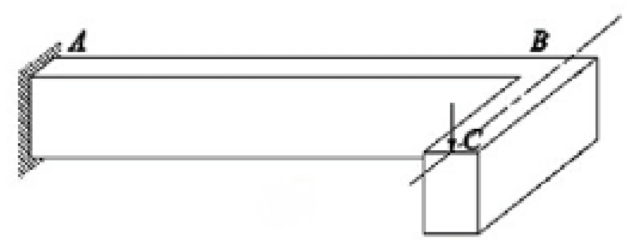
(b)



(c)







5.1 TORSIONAL DEFORMATION OF A CIRCULAR SHAFT

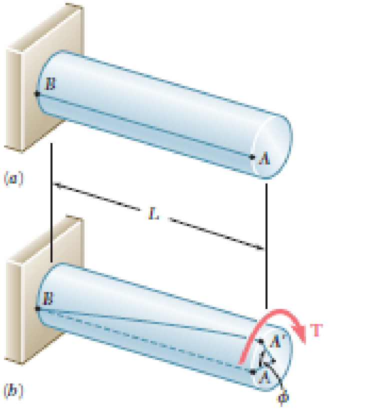
Deformations in a Circular Shaft

Torque is a moment that twists a member about its longitudinal axis. Units: N.m

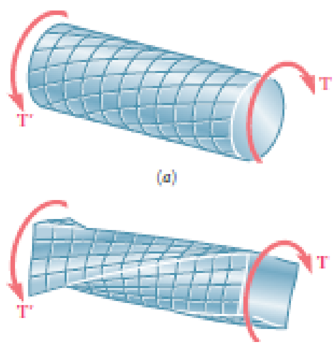
Angle of twist (Φ) is defined as the rotation of a radial line from a fixed end to a cross section some distance x from the end.

1. The longitudinal axis of the shaft remains straight
2. The shaft does not increase or decrease in length
3. Radial lines remain straight and radial as the cross section rotates
4. Cross sections rotate about the axis of the member

The right end will rotate with respect to the left end of the bar. The angle of rotation = Angle of twist ϕ. It changes along the length L of the bar linearly.

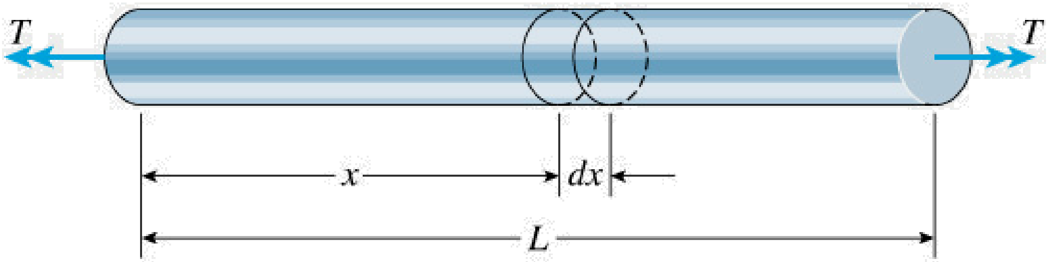


Shaft with fixed support and line AB drawn showing deformation under torsion loading: (a) unloaded; (b) loaded

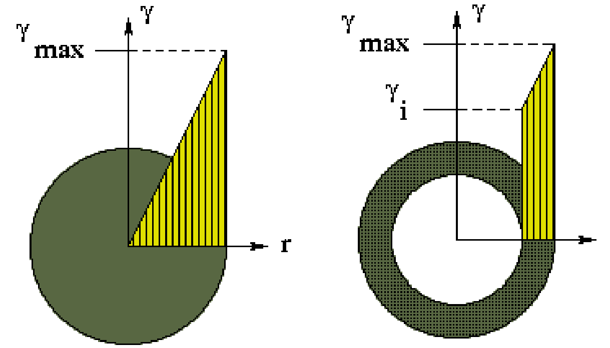


Comparison of deformations in (a) circular and (b) square shafts.

Shearing strain :

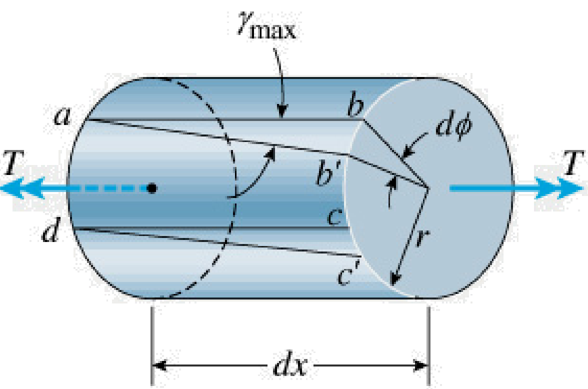


(a)

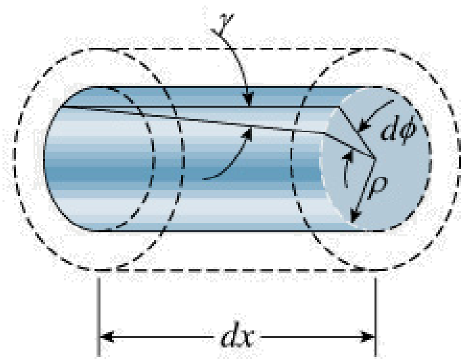


Solid Shaft

Hollow Shaft



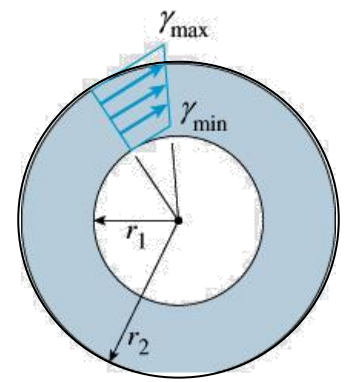
(b)



(c)

Deformation of an element of length *dx* cut from a bar in torsion

Shear strain in a circular bar varies linearly with the radial distance ρ, from zero at the center (ρ = 0) to maximum at outer surface ($\rho = r$)



Shear strains in a circular tube

$$\gamma = \frac{\rho\phi}{L}$$

The max. and min. shear strains are

$$\gamma_{max} = r_2\frac{\phi}{L}$$

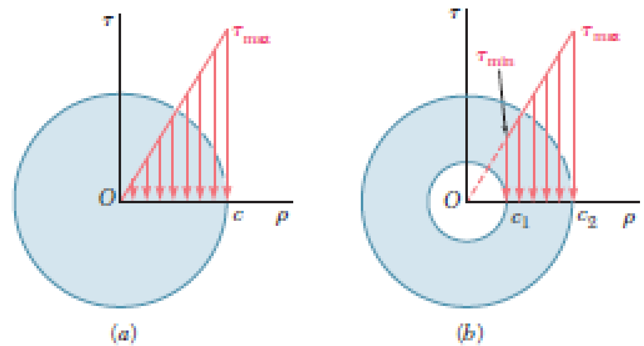
$$\gamma_{min} = \frac{r_1}{r_2}\gamma_{max} = r_1\frac{\phi}{L}$$

5.2 Torsion Formula

When the torque **T** is such that all shearing stresses in the shaft remain below the yield strength , the stresses in the shaft will remain below both the proportional limit and the elastic limit. Thus, Hooke's law will apply, and there will be no permanent deformation.

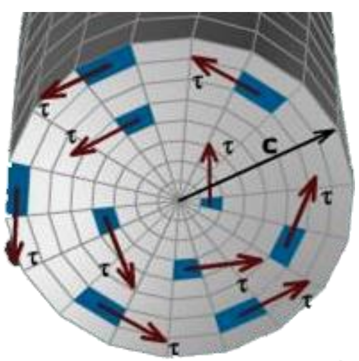
$$\tau = G\gamma$$

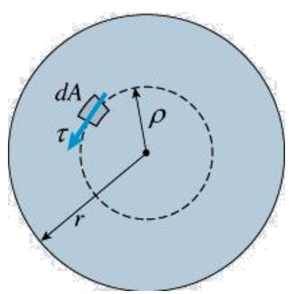
$$\tau = G\gamma = G\frac{\rho}{r}\gamma_{max} = \frac{\rho}{r}\tau_{max}$$

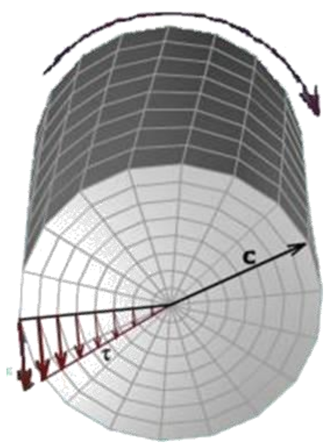


Distribution of shearing stresses in a torqued shaft: (a) Solid shaft, (b) Hollow shaft.

The Torsion Formula







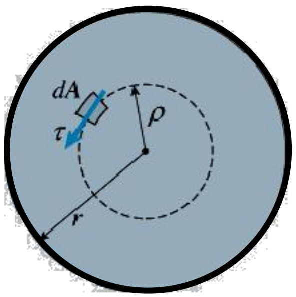
$$dM = (\rho)(\tau)dA = \rho^2 \frac{\tau_{max}}{r} dA$$

$$T = \int_A dM = \frac{\tau_{max}}{r}\int_A \rho^2 dA = \frac{\tau_{max}}{r} I_p$$

$$\tau_{max} = \frac{Tr}{I_p}, \qquad \tau = \frac{T\rho}{I_p}$$

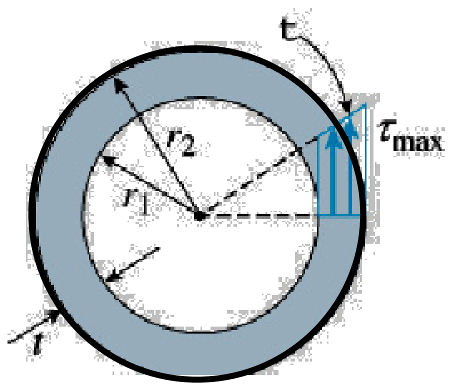
$$I_p = \int_A \rho^2 dA = \text{Polar Moment of Inertia}$$

The polar moment of inertia of a circle of radius r



$$I_P = \frac{\pi r^4}{2} = \frac{\pi d^4}{32}$$

The polar moment of inertia of a hollow circular section



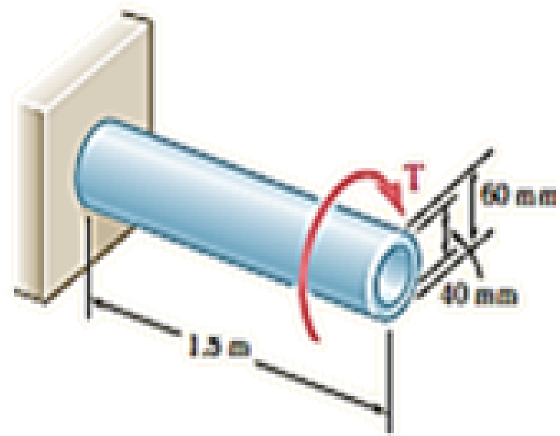
$$I_P = \frac{\pi}{2}\left(r_2^4 - r_1^4\right) = \frac{\pi}{32}\left(d_2^4 - d_1^4\right)$$
$$= \frac{\pi r t}{2}\left(4r^2 + t^2\right) = \frac{\pi d t}{4}\left(d^2 + t^2\right)$$

where $r = \dfrac{r_1 + r_2}{2}$ and $d = \dfrac{d_1 + d_2}{2}$

If $d_1 \approx d_2$, i.e., $t << d$, then

$$I_P \approx 2\pi r^3 t = \frac{\pi d^3 t}{4}$$

What is the largest torque that can be applied to the shaft if the shearing stress is not to exceed 120 MPa.
What is the corresponding minimum value of the shearing stress in the shaft.



$\tau_{all} = 120\ MPa$

$\tau = \frac{Tr}{J}$

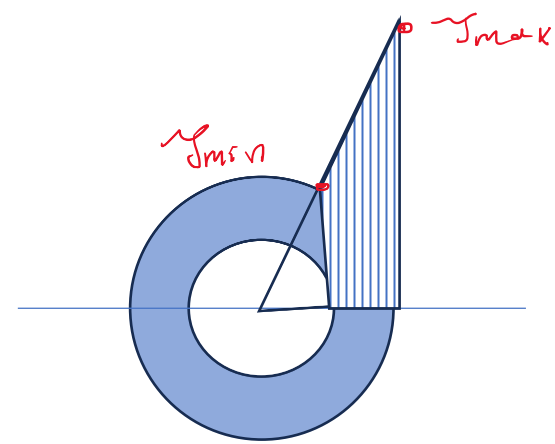
$J = \frac{\pi (30^4 - 20^4)}{2} = 1.02 \times 10^6\ mm^4$

$120 = \frac{(T)(30)}{1.02 \times 10^6} \Rightarrow$

$T = 4.08 \times 10^6\ N.mm = \underline{4.08\ kN.m}$

$\tau_{min} = \frac{T r_2}{J} = \frac{(4.08 \times 10^6)(20)}{1.02 \times 10^6}$

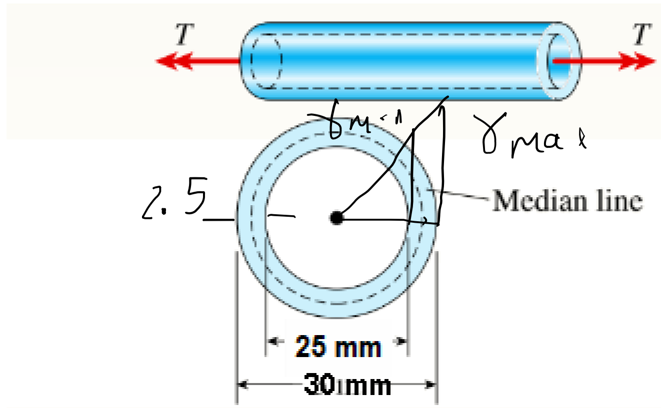
$= 80\ MPa$

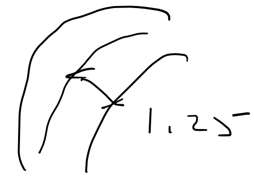


A circular tube is subjected to torque T at its ends. The resulting maximum shear strain in the tube is 0.005. Calculate the minimum shear strain in the tube and the shear strain at the median line of the tube section.

$\gamma_{max} = 0.005$

$\gamma_{min} = ?$





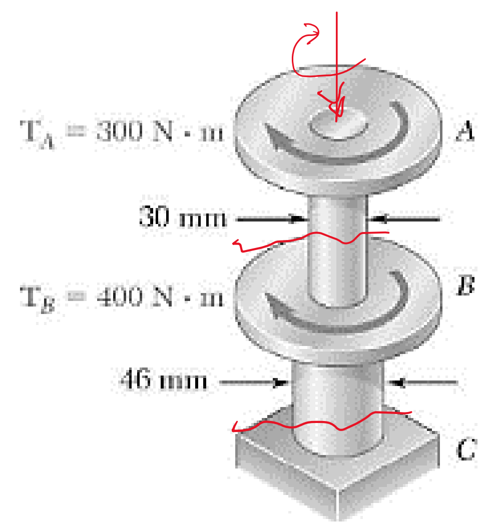
$$\gamma = \frac{r\Phi}{L}$$

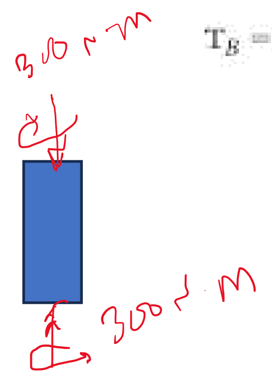
$$\gamma_{min} = \frac{r_i \gamma_{max}}{r_o} = \frac{(12.5)(0.005)}{15} = 0.0042$$

@ median

$$\gamma_{med} = \frac{r_{med}\,\gamma_{max}}{r_o} = \frac{(12.5+1.25)(0.005)}{15} = 0.00458$$

The torques shown are exerted on pulleys A and B. Knowing that both shafts are solid, determine the maximum shearing stress (a) in shaft AB, (b) in shaft BC





(a) Shaft AB: $T_{AB} = 300 \text{ N}\cdot\text{m},\ d = 0.030 \text{ m},\ c = 0.015 \text{ m}$

$$\tau_{max} = \frac{Tc}{J} = \frac{2T}{\pi c^3} = \frac{(2)(300)}{\pi(0.015)^3}$$

$$= 56.588 \times 10^6 \text{Pa}$$

$$\tau_{max} = 56.6 \text{ MPa}$$

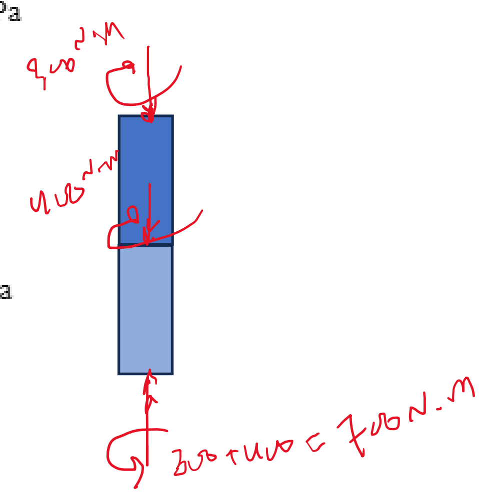
(b) Shaft BC: $T_{BC} = 300 + 400 = 700 \text{ N}\cdot\text{m}$

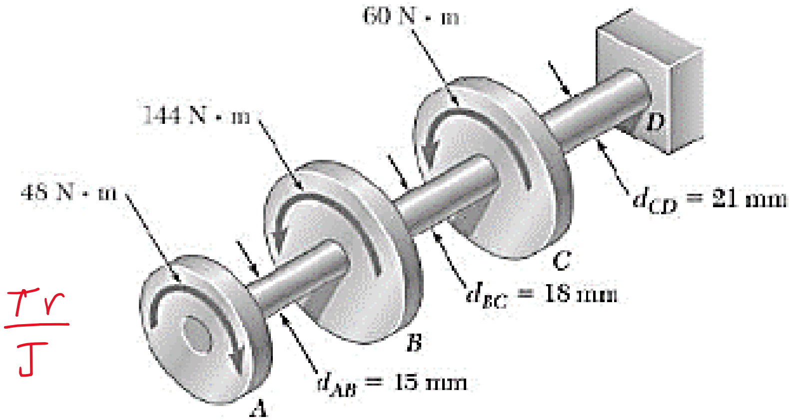
$$d = 0.046 \text{ m},\ c = 0.023 \text{ m}$$

$$\tau_{max} = \frac{Tc}{J} = \frac{2T}{\pi c^3} = \frac{(2)(700)}{\pi(0.023)^3}$$

$$= 36.626 \times 10^6 \text{ Pa}$$

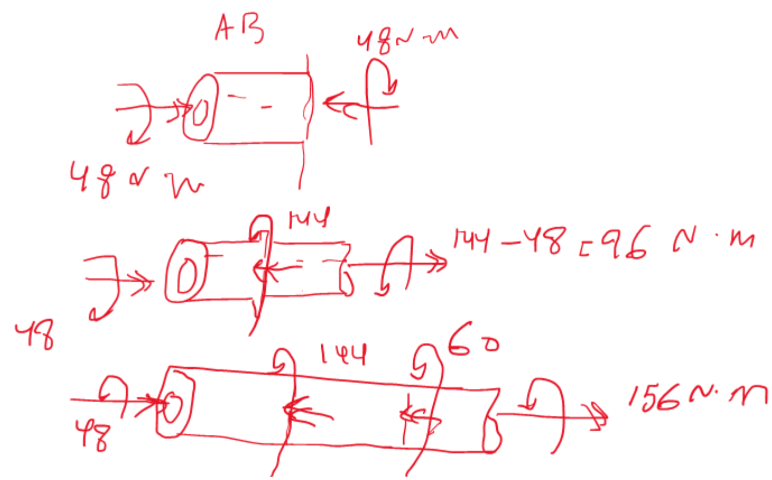
$$\tau_{max} = 36.6 \text{ MPa}$$





Knowing that an 8-mm-diameter hole has been drilled through each of the shafts *AB*, *BC*, and *CD*, determine (*a*) the shaft in which the maximum shearing stress occurs, (*b*) the magnitude of that stress.

$\tau = \frac{Tr}{J}$



shaft AB

$$\tau_{max} = \frac{48000 \times 7.5}{\frac{\pi(7.5^4 - 4^4)}{2}} = 78.8 \text{ MPa}$$

Shaft BC

$$\tau_{max} = \frac{96000 \times 9}{\frac{\pi(9^4 - 4^4)}{2}} = 87.3 \text{ MPa}$$

Shaft CD

$$\tau_{max} = \frac{156000 \times 10.5}{\frac{\pi(10.5^4 - 4^4)}{2}} = 87.64 \text{ MPa} \leftarrow * \text{Max}$$

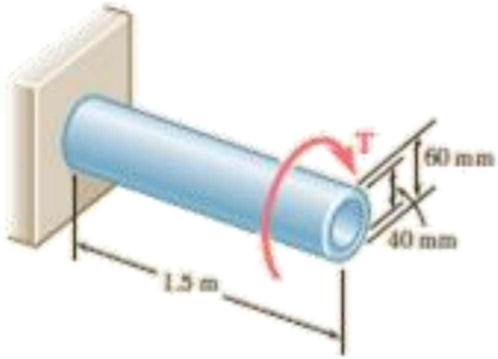


Fig. 3.15 Hollow, fixed-end shaft having torque *T* applied at end.

A hollow cylindrical steel shaft is 1.5 m long and has inner and outer diameters respectively equal to 40 and 60 mm (Fig. 3.15). (*a*) What is the largest torque that can be applied to the shaft if the shearing stress is not to exceed 120 MPa? (*b*) What is the corresponding minimum value of the shearing stress in the shaft?

The largest torque **T** that can be applied to the shaft is the torque for which $\tau_{max} = 120$ MPa. Since this is less than the yield strength for any steel, use Eq. (3.9). Solving this equation for *T*,

$\tau_{all} = 120$ MPa

$$\tau = \frac{Tr}{J}$$

$$J = \frac{\pi(30^4 - 20^4)}{2} = 1.02 \times 10^6 \text{ mm}^4$$

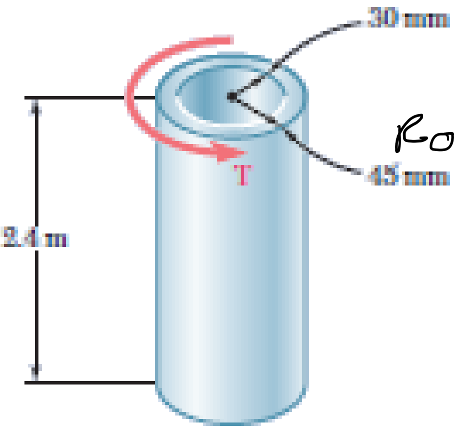
$$120 = \frac{(T)(30)}{1.02 \times 10^6} \Rightarrow$$

$$T = 4.08 \times 10^6 \text{ N.mm} = \underline{4.08 \text{ kN.m}}$$

$$\tau_{min} = \frac{T r_2}{J} = \frac{(4.08 \times 10^6)(20)}{1.02 \times 10^6}$$

$$= 80 \text{ MPa}$$

Determine the torque **T** that causes a maximum shearing stress of 45 MPa in the hollow cylindrical steel shaft shown. (*b*) Determine the maximum shearing stress caused by the same torque **T** in a solid cylindrical shaft of the same cross-sectional area.



$$J = \frac{\pi(45^4 - 30^4)}{2} = 5168901.7 \text{ mm}^4$$

$$\tau = \frac{Tr}{J} = 45 = \frac{(T)(45)}{5168901.7} \Longrightarrow$$

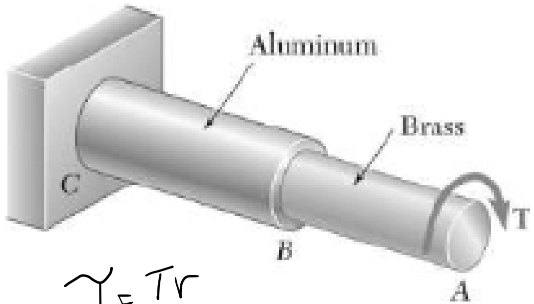
$$T = 5.1689 \text{ kN}\cdot\text{m}$$

solid shaft of the same area

$$\pi r^2 = \pi(45^2 - 30^2) \Rightarrow r = 33.541 \text{ mm} \Longrightarrow J = \frac{\pi \, 33.541^4}{2}$$

$$\tau = \frac{Tr}{J} = \frac{(5.1689\times10^6)(33.541)}{1988034.44} = 87.21 \text{ MPa}$$

$$J_{solid} = 1988034.44 \text{ mm}^4$$



The allowable stress is 50 MPa in the brass rod AB and 25 MPa in the aluminum rod BC. Knowing that a torque of magnitude $T = 1250$ N · m is applied at A, determine the required diameter of (*a*) rod AB, (*b*) rod BC.

$$\tau = \frac{Tr}{J}$$

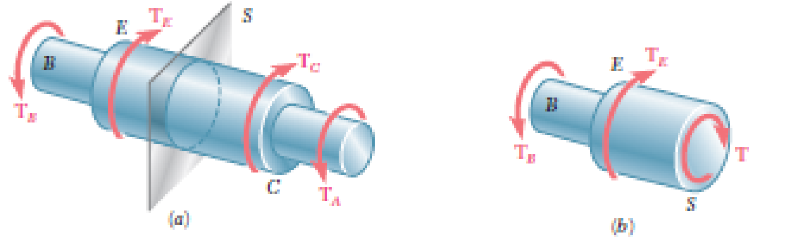
Brass

$$50 = \frac{1250 \times 10^3\, r}{\frac{\pi r^4}{2}} \Longrightarrow r = 25.16 \Longrightarrow D_{AB} = 50.3 \text{ mm}$$

Alluminum

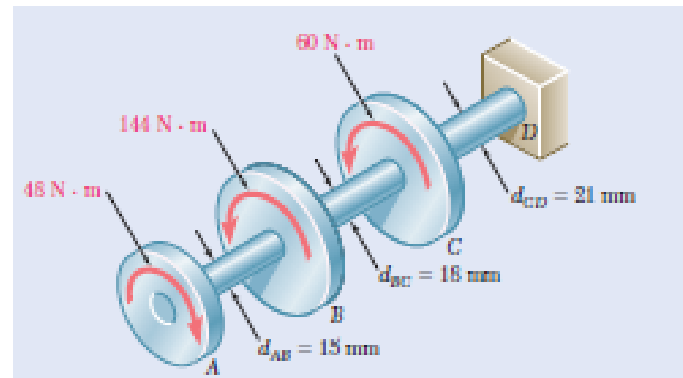
$$25 = \frac{1250 \times 10^3\, r}{\frac{\pi r^4}{2}} \Longrightarrow r = 31.69 \text{ mm} \Longrightarrow D_{BC} = 63.4 \text{ mm}$$

Shaft of variable cross section or for a shaft subjected to torques at locations other than its ends



Shaft with variable cross section. (a) With applied torques and section S. (b) Free-body diagram of sectioned shaft.

Knowing that each of the shafts *AB*, *BC*, and *CD* consist of a solid circular rod, determine (*a*) the shaft in which the maximum shearing stress occurs, (*b*) the magnitude of that stress.



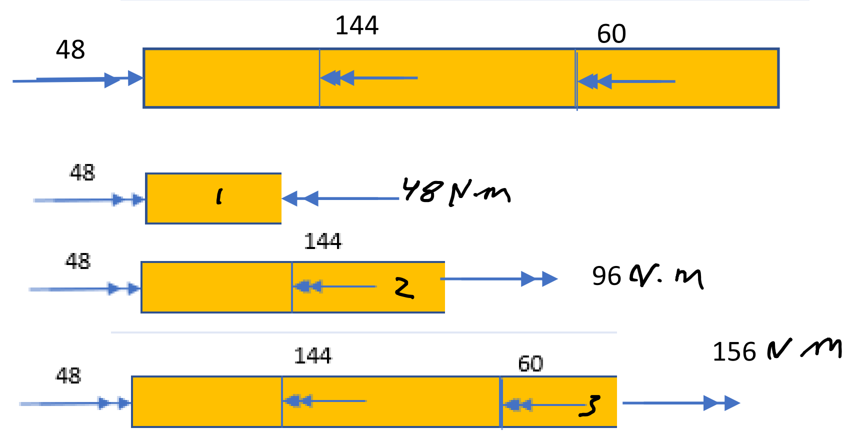
you have to check the shear stress at each section : $\tau = \frac{Tr}{J}$

$$\tau_1 = \frac{(48000)(7.5)\times 2}{\pi(7.5)^4} = 72.43 \text{ MPa}$$

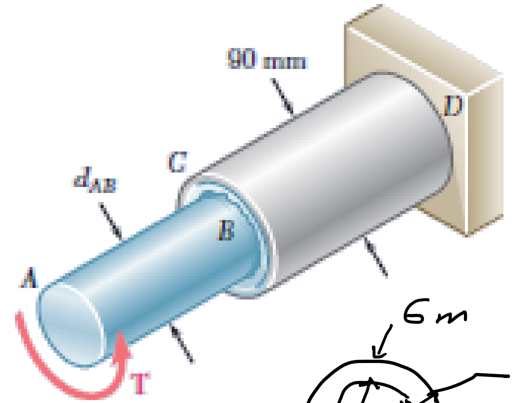
$$\tau_2 = \frac{(96000)(9)\times 2}{\pi\, 9^4} = 83.83 \text{ MPa}$$

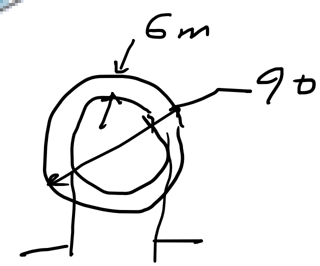
$$\tau_3 = \frac{(156000)(10.5)(2)}{\pi(10.5)^4} = 85.79 \text{ MPa}$$

Max in shaft CD



The solid rod *AB* has a diameter d_{AB} **= 60 mm** and is made of a steel for which the allowable **shearing stress is 85 MPa.** The pipe *CD*, which has an **outer diameter of 90 mm** and a wall **thickness of 6 mm**, is made of an aluminum for which the **allowable shearing stress is 54 MPa**. Determine the largest **torque T that can be applied at *A***.





90 − 12 = 78

AB

$$\tau_{all} = 85 = \frac{2T(30)}{\pi(30)^4} \Rightarrow T = 3.603 \text{ KN.m}$$

CD

$$J = \frac{\pi(90^4 - 78^4)}{32} = 2807302.06 \text{ mm}^4$$

$$\tau_{all} = 54 = \frac{(T)(45)}{2867302.06} \Rightarrow \underline{\underline{T = 3.3688 \text{ KN.m}}}$$

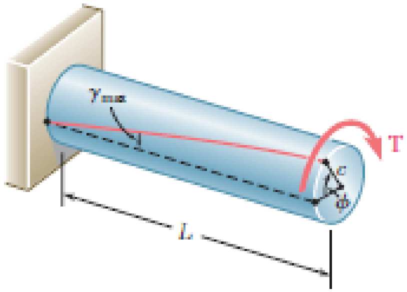
Take the smaller

5.4 ANGLE OF TWIST

$$\gamma_{max} = \frac{c\phi}{L}$$

But in the elastic range, the yield stress is not exceeded anywhere in the shaft. Hooke's law applies, and $\gamma_{max} = \tau_{max}/G$.

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG}$$



Torque applied to fixed end shaft resulting in angle of twist ϕ.

$$\boxed{\phi = \frac{TL}{JG}}$$

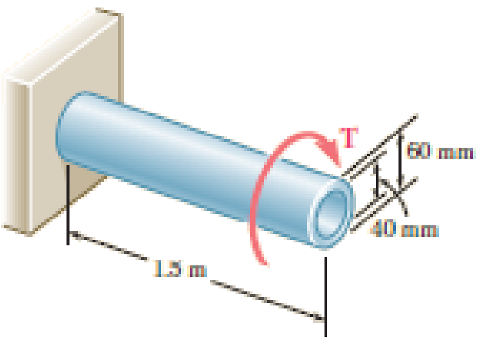
ϕ is in radians.

What torque should be applied to the end of the, to produce a twist of 2°? Use the value G = 77 GPa for the modulus of rigidity of steel.

$$\phi = \frac{2 \times \pi}{180} = 0.03491 \text{ rad}$$

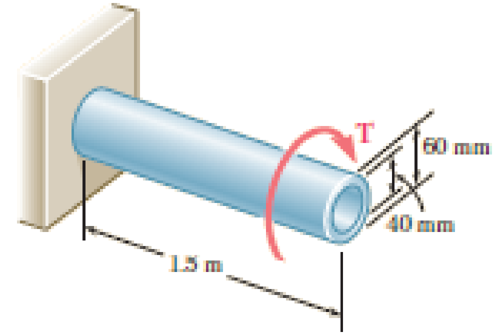
$$\phi = \frac{TL}{GJ} \quad , \quad J = \frac{\pi(60^4 - 40^4)}{32} = 1021017.6 \text{ mm}^4$$

$$0.03491 = \frac{(T)(1.5 \times 1000)}{(77000)(1021017.6)} \Rightarrow T = 18.297 \text{ kN.m}$$



Hollow, fixed-end shaft having torque T applied at end.

What angle of twist will create a shearing stress of 70 MPa on the inner surface of the hollow steel shaft. Determine the angle of twist ϕ corresponding to the value of *T, G= 77 GPa*



Hollow, fixed-end shaft having torque T applied at end.

$$\phi = \frac{TL}{GJ} \quad , \quad \tau = \frac{Tr}{J}$$

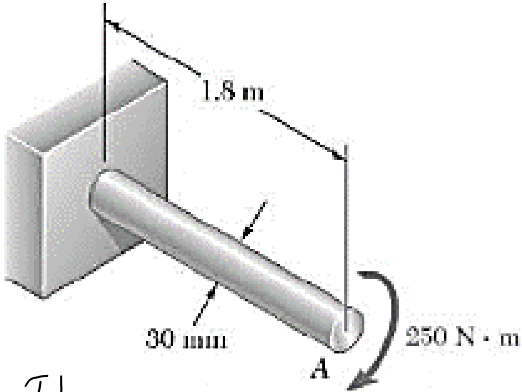
$$J = \frac{\pi(60^4 - 40^4)}{32}$$
$$= 1021017.6 \ mm^4$$

$$\tau = \frac{Tr}{J} = 70 = \frac{20T}{1021017.6} \Rightarrow$$

$$T = 3573561.6 \ N\cdot mm$$

$$\phi = \frac{(3573561.6)(1.5\times10^3)}{(77000)(1021017.6)} = 0.0682 \ rad = \frac{0.0682}{\pi} \times 180$$

$$\underline{\phi = 3.91^\circ}$$



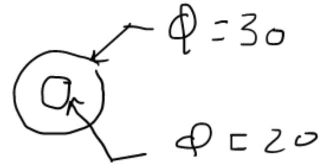
(*a*) For the solid steel shaft shown (G = 77 GPa), determine the angle of twist at A. (*b*) Solve part *a*, assuming that the steel shaft is hollow with a 30-mm-outer diameter and a 20-mm-inner diameter.

$$\phi = \frac{TL}{GJ}$$

2

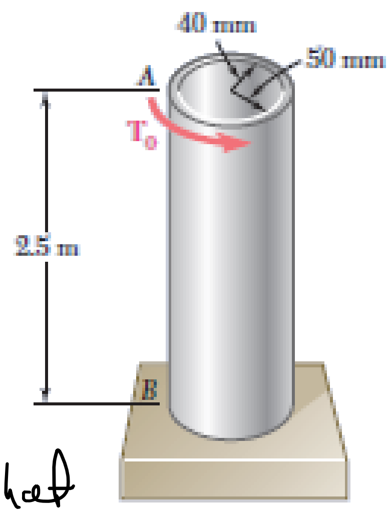
$$\phi_A = \frac{(250\times10^3)(1.8\times1000)}{\underline{\pi\times15^4}\times 77\times10^3} = 0.074 \text{ rad} = \underline{4.21^\circ}$$

If the shaft is Hollow



$$\phi = \frac{(250\times10^3)(1.8\times10^3)}{(77\times10^3)\,\pi\,\frac{(15^4-10^4)}{2}} = 0.0916 \text{ rad} = \underline{5.25^\circ}$$

(*a*) For the aluminum pipe shown (G = 27 GPa), determine the torque $\mathbf{T}_0$ causing an angle of twist of 2º. (*b*) Determine the angle of twist if the same torque $\mathbf{T}_0$ is applied to a solid cylindrical shaft of the same length and cross-sectional area.



$$\phi = \frac{2}{180} \times \pi = 0.035 \text{ rad}$$

$$J = \frac{\pi (50^4 - 40^4)}{2} = 5793300 \text{ mm}^4$$

$$\phi = \frac{TL}{GJ} = \frac{2500T}{(27\times10^3)(5793300)} = 0.035$$

$$\underline{T = 2.19 \text{ KN.M}}$$

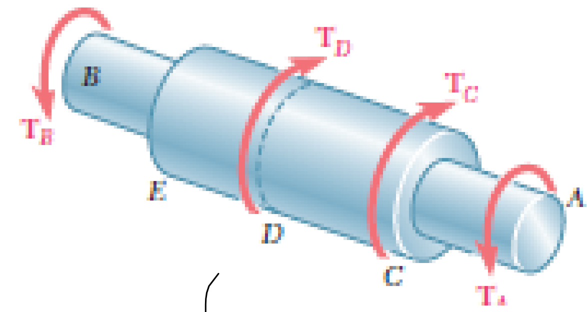
$$\text{Area of shaft} = \pi (50^2 - 40^2) = 2826 \text{ mm}^2 = A_{solid\ shaft}$$

$$\pi r^2 = 2826 \Rightarrow r_{solid} = 30 \text{ mm}$$

$$J = \frac{(\pi)(30)^4}{2} = 1271700 \text{ mm}^4$$

$$\phi_{solid} = \frac{(2.19\times10^6)(2.5\times10^3)}{(27\times10^3)(1271700)} = 0.1595 \text{ rad} = \underline{9.14^0}$$

Shaft with multiple cross-section dimensions and multiple loads.

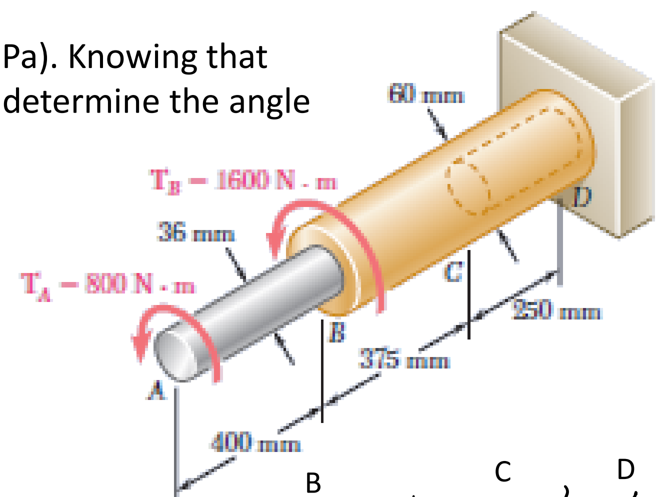


$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

If the shaft is subjected to torques at locations other than its ends or if it has several portions with various cross sections and possibly of different materials, it must be divided into parts

---

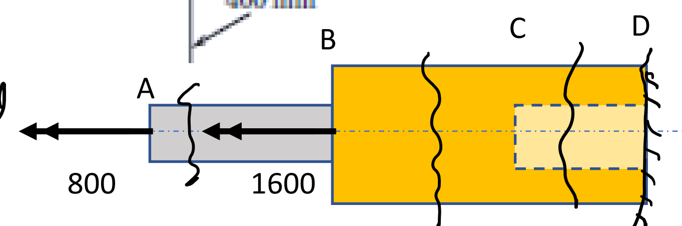
The aluminum rod *AB* (*G* = 27 GPa) is bonded to the brass rod *BD* (*G* = 39 GPa). Knowing that portion *CD* of the brass rod is hollow and has an inner diameter of 40 mm, determine the angle of twist at *A*.



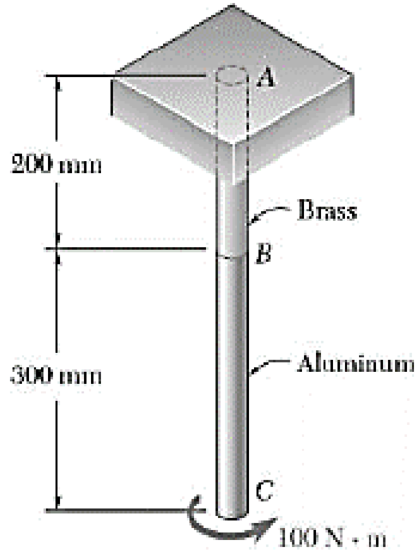
$$\phi_{AB} = \frac{(800)(400)\times 1000}{(27\times10^3)\left(\frac{\pi\, 36^4}{32}\right)} = 0.0719 \text{ rad}$$

$$\phi_{BC} = \frac{(2400)(375)\times 1000}{(39\times10^3)\left(\frac{\pi\, 60^4}{32}\right)} = 1.815\times10^{-2} \text{ rad}$$

$$\phi_{CD} = \frac{(2400)(250)\times 1000}{(39\times10^3)\left(\frac{\pi(60^4-40^4)}{32}\right)} = 1.5676\times10^{-2} \text{ rad}$$



$$\phi_A = \sum \phi = 0.105 \times (180/\pi) = \underline{6.02^\circ}$$



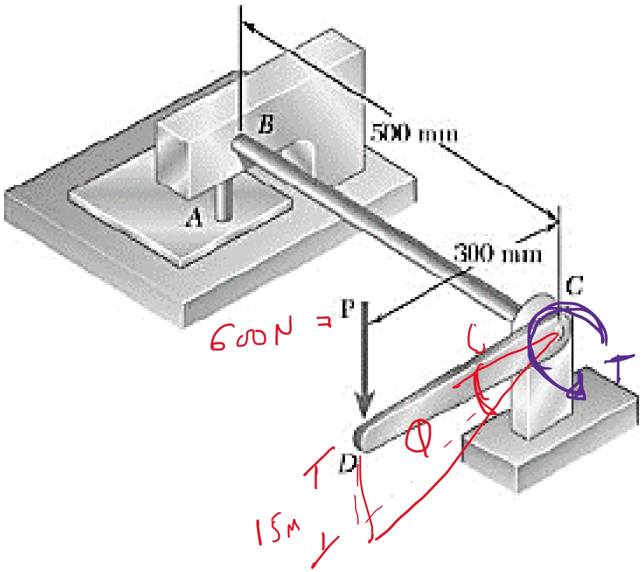
The aluminum rod BC ($G = 26$ GPa) is bonded to the brass rod AB ($G = 39$ GPa). Knowing that each rod is solid and has a diameter of 12 mm, determine the angle of twist (*a*) at B, (*b*) at C.

$$\phi = \frac{TL}{GJ}$$

$$\phi_{B/A} = \frac{(10000)(200)}{(39000)\left(\frac{\pi \times 6^4}{2}\right)} = 0.252\,\text{rad} = \underline{14.43^\circ}$$

$$\phi_{C/A} = \phi_{B/A} + \phi_{C/B}$$

$$= 0.25 + \frac{(10000)(300)}{(26000)\left(\frac{\pi \times 6^4}{2}\right)} = 0.25 + 0.567 = 0.817\,\text{rad} = \underline{46.8^\circ}$$



A hole is punched at A in a plastic sheet by applying a 600-N force **P** to end D of lever CD, which is rigidly attached to the solid cylindrical shaft BC. Design specifications require that the displacement of D should not exceed 15 mm from the time the punch first touches the plastic sheet to the time it actually penetrates it. Determine the required diameter of shaft BC if the shaft is made of a steel with $G = 77$ GPa and $\tau_{all} = 80$ MPa.

$T = (600)(300) = 180 \times 10^3$ N·mm

$\Phi_{all} = \frac{15}{300} = 0.05$ rad.

Allowable stress

$$\tau = \frac{Tr}{J} = \frac{(180 \times 10^3)(r)}{\frac{\pi r^4}{2}} = 80 \Rightarrow r = 11.27 \Rightarrow d = 22.54 \text{ mm}$$

Take Larger

Allowable Φ

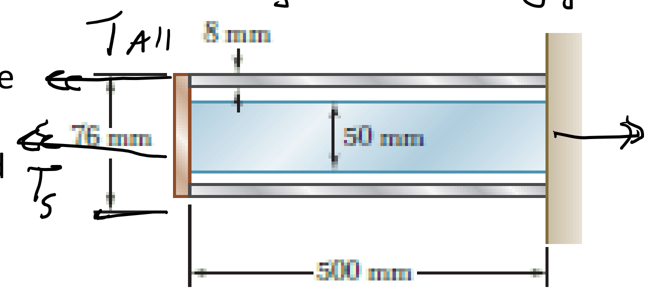
$$\Phi = \frac{TL}{GJ} = 0.05 = \frac{(180000)(500)(32)}{(77000)(\pi d^4)} \Rightarrow d = 22.1 \text{ mm}$$

5.5 STATICALLY INDETERMINATE SHAFTS

The equilibrium equations must be complemented by relations involving the deformations of the shaft and obtained by the geometry of the problem.

A steel shaft and an aluminum tube are connected to a fixed support and to a rigid disk as shown in the cross section. Knowing that the initial stresses are zero, **determine the maximum torque T_0 that can be applied to the disk if the allowable stresses are 120 MPa in the steel** shaft **and 70 MPa in the aluminum tube**. Use ***G*** **= 77 GPa** for steel and ***G*** **= 27 GPa** for aluminum.

$\tau = \frac{Tr}{J}$, $\phi = \frac{TL}{GJ}$



$$T_0 = T_s + T_{All} \quad , \quad \phi_s = \phi_{All}$$

$$\frac{500\, T_s \times 2}{(77\times10^3)\,\pi\,(25)^4} = \frac{(500\, T_{All})(32)}{(27\times10^3)\,\pi\,(76^4-60^4)} \Rightarrow 1.06\times10^{-8}\, T_s = 9.25\times10^{-9}\, T_{All}$$

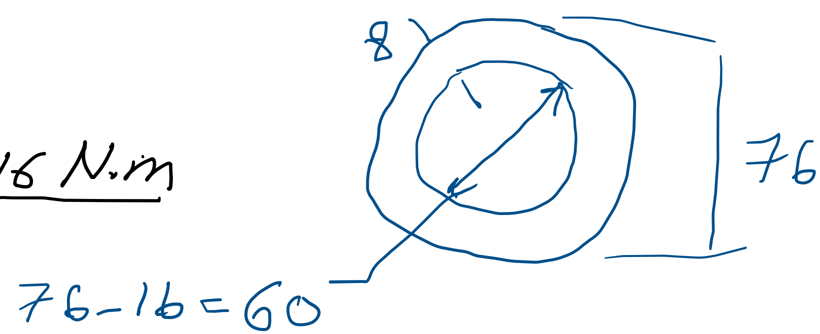
$$T_s = 0.873\, T_{All}$$

$$\tau_{s\,all} = \frac{T_s\, r}{J} = 120 = \frac{(T_s)(25)\times 2}{\pi\,(25)^4}$$

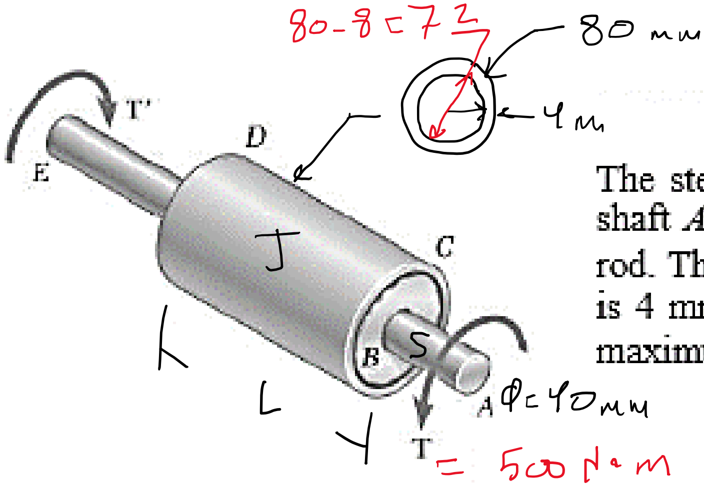
$T_s = 2943750$ N.mm $= 2944$ N.m $\Rightarrow$ check all stress in All

$$T_{All} = 2944/0.873 = 3372.3 \text{ N.m}$$

$$\tau_{All} = \frac{(3372.3\times10^3)(76/2)(32)}{\pi\,(76^4-60^4)} = 64.1 \text{ MPa} < 70 \text{ MPa} \quad \text{o.k}$$



$$T_0 = 2944 + 3372.3 = 6316 \text{ N.m}$$



The steel jacket *CD* has been attached to the 40-mm-diameter steel shaft *AE* by means of *rigid* flanges welded to the jacket and to the rod. The outer diameter of the jacket is 80 mm and its wall thickness is 4 mm. If 500 N · m torques are applied as shown, determine the maximum shearing stress in the jacket.

$$\phi = \frac{TL}{GJ} , \quad \tau = \frac{Tr}{J}$$

$$T = 500\times10^3 = T_J + T_s \quad \text{— (1)}$$

$$\phi_J = \phi_s$$

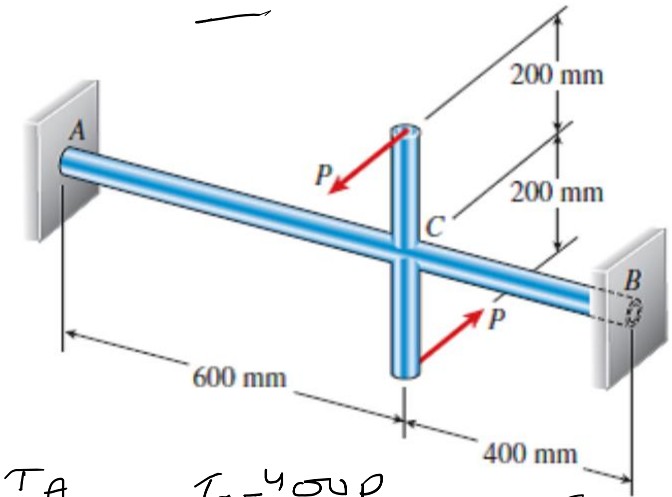
$$\frac{T_J\,(L)\times 32}{\pi G\,(80^4 - 72^4)} = \frac{T_s\,(L)(32)}{\pi G\,(40)^4} \Rightarrow T_J = 5.502\,T_s \quad \text{— (2)}$$

$$5.502\,T_s + T_s = 500\times10^3 \Rightarrow T_s = 76.9\times10^3 \text{ N.mm}$$

$$T_J = 76.9\times10^3 \times 5.502 = 423.1\times10^3 \text{ N-mm}$$

$$\tau_{max\,jacket} = \frac{(423.1\times10^3)(40)\,32}{\pi\,(80^4 - 72^4)} = \underline{12.24 \text{ MPa}}$$

A hollow steel shaft *ACB* of outside diameter 50 mm and inside diameter 40 mm is held against rotation at ends *A* and *B*. Determine the allowable value of the forces *P* if the maximum permissible shear stress in the shaft is 45 MPa

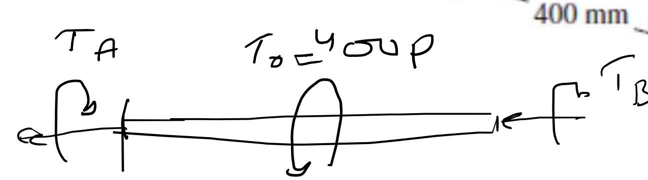


$$T_A = \frac{T_0 L_B}{L} \qquad T_B = \frac{T_0 L_A}{L}$$

$J = \pi(25^4 - 20^4)/2 = 362264.9 \text{ mm}^4$

$T_0 = 400P \qquad \tau_{all} = 45 \text{ MPa}$

$\phi_{AC} = \phi_{CB} \quad , \quad T_0 = T_A + T_B$



$$\frac{600 T_A}{GJ} = \frac{400 T_B}{GJ} \Rightarrow T_A = 0.67 T_B$$

$$400P = 0.67T_B + T_B \Rightarrow T_B = 239.5P \text{ mm}$$

$$T_A = 160.5P \text{ N.mm}$$

AC $\quad 45 = \dfrac{(160.5P)(25)}{362264.9} \Rightarrow P = 4090.9 \text{ N}$

CB $\quad 45 = \dfrac{(239.5P)(25)}{362264.9} \Rightarrow P = 2722.7 \text{ N}$ ⟵ Take smaller

Structural Mechanics

Chapter 6

Bending

R.C. Hibbler, Mechanics of Materials,

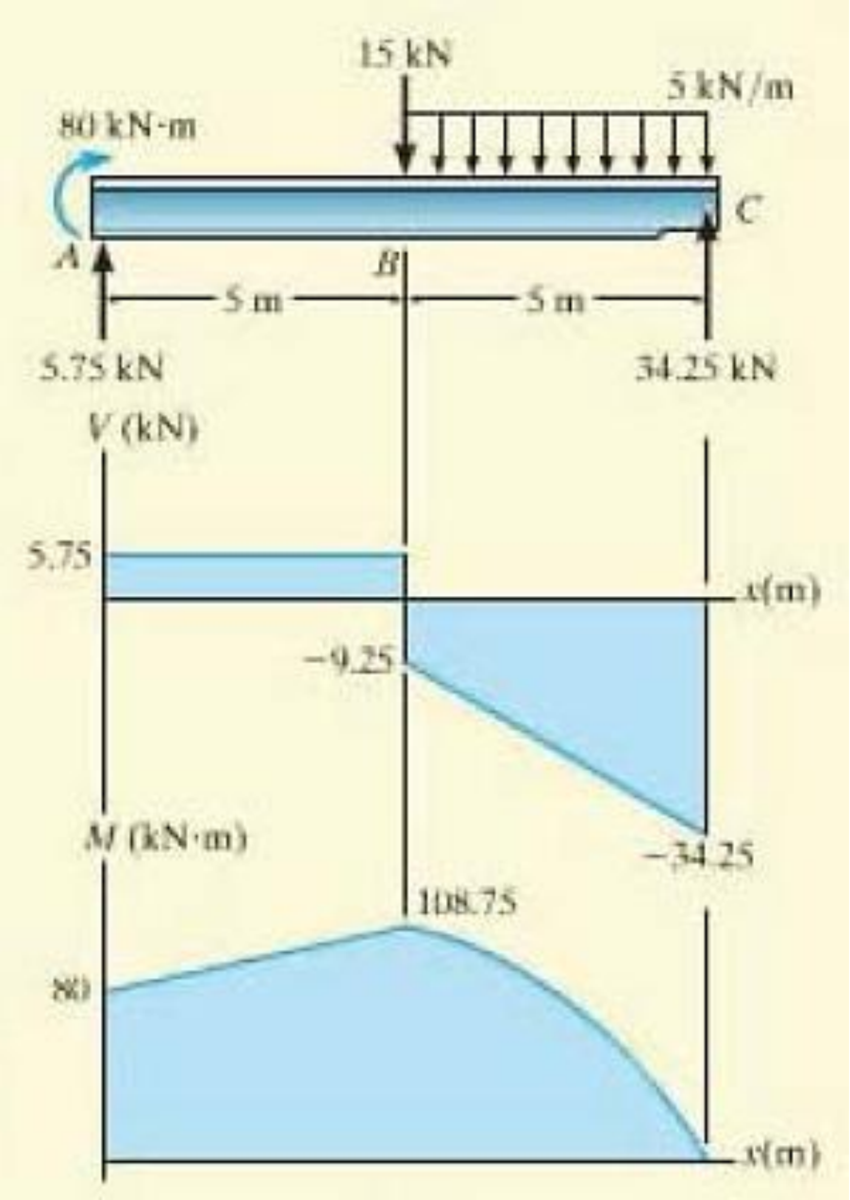
Lee, D.V., Beer, Johnston, DeWolf & Mazurek, " Mechanics of Materials, McGraw Hill

I. Marie

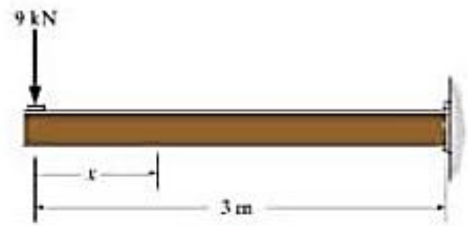
6.1 + 6.2 Shear and Moment Diagrams

Go back to your static book

Draw shear and bending moment diagrams

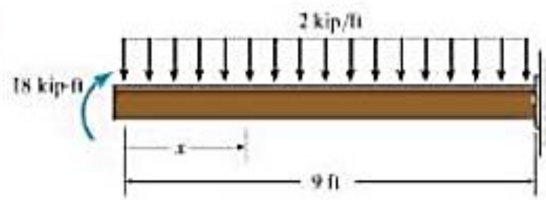


F6–1. Express the shear and moment functions in terms of x, and then draw the shear and moment diagrams for the cantilever beam.



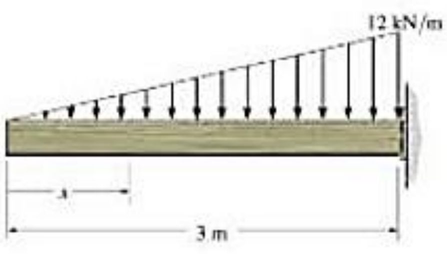
F6–1

F6–2. Express the shear and moment functions in terms of x, and then draw the shear and moment diagrams for the cantilever beam.

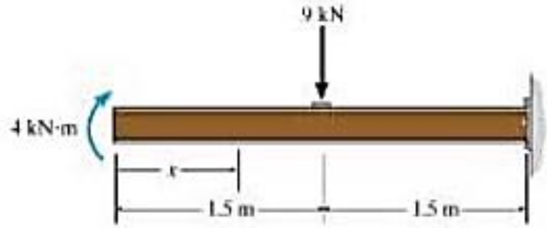


F6–2

F6–3. Express the shear and moment functions in terms of x, and then draw the shear and moment diagrams for the cantilever beam.

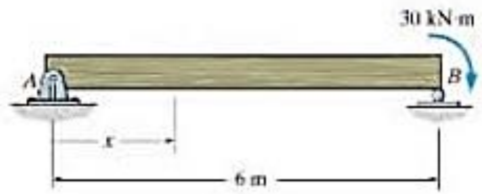


F6–4. Express the shear and moment functions in terms of x, where $0 < x < 1.5$ m and $1.5 \text{ m} < x < 3$ m, and then draw the shear and moment diagrams for the cantilever beam.



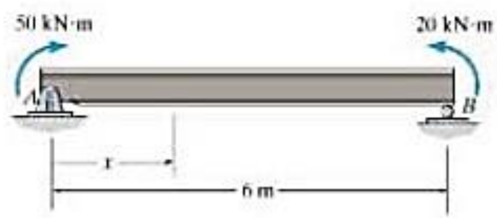
F6–4

F6–5. Express the shear and moment functions in terms of x, and then draw the shear and moment diagrams for the simply supported beam.



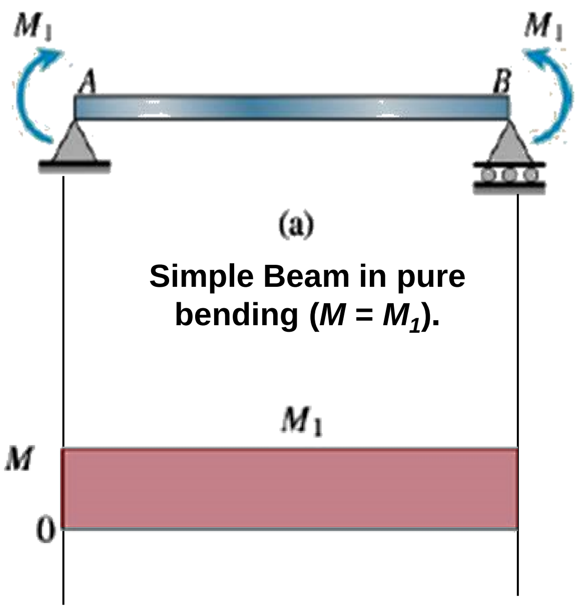
F6–5

F6–6. Express the shear and moment functions in terms of x, and then draw the shear and moment diagrams for the simply supported beam.

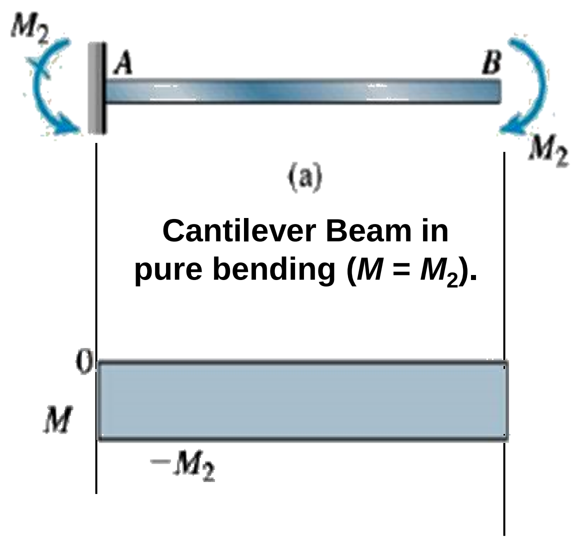


6.3 Bending deformation of straight members
6.4 flexure formula

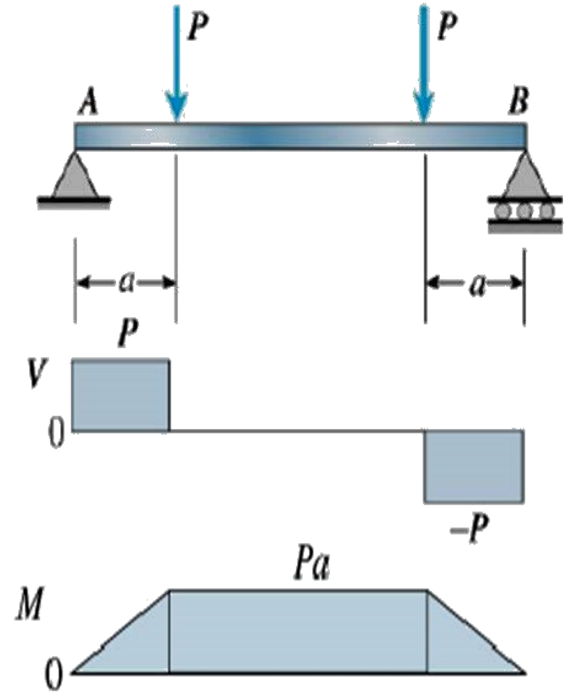
Pure bending = No shear,



Simple Beam in pure bending ($M = M_1$).

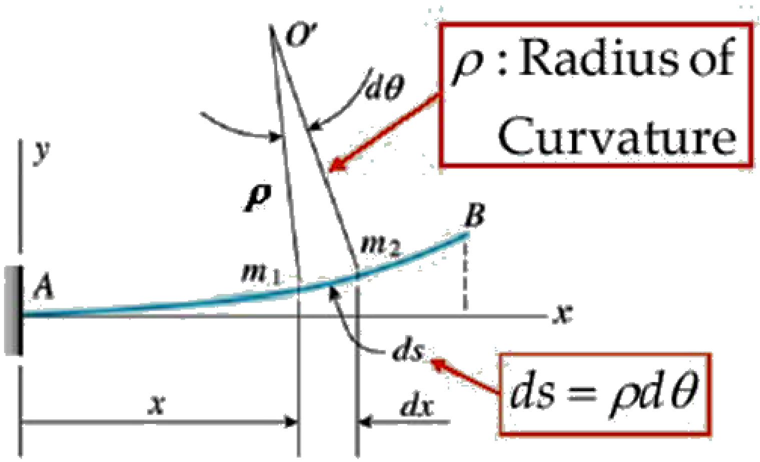


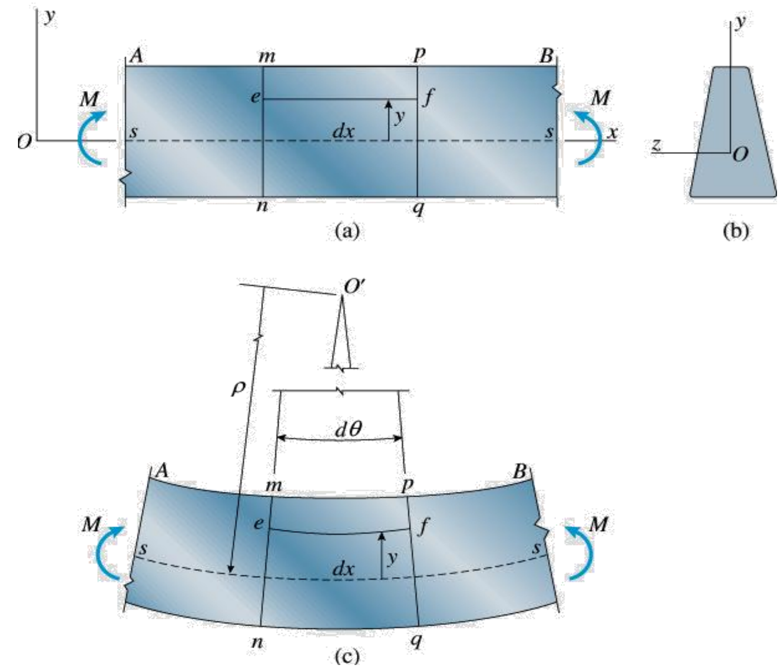
Cantilever Beam in pure bending ($M = M_2$).



Simple beam with central region in pure bending

Curvature of a Beam





Radius of Curvature : ρ

Curvature : κ

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{ds}$$

For Infinitesimal Deformation

$$ds \approx dx \quad \Rightarrow \quad \kappa = \frac{1}{\rho} \approx \frac{d\theta}{dx}$$

1. Cross sections (mn and pq) remain plane

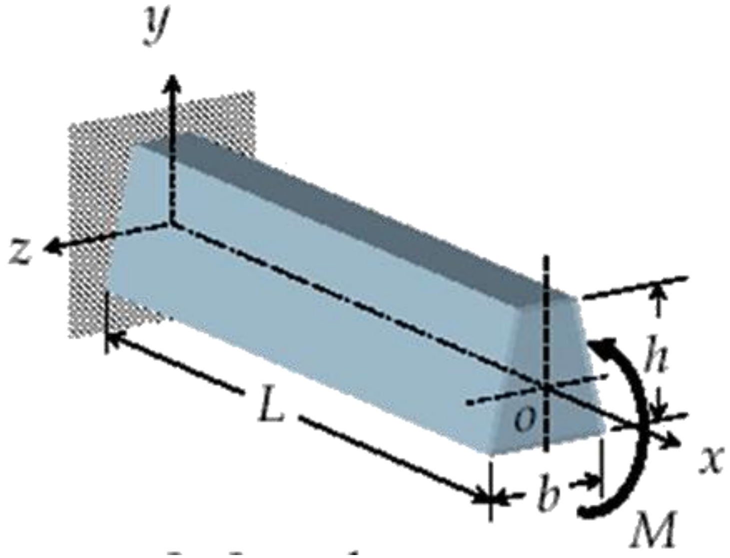
2. Cross sections remain perpendicular to the axis of the beam

3. For positive moments (hence positive curvature), lines on the lower part of the beam (nq) are elongated; those on the upper part (mp) are shortened

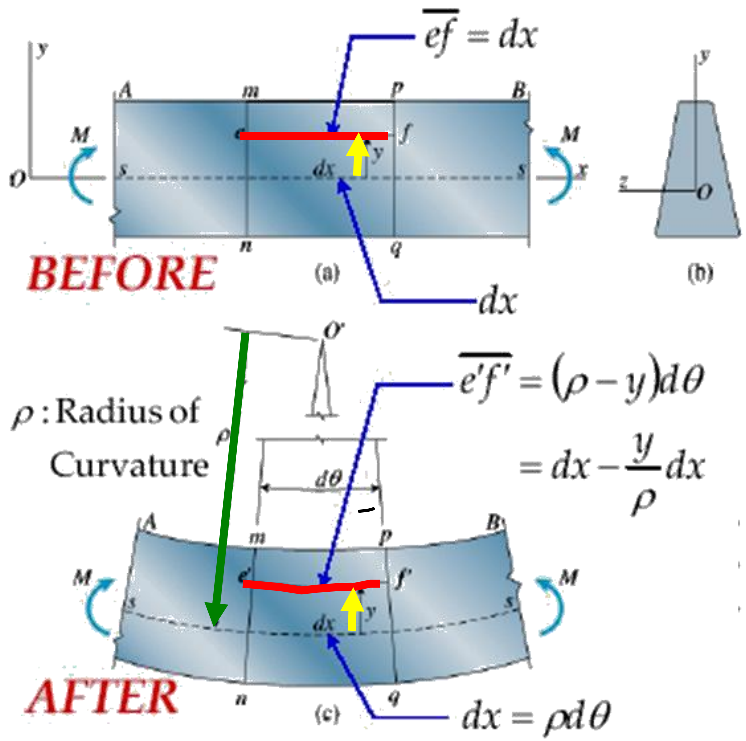
4. Somewhere between top and bottom there is a line whose length does not change, and is called "Neutral Axis"

Longitudinal Strains in beams

- *xy* plane is a plane of symmetry
- Loading is applied in *xy* plane
- Beam deflects in *xy* plane
- Thickness of the beam, *h, remains* unchanged
- Axis of the beam coincides with the centroidal line of the cross section



6.3 bending deformation of a straight bar
Normal Strain Due to Bending



dx/ρ

$$\boxed{\varepsilon_x = \frac{\overline{e'f'} - \overline{ef}}{\overline{ef}} = \frac{(\rho - y)d\theta - dx}{dx}}$$

$$\boxed{= \frac{\left[dx - \frac{y}{\rho}dx\right] - dx}{dx} = -\frac{y}{\rho}}$$

$$\boxed{\varepsilon_x = -\frac{y}{\rho} = -\kappa y}$$

The minus sign is because it is assumed the bending moment is positive, and thus the beam is concave upward.

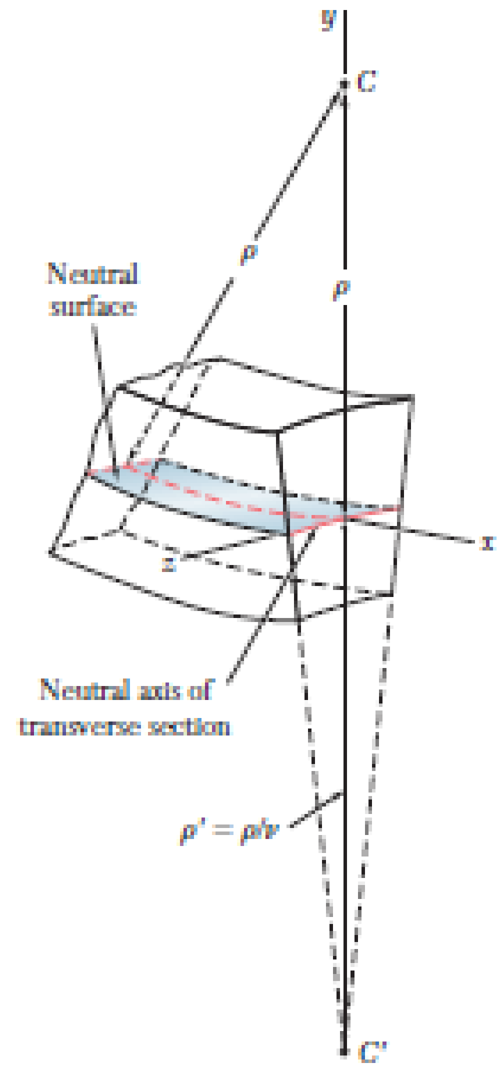
- ***Strains vary linearly with y***
- ***Along x-axis (y = 0) strain is zero***
- ***For a positive curvature, strains on upper part of the beam (y > 0) are negative (in compression) and those on lower part (y < 0) are positive (in tension)***

DEFORMATIONS IN A TRANSVERSE CROSS SECTION

$$\epsilon_y = -\nu\epsilon_x \qquad \epsilon_z = -\nu\epsilon_x$$

$$\epsilon_y = \frac{\nu y}{\rho} \qquad \epsilon_z = \frac{\nu y}{\rho}$$

$$\boxed{\frac{1}{\rho} = \frac{M}{EI}}$$



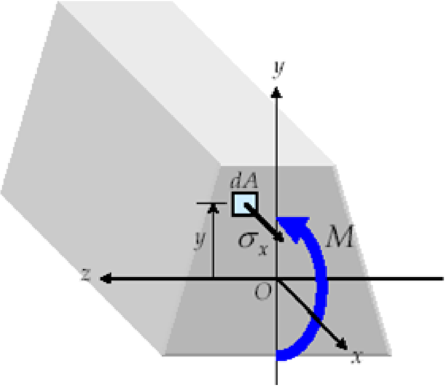
Deformation of a transverse cross section.

6.4 flexure formula

$$\sigma_x = E\varepsilon_x = -\frac{Ey}{\rho} = -E\kappa y$$

Moment due to $\sigma_x dA$:

$$dM = (\sigma_x dA)y = -E\kappa y^2 dA$$



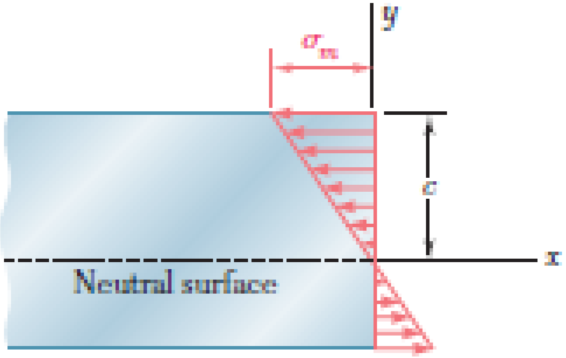
The resultant moment of the normal stress over the cross section must equal to the applied moment *M*

$$\boxed{M = -\int_A \sigma_x y dA = \kappa E \int_A y^2 dA = \kappa E I_z \quad \Rightarrow \quad \kappa = \frac{1}{\rho} = \frac{M}{EI_z}}$$

$$\boxed{\sigma_x = -\kappa E y = -\left(\frac{M}{EI_z}\right)(Ey) = -\frac{My}{I_z}}$$

$$\boxed{\sigma_x = -\frac{My}{I_z}}$$

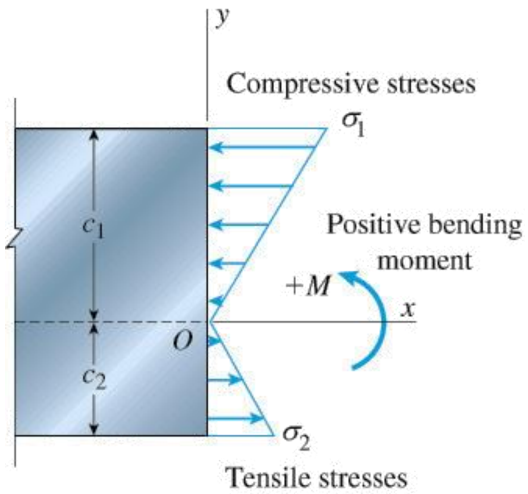
flexural stress.



Bending stresses vary linearly with distance from the neutral axis.

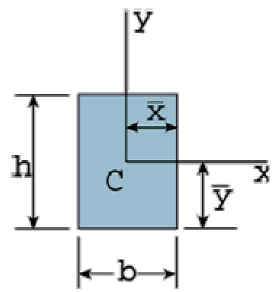
$$I_z = \int_A y^2 dA = \text{Moment of Inertia}$$

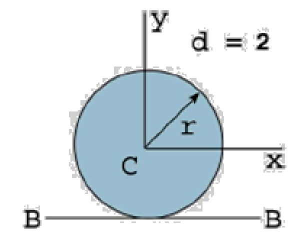
Maximum Stresses at a Cross Section



(a)

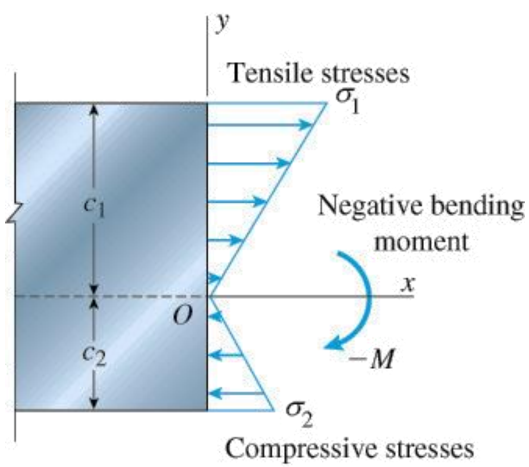
$$I_x = \int_A y^2 dA, \quad I_y = \int_A x^2 dA$$





$$I_x = \frac{bh^3}{12}, \quad I_y = \frac{hb^3}{12}$$

$$I_x = I_y = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}$$



(b)

$$\kappa = \frac{1}{\rho} = \frac{M}{EI}$$

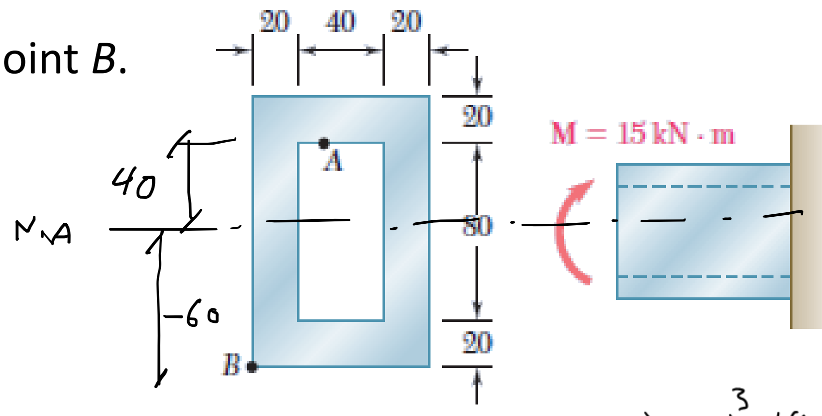
$$\sigma_x = -\frac{My}{I}$$

(1) Centroid, (2) Moment of Inertia I_z

Important Points

- The cross section of a straight beam *remains plane* when the beam deforms due to bending. This causes tensile stress on one portion of the cross section and compressive stress on the other portion. In between these portions, there exists the *neutral axis* which is subjected to *zero stress*.
- Due to the deformation, the *longitudinal strain* varies *linearly* from zero at the neutral axis to a maximum at the outer fibers of the beam. Provided the material is homogeneous and linear elastic, then the *stress* also varies in a *linear* fashion over the cross section.
- The neutral axis passes through the *centroid* of the cross-sectional area. This result is based on the fact that the resultant normal force acting on the cross section must be zero.
- The flexure formula is based on the requirement that the resultant internal moment on the cross section is equal to the moment produced by the normal stress distribution about the neutral axis.

determine the stress at (*a*) point *A*, (*b*) point *B*.



Dimensions in mm

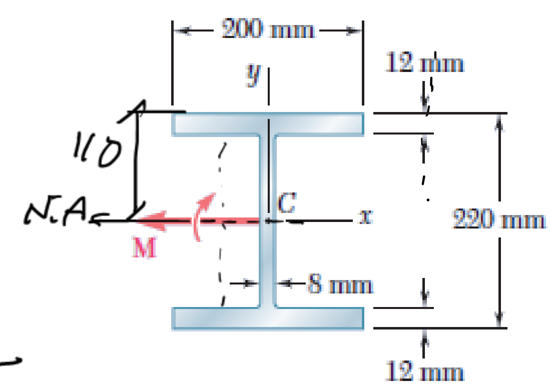
$$\sigma_A = -\frac{My}{I}$$

$$= -\frac{(15\times10^6)(40)}{9.813\times10^6} = -61.14 \text{ MPa} \quad (C)$$

$$I = \frac{(80)(120)^3}{12} - \frac{(40)(80)^3}{12} \simeq 9.813\times10^6 \text{ mm}^4$$

$$\sigma_B = -\frac{(15\times10^6)(-60)}{9.813\times10^6} = 91.72 \text{ MPa} \quad (T)$$

Using an **allowable stress of 155 MPa**, determine the largest bending moment **M** that can be applied to the wide-flange beam shown. Neglect the effect of fillets.



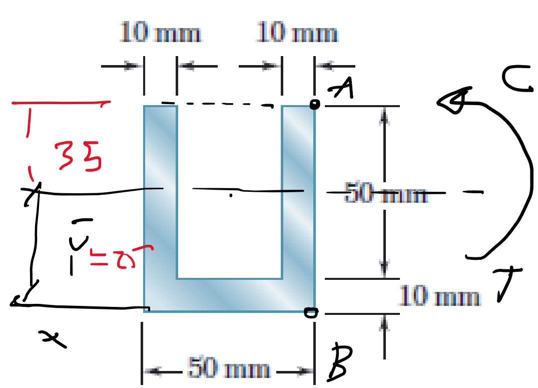
$$I = \frac{(200)(220)^3}{12} - (2)\frac{(96)(196)^3}{12} = 56.9941\times10^6 \text{ mm}^4$$

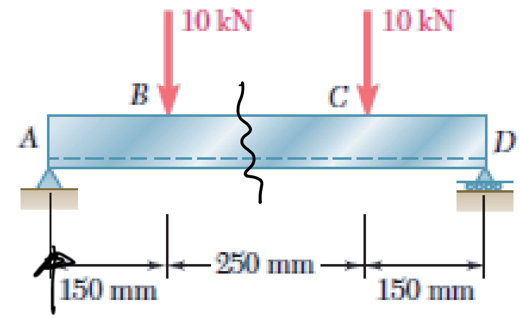
$$\sigma = \frac{My}{I} = 155 = \frac{M\times110}{56.9941\times10^6} \Rightarrow$$

$$M = 80.301\times10^6 \text{ N mm} = \underline{80.301 \text{ kN.m}}$$

Determine the maximum tensile and compressive stresses in portion *BC* of the beam.

Cross-section

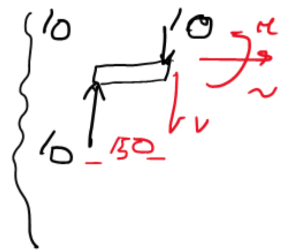




- centroid: $\bar{Y}_c = \dfrac{\Sigma y_i A_i}{\Sigma A_i}$

$$\bar{Y}_c = \frac{(50)(60)(30) - (30)(50)(35)}{1500}$$

$= 25\ \text{mm}$



$$I = \Sigma I_x' + \Sigma A D_y^2$$

$$I = \left[\frac{(50)(60)^3}{12} - \frac{(50)(50)^3}{12}\right] + \left[(50\times 60)(30-25)^2 - 1500(35-25)^2\right]$$

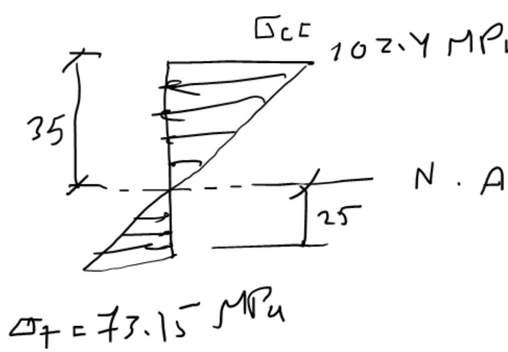
$= 512.5 \times 10^3\ \text{mm}^4$

$M = 1500\ \text{kN·mm}$
$= 1.5\ \text{kN·m}$

$$\sigma = \frac{-My}{I}$$

$$\sigma_A = \frac{-(1500\times 10^3)(35)}{512.5\times 10^3} = -102.4\ \text{MPa (C)}$$

$$\sigma_B = \frac{-(1500\times 10^3)(-25)}{512.5\times 10^3} = 73.17\ \text{MPa (T)}$$



Determine the minimum height h of the beam shown , if the flexural stress is not to exceed 20 MPa

$\sigma_{all} = 20 MPa$ | $M_{max} = ?$

$\sigma = \frac{My}{I}$

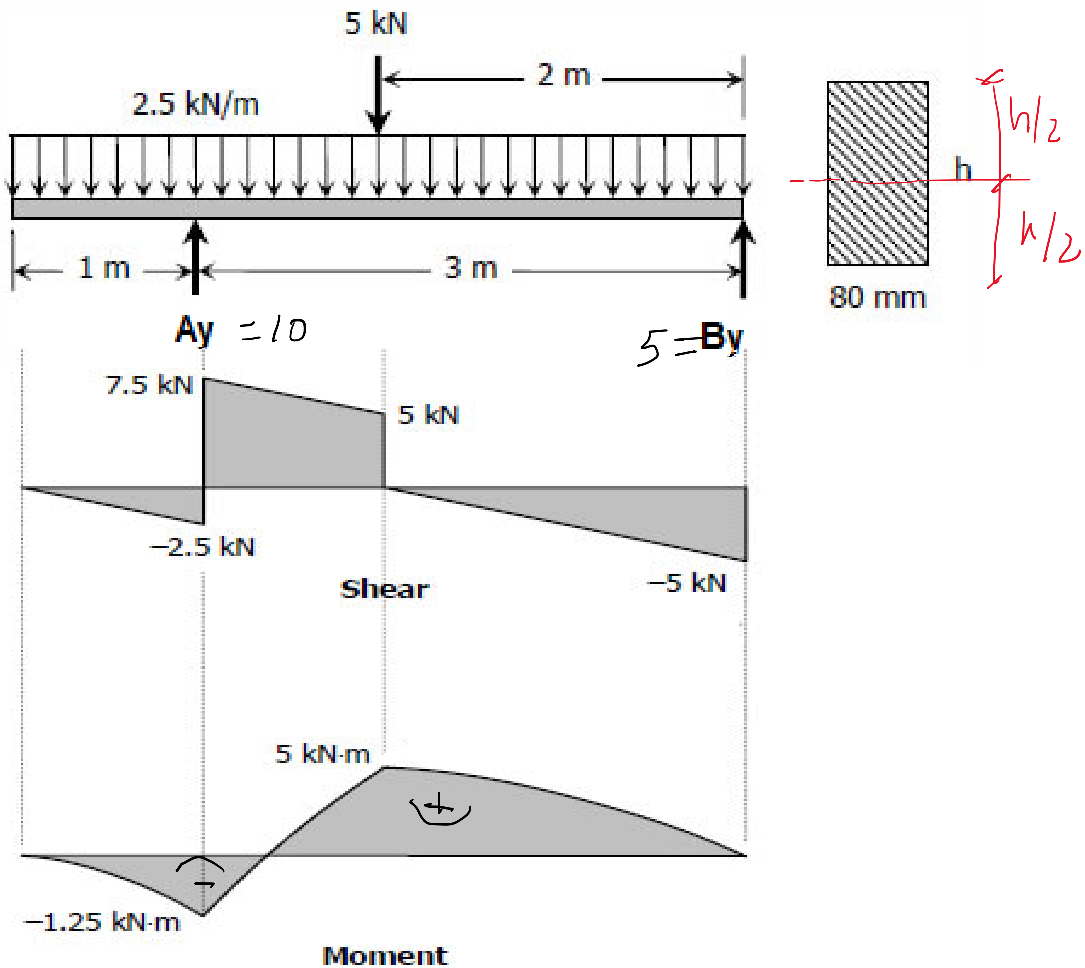
- Find Reaction
- Draw M.D $\Longrightarrow M_{max}$

$M_{max} = +5\ kN\ m$ (+)

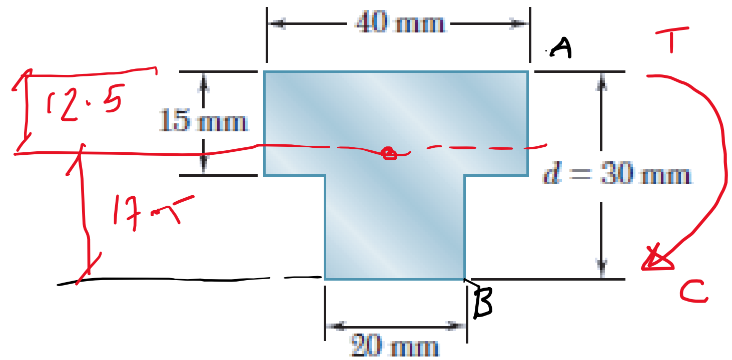
$\sigma = \frac{My}{I}$: $y = h/2$

$20 = \frac{(5\times10^6)(h/2)12}{80\ h^3} \Longrightarrow$

$h = 136.9\ mm \approx 137\ mm$



The beam shown is made of a nylon for which the allowable stress is 24 MPa in tension and 30 MPa in compression. Determine the largest couple **M** that can be applied to the beam.



$$\bar{Y} = \frac{(40\times 15)(22.5) + (300)(7.5)}{900} = 17.5 \text{ mm}$$

$$I = 61.875\times 10^3 \text{ mm}^4$$

$$\sigma_A = \frac{(M)(12.5)}{61.875\times 10^3} = 24 \Rightarrow M = 118\,800 \text{ N}\cdot\text{mm} = 118.8 \text{ N}\cdot\text{m}$$

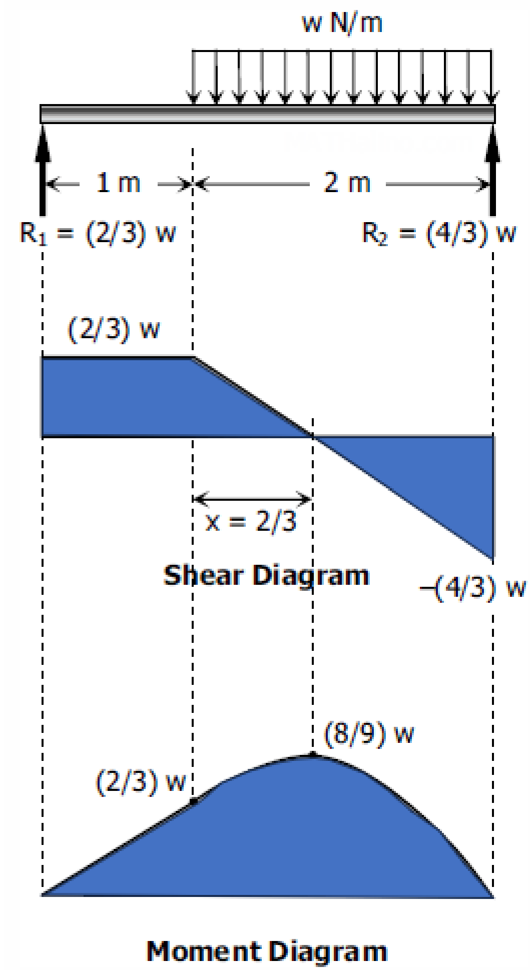
$$\sigma_B = \frac{(M)(17.5)}{61.875\times 10^3} = 30 \Rightarrow M = 106.07 \text{ N m}$$

chose smaller

A 50-mm diameter bar is used as a simply supported beam 3 m long. Determine the largest uniformly distributed load (w) if the flexural stress is limited to 50 MPa

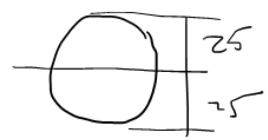
$d = 50\,mm$

$$I = \frac{\pi (25)^4}{4} = 306796.2\ mm^4$$



- Find Support Readings in terms of w

$M_{max} = 8/9\,w$ (N-m)

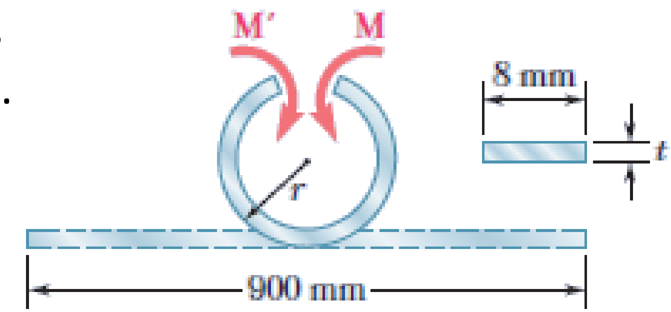


$$\sigma = \frac{My}{I}$$

$$= \frac{(8w \times 1000)\,25}{9\,(306796.2)} = 50$$

$$w = 690.29\ N/m$$

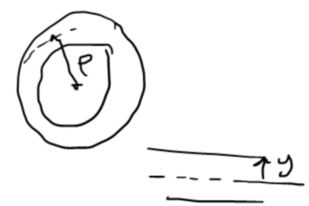
900-mm strip of steel is bent into a full circle by two couples applied as shown. Determine (*a*) the maximum **thickness *t*** of the strip if the allowable stress of the steel is **420 MPa**, (*b*) the corresponding **moment *M*** of the couples. Use ***E*** **=200 GPa.**



$$\frac{1}{\rho} = \frac{M}{EI} \quad , \quad \sigma = \frac{My}{I} \quad , \quad \epsilon = \frac{y}{\rho}$$

$\boxed{a}$

$$2\pi\rho = 900 \Rightarrow \rho = 143.3 \text{ mm}$$



$$\sigma = E\epsilon \Rightarrow 420 = 200\times10^{3}\,\epsilon \Rightarrow \epsilon = 2.1\times10^{-3}$$

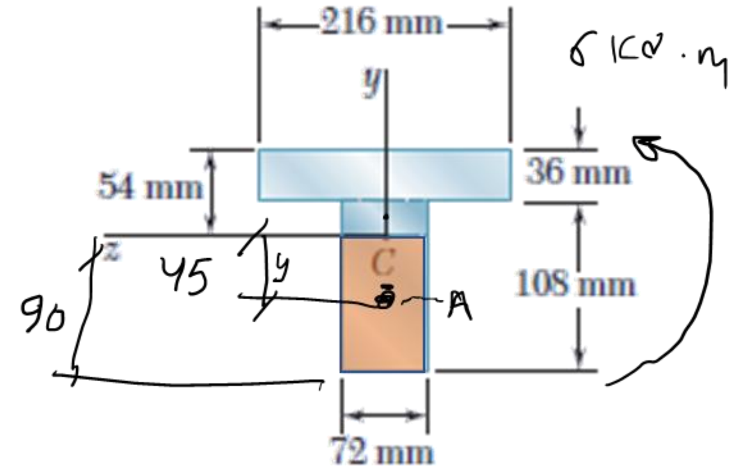
$$\epsilon = y/\rho = 2.1\times10^{-3} = y/143.3 \Rightarrow y = 0.3 \text{ mm}$$

$$t = 2y = (2)(0.3) = \underline{0.6 \text{ mm}} \quad \Big\} \quad I = \frac{(8)(0.6)^3}{12} = 0.144 \text{ mm}^4$$

$\boxed{b}$

$$\sigma = \frac{My}{I} = 420 = \frac{0.3M}{0.144} \Rightarrow M = 201.6 \text{ N.mm} = \underline{0.202 \text{ N m}}$$

Knowing that a beam of the cross section shown is bent about a horizontal axis and that the bending **moment is 6 kN.m,** determine the ***total force acting on the shaded portion*** of the web.



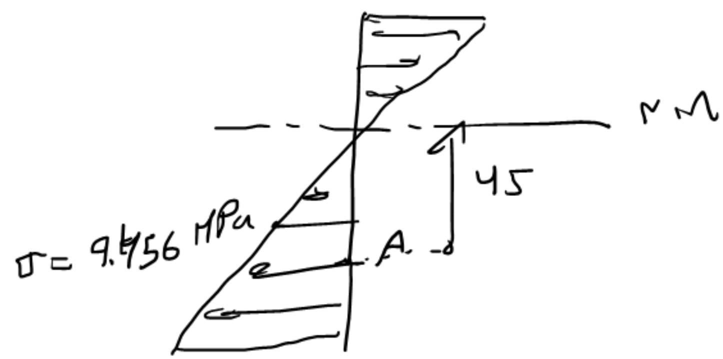
$$I = \Sigma I' + \Sigma A d_y^2$$

$$= \left[\frac{72 \times 108^3}{12} + \frac{216 \times 36^3}{12}\right] + \left[7776 \times 36^2 + 7776 \times 36^2\right]$$

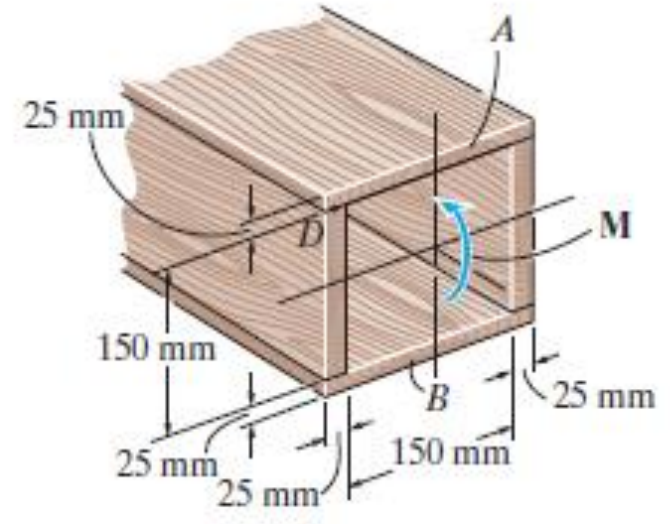
$$= 28.553 \times 10^6 \ mm^4$$

$$\sigma_A = \frac{My}{I} = \frac{(6 \times 10^6)(45)}{28.553 \times 10^6} = 9.456 \ MPa$$

$$F = \sigma A = (9.456)(72 \times 90) = \underline{61.274 \ kN}$$



Determine the moment **M** that should be applied to the beam in order to create a compressive stress at poin of 30 MPa , Also sketch the stress distribution acting over the cross section and compute the maximum stress developed in the beam.



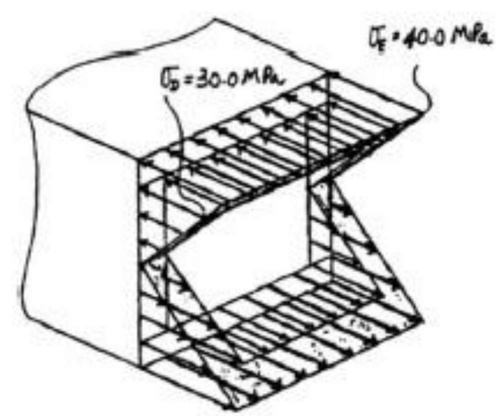
$$I = \frac{1}{12}(0.2)\left(0.2^3\right) - \frac{1}{12}(0.15)\left(0.15^3\right) = 91.14583\left(10^{-6}\right)\ \mathrm{m}^4$$

$$\sigma = \frac{My}{I}$$

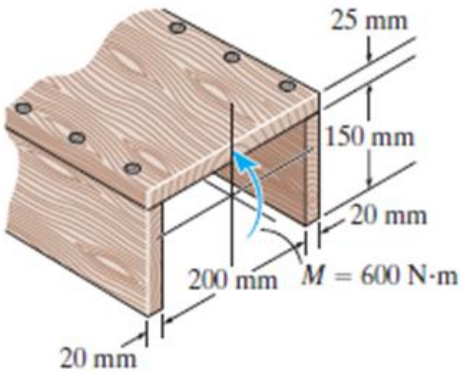
$$30\left(10^6\right) = \frac{M(0.075)}{91.14583(10^{-6})}$$

$$M = 36458\ \mathrm{N\cdot m} = 36.5\ \mathrm{kN\cdot m}$$

$$\sigma_{\max} = \frac{Mc}{I} = \frac{36458(0.1)}{91.14583(10^{-6})} = 40.0\ \mathrm{MPa}$$



The beam is made from three boards nailed together as shown. If the moment acting on the cross section is 600 N.m, determine the maximum bending stress in the beam. Sketch a three-dimensional view of the stress distribution acting over the cross section.

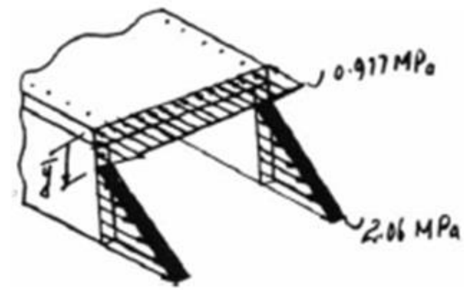


$$\bar{y} = \frac{(0.0125)(0.24)(0.025) + 2\,(0.1)(0.15)(0.2)}{0.24\,(0.025) + 2\,(0.15)(0.02)} = 0.05625 \text{ m}$$

$$I = \frac{1}{12}(0.24)(0.025^3) + (0.24)(0.025)(0.04375^2)$$

$$+ 2\left(\frac{1}{12}\right)(0.02)(0.15^3) + 2(0.15)(0.02)(0.04375^2)$$

$$= 34.53125\,(10^{-6}) \text{ m}^4$$



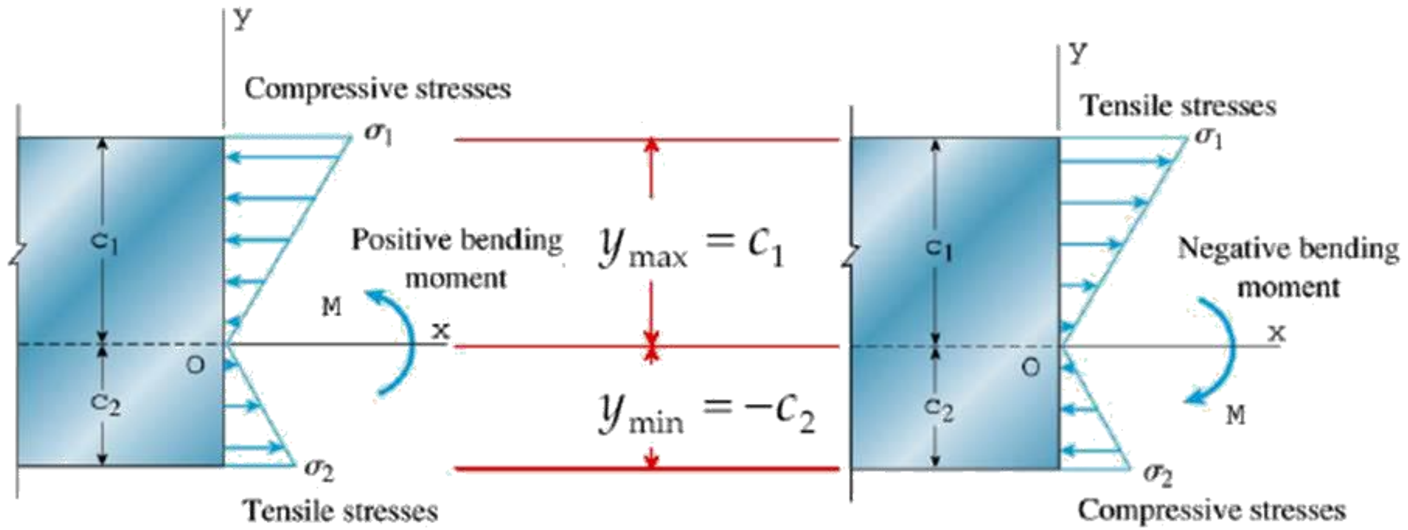
$$\sigma_{max} = \sigma_B = \frac{Mc}{I}$$

$$= \frac{600\,(0.175 - 0.05625)}{34.53125\,(10^{-6})}$$

$$= 2.06 \text{ MPa}$$

$$\sigma_C = \frac{My}{I} = \frac{600\,(0.05625)}{34.53125\,(10^{-6})} = 0.977 \text{ MPa}$$

Maximum Stresses at a Cross Section



$$\boxed{\sigma_x = -\frac{My}{I}}$$

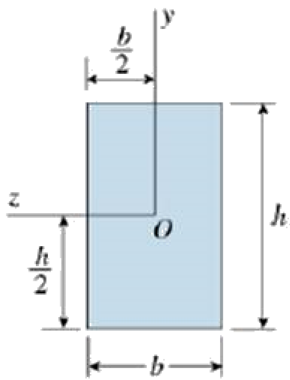
$$\sigma_1 = -\frac{Mc_1}{I} = -\frac{M}{S_1}, \quad S_1 = \frac{I}{c_1}$$

$$\sigma_2 = -\frac{M(-c_2)}{I} = \frac{M}{S_2}, \quad S_2 = \frac{I}{c_2}$$

S_1 and S_2 are known as the "Section Muduli" of the cross-sectional area. (See Appendix E)

Section Moduli for Doubly Symmetric Shapes

Rectangular:

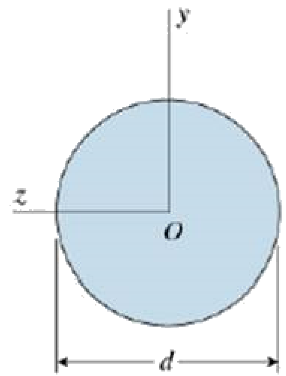


$$I = \frac{bh^3}{12}$$

$$c_1 = c_2 = \frac{h}{2}$$

$$S = \frac{bh^2}{6} = \frac{Ah}{6}$$

Circular:



$$I = \frac{\pi d^4}{64}$$

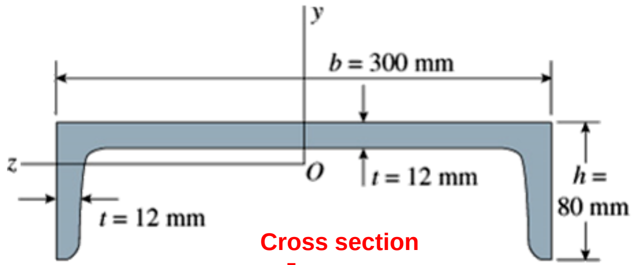
$$c_1 = c_2 = \frac{d}{2}$$

$$S = \frac{\pi d^3}{32} = \frac{Ad}{8}$$

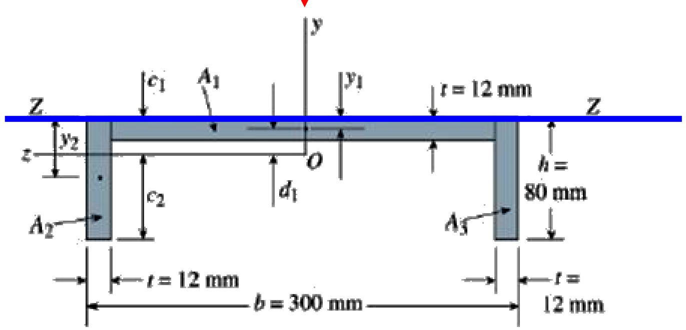
$$c_1 = c_2 \quad \Rightarrow \quad \sigma_1 = -\sigma_2 = -\frac{Mc}{I} = -\frac{M}{S}$$

$$\sigma_{\max} = \frac{M}{S}, \quad S = \frac{I}{c}$$

Example 5-4: Determine the maximum tensile and compressive stress in the beam

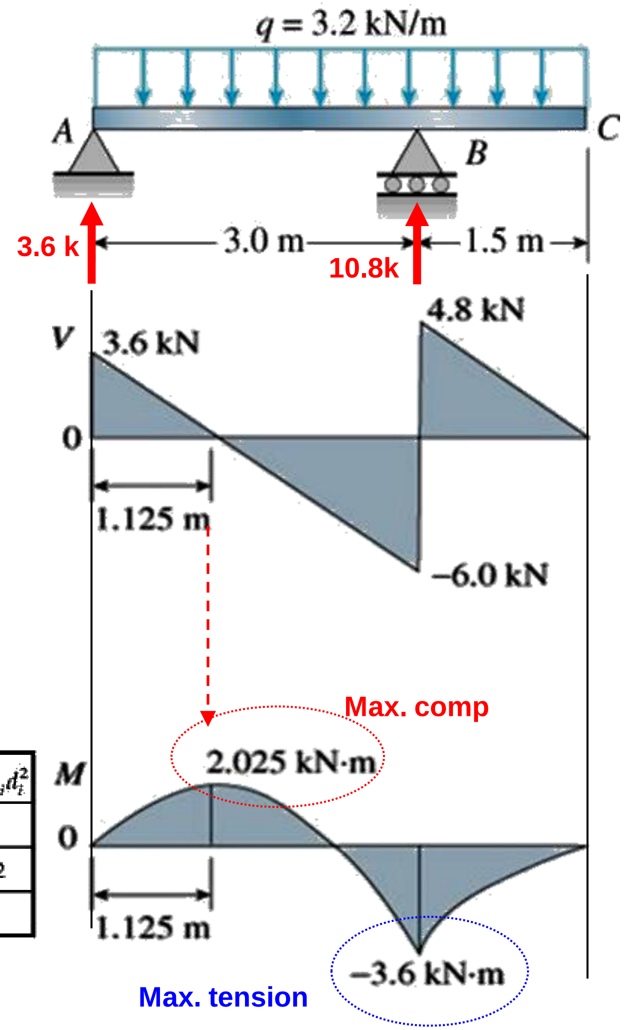


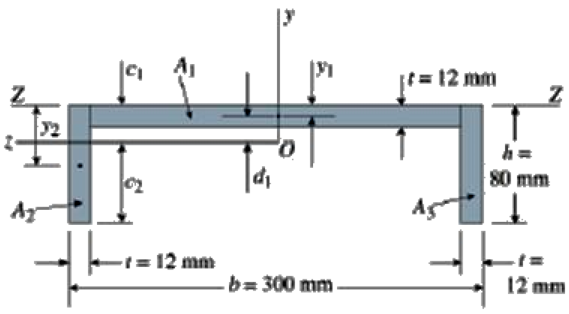
Cross section



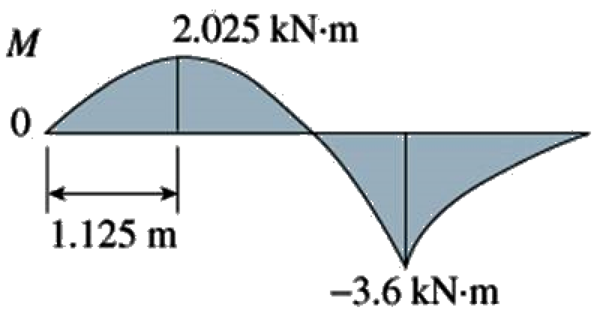
	A_i	$\overline{y}_i$	Q_i	$I_{zi}^{(i)} = bh^3/12$	d_i	$A_i d_i^2$	$I_z^{(i)} = I_{zi}^{(i)} + A_i d_i^2$
①	3,312	6	19,872	39,744	12.48	515,845	555,589
②, ③	960 × 2	40	38,400 × 2	512,000 × 2	- 21.52	444,586 × 2	956,586 × 2
$\sum$	5,232		97,672				2,468,761

$$\overline{y} = c_1 = \frac{\sum Q_i}{\sum A_i} = 18.48\,\text{mm}, \quad c_2 = h - c_1 = 61.52\,\text{mm}$$





(b)



$$I_z = 2.469 \times 10^6 \text{ mm}^4$$

$$c_1 = 18.48 \text{ mm} \quad \Rightarrow \quad S_1 = \frac{I_z}{c_1} = 133{,}600 \text{ mm}^3$$

$$c_2 = 61.52 \text{ mm} \quad \Rightarrow \quad S_2 = \frac{I_z}{c_2} = 40{,}100 \text{ mm}^3$$

at $\boxed{x = 1.125 \text{ m}, \quad M = 2.025 \text{ kN}\cdot\text{m}}$

$$\sigma_1 = -\frac{M}{S_1} = -\frac{2.025 \text{ kN}\cdot\text{m}}{133{,}600 \text{ mm}^3} = -15.2 \text{ MPa}$$

$$\sigma_2 = \frac{M}{S_2} = \frac{2.025 \text{ kN}\cdot\text{m}}{40{,}100 \text{ mm}^3} = 50.5 \text{ MPa}$$

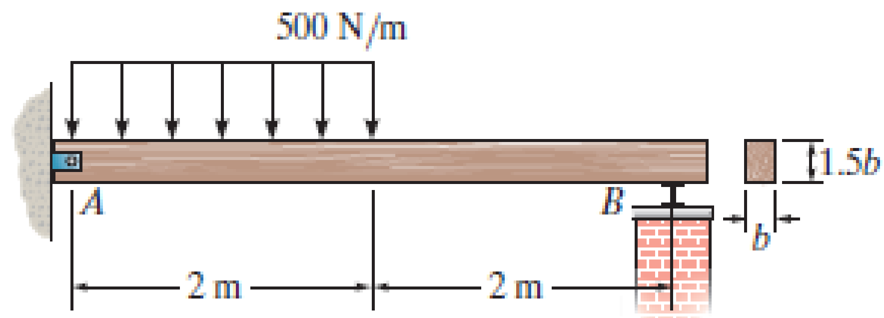
at $\boxed{x = 3.0 \text{ m}, \quad M = -3.6 \text{ kN}\cdot\text{m}}$

$$\sigma_1 = -\frac{M}{S_1} = -\frac{-3.6 \text{ kN}\cdot\text{m}}{133{,}600 \text{ mm}^3} = 26.9 \text{ MPa}$$

$$\sigma_2 = \frac{M}{S_2} = \frac{-3.6 \text{ kN}\cdot\text{m}}{40{,}100 \text{ mm}^3} = -89.8 \text{ MPa}$$

$$\boxed{\begin{aligned} (\sigma_{tensile})_{\max} &= 50.5 \text{ MPa} \\ (\sigma_{compressive})_{\max} &= -89.8 \text{ MPa} \end{aligned}}$$

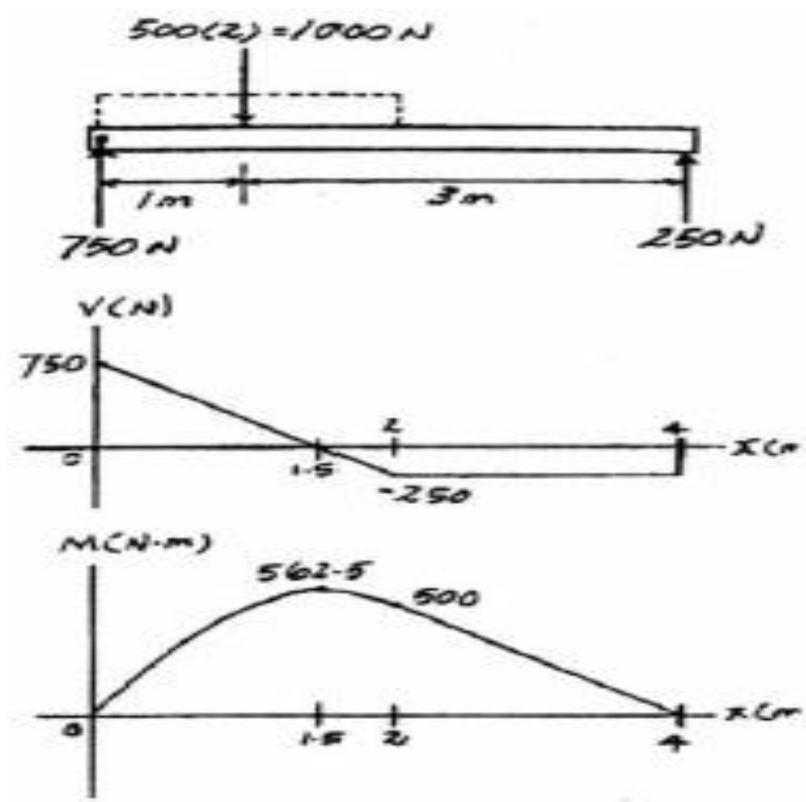
The wood beam has a rectangular cross section in the proportion shown. Determine its required dimension b if the allowable bending stress is $\sigma_{allow} = 10$ MPa.



$$\sigma_{max} = \sigma_{allow} = \frac{M_{max}\, c}{I}$$

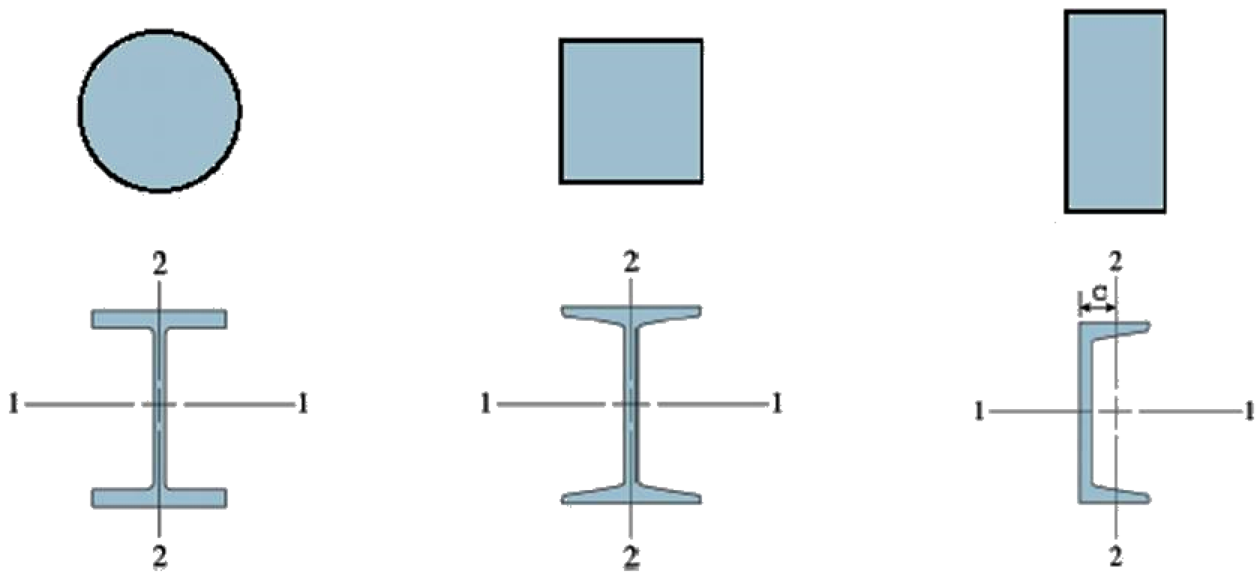
$$10\left(10^6\right) = \frac{562.5(0.75b)}{\frac{1}{12}(b)(1.5b)^3}$$

$$b = 0.05313 \text{ m} = 53.1 \text{ mm}$$



Design of Beams for Bending Stresses

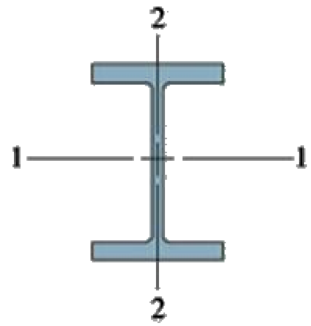
$$S = \frac{M_{\max}}{\sigma_{\text{allow}}}$$



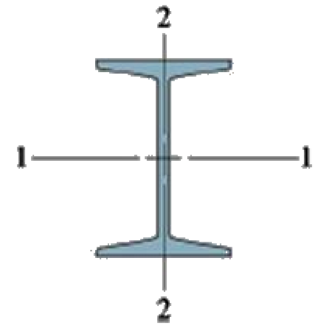
Which cross section is the most efficient one?

Properties of Structural-Steel Shapes

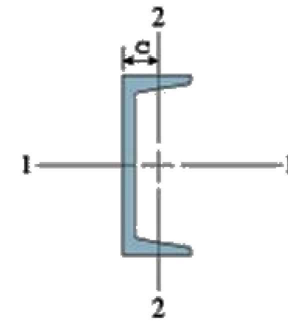
Appendix E, pp. 897 - 902



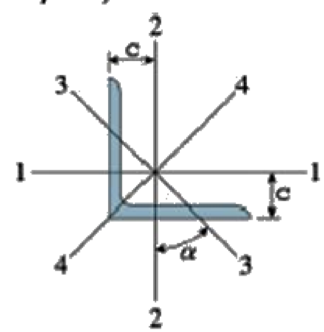
Wide-Flange Sections
(W Shapes)



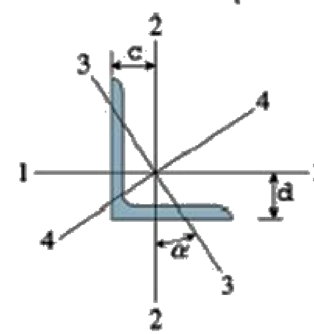
I-Beam Sections
(S Shapes)



Channel Sections
(C Shapes)

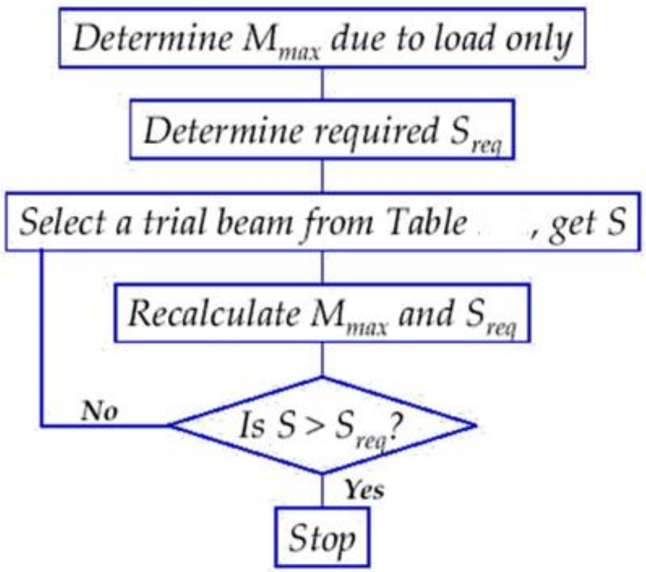


Angle Sections with Equal Legs
(L Shapes)



Angle Sections with Unequal Legs
(L Shapes)

Design of Beams for Bending Stresses

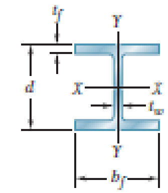


Appendix C Properties of Rolled-Steel Shapes

(SI Units)

Continued from page A19

W Shapes
(Wide-Flange Shapes)



Designation*	Area A, mm²	Depth d, mm	Flange Width b_f, mm	Flange Thickness t_f, mm	Web Thickness t_w, mm	Axis X-X I_x 10^6 mm⁴	Axis X-X S_x 10^3 mm³	Axis X-X r_x mm	Axis Y-Y I_y 10^6 mm⁴	Axis Y-Y S_y 10^3 mm³	Axis Y-Y r_y mm
W310 × 143	18200	323	310	22.9	14.0	347	2150	138	112	728	78.5
107	13600	312	305	17.0	10.9	248	1600	135	81.2	531	77.2
74	9420	310	205	16.3	9.40	163	1050	132	23.4	228	49.8
60	7550	302	203	13.1	7.49	128	844	130	18.4	180	49.3
52	6650	318	167	13.2	7.62	119	747	133	10.2	122	39.1
44.5	5670	312	166	11.2	6.60	99.1	633	132	8.45	102	38.6
38.7	4940	310	165	9.65	5.84	84.9	547	131	7.20	87.5	38.4
32.7	4180	312	102	10.8	6.60	64.9	416	125	1.94	37.9	21.5
23.8	3040	305	101	6.73	5.59	42.9	280	119	1.17	23.1	19.6
W250 × 167	21200	290	264	31.8	19.2	298	2060	118	98.2	742	68.1
101	12900	264	257	19.6	11.9	164	1240	113	55.8	433	65.8
80	10200	257	254	15.6	9.4	126	983	111	42.9	338	65.0
67	8580	257	204	15.7	8.89	103	805	110	22.2	218	51.1
58	7420	252	203	13.5	8.00	87.0	690	108	18.7	185	50.3
49.1	6260	247	202	11.0	7.37	71.2	574	106	15.2	151	49.3
44.8	5700	267	148	13.0	7.62	70.8	531	111	6.95	94.2	34.8
32.7	4190	259	146	9.14	6.10	49.1	380	108	4.75	65.1	33.8
28.4	3630	259	102	10.0	6.35	40.1	308	105	1.79	35.1	22.2
22.3	2850	254	102	6.86	5.84	28.7	226	100	1.20	23.8	20.6
W200 × 86	11000	222	209	20.6	13.0	94.9	852	92.7	31.3	300	53.3
71	9100	216	206	17.4	10.2	76.6	708	91.7	25.3	246	52.8
59	7550	210	205	14.2	9.14	60.8	582	89.7	20.4	200	51.8
52	6650	206	204	12.6	7.87	52.9	511	89.2	17.7	174	51.6
46.1	5880	203	203	11.0	7.24	45.8	451	88.1	15.4	152	51.3
41.7	5320	205	166	11.8	7.24	40.8	398	87.6	9.03	109	41.1
35.9	4570	201	165	10.2	6.22	34.4	342	86.9	7.62	92.3	40.9
31.3	3970	210	134	10.2	6.35	31.3	298	88.6	4.07	60.8	32.0
26.6	3390	207	133	8.38	5.84	25.8	249	87.1	3.32	49.8	31.2
22.5	2860	206	102	8.00	6.22	20.0	193	83.6	1.42	27.9	22.3
19.3	2480	203	102	6.48	5.84	16.5	162	81.5	1.14	22.5	21.4
W150 × 37.1	4740	162	154	11.6	8.13	22.2	274	68.6	7.12	91.9	38.6
29.8	3790	157	153	9.27	6.60	17.2	220	67.6	5.54	72.3	38.1
24	3060	160	102	10.3	6.60	13.4	167	66.0	1.84	36.1	24.6
18	2290	153	102	7.11	5.84	9.20	120	63.2	1.24	24.6	23.3
13.5	1730	150	100	5.46	4.32	6.83	91.1	62.7	0.916	18.2	23.0
W130 × 28.1	3590	131	128	10.9	6.86	10.9	167	55.1	3.80	59.5	32.5
23.8	3040	127	127	9.14	6.10	8.91	140	54.1	3.13	49.2	32.0
W100 × 19.3	2470	106	103	8.76	7.11	4.70	89.5	43.7	1.61	31.1	25.4

*A wide-flange shape is designated by the letter W followed by the nominal depth in millimeters and the mass in kilograms per meter.

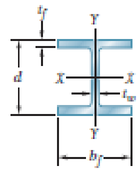
Appendix C Properties of Rolled-Steel Shapes

(SI Units)

Continued from page A19

W Shapes

(Wide-Flange Shapes)



Designation†	Area A, mm^2	Depth d, mm	Flange Width b_f, mm	Flange Thickness t_f, mm	Web Thickness t_w, mm	Axis X-X I_x 10^6 mm^4	Axis X-X S_x 10^3 mm^3	Axis X-X r_x mm	Axis Y-Y I_y 10^6 mm^4	Axis Y-Y S_y 10^3 mm^3	Axis Y-Y r_y mm
W310 × 143	18200	323	310	22.9	14.0	347	2150	138	112	728	78.5
107	13600	312	305	17.0	10.9	248	1600	135	81.2	531	77.2
74	9420	310	205	16.3	9.40	163	1050	132	23.4	228	49.8
60	7550	302	203	13.1	7.49	128	844	130	18.4	180	49.3
52	6650	318	167	13.2	7.62	119	747	133	10.2	122	39.1
44.5	5670	312	166	11.2	6.60	99.1	633	132	8.45	102	38.6
38.7	4940	310	165	9.65	5.84	84.9	547	131	7.20	87.5	38.4
32.7	4180	312	102	10.8	6.60	64.9	416	125	1.94	37.9	21.5
23.8	3040	305	101	6.73	5.59	42.9	280	119	1.17	23.1	19.6
W250 × 167	21200	290	264	31.8	19.2	298	2060	118	98.2	742	68.1
101	12900	264	257	19.6	11.9	164	1240	113	55.8	433	65.8
80	10200	257	254	15.6	9.4	126	983	111	42.9	338	65.0
67	8580	257	204	15.7	8.89	103	805	110	22.2	218	51.1
58	7420	252	203	13.5	8.00	87.0	690	108	18.7	185	50.3
49.1	6260	247	202	11.0	7.37	71.2	574	106	15.2	151	49.3
44.8	5700	267	148	13.0	7.62	70.8	531	111	6.95	94.2	34.8
32.7	4190	259	146	9.14	6.10	49.1	380	108	4.75	65.1	33.8
28.4	3630	259	102	10.0	6.35	40.1	308	105	1.79	35.1	22.2
22.3	2850	254	102	6.86	5.84	28.7	226	100	1.20	23.8	20.6
W200 × 86	11000	222	209	20.6	13.0	94.9	852	92.7	31.3	300	53.3
71	9100	216	206	17.4	10.2	76.6	708	91.7	25.3	246	52.8
59	7550	210	205	14.2	9.14	60.8	582	89.7	20.4	200	51.8
52	6650	206	204	12.6	7.87	52.9	511	89.2	17.7	174	51.6
46.1	5880	203	203	11.0	7.24	45.8	451	88.1	15.4	152	51.3
41.7	5320	205	166	11.8	7.24	40.8	398	87.6	9.03	109	41.1
35.9	4570	201	165	10.2	6.22	34.4	342	86.9	7.62	92.3	40.9
31.3	3970	210	134	10.2	6.35	31.3	298	88.6	4.07	60.8	32.0
26.6	3390	207	133	8.38	5.84	25.8	249	87.1	3.32	49.8	31.2
22.5	2860	206	102	8.00	6.22	20.0	193	83.6	1.42	27.9	22.3
19.3	2480	203	102	6.48	5.84	16.5	162	81.5	1.14	22.5	21.4
W150 × 37.1	4740	162	154	11.6	8.13	22.2	274	68.6	7.12	91.9	38.6
29.8	3790	157	153	9.27	6.60	17.2	220	67.6	5.54	72.3	38.1
24	3060	160	102	10.3	6.60	13.4	167	66.0	1.84	36.1	24.6
18	2290	153	102	7.11	5.84	9.20	120	63.2	1.24	24.6	23.3
13.5	1730	150	100	5.46	4.32	6.83	91.1	62.7	0.916	18.2	23.0
W130 × 28.1	3590	131	128	10.9	6.86	10.9	167	55.1	3.80	59.5	32.5
23.8	3040	127	127	9.14	6.10	8.91	140	54.1	3.13	49.2	32.0
W100 × 19.3	2470	106	103	8.76	7.11	4.70	89.5	43.7	1.61	31.1	25.4

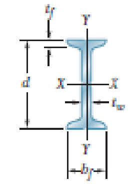
†A wide-flange shape is designated by the letter W followed by the nominal depth in millimeters and the mass in kilograms per meter.

Appendix C Properties of Rolled-Steel Shapes

(SI Units)

S Shapes

(American Standard Shapes)



Designation†	Area A, mm^2	Depth d, mm	Flange Width b_f, mm	Flange Thickness t_f, mm	Web Thickness t_w, mm	Axis X-X I_x 10^6 mm^4	Axis X-X S_x 10^3 mm^3	Axis X-X r_x mm	Axis Y-Y I_y 10^6 mm^4	Axis Y-Y S_y 10^3 mm^3	Axis Y-Y r_y mm
S610 × 180	22900	622	204	27.7	20.3	1320	4230	240	34.5	338	38.9
158	20100	622	200	27.7	15.7	1220	3930	247	32.0	320	39.9
149	18900	610	184	22.1	18.9	991	3260	229	19.7	215	32.3
134	17100	610	181	22.1	15.9	937	3060	234	18.6	205	33.0
119	15200	610	178	22.1	12.7	874	2870	241	17.5	197	34.0
S510 × 143	18200	516	183	23.4	20.3	695	2700	196	20.8	228	33.8
128	16300	516	179	23.4	16.8	653	2540	200	19.4	216	34.5
112	14200	508	162	20.2	16.1	533	2100	194	12.3	152	29.5
98.2	12500	508	159	20.2	12.8	495	1950	199	11.4	144	30.2
S460 × 104	13200	457	159	17.6	18.1	384	1690	170	10.0	126	27.4
81.4	10300	457	152	17.6	11.7	333	1460	180	8.62	113	29.0
S380 × 74	9480	381	143	15.8	14.0	202	1060	146	6.49	90.6	26.2
64	8130	381	140	15.8	10.4	186	973	151	5.95	85.0	26.9
S310 × 74	9420	305	139	16.7	17.4	126	829	116	6.49	93.2	26.2
60.7	7680	305	133	16.7	11.7	112	739	121	5.62	84.1	26.9
52	6580	305	129	13.8	10.9	94.9	624	120	4.10	63.6	24.9
47.3	6010	305	127	13.8	8.89	90.3	593	123	3.88	61.1	25.4
S250 × 52	6650	254	125	12.5	15.1	61.2	482	96.0	3.45	55.1	22.8
37.8	4810	254	118	12.5	7.90	51.2	403	103	2.80	47.4	24.1
S200 × 34	4360	203	106	10.8	11.2	26.9	265	78.5	1.78	33.6	20.2
27.4	3480	203	102	10.8	6.88	23.9	236	82.8	1.54	30.2	21.0
S150 × 25.7	3260	152	90.7	9.12	11.8	10.9	143	57.9	0.953	21.0	17.1
18.6	2360	152	84.6	9.12	5.89	9.16	120	62.2	0.749	17.7	17.8
S130 × 15	1890	127	76.2	8.28	5.44	5.12	80.3	52.1	0.495	13.0	16.2
S100 × 14.1	1800	102	71.1	7.44	8.28	2.81	55.4	39.6	0.369	10.4	14.3
11.5	1460	102	67.6	7.44	4.90	2.52	49.7	41.7	0.311	9.21	14.6
S75 × 11.2	1420	76.2	63.8	6.60	8.86	1.21	31.8	29.2	0.241	7.55	13.0
8.5	1070	76.2	59.2	6.60	4.32	1.04	27.4	31.2	0.186	6.28	13.2

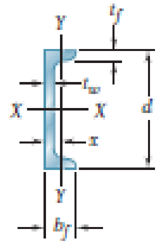
†An American Standard Beam is designated by the letter S followed by the nominal depth in millimeters and the mass in kilograms per meter.

Appendix C Properties of Rolled-Steel Shapes

(SI Units)

C Shapes

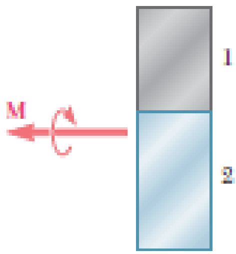
(American Standard Channels)



			Flange		Web	Axis X-X			Axis Y-Y			
Designation[†]	Area A, mm²	Depth d, mm	Width b_f, mm	Thickness t_f, mm	Thickness t_w, mm	I_x 10^6 mm⁴	S_x 10^3 mm³	r_x mm	I_y 10^6 mm⁴	S_y 10^3 mm³	r_y mm	x mm
C380 × 74	9480	381	94.5	16.5	18.2	168	882	133	4.58	61.8	22.0	20.3
60	7610	381	89.4	16.5	13.2	145	762	138	3.82	54.7	22.4	19.8
50.4	6450	381	86.4	16.5	10.2	131	688	143	3.36	50.6	22.9	20.0
C310 × 45	5680	305	80.5	12.7	13.0	67.4	442	109	2.13	33.6	19.4	17.1
37	4740	305	77.5	12.7	9.83	59.9	393	113	1.85	30.6	19.8	17.1
30.8	3920	305	74.7	12.7	7.16	53.7	352	117	1.61	28.2	20.2	17.7
C250 × 45	5680	254	77.0	11.1	17.1	42.9	339	86.9	1.64	27.0	17.0	16.5
37	4740	254	73.4	11.1	13.4	37.9	298	89.4	1.39	24.1	17.1	15.7
30	3790	254	69.6	11.1	9.63	32.8	259	93.0	1.17	21.5	17.5	15.4
22.8	2890	254	66.0	11.1	6.10	28.0	221	98.3	0.945	18.8	18.1	16.1
C230 × 30	3790	229	67.3	10.5	11.4	25.3	221	81.8	1.00	19.2	16.3	14.8
22	2850	229	63.2	10.5	7.24	21.2	185	86.4	0.795	16.6	16.7	14.9
19.9	2540	229	61.7	10.5	5.92	19.9	174	88.6	0.728	15.6	16.9	15.3
C200 × 27.9	3550	203	64.3	9.91	12.4	18.3	180	71.6	0.820	16.6	15.2	14.4
20.5	2610	203	59.4	9.91	7.70	15.0	148	75.9	0.633	13.9	15.6	14.1
17.1	2170	203	57.4	9.91	5.59	13.5	133	79.0	0.545	12.7	15.8	14.5
C180 × 18.2	2320	178	55.6	9.30	7.98	10.1	113	66.0	0.483	11.4	14.4	13.3
14.6	1850	178	53.1	9.30	5.33	8.82	100	69.1	0.398	10.1	14.7	13.7
C150 × 19.3	2460	152	54.9	8.71	11.1	7.20	94.7	54.1	0.437	10.5	13.3	13.1
15.6	1990	152	51.6	8.71	7.98	6.29	82.6	56.4	0.358	9.19	13.4	12.7
12.2	1540	152	48.8	8.71	5.08	5.45	71.3	59.4	0.286	8.00	13.6	13.0
C130 × 13	1700	127	48.0	8.13	8.26	3.70	58.3	46.5	0.260	7.28	12.3	12.1
10.4	1270	127	44.5	8.13	4.83	3.11	49.0	49.5	0.196	6.10	12.4	12.3
C100 × 10.8	1370	102	43.7	7.52	8.15	1.91	37.5	37.3	0.177	5.52	11.4	11.7
8	1020	102	40.1	7.52	4.67	1.60	31.5	39.6	0.130	4.54	11.3	11.6
C75 × 8.9	1140	76.2	40.6	6.93	9.04	0.862	22.6	27.4	0.125	4.31	10.5	11.6
7.4	948	76.2	38.1	6.93	6.55	0.770	20.2	28.4	0.100	3.74	10.3	11.2
6.1	774	76.2	35.8	6.93	4.32	0.687	18.0	29.7	0.0795	3.21	10.1	11.1

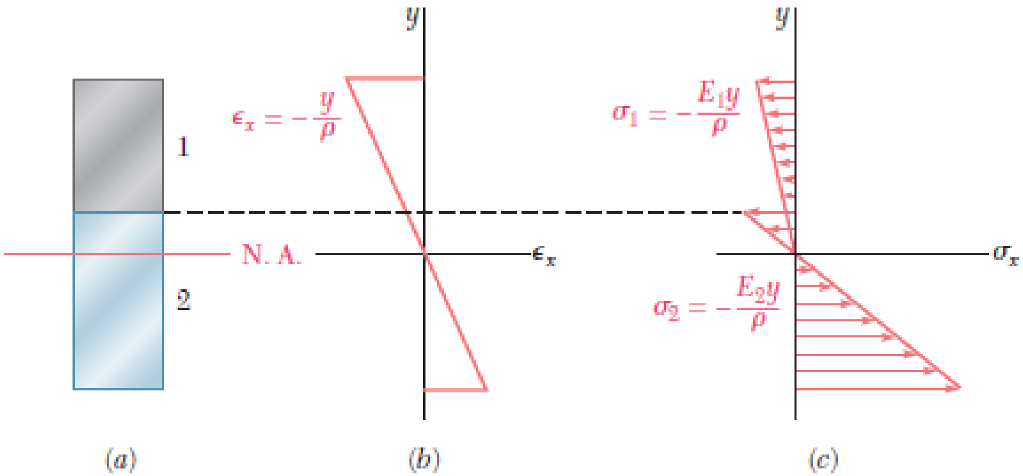
[†]An American Standard Channel is designated by the letter C followed by the nominal depth in millimeters and the mass in kilograms per meter.

6.6 Composite Beam



Cross section made with different materials

$$\epsilon_x = -\frac{y}{\rho}$$



Stress and strain distributions in bar Made of two materials. (*a*) Neutral axis shifted from centroid. (*b*) Strain distribution. (*c*) Corresponding stress distribution.

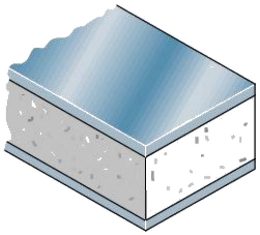
$$\sigma_1 = E_1 \epsilon_x = -\frac{E_1 y}{\rho}$$

$$\sigma_2 = E_2 \epsilon_x = -\frac{E_2 y}{\rho}$$

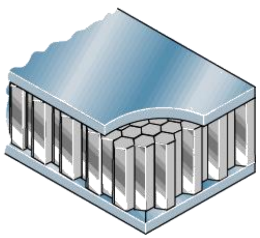
Composite Beams

A *composite* beam is composed of two or more elemental structural forms, or different materials, bonded, knitted, or otherwise joined together. *Composite materials or forms* include such heavy handed stuff as concrete (one material) reinforced with steel bars (another material); high-tech developments such as tubes built up of graphite fibers embedded in an epoxy matrix; sports structures like *laminated* skis, the poles for vaulting, even a golf ball can be viewed as a *filament wound* structure encased within another material.

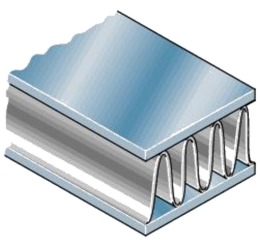
Sandwich beams with: (a) plastic core, (b) honeycomb core, and (c) corrugated core.



(a)

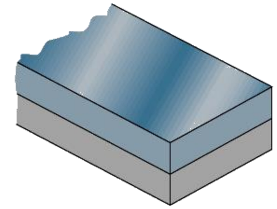


(b)

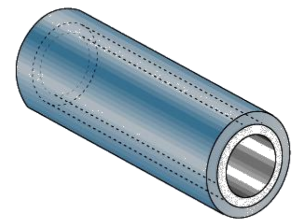


(c)

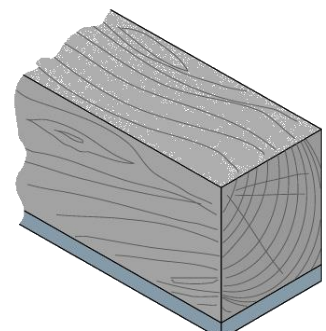
Examples of composite beams: (a) bimetallic beam, (b) plastic-coated steel pipe, and (c) wood beam reinforced with a steel plate.



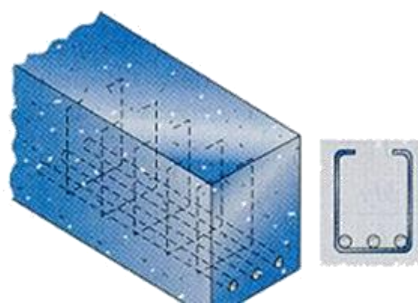
(a)



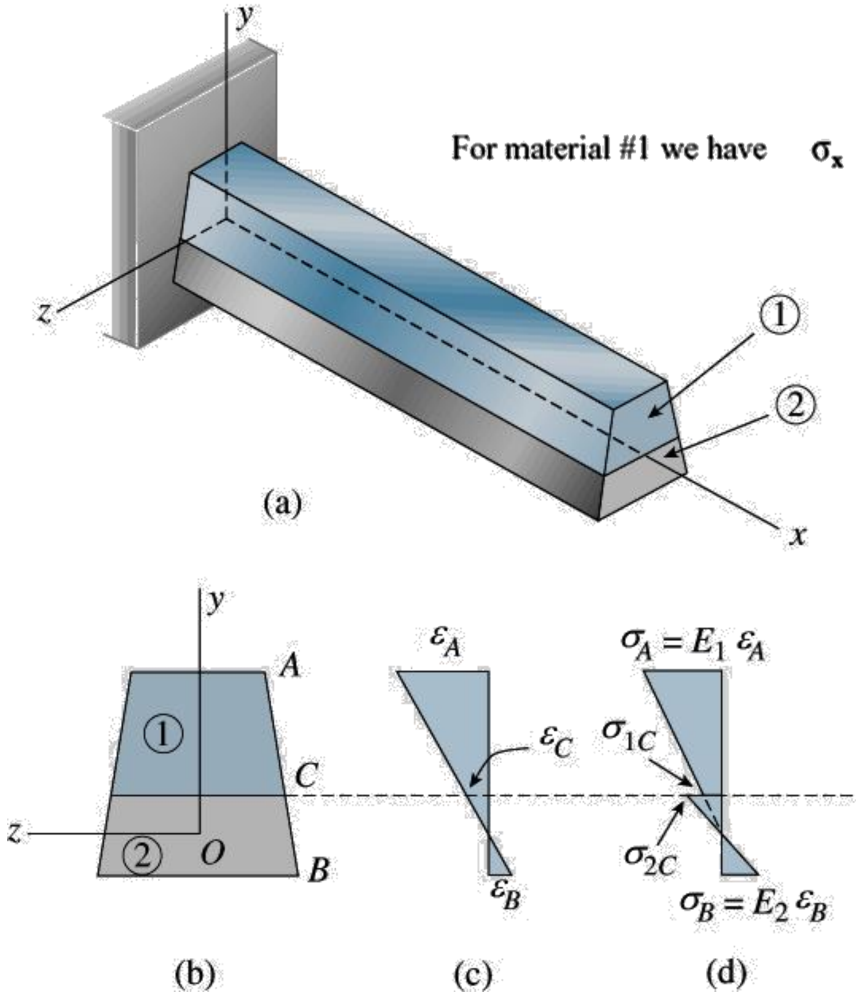
(b)



(c)



For material #1 we have $\sigma_x = -E_1 \cdot (y/\rho)$ while for #2 $\sigma_x = -E_2 \cdot (y/\rho)$



$$\boxed{\varepsilon_x(y) = -y \cdot \left(\frac{d\phi}{ds}\right) = -(y/\rho)}$$

(a) Composite beam of two materials, (b) cross section of beam, (c) distributions of strains of ε_x throughout the height of the beam, and (d) distributions of stresses σ_x in the beam for the case where $\mathbf{E_2 > E_1}$.

Transformed Section Method

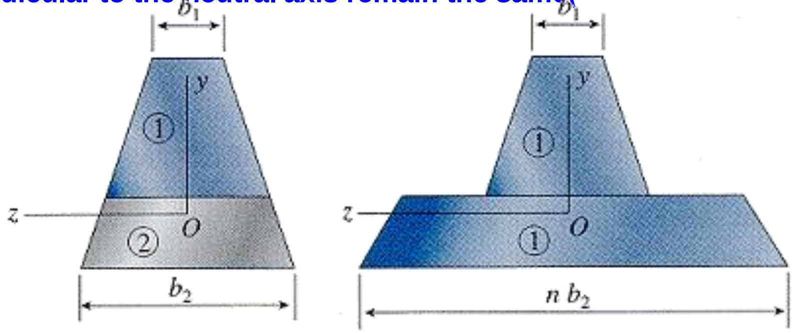
1. Transform the cross section of a composite beam into an equivalent cross section (of an imaginary beam composed of only one material) is called the transformed section

2. Analyze the transformed section as customary for a beam of one material .

3. Convert the stresses back to the original beam .

4. Modular ratio $$n = \frac{E_2}{E_1}$$

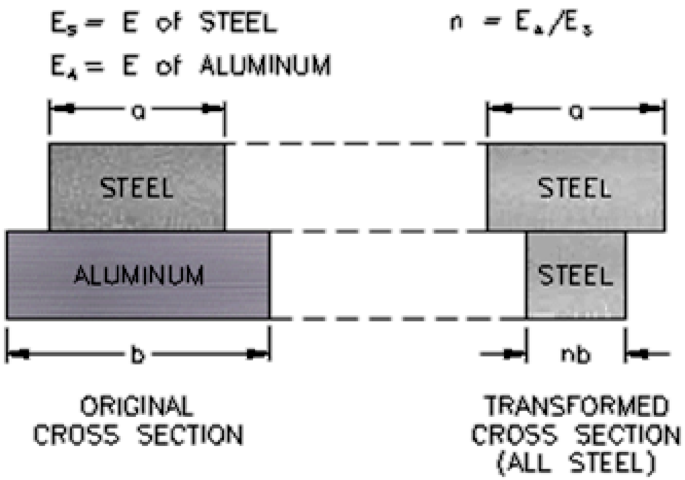
5. The dimensions of area 1 remain unchanged, and the width of area 2 is multiplied by (*n*) .all dimensions perpendicular to the neutral axis remain the same(

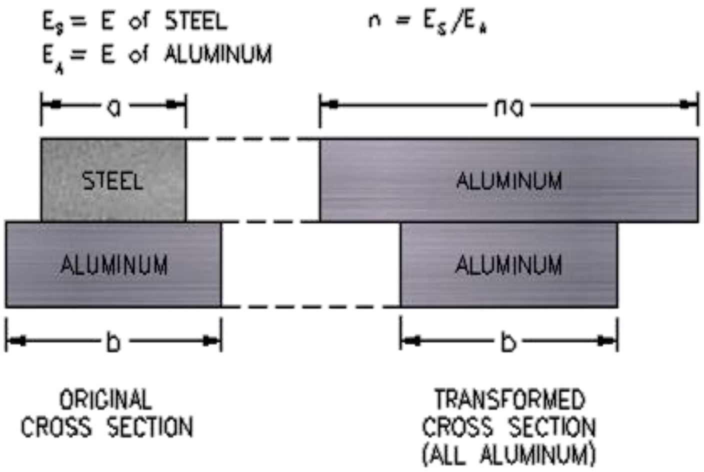


A similar procedure can be used to transform the beam into material 2 or a completely different material. One can also extend this technique to cover beams of more than two materials.

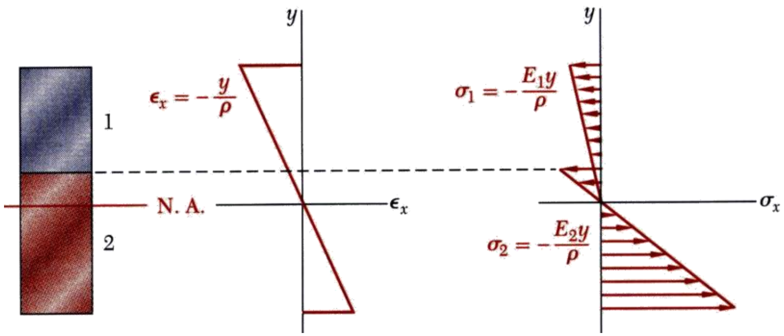
Flexure Formula

$$\sigma_{x1} = -\frac{My}{I_T} \qquad \sigma_{x2} = -\frac{My}{I_T}n$$





After the section is transformed all calculations are made using the transformed cross section, just as they would be on a beam of one material.
The neutral axis of bending is at the centroid of the transformed section and flexure stresses are calculated with the flexure stress formula .
One final step is required to return to the original cross section. If in going from the stress state in the transformed material we find a reduction in area then we must increase the stresses accordingly to carry the same load. Conversely if we increase area then we reduce stress. Those portions of the cross section which were unaltered in the transformation process carry the same stresses on both the original and transformed sections .



- Consider a composite beam formed from two materials with E_1 and E_2.

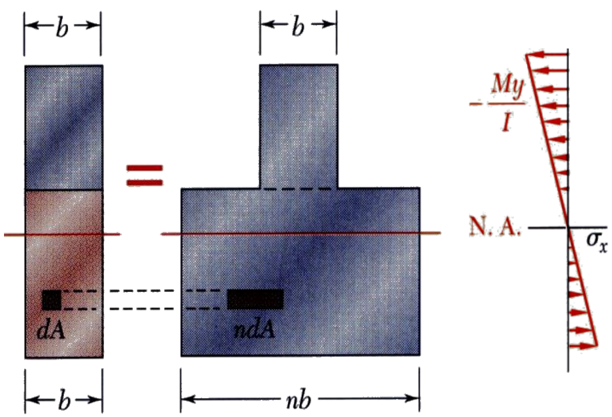
  - Normal strain varies linearly.

$$\varepsilon_x = -\frac{y}{\rho}$$

- Piecewise linear normal stress variation.

$$\sigma_1 = E_1 \varepsilon_x = -\frac{E_1 y}{\rho} \qquad \sigma_2 = E_2 \varepsilon_x = -\frac{E_2 y}{\rho}$$

Neutral axis does not pass through section centroid of composite section.



$$\sigma_x = -\frac{My}{I}$$

$$\sigma_1 = \sigma_x \qquad \sigma_2 = n\sigma_x$$

- Elemental forces on the section are

$$dF_1 = \sigma_1 dA = -\frac{E_1 y}{\rho} dA \quad dF_2 = \sigma_2 dA = -\frac{E_2 y}{\rho} dA$$

- Define a transformed section such that

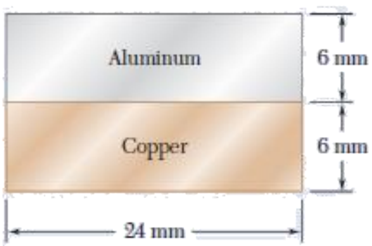
$$dF_2 = -\frac{(nE_1)y}{\rho} dA = -\frac{E_1 y}{\rho}(n\,dA) \qquad n = \frac{E_2}{E_1}$$

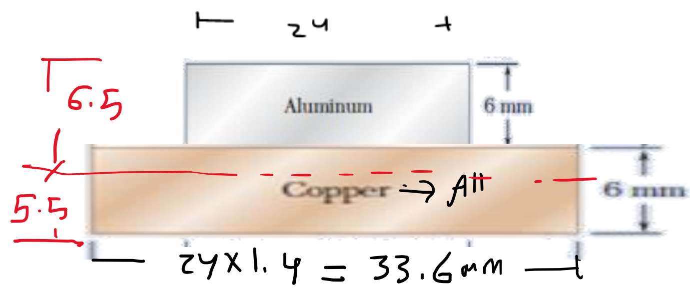
A copper strip (E_c=105 GPa) and an aluminum strip (E_a =75 GPa) are bonded together to form the composite beam shown. Knowing that the beam is bent about a horizontal axis by a couple of moment M =35 N.m, determine the maximum stress in (a) the aluminum strip, (b) the copper strip.

Copper → All ⇒ $n = \frac{E_c}{E_{Al}} = \frac{105}{75} = 1.4$

$$\bar{Y}_T = \frac{(24\times6)(9)+(33.6)(6)(3)}{345.6}$$

$= 5.5$ mm





Transformed section

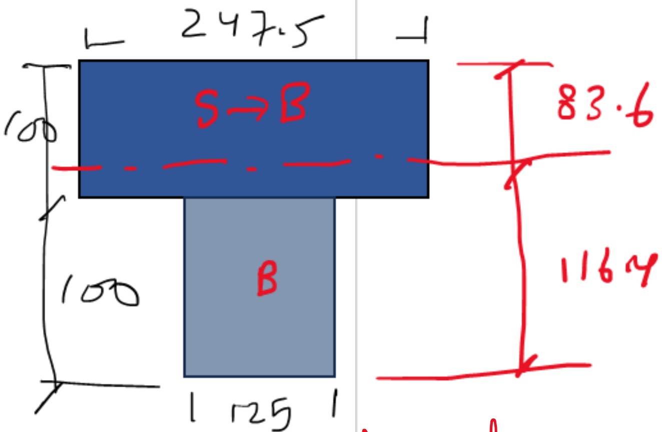
$$I_T = \Sigma I' + \Sigma A D_y^2 = \left[\frac{33.6\times6^3}{12} + \frac{24\times6^3}{12}\right] + \left[201.6\,(5.5-3)^2 + (144)(9-5.5)^2\right]$$

$I_T = 4.061\ 10^3\ mm^4$

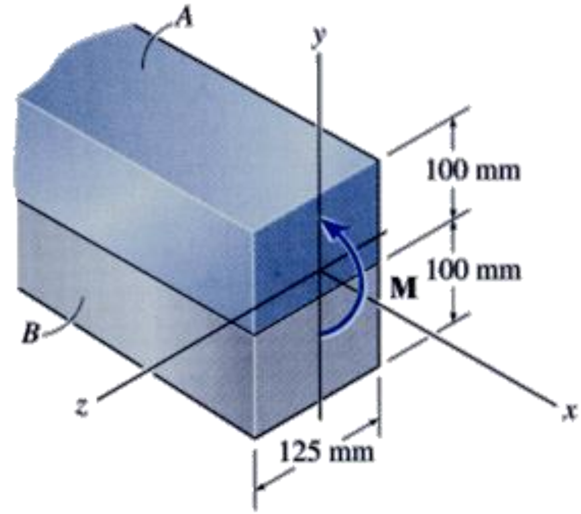
$$\sigma_{Al} = \frac{(35\times10^3)(6.5)}{4.061\times10^3} = -\underline{56\,MPa}$$

$$\sigma_{brass} = \frac{nMy}{I} = \frac{(1.4)(35\times10^3)(5.5)}{4.061\ \times10^3} = \underline{66.4\ MPa}$$

The composite beam is made of steel (A) and brass (B). If the allowable bending stress for the steel is $\sigma_s = 180$ MPa and for the brass $\sigma_b = 60$ MPa, determine the maximum moment M that can be applied to the beam. Assume $E_s = 200$ GPa and $E_b = 101$ MPa.



Transferred section



steel ⟹ Brass

$$n = \frac{E_{st}}{E_B} = \frac{200}{101} = 1.98$$

$$\text{Centroid} = \bar{Y} = \frac{(125)(100)(50) + (247.5)(100)(150)}{(100)(247.5) + (125)(100)} = \underline{116.4\text{ mm}}$$

$$I_T' = \frac{(125)(100)^3}{12} + (125\times100)(116.4-50)^2 + \frac{247.5\times100^3}{12} + (247.5\times100)(150-116.4)^2$$

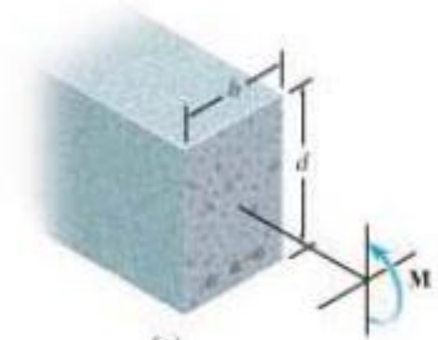
Centroid and I for the Transformed Section

$$I_T = 114.1\times10^6\text{ mm}^4$$

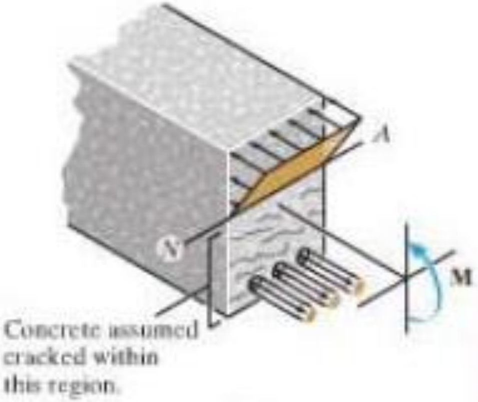
$$\sigma_b = 60 = \frac{M\times116.4}{114.1\times10^6} \Rightarrow \boxed{M = 58.8\text{ kN.m}}$$ ← Take Min

$$\sigma_s = 180 = 1.98\,\frac{M\times83.6}{114.1\times10^6} \Rightarrow M = 124\text{ kNm}$$

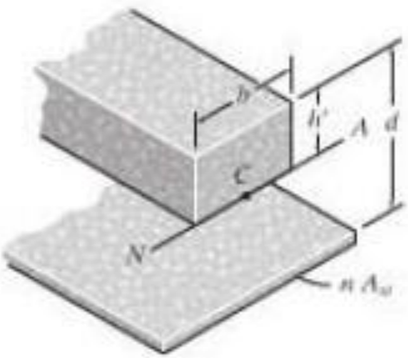
6.7 Reinforced concrete Beams



(a)



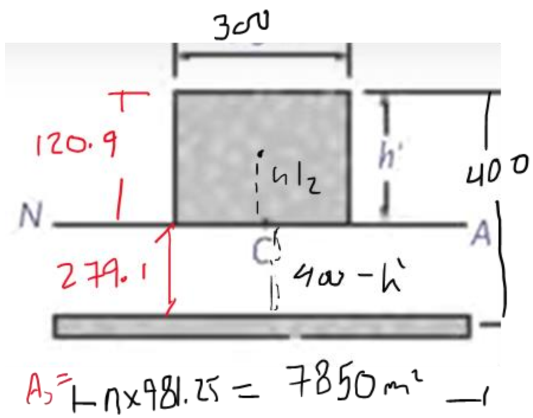
(b)



(c)

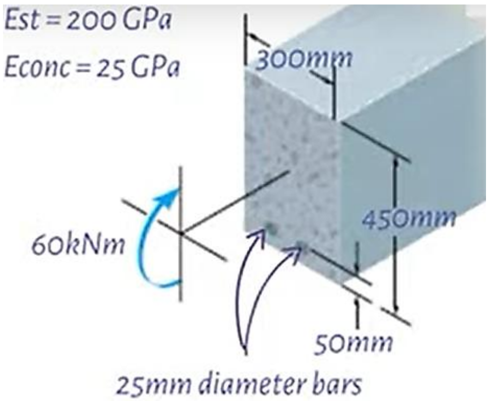
For the composite concrete beam, calculate the stresses in the concrete and steel.

2 Steel Bars $\quad \text{Area} = 2(\pi\, 25^2/4) = 981.25\ mm^2$



SOLUTION:

- Transform to a section made entirely of concrete.
- Evaluate geometric properties of transformed section.
- Calculate the maximum stresses in the concrete and steel.



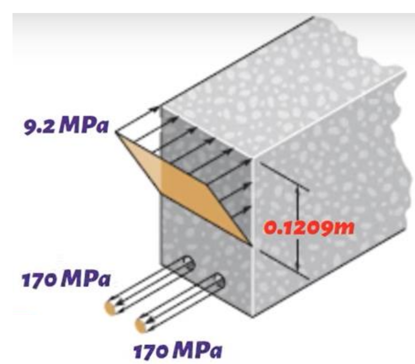
Transformed section $\quad st \rightarrow concrete \Rightarrow \dfrac{E_s}{E_c} = \dfrac{200}{25} = 8$

Location of the neutral axis $\Rightarrow (300\,h' \times h'/2) - (7858)(400 - h') = 0 \Rightarrow h' = 120.9\ mm$

$$I_T = \frac{(300)(120.9)^3}{12} + (300)(120.9)(120.9/2)^2 + (7850)(279.1)^2 = 788.52 \times 10^6\ mm^4$$

$$\sigma_c = \frac{My}{I_T}\ (\text{Compression}) = \frac{(60 \times 10^6)(120.9)}{788.52 \times 10^6} = 9.2\ MPa$$

$$\sigma_{st} = \frac{n\,My}{I_T} = \frac{(8)(60 \times 10^6)(279.1)}{788.52 \times 10^6} = 169.9\ MPa\ \ (\text{Tension})$$



Structural Mechanics

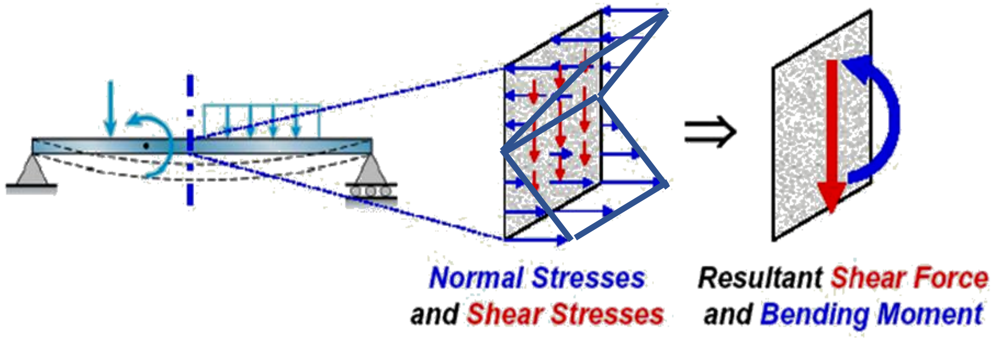
Chapter 7

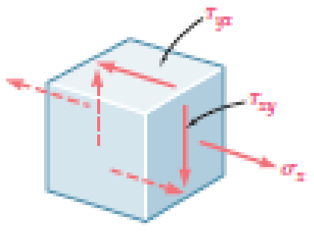
Transverse Shear

I. Marie

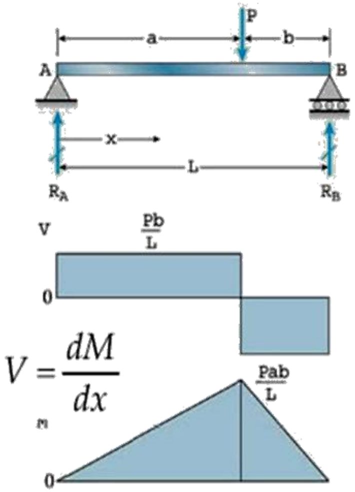
7.1 Introduction

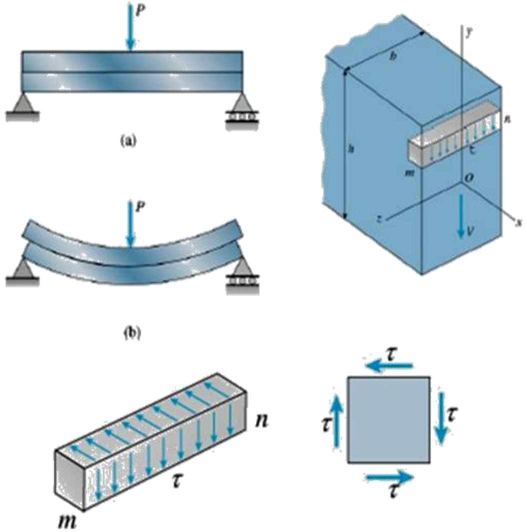
Bending Moments and Shear Forces in Beams

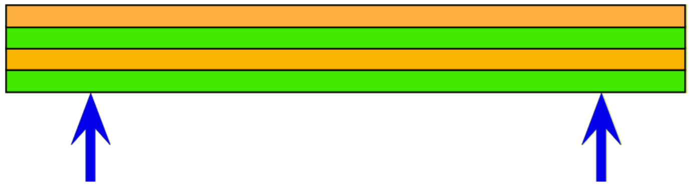




Stress element
from section of a transversely
loaded beam.





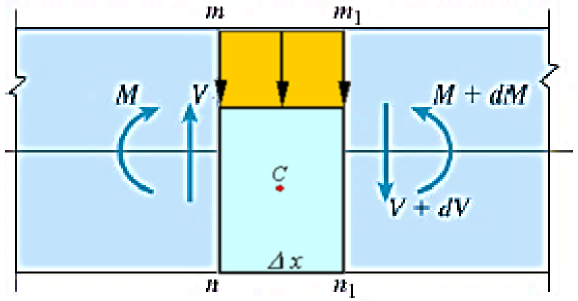


INITIAL SHAPE OF PLANKS

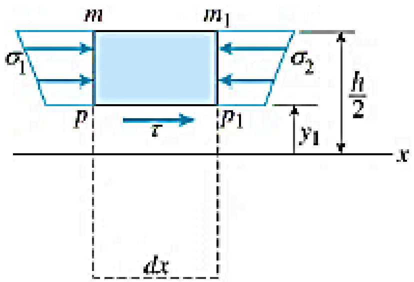
https://limitstatelessons.blogspot.com/2015/11/Shear-stress-in-beams.html

7.2 The Shear Formula

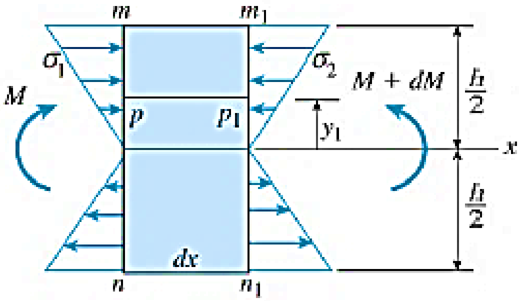
Shear Stresses in Beams



Side view of beam

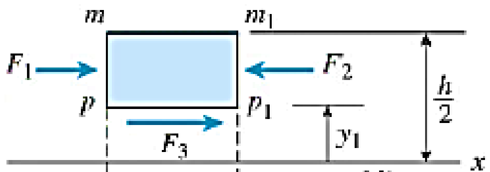


Side view of subelement



Side view of element

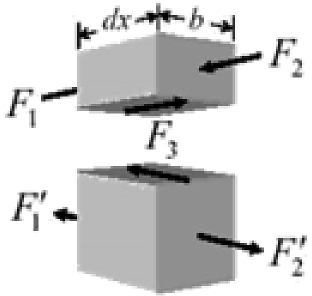
$$\sigma_1 = -\frac{My}{I} \qquad \sigma_2 = -\frac{(M+dM)y}{I}$$



$$F_1 = \int_{y_1}^{h/2} \sigma_1 dA = \int_{y_1}^{h/2} \frac{My}{I} dA \qquad F_2 = \int_{y_1}^{h/2} \sigma_2 dA = \int_{y_1}^{h/2} \frac{(M+dM)y}{I} dA$$

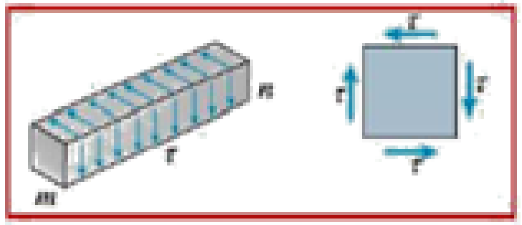
$$\boxed{F_3 = F_2 - F_1 = \int_{y_1}^{h/2} \frac{(dM)y}{I} dA}$$

$$F_3 = F_2 - F_1 = \frac{dM}{I} \int_{y_1}^{h/2} y dA$$

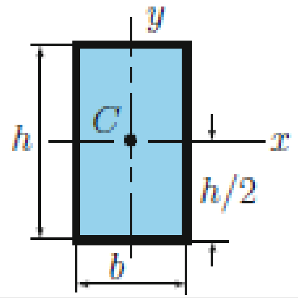


$$\text{Shear Stress}: \tau = \tau_{ave} = \frac{F_3}{\text{Bottom Area of the sub-element}}$$

$$\tau_{ave} = \frac{F_3}{bdx} = \left(\frac{dM}{dx}\right)\frac{1}{bI}\int_{y_1}^{h/2} y dA = \frac{V}{bI}\int_{y_1}^{h/2} y dA$$



$$\text{Let } Q = \int_{y_1}^{h/2} y dA \qquad \boxed{\tau = \frac{VQ}{bI}}$$

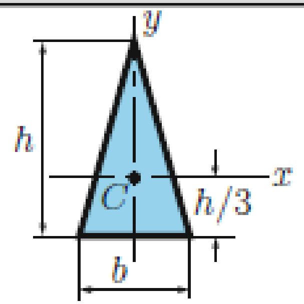


$$A = bh$$

$$I_{xx} = \frac{bh^3}{12}$$

$$I_{yy} = \frac{b^3h}{12}$$

$$I_C = \frac{bh}{12}(b^2 + h^2)$$

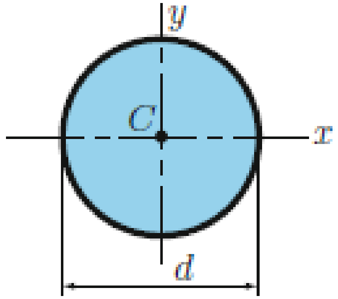


$$A = \frac{bh}{2}$$

$$I_{xx} = \frac{bh^3}{36}$$

$$I_{yy} = \frac{b^3h}{36}$$

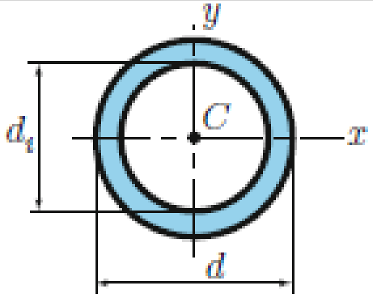
$$I_C = \frac{bh}{36}(b^2 + h^2)$$



$$A = \frac{\pi d^2}{4}$$

$$I_{xx} = I_{yy} = \frac{\pi d^4}{64}$$

$$I_C = \frac{\pi d^4}{32}$$



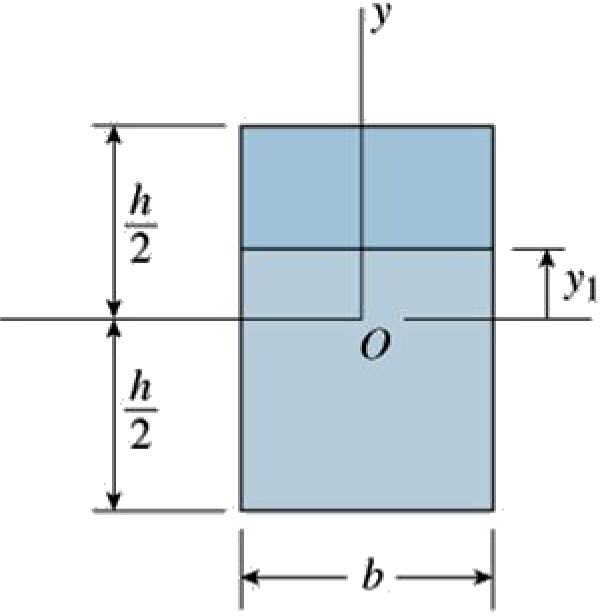
$$A = \frac{\pi}{4}(d^2 - d_i^2)$$

$$I_{xx} = I_{yy} = \frac{\pi}{64}(d^4 - d_i^4)$$

$$I_C = \frac{\pi}{32}(d^4 - d_i^4)$$

6.1B Shearing Stresses in a Beam

Shear Stresses in Rectangular Beams

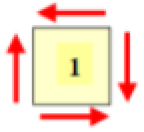


$$Q = \int_{y_1}^{h/2} y\,dA = \text{First moment of } A_1 \text{ w.r.t the z-axis}$$

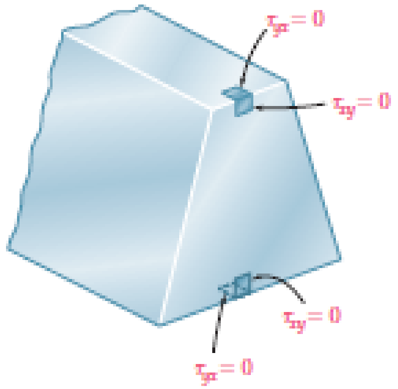
$$\tau = \frac{VQ}{bI}$$

$$\boxed{\tau_{\max} = 1.5\frac{V}{A}}$$

$$\tau_{(y_1=0)} = \tau_{\max} = \frac{Vh^2}{8I} = \frac{Vh^2}{8}\left(\frac{12}{bh^3}\right) = \frac{3}{2}\frac{V}{bh} = 1.5\tau_{ave}$$

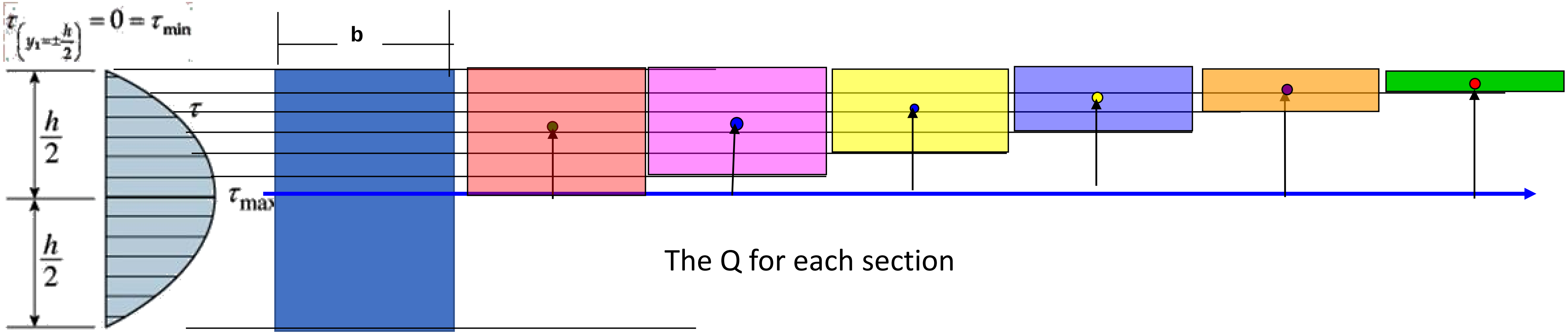


$\tau_{yx} = 0$ on the upper and lower faces of the beam



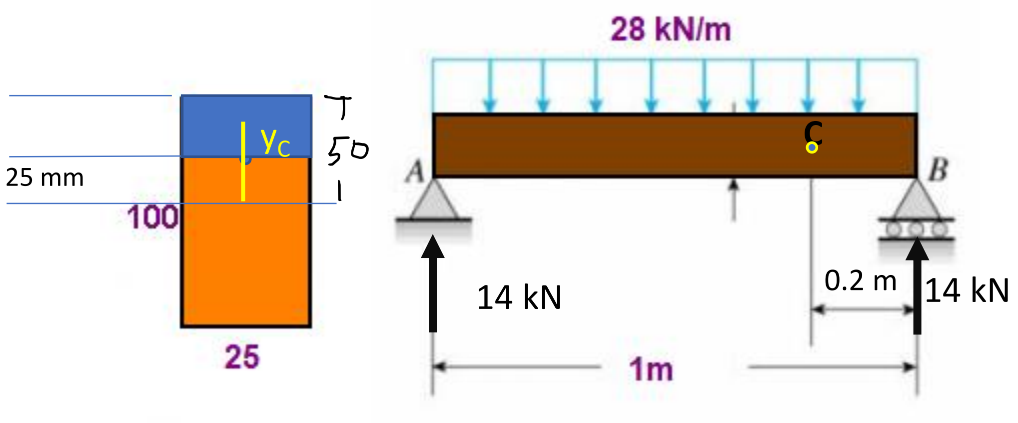
Beam cross section showing that the shearing stress is zero at the top and bottom of the beam.

$$\tau_{\left(y_1=\pm\frac{h}{2}\right)} = 0 = \tau_{\min}$$



Parabolic Distribution

Determine the bending stress and shear stress at **point C** and draw the stress element.



M_C = 2.24 kN.m
V_C = - 8.4 kN

$$I = \frac{b\,h^3}{12} = \frac{1}{12} \text{ x } 25 \text{ x } 100^3 = 2{,}083 \text{ x } 10^3 \text{ mm}^4$$

normal stress at C is

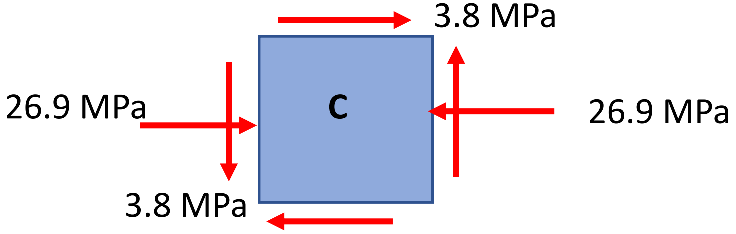
$$\sigma_C = -\frac{M\,y}{I} = -\frac{2.24 \text{ x } 106 \text{ N-mm x } 25 \text{ mm}}{2{,}083 \text{ x } 10^3 \text{ mm}^4} = -26.9 \text{ MPa}$$

shear stress at C, calculate Q_C first

$$A_C = 25 \text{ x } 25 = 625 \text{ mm}^2 \qquad y_C = 37.5 \text{ mm}$$

$$Q_C = A_C\,y_C = 23{,}400 \text{ mm}^3$$

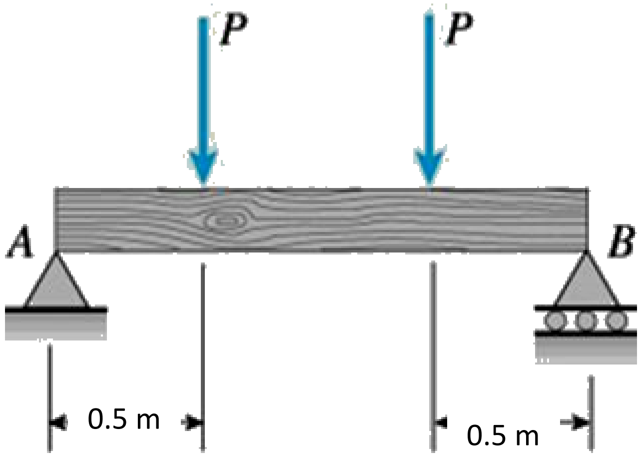
$$\tau_C = \frac{V_C\,Q_C}{I\,b} = \frac{8{,}400 \text{ x } 23{,}400}{2{,}083 \text{ x } 10^3 \text{ x } 25} = 3.8 \text{ MPa}$$



Stress element

What is the Maximum Permissible Load

$$\sigma_{allow} = 11\,\text{MPa} \quad \tau_{allow} = 1.2\,\text{MPa}$$

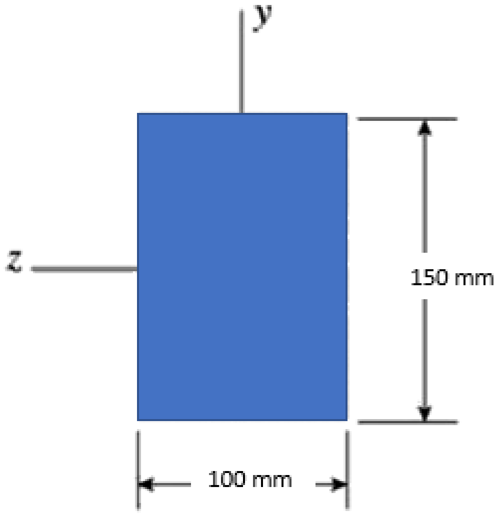


$$\sigma_{max} = \frac{My}{I} = \frac{Mx75}{\dfrac{100\text{ x, }(150^3)}{12}} = 11$$

$$M = 500P = 4125000$$

P = 8250 N

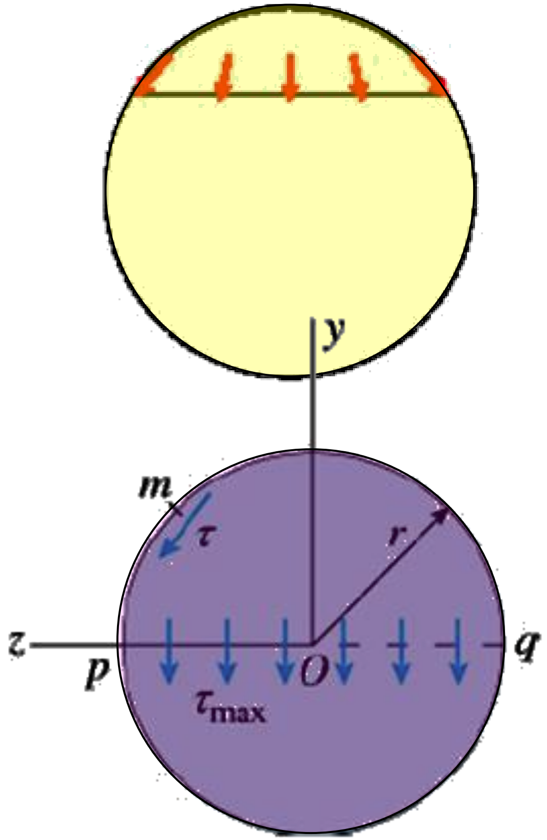
The smaller

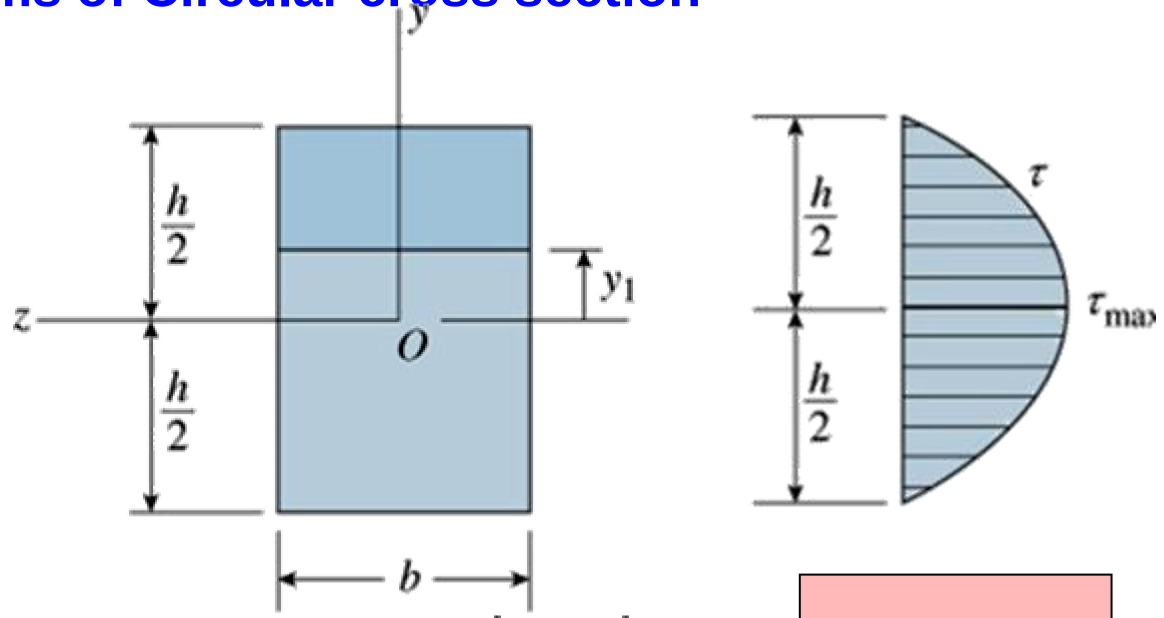


$$\tau = \frac{vQ}{Ib} = 1 \cdot 5v_{avc} = \frac{1.5P}{100 \times 150} = 1.2$$

P = 12000 N

Shear Stresses in beams of Circular cross section





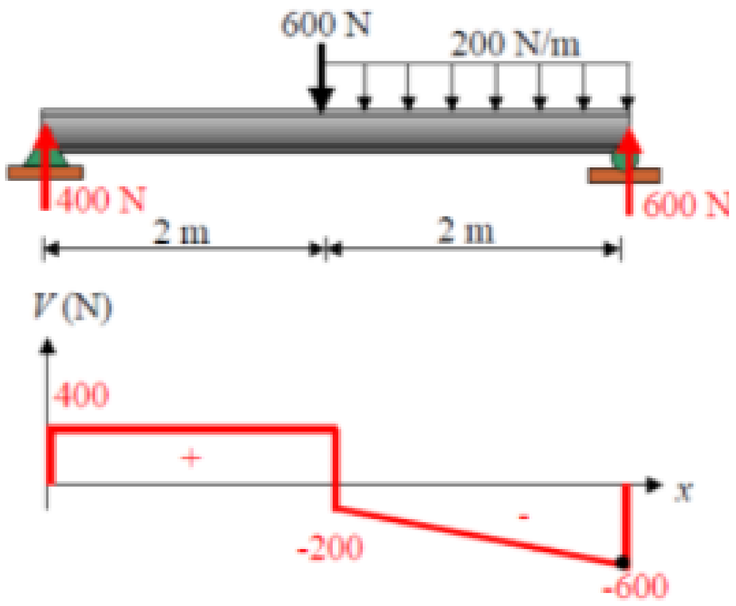
$$\tau_{(y_1=0)} = \tau_{max} = \frac{Vh^2}{8I} = \frac{Vh^2}{8}\left(\frac{12}{bh^3}\right) = \boxed{\frac{3}{2}\frac{V}{bh} = 1.5\tau_{ave}}$$

$$I = \frac{\pi r^4}{4}, \quad Q = A\bar{y} = \left(\frac{\pi r^2}{2}\right)\left(\frac{4r}{3\pi}\right) = \frac{2r^3}{3}$$

$$\tau_{max} = \frac{VQ}{bI} = \frac{4V}{3\pi r^2} = \boxed{\frac{4V}{3A} = 1.33\tau_{ave}}$$

The exact distribution of shear stress in a beam of circular cross section is very complicated and only that along the neutral axis can be determined relatively easily.

Draw the shear stress distribution along the depth of the cross section of the beam



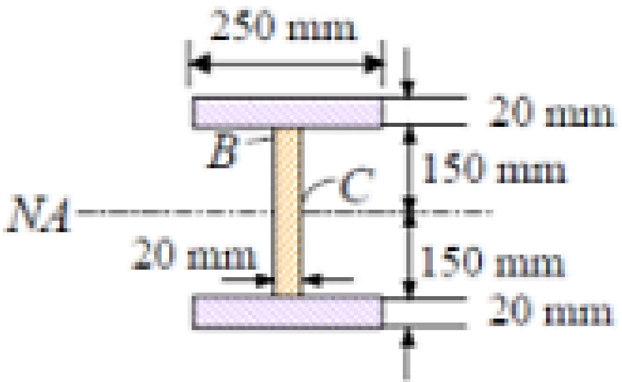
$$I = \frac{(250)\,(340)^3}{12} - \frac{230\ x\ 300^3}{12} = 301.3\ \mathrm{x}10^6 \mathrm{mm}^2$$

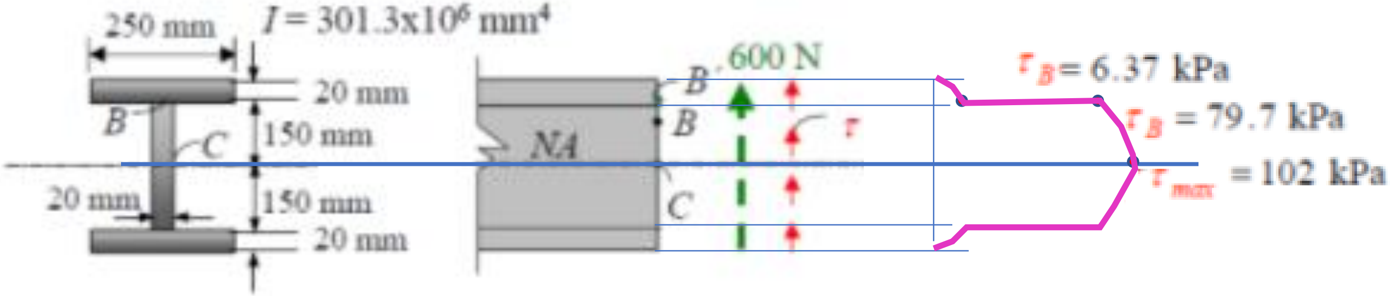
$$\tau = \frac{vQ}{Ib}$$

$$\tau_B = \frac{600x\ 20x250\ x160}{301.3x10^6\ x\ 20} = 6.37\ \mathrm{x}10^{-3}\mathrm{MPa} = 6.37\ \mathrm{kPa}$$

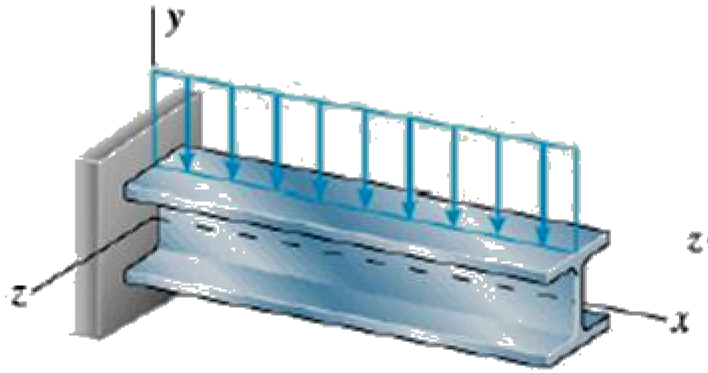
$$\tau_{B'} = \frac{600x\ 20x250\ x160}{301.3x10^6\ x250} = 0.0797\mathrm{x}10^{-3}\mathrm{MPa} = 79.7\ \ \mathrm{kPa}$$

$$\tau_{max} = \frac{600x((20x250\ x160) + (150x20x75))}{301.3x10^6\ x20} = 0.102\mathrm{MPa} = 102\ \ \mathrm{kPa}$$

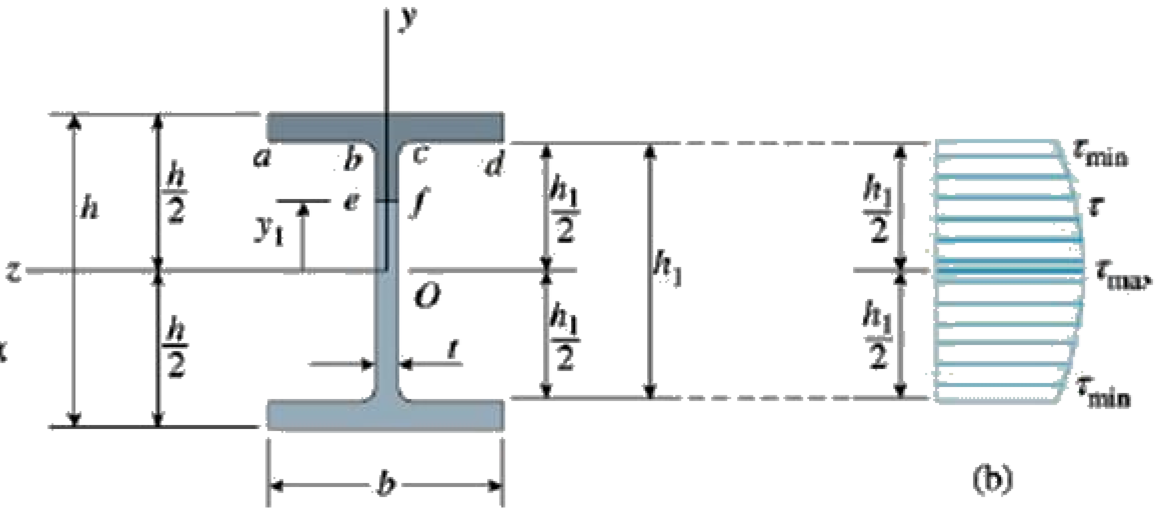




Shear Stresses in the Webs of Beams with Flanges

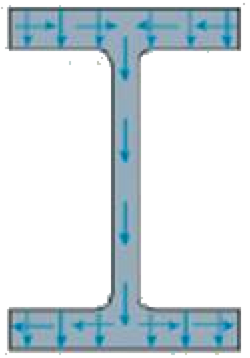


(a)



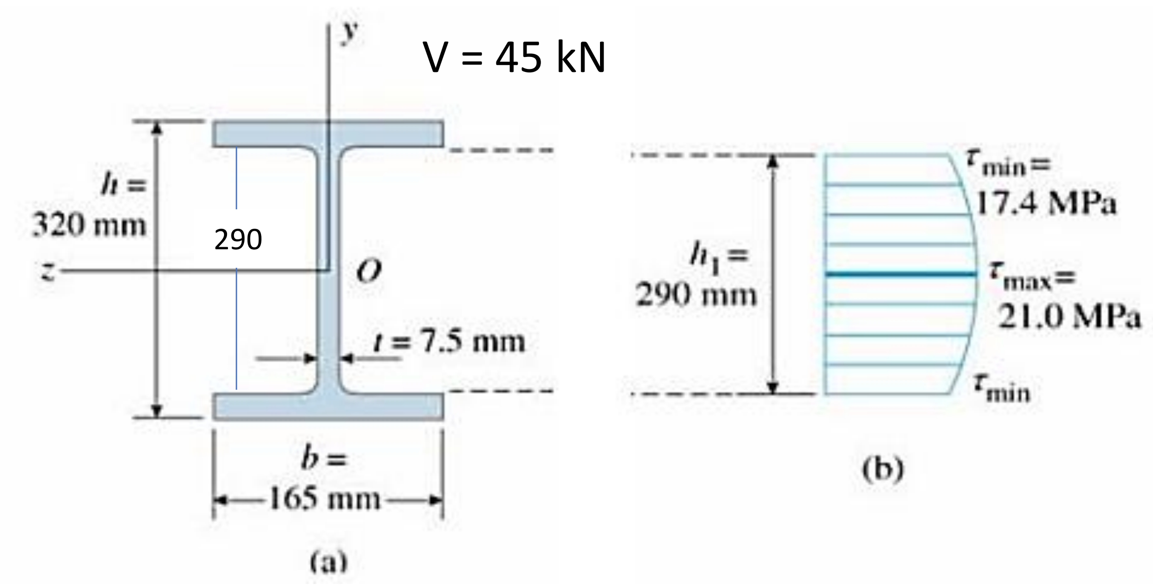
(b)

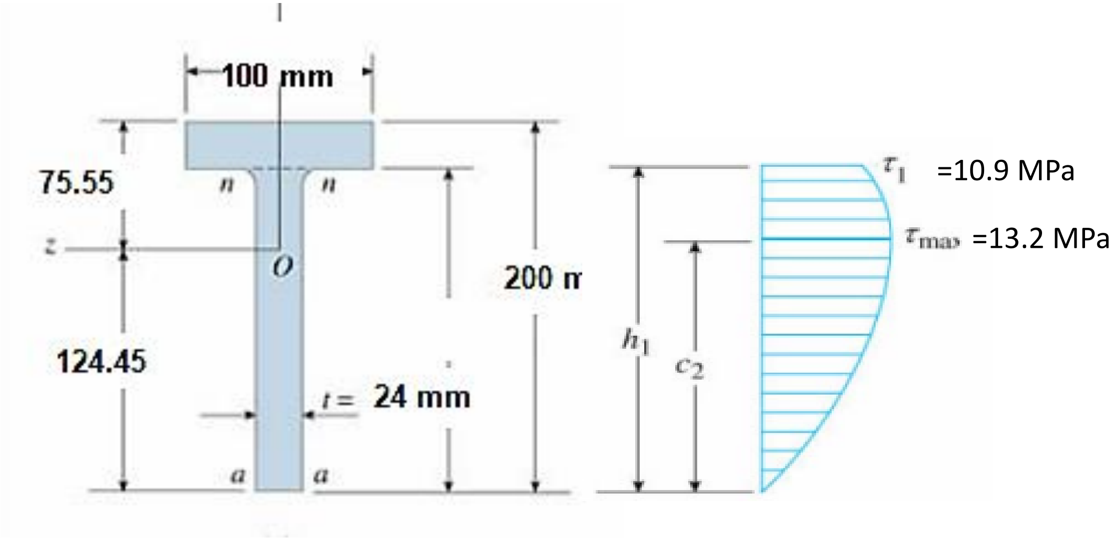
(a)


(b)

$$\tau_{web} = \frac{VQ}{It} = \frac{V}{8It}\left[b\left(h^2 - h_1^2\right) + t\left(h_1^2 - 4y_1^2\right)\right]$$

$$I = \frac{bh^3}{12} - \frac{(b-t)h_1^3}{12} = \frac{1}{12}\left(bh^3 - bh_1^3 + th_1^3\right)$$





V = 45 kN

Shear Force in the Web: **= (The area of the shear stress diagram) x(the thickness of the web)**

$$V = [\tau_{min}\, h_1 + 2/3\, h_1\, (\tau_{max} - \tau_{min})]t$$

$$= t\, h_1/3\, (2\, \tau_{max} + \tau_{min})$$

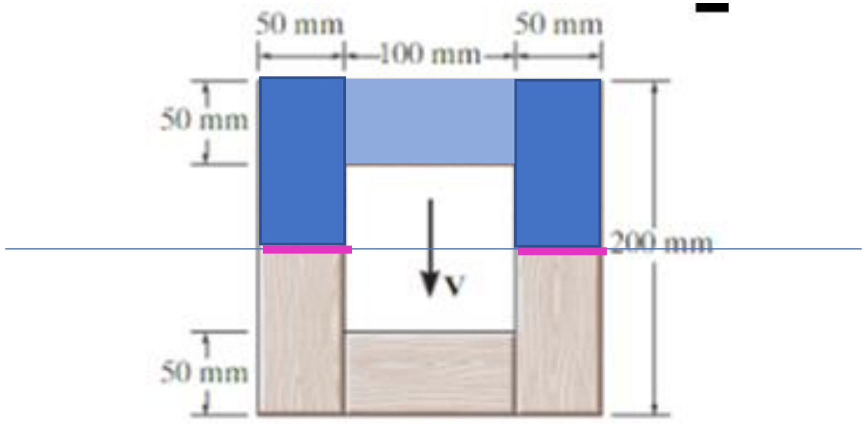
Shear force in the web is 90% -98% of the total shear force V acting on the cross section

Assuming the web carries

all of the shear force…

$$\tau_{avg} = V / th_1$$

The wood beam has an allowable shear stress of $\tau_{allow} = 7$ MPa. Determine the maximum shear force V that can be applied to the cross section.



$$I = \frac{1}{12}(0.2)(0.2)^3 - \frac{1}{12}(0.1)(0.1)^3 = 125(10^{-6})\ \text{m}^4$$

$$\tau_{all} = \frac{vQ}{Ib}$$

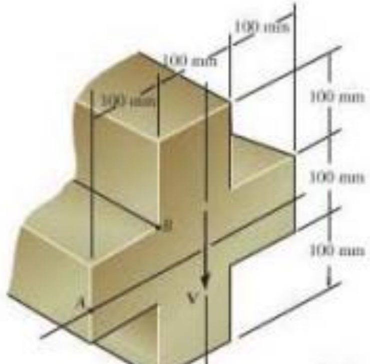
$$7(10^6) = \frac{V[(0.075)(0.1)(0.05) + 2(0.05)(0.1)(0.05)]}{125(10^{-6})(0.1)}$$

$V = 100$ kN

F7–1. If the beam is subjected to a shear force of $V = 100$ kN, determine the shear stress developed at point A. Represent the state of stress at A on a volume element.

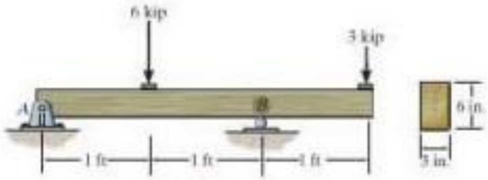
F7–1

F7–2. Determine the shear stress at points A and B on the beam if it is subjected to a shear force of $V = 600$ kN.



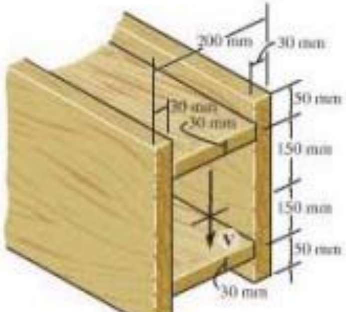
F7–2

F7–3. Determine the absolute maximum shear stress developed in the beam.



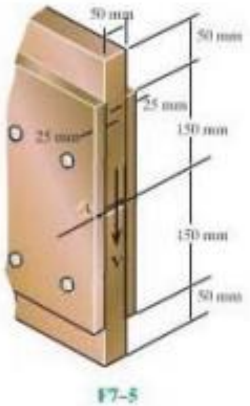
F7–3

F7–4. If the beam is subjected to a shear force of $V = 20$ kN, determine the maximum shear stress developed in the beam.



F7–4

F7–5. If the beam is made from four plates and subjected to a shear force of $V = 20$ kN, determine the maximum shear stress developed in the beam.



F7–5

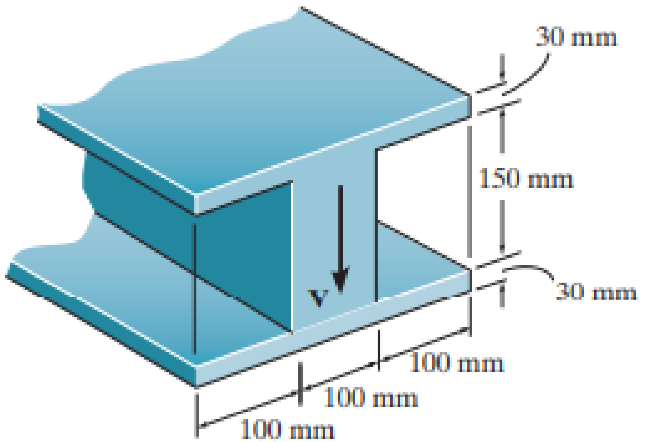
Determine the maximum shear stress in the strut if it is subjected to a shear force of V = 600 kN.

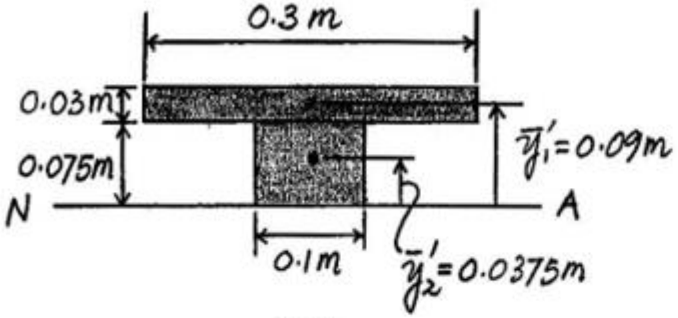
$$I = \frac{(300)(210)^3}{12} - \frac{(200)(150)^3}{12} = 175275000 \text{ mm}^4$$

$$Q = \Sigma \bar{y}' A' = (300 \times 30)(90) + (75 \times 100)(37.5)$$

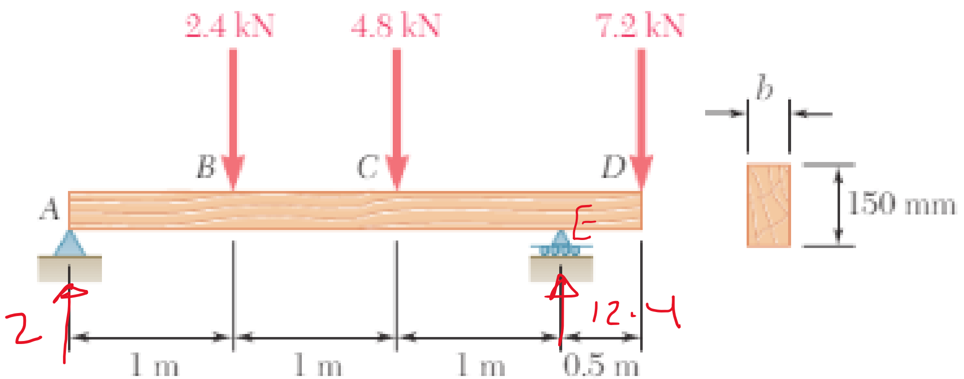
$$= 1091250 \text{ mm}^3$$

$$\tau_{max} = \frac{VQ}{Ib} = \frac{(600 \times 10^3)(1091250)}{(175275000)(100)} = \underline{37.4 \text{ MPa}}$$





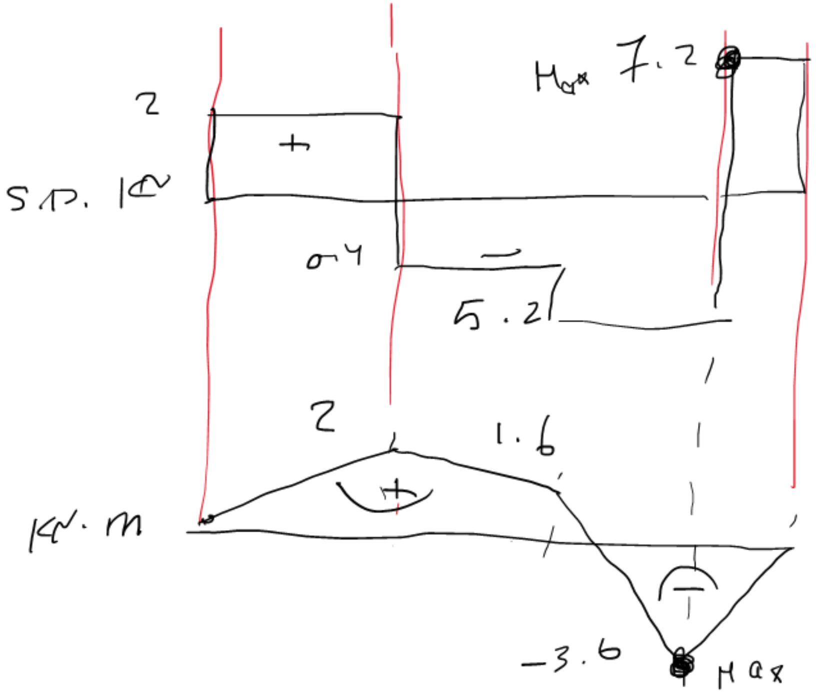
$$\sigma = \frac{My}{I}, \quad \tau = \frac{VQ}{Ib}, \quad \tau_{max} = \frac{3}{2}\frac{V}{A}$$



For the beam and loading shown, determine the minimum required width b, knowing that for the grade of timber used, $\sigma_{all} = 12$ MPa and $\tau_{all} = 825$ kPa.

$$\circlearrowleft \Sigma M_A = 0 \Rightarrow -(2.4)(1) - (4.8)(2) - (7.2)(3.5) + 3E_y = 0$$

$$E_y = 12.4 \text{ kN} \Rightarrow A_y = 2$$



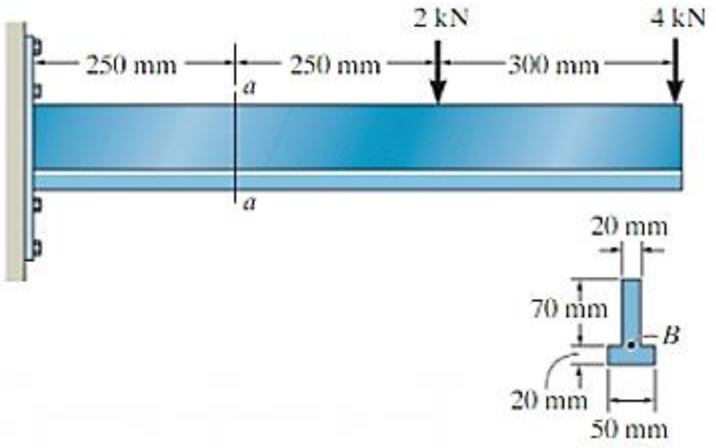
$$I = \frac{b(150)^3}{12} = 281250\,b$$

$$\sigma = 12 = \frac{(3.6\times10^6)(75)}{281250\,b} \Rightarrow b = 80 \text{ mm}$$

$$\tau = 825\times10^{-3} = \frac{(1.5)(7.2\times1000)}{150\,b} \Rightarrow \boxed{b = 87.3 \text{ mm}}$$

Take larger

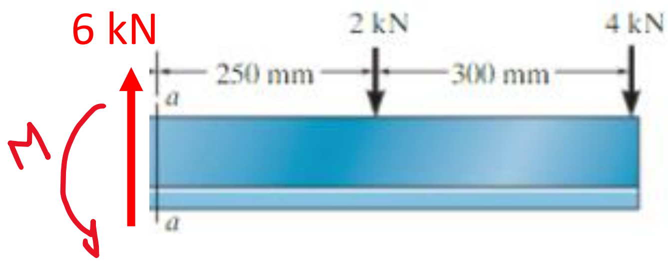
Determine the shear stress at point B on the web of the cantilever strut at section a-a



$$\bar{Y} = \frac{(20)(70)(55) + (20\times 50\times 10)}{(20\times 70 + 20\times 50)} = 36.25 \text{ mm}$$

$$I = \left[\frac{50\times 20^3}{12} + \frac{20\times 70^3}{12}\right] + \left(20\times 50\times (36.5-10)^2 + (20\times 70)(36.5-55)^2\right]$$

$$= 1.78625\times 10^6 \text{ mm}^4$$

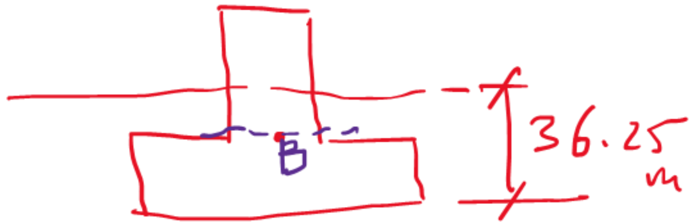


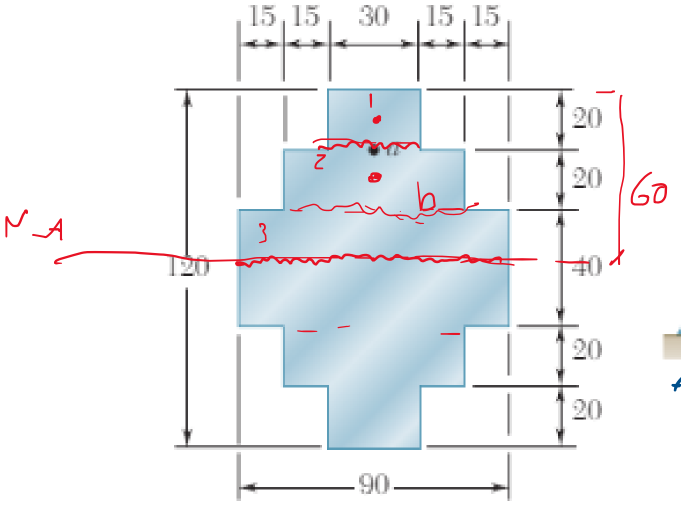
stresses at B.

$$\tau = \frac{VQ}{Ib}$$

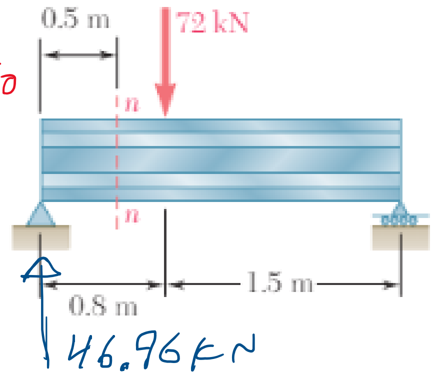
$$Q = (20\times 50)(36.25-10) = 26250 \text{ mm}^3$$

$$\tau_B = \frac{(6000)(26250)}{(1.78625\times 10^6)(20)} = \underline{4.41 \text{ MPa}}$$





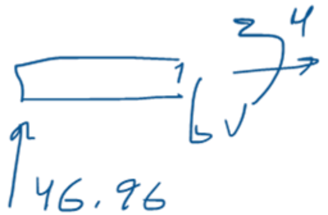
Dimensions in mm



For the beam and loading shown, consider section n-n and determine (*a*) the largest shearing stress in that section, (*b*) the shearing stress at point *a*.

$$\tau = \frac{VQ}{I}$$

at Section n-n



$$\Sigma F_y = 0$$

$$\underline{V = 46.96 \text{ kN}}$$

$$I = \frac{(90)(40)^3}{12} + (2)\frac{(60)(20)^3}{12} + 2\left(\frac{30\times 20^3}{12}\right) + 2(20)(60)(30)^2 + (2)(30)(20)(50)^2 = 5760000 \text{ mm}^4$$

τ at the neutral Axis : $Q = (20)(90)(10) + (20)(60)(30) + (30)(20)(50) = 84000 \text{ mm}^3$

$$\tau_{NA} = \frac{(46.96\times 10^3)(84000)}{(576\times 10^4)(90)} = 7.61 \text{ MPa}$$

τ at a

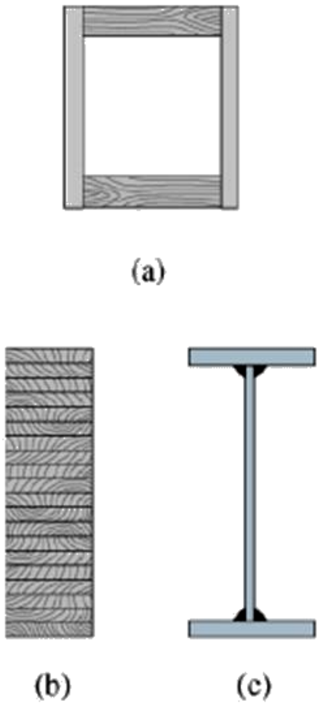
$$\tau_a = \frac{(46.96\times 10^3)(30\times 20\times 50)}{(576\times 10^4)(30)} = 8.15 \text{ MPa} \leftarrow \text{Max}$$

check also τ @ b

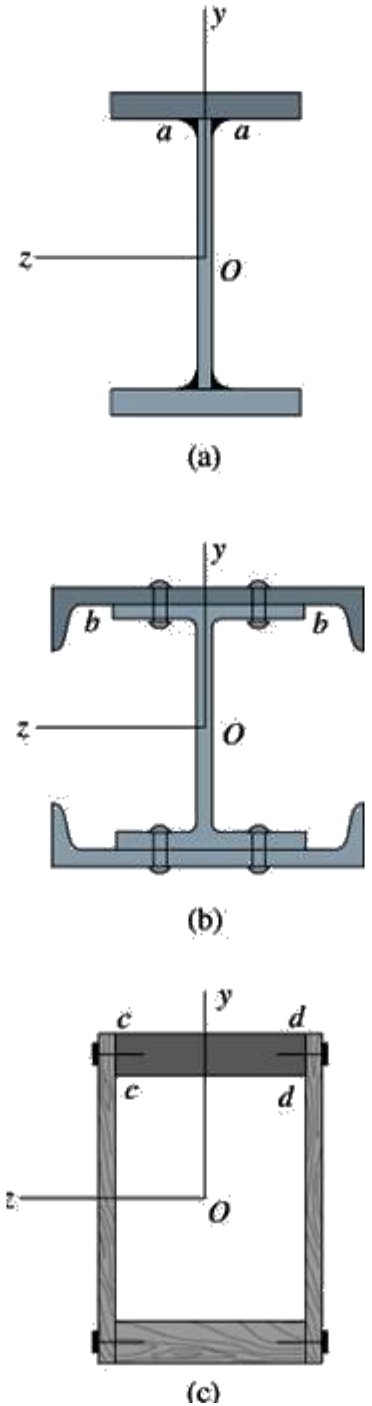
7.3 Shear Flow in Built-Up Members

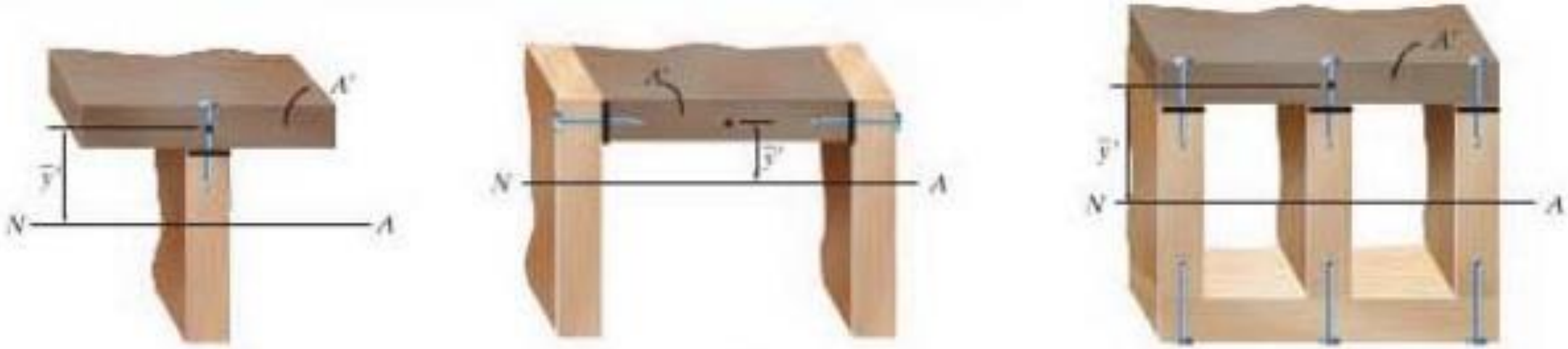
Shear Force in Fasteners :

In many applications, beam sections consist of several pieces of material that are attached together in a number ways: bolts, rivets, nails, glue, weld, etc. In such so called built-up sections we are interested in knowing the amount of shear stress and the resulting shear force at the cross section of fasteners or over the glued surface .



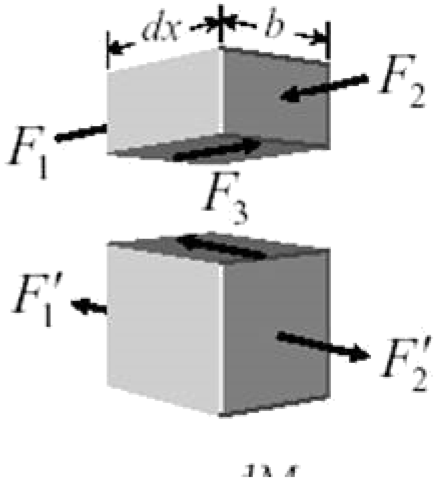
$$f = \frac{VQ}{I}$$



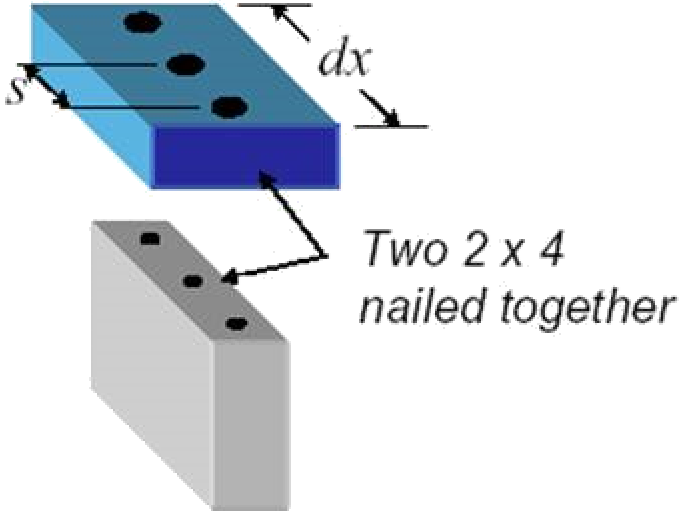


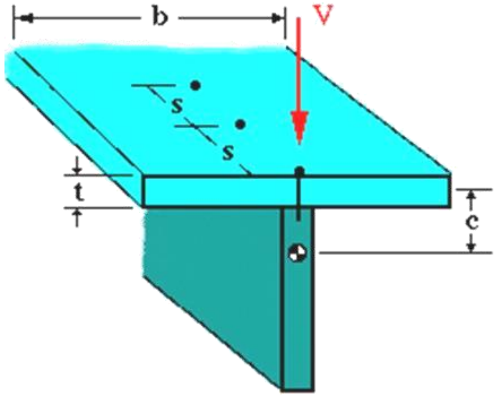
(a) (b) (c)

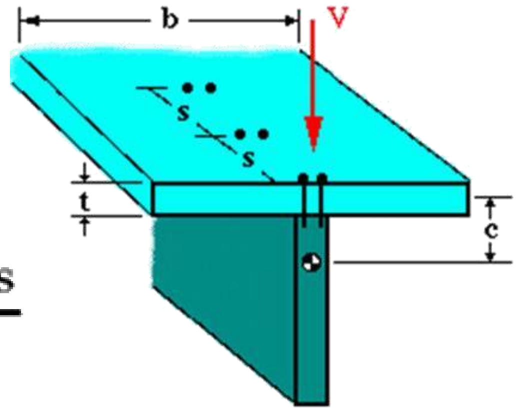
Shear Flow



Built-Up Beams







Shear Force : $F_3 = \dfrac{dM}{I}\int y\,dA$

Shear Stress : $\tau = \dfrac{F_3}{bdx} = \dfrac{dM}{dxbI}\int y\,dA = \dfrac{VQ}{bI}$

Shear Flow : $f = \dfrac{F_3}{dx} = \dfrac{dM}{dx}\dfrac{1}{I}\int y\,dA = \dfrac{VQ}{I}$

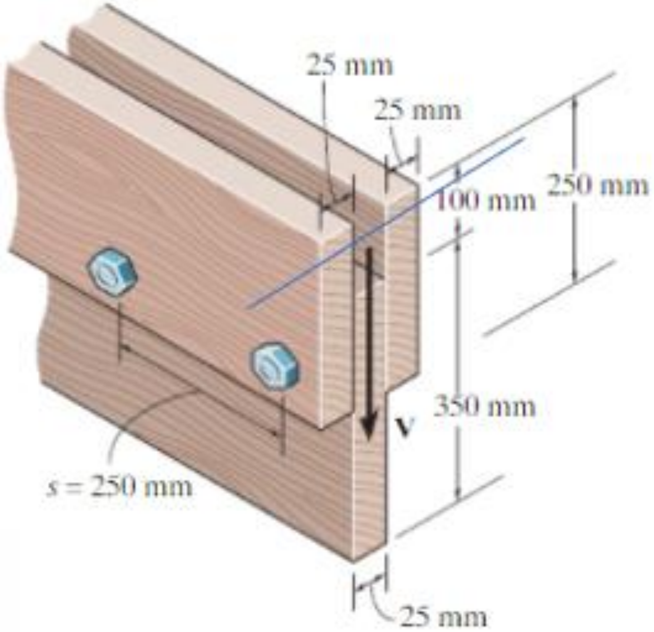
n = Number of rows of nails

F = Strength of each nail

$$f = \frac{nF}{s} = \frac{\text{Shear force provided by nails}}{\text{Nail Spacing}}$$

Nail Spacing : $s = \dfrac{nF}{f}$

A beam is constructed from three boards bolted together as shown. Determine the shear force developed in each bolt if the bolts are spaced $s = 250$ mm apart and the applied shear is $V = 35$ kN.



$$\bar{y} = \frac{2\,(0.125)(0.25)(0.025) + 0.275\,(0.35)(0.025)}{2\,(0.25)(0.025) + 0.35\,(0.025)} = 0.18676\text{ m}$$

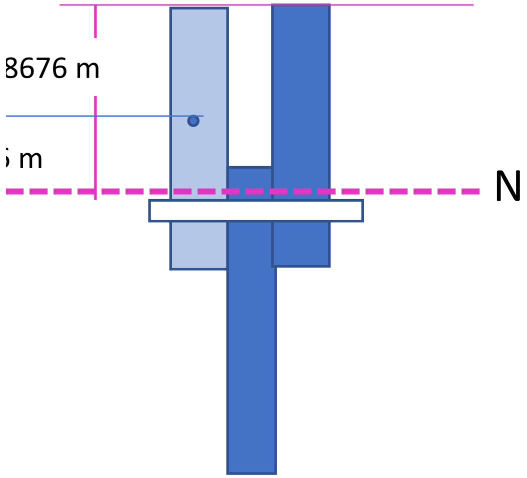
$$I = (2)\left(\frac{1}{12}\right)(0.025)(0.25^3) + 2\,(0.025)(0.25)(0.18676 - 0.125)^2$$

$$+ \frac{1}{12}(0.025)(0.35)^3 + (0.025)(0.35)(0.275 - 0.18676)^2$$

$$= 0.270236\,(10^{-3})\text{ m}^4$$

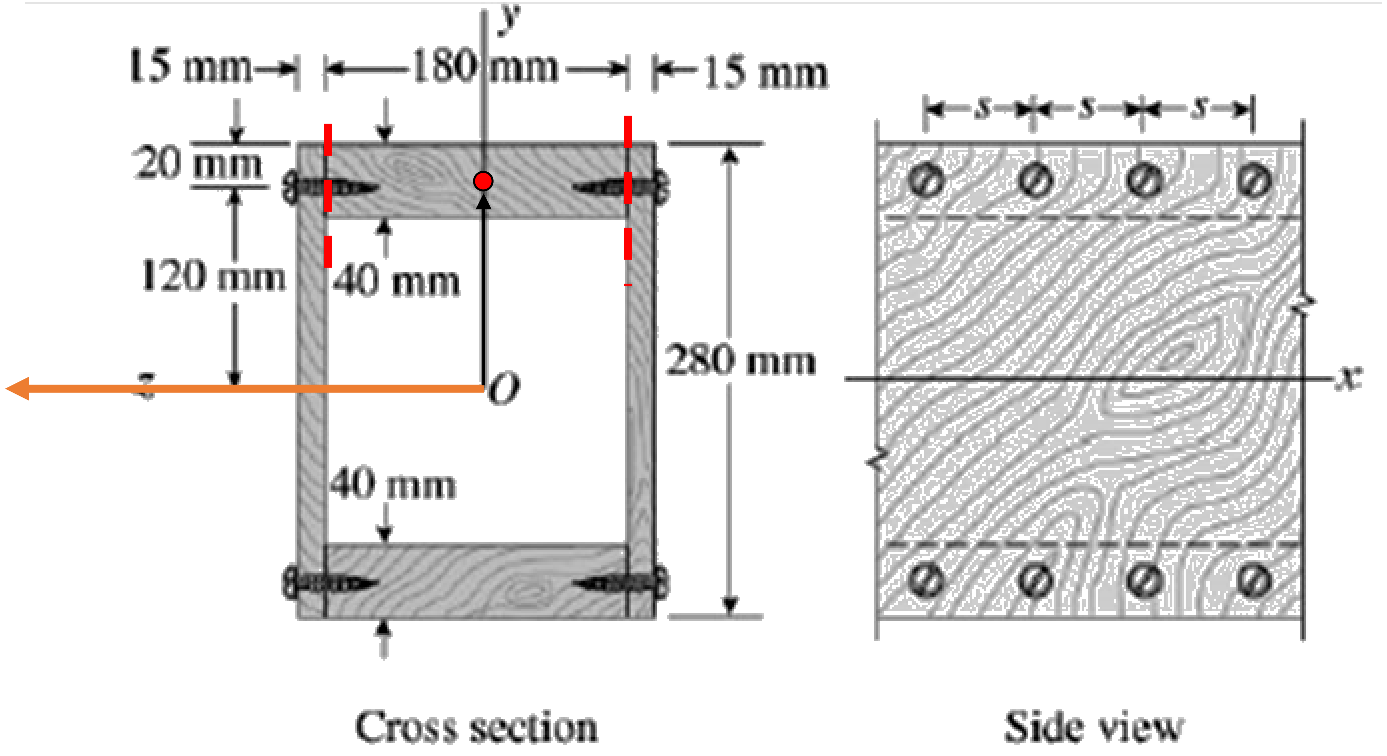
$$Q = \bar{y}'A' = 0.06176(0.025)(0.25) = 0.386(10^{-3})\text{ m}^3$$

$$q = \frac{VQ}{I} = \frac{35\,(0.386)(10^{-3})}{0.270236\,(10^{-3})} = 49.997\text{ kN/m}$$



F=f s = 49.997 (0.25) =12.5 kN

Ex. 5-16 The plywood is fastened to the flanges by wood screws having an allowable load in shear of F = 800 N each if the shear force V acting on the cross section = 10.5 kN.
Determine the max. permissible longitudinal spacing s of the screws.



$$I = \frac{(210)(280)^3}{12} - \frac{(180)(200)^3}{12}$$
$$= 264.2\times10^6 \text{ mm}^4$$

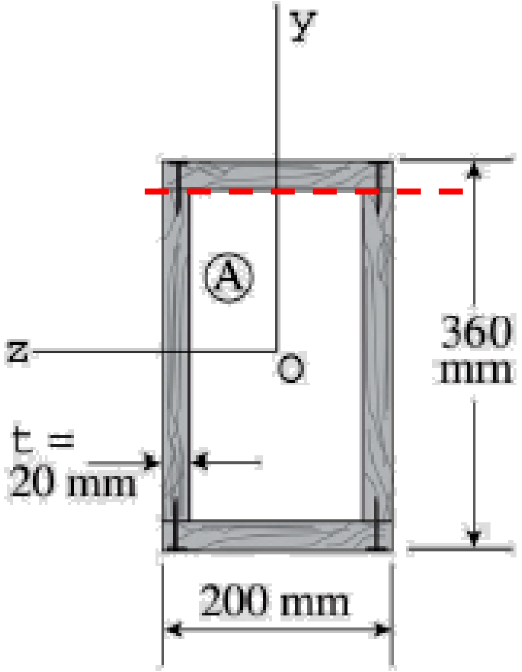
$$Q = (180)(40)(140-20)$$
$$= 864\times10^3 \text{ mm}^3$$

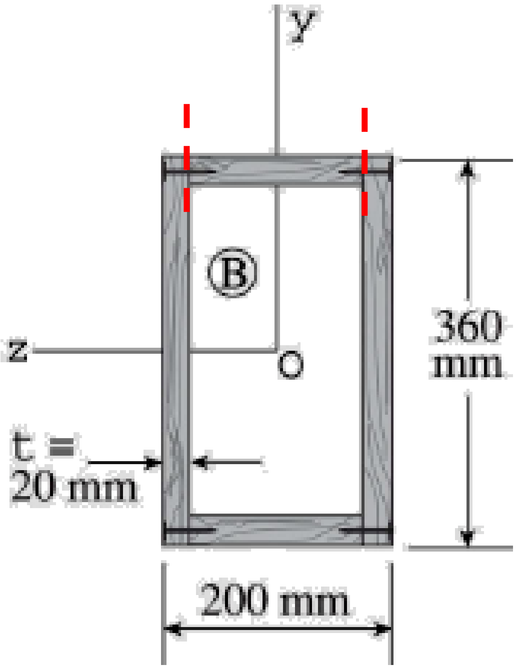
$$f = \frac{VQ}{I} = \frac{(10.5\times10^3)(864\times10^3)}{264.16\times10^6} = 34.3\frac{\text{N}}{\text{mm}}$$

$$s = \frac{2F}{f} = \frac{2(800)}{34.3} = 46.6 \text{ mm}$$

Use 45 mm

Find the spacing for each case





$V = 3.2\ \text{kN} \qquad F = 250\ \text{N}$

$$I = \frac{(200)(360)^3}{12} - \frac{(160)(320)^3}{12} = 340.69 \times 10^6\ \text{mm}^4$$

$$Q = (200)(20)(180-10) = 680 \times 10^3\ \text{mm}^3$$

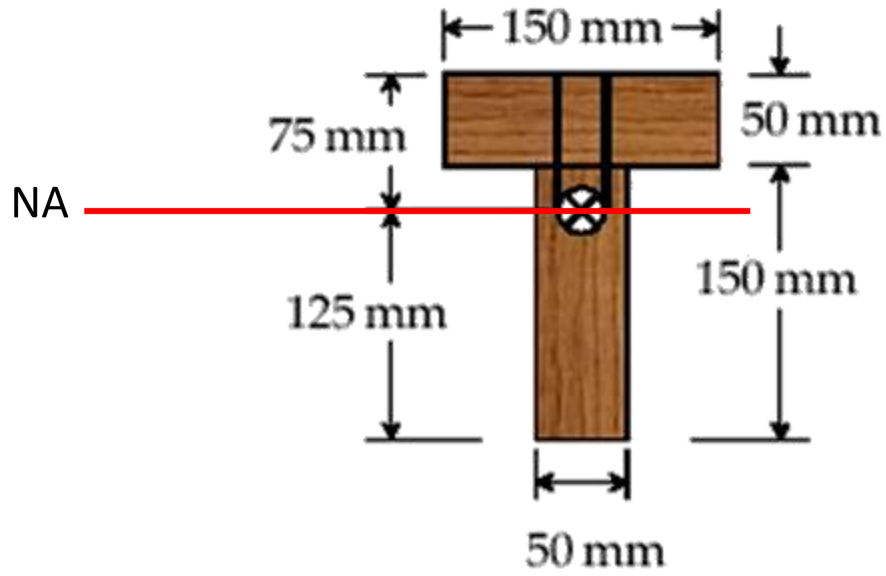
$$s = \frac{2FI}{VQ} = 78.3\ \text{mm}$$

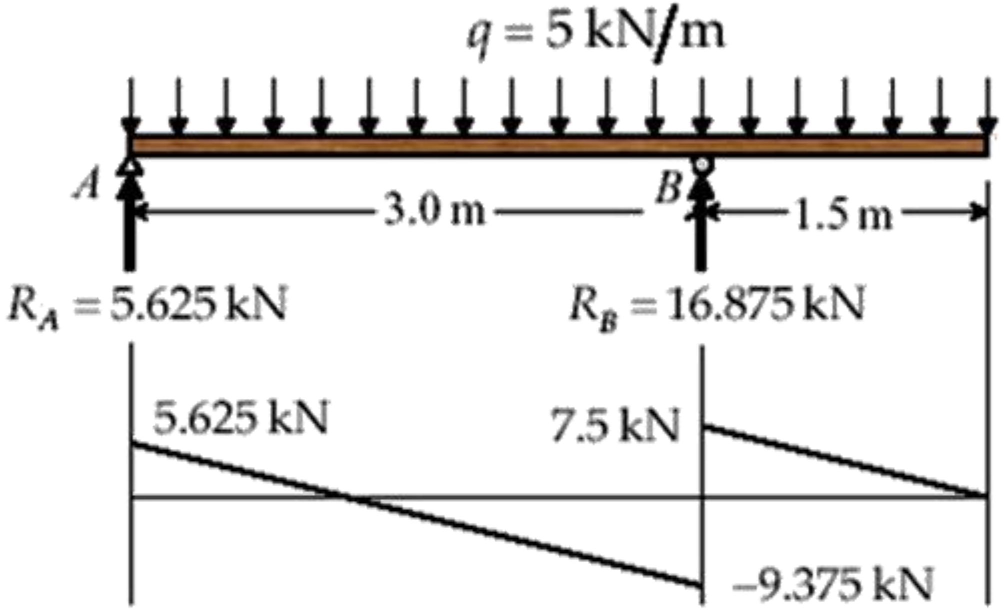
$$Q = (160)(20)(180-10) = 544 \times 10^3\ \text{mm}^3$$

$$s = \frac{2FI}{VQ} = 97.9\ \text{mm}$$

Allowable shear load per each nail (F) =2000 N
What is the maximum nail spacing

$I = 53.125 \times 10^{-6}\ m^4$





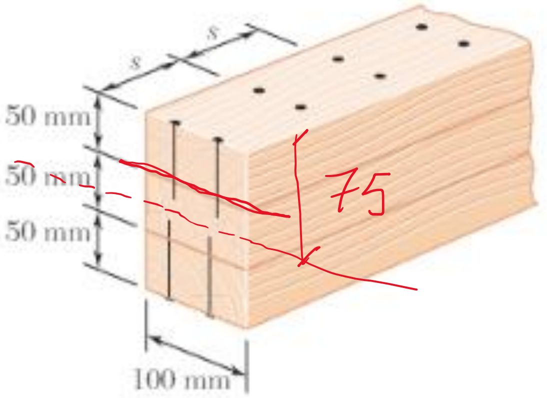
$$f = \frac{VQ}{I} = \frac{(9{,}375)(375.0\times10^{-6})}{53.125\times10^{-6}} = 66.18\times10^{3}\ \frac{\mathrm{N}}{\mathrm{m}}$$

$$Q = (150)(50)(75-25)$$
$$= 375.0\times10^{3}\ \mathrm{mm}^{3}$$

$$s = \frac{2F}{f} = \frac{2\times2{,}000}{66.18\times10^{3}} = 0.0604\ \mathrm{m} = 60.4\ \mathrm{mm}$$

Full size 50x100 mm boards are nailed together to form a beam that is subjected to a vertical shear of 1500 N. The allowable shearing force in each anil is 400 N. Determine the largest longitudinal spacing S that can be used between each pair of nails.

V F



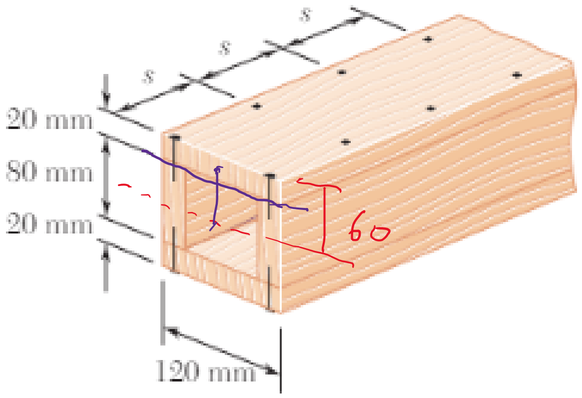
$S = \frac{nF}{f}$; $n = 2$ — $f = \frac{VQ}{I}$:

$I = \frac{(100)(150)^3}{12} = 28125000 \text{ mm}^4$

$f = \frac{(1500)(250000)}{28125000} = 13.33 \text{ N/m}$

$Q = (50)(100)(50) = 250000 \text{ mm}^3$

$S = \frac{(2)(400)}{13.33} = \underline{60 \text{ mm}}$



A square box beam is made of two 20×80-mm planks and two 20×120-mm planks nailed together as shown. Knowing that the spacing between the nails is $s = 30$ mm and that the vertical shear in the beam is $V = 1200$ N, determine (*a*) the shearing force in each nail, (*b*) the maximum shearing stress in the beam.

$F = ?$

τ_{max}

(a)

$S = \frac{nF}{f}$, $f = \frac{VQ}{I}$, $n = 2$

$I = \frac{(120)(120)^3}{12} - \frac{(80)(80)^3}{12} = 13866666.67 \text{ mm}^4$

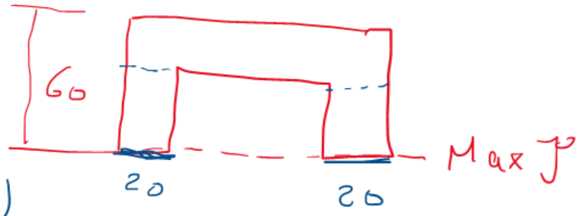
$Q = (20)(120)(50) = 120000$

$f = \frac{(1200)(120000)}{13866666.67} = 10.38 \text{ N/mm}$

$S = \frac{nF}{f} = 30 = \frac{2F}{10.38} \Longrightarrow F = 155.8 \text{ N}$

(b)

$\tau = \frac{VQ}{Ib}$, $b = 40$



$Q = (120)(20)(50) + (2)(20)(40)(20)$

$= 152000$

$\tau_{max} = \frac{(1200)(152000)}{(13866666.67)(40)} = 0.329 \text{ MPa}$

6.1 Three full-size 50 × 100-mm boards are nailed together to form a beam that is subjected to a vertical shear of 1500 N. Knowing that the allowable shearing force in each nail is 400 N, determine the largest longitudinal spacing s that can be used between each pair of nails.

6.2 For the built-up beam of Prob. 6.1, determine the allowable shear if the spacing between each pair of nails is $s = 45$ mm.

6.3 Three boards, each 2 in. thick, are nailed together to form a beam that is subjected to a vertical shear. Knowing that the allowable shearing force in each nail is 150 lb, determine the allowable shear if the spacing s between the nails is 3 in.

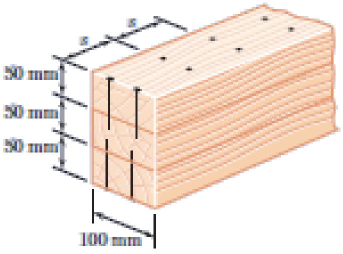


Fig. P6.1

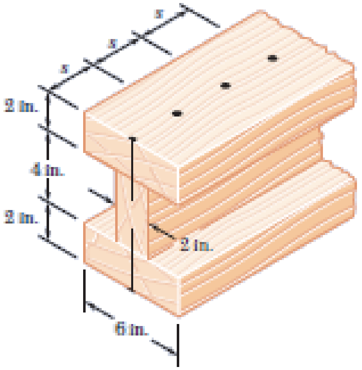


Fig. P6.3

6.4 A square box beam is made of two 20 × 80-mm planks and two 20 × 120-mm planks nailed together as shown. Knowing that the spacing between the nails is $s = 30$ mm and that the vertical shear in the beam is $V = 1200$ N, determine (*a*) the shearing force in each nail, (*b*) the maximum shearing stress in the beam.

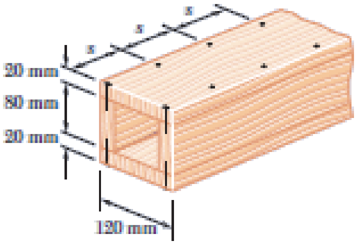


Fig. P6.4

6.5 The American Standard rolled-steel beam shown has been reinforced by attaching to it two 16 × 200-mm plates, using 18-mm-diameter bolts spaced longitudinally every 120 mm. Knowing that the average allowable shearing stress in the bolts is 90 MPa, determine the largest permissible vertical shearing force.

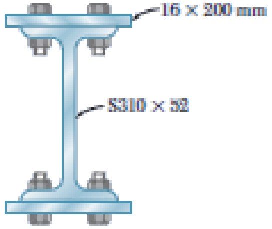


Fig. P6.5

6.13 and *6.14* For a beam having the cross section shown, determine the largest allowable vertical shear if the shearing stress is not to exceed 60 MPa.

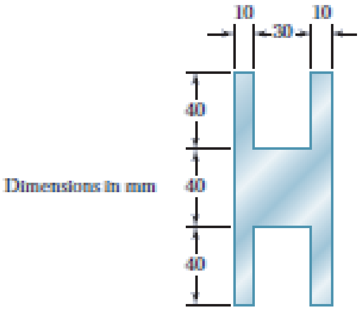


Fig. P6.13

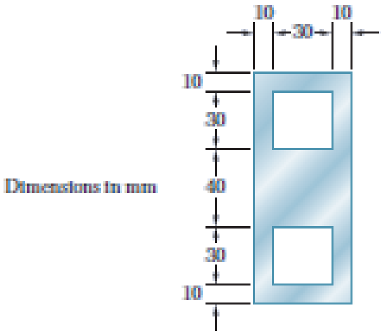


Fig. *P6.14*

6.18 For the beam and loading shown, determine the minimum required width b, knowing that for the grade of timber used, $\sigma_{all} = 12$ MPa and $\tau_{all} = 825$ kPa.

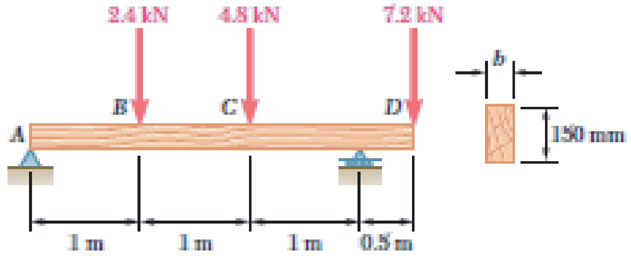


Fig. P6.18

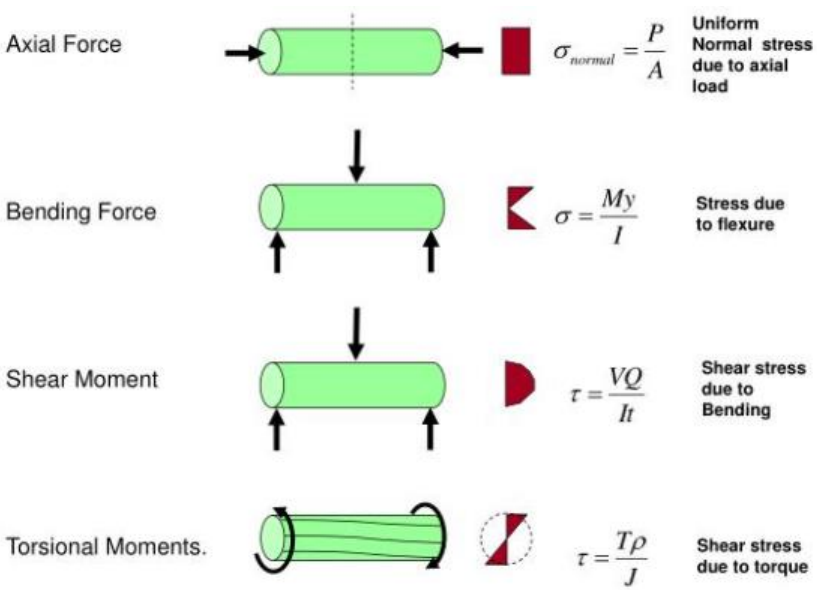
Structural Mechanics

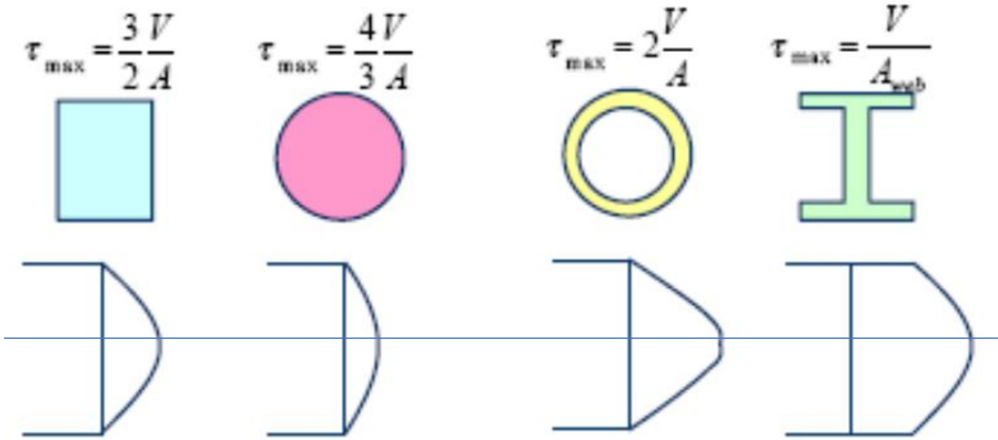
Chapter 8

Combined Loading

I. Marie

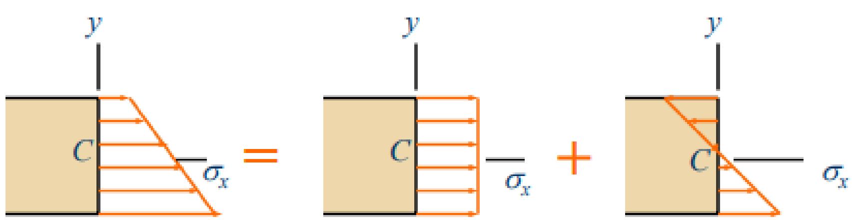
8.2 State of stress caused by combined loading





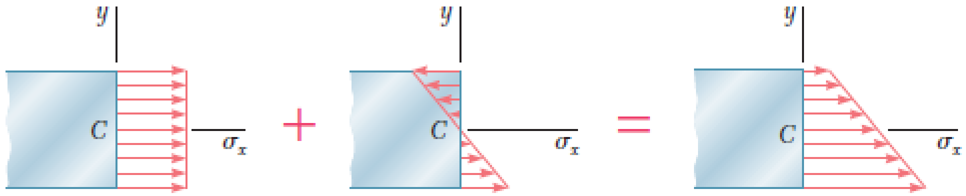
Superposition

- Determine normal and shear stress for each loading . Use superposition principle to determine the resultant normal and shear stresses.
- Show the results as a distribution of stresses acting over the member cross-sectional area
- Represent the results on am element located at the point of interest

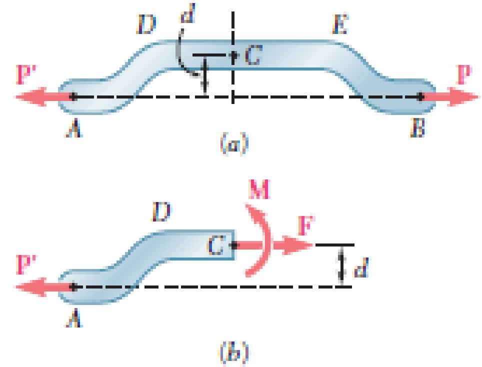


$$\sigma_x = (\sigma_x)_{\text{centric}} + (\sigma_x)_{\text{bending}}$$

ECCENTRIC AXIAL LOADING IN A PLANE OF SYMMETRY

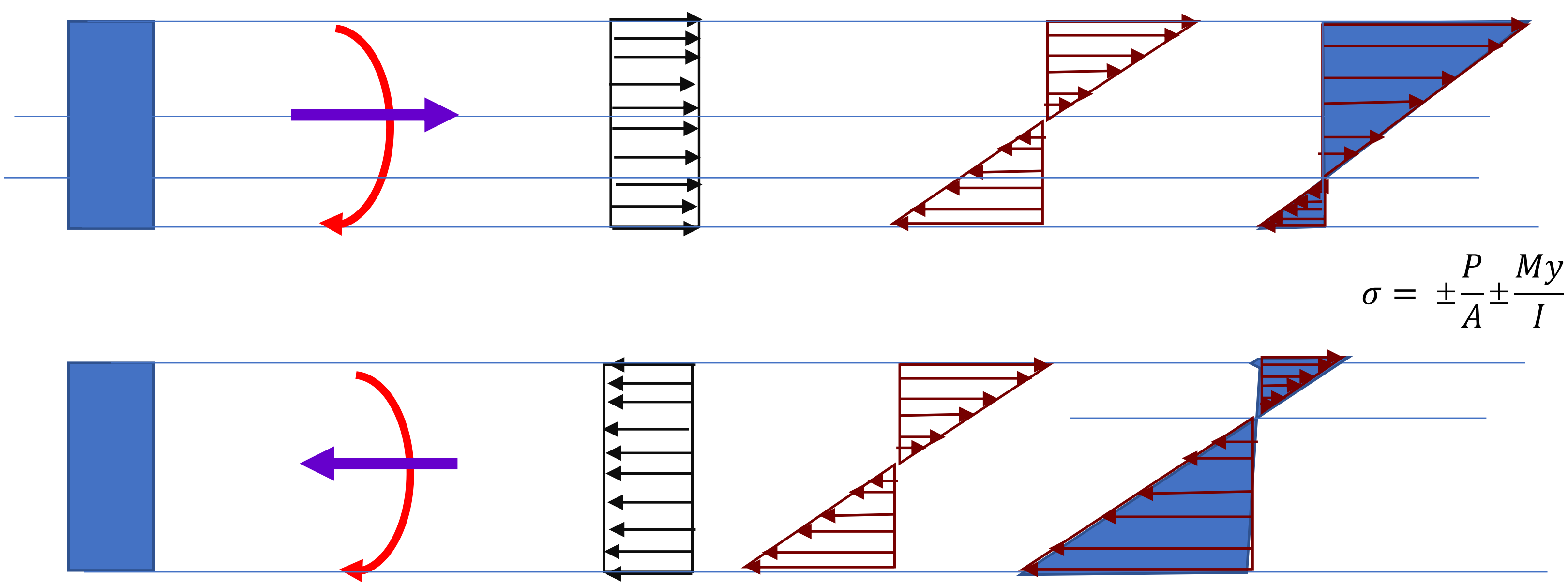


Stress distribution for eccentric loading is obtained by superposing the axial and pure bending distributions.

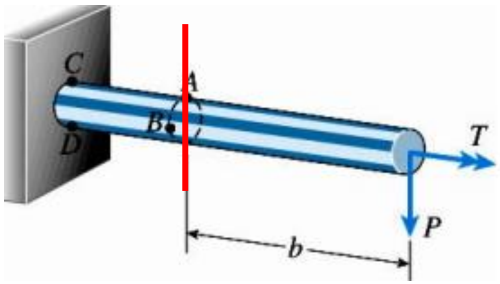


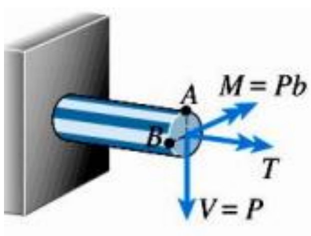
(a) Member with eccentric loading. (b) Free-body diagram of the member with internal loads at section C.

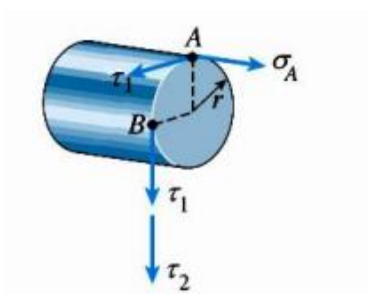
$$\sigma_x = \frac{P}{A} - \frac{My}{I}$$



$$\sigma = \pm\frac{P}{A}\pm\frac{My}{I}$$



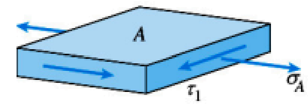


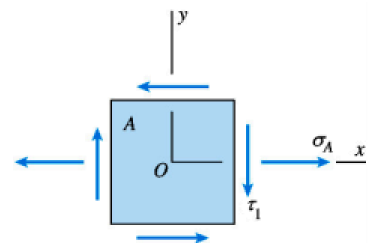


$$\tau_{torsion} = \frac{Tr}{I_{Polar}} = \frac{2T}{\pi r^3}$$

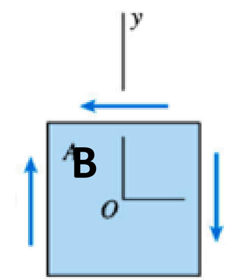
$$\sigma_{bending} = \frac{Mr}{I} = \frac{4M}{\pi r^3}$$

$$\tau_{shear} = \frac{VQ}{Ib} = \frac{4V}{3A}$$



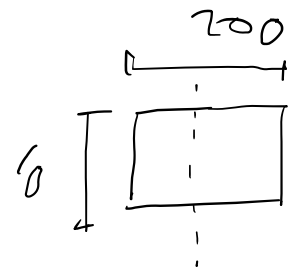


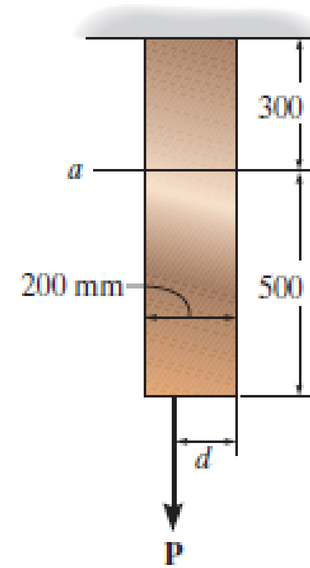
$$\sigma = \pm\frac{P}{A} \pm \frac{My}{I}$$



$\tau_1 + \tau_2$

The vertical force **P** acts on the bottom of the plate having a negligible weight. Determine the shortest distance *d* to the edge of the plate at which it can be applied so that it produces no compressive stresses on the plate at section *a*–*a*. The plate has a thickness of 10 mm and **P** acts along the center line of this thickness.



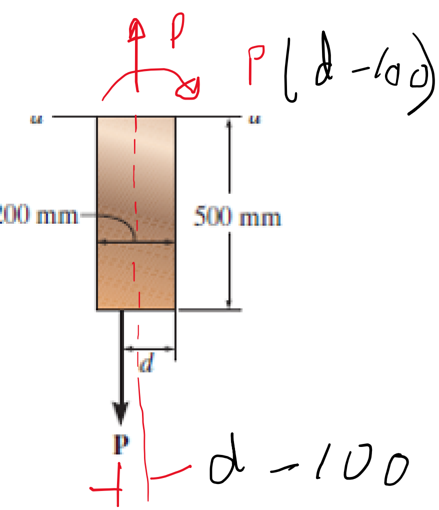


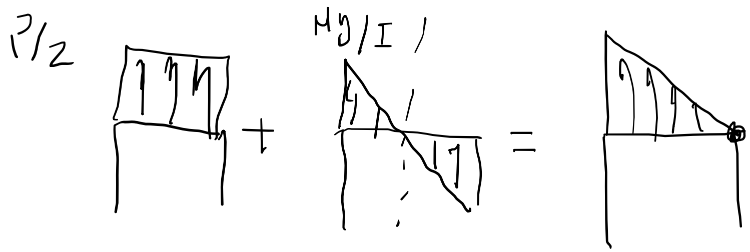
$$\sigma = \sigma_a - \sigma_b = 0$$

$$= \frac{P}{10 \times 200} - \frac{P(d-100)(100)12}{(10)(200)^3} = 0$$

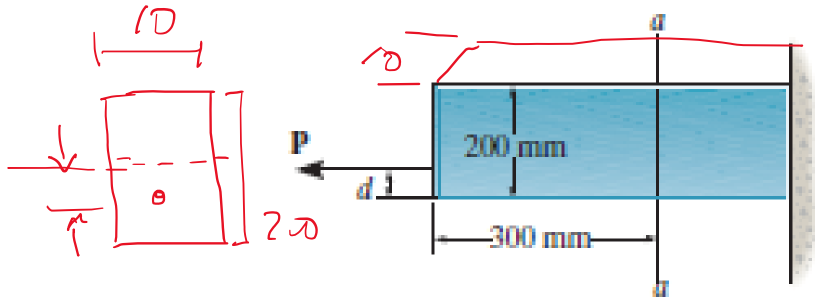
$$5 \times 10^{-4} - 1.5 \times 10^{-5} d + 1.5 \times 10^{-3} = 0$$

$$d = 133.3 \text{ mm}$$





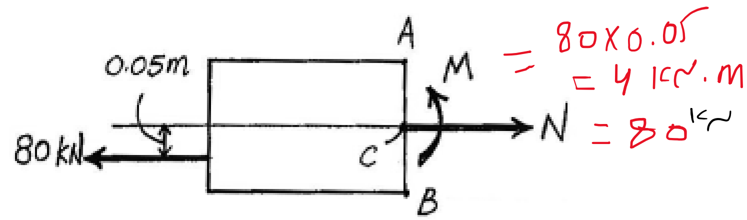
The horizontal force of acts at the end of the plate. The plate has a thickness of 10 mm and **P = 80kN** acts along the centerline of this thickness such that d=50mm
Plot the distribution of normal stress acting along section *a*–*a*.



$$A = 2000 \text{ mm}^2$$

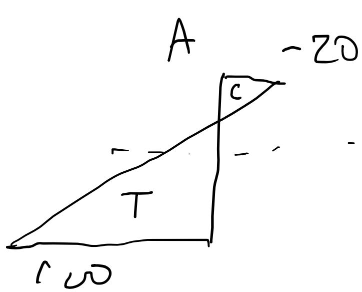
$$I = \frac{10 \times 200^3}{12} = 6.667 \times 10^6 \text{ mm}^4$$

$$\sigma = \pm\frac{P}{A} \pm \frac{My}{I}$$



$$\sigma_A = \frac{(80 \times 10^3)}{2000} - \frac{(4 \times 10^6)(100)}{6.667 \times 10^6} = -20 \text{ MPa (C)}$$

$$\sigma_B = \frac{80 \times 10^3}{2000} + \frac{(4 \times 10^6)(100)}{6.67 \times 10^6} = 100 \text{ MPa (T)}$$



A link in a machine is designed so that its cross-sectional area is reduced one half at section A-B as shown. If the thickness of the link is **50 mm**, compute the maximum force P that can be applied if the maximum normal stress on section A-B is limited to 80 MPa.

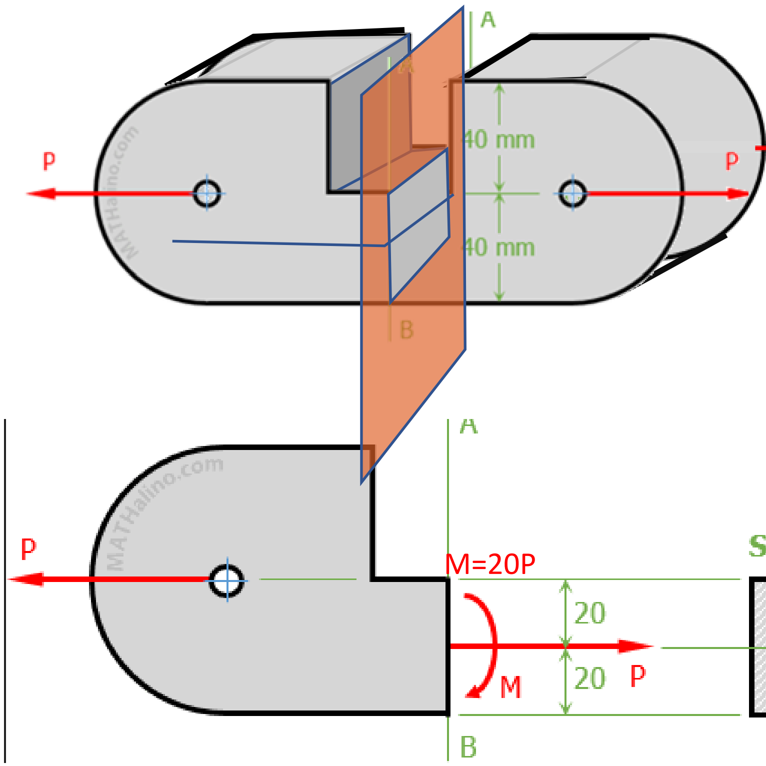
$$I = \frac{50 \times 40^3}{12} = 266.67x10^3\ mm\ 4$$

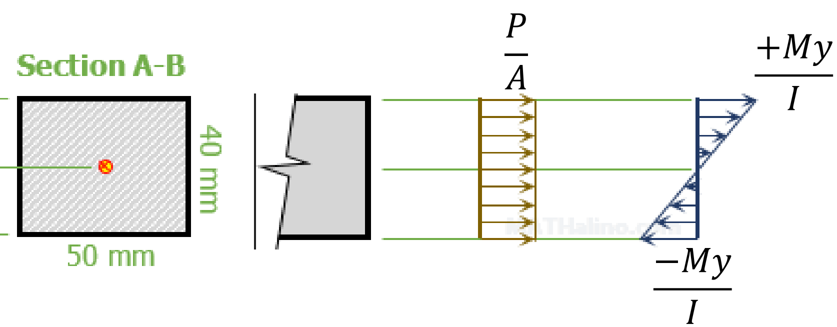
$$\sigma = \pm\frac{P}{A}\pm\frac{My}{I}$$

A = 50x40 = 2000 mm [2]

$$\sigma = +\frac{P}{2000} + \frac{20Px20}{266.67x10^3} = 80\text{ MPa}$$

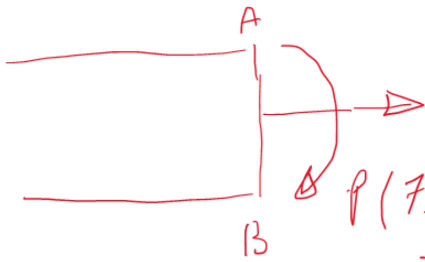
P= **40,000 N= 40kN**





https://mathalino.com/reviewer/strength-materials/problem-904-combined-axial-and-bending

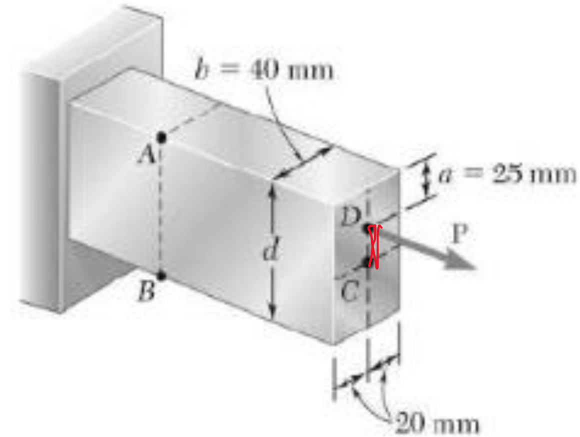
Determine the maximum stresses at A and B, if d=75 mm, P = 24kN



$P(75-25) = 50P$

$= (50)(24\times10^3)$

$= 1200{,}000\ \text{N mm}$



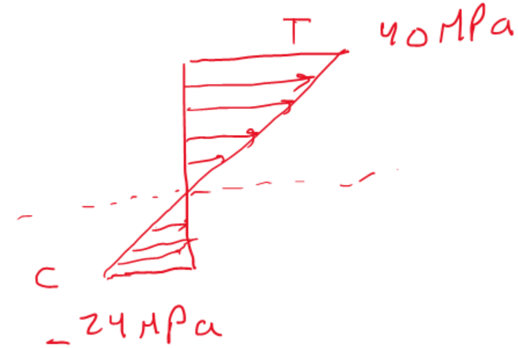
$$\sigma_A = \frac{My}{I} + \frac{P}{A}$$

$$\sigma_A = \frac{(1200\times10^3)(37.5)}{1406250} + \frac{24\times10^3}{(40)(75)} = 40\ \text{MPa}$$

$$I = \frac{40(75)^3}{12} = 1406250\ \text{mm}^4$$

$$\sigma_B = -\frac{My}{I} + \frac{P}{A}$$

$$= -\frac{(1200\times10^3)(37.5)}{1406250} + \frac{24000}{(40)(75)} = -24\ \text{MPa}$$



Determine the stress at points A and B,

(a) for the loading shown,

(b) if the 60-kN loads are applied at points 1 and 2 only.

(*a*) Loading is centric.

$$P = 180 \text{ kN} = 180\times 10^3 \text{ N}$$

$$A = (90)(240) = 21.6\times 10^3 \text{mm}^2 = 21.6\times 10^{-6} \text{m}^2$$

At A and B:

$$\sigma = -\frac{P}{A} = \frac{180\times 10^3}{21.6\times 10^{-3}} = -8.33\times 10^6 \text{ Pa}$$

$$\sigma_A = \sigma_B = -8.33 \text{ MPa} \blacktriangleleft$$

(*b*) Eccentric loading.

$$P = 120 \text{ kN} = 120\times 10^3 \text{N}$$

$$M = (60\times 10^3)(150\times 10^{-3}) = 9.0\times 10^3 \text{ N}\cdot\text{ m}$$

$$I = \frac{1}{12}bh^3 = \frac{1}{12}(90)(240)^3 = 103.68\times 10^6 \text{mm}^4 = 103.68\times 10^{-6} \text{m}^4$$

$$c = 120 \text{ mm} = 0.120 \text{ m}$$

At A:

$$\sigma_A = -\frac{P}{A} - \frac{Mc}{I} = -\frac{120\times 10^3}{21.6\times 10^{-3}} - \frac{(9.0\times 10^3)(0.120)}{103.68\times 10^{-6}} = -15.97\times 10^6 \text{ Pa}$$

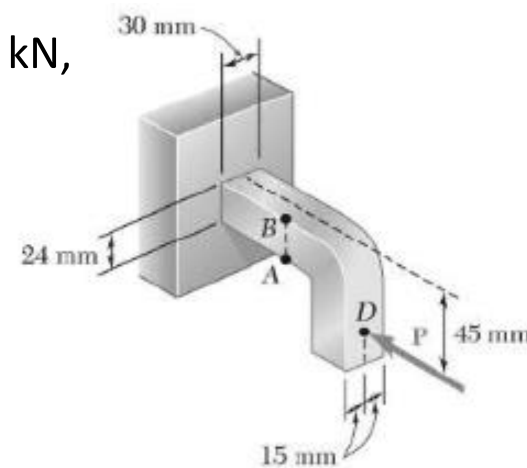
At B:

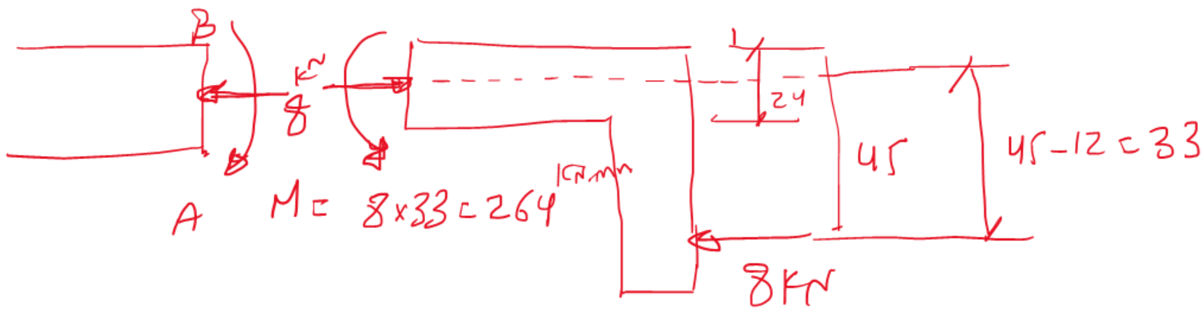
$$\sigma_B - \frac{P}{A} + \frac{Mc}{I} = -\frac{120\times 10^3}{21.6\times 10^{-3}} + \frac{(9.0\times 10^3)(0.120)}{103.68\times 10^{-6}} = 4.86\times 10^6 \text{Pa}$$

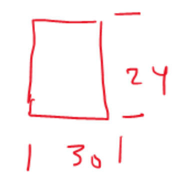
$$\sigma_A = -15.97 \text{ MPa} \blacktriangleleft$$

$$\sigma_B = 4.86 \text{ MPa} \blacktriangleleft$$

Knowing that the magnitude of the horizontal force P is 8 kN, determine the stress at (a) point A, (b) point B







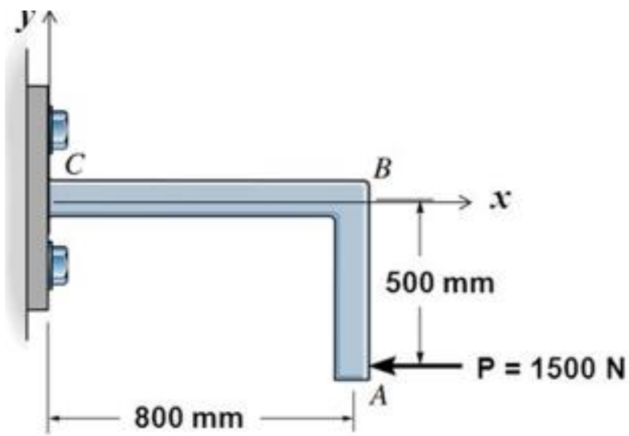
$$I = \frac{(30)(24)^3}{12} = 34560 \text{ mm}^4$$

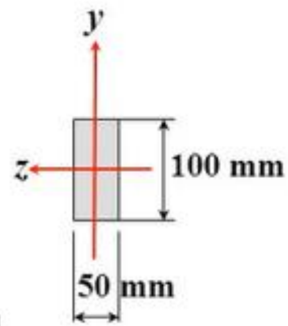
$$\sigma = \pm\frac{My}{I} \pm \frac{P}{A}$$

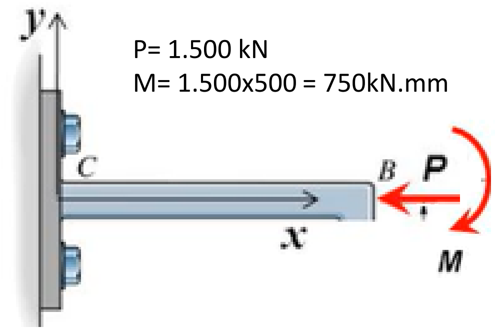
$$\sigma_A = -\frac{(264\times10^3)(12)}{34560} - \frac{8\times1000}{30\times24} = \underline{102.8 \text{ MPa}}$$

$$\sigma_B = \frac{264\times10^3\times12}{34560} - \frac{8000}{30\times24} = \underline{80.6 \text{ MPa}}$$

Determine the maximum tensile and compressive stresses in segment BC

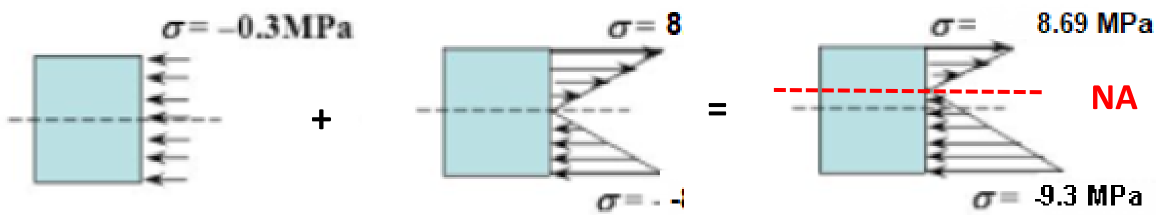






P= 1.500 kN
M= 1.500x500 = 750kN.mm

$$\boldsymbol{\sigma} = \pm\frac{\boldsymbol{P}}{\boldsymbol{A}} \pm \frac{\boldsymbol{My}}{\boldsymbol{I}}$$



$$A = (50)(100) = 5000 \text{ mm}^2$$

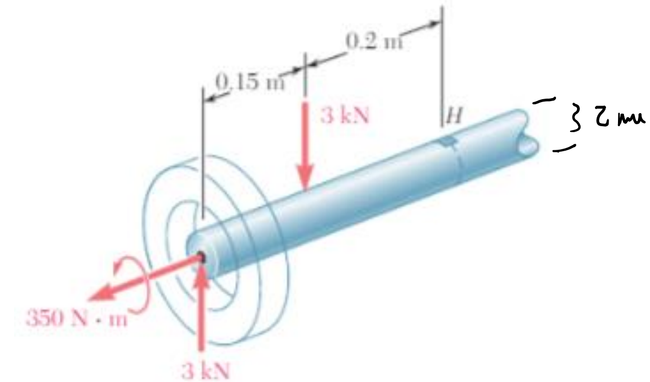
$$I = \frac{bh^3}{12} = \frac{(50)(100)^3}{12} = 4.167\text{x}10^6 \text{ mm}^4$$

$$\sigma = -\frac{1500}{5000} - \frac{750x10^3 \ x\ 50}{4.167x10^6}$$ = -0.3 -8.99 =9.3 MPa (C)

$$\sigma = -\frac{1500}{5000} + \frac{750x10^3 \ x\ 50}{4.167x10^6}$$ = -0.3 +8.99 = 8.69 MPa (T)

Location of the NA
8.69/C1 =9.3/ (100-C1)
C1 = 48.3 mm

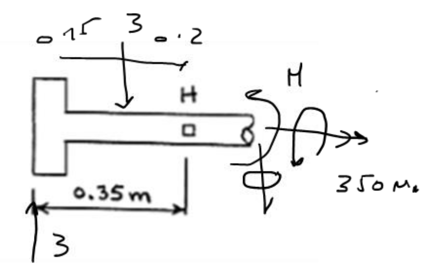
The axle of an automobile is acted upon by the forces and couple shown. Knowing that the diameter of the solid axle is 32 mm, determine (*a*) the principal planes and principal stresses at point *H* located on top of the axle, (*b*) the maximum shearing stress at the same point.



$$\sigma = \frac{My}{I} \quad , \quad \tau = \frac{Tr}{J} \qquad ; r = 32/2 = 16$$

$$I = \frac{\pi r^4}{4} = \frac{(3.14)(32/2)^4}{4} = 51445.76 \text{ mm}^4$$

$$J = \frac{\pi r^4}{2} = \frac{(3.14)(32/2)^4}{2} = 102891.52 \text{ mm}^4$$



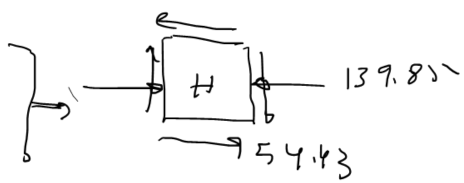
$\circlearrowleft \Sigma M = 0$

$-(3)(0.35) + (3)(0.2) + M = 0$

$M = 0.45 \text{ kN·m}$

$$\sigma = \frac{(0.45 \times 10^6)(16)}{51445.76} = 139.95 \text{ MPa}$$

$$\tau = \frac{(350000)(16)}{102891.52} = 54.43 \text{ MPa}$$



$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm R$$

$$\tau = R$$

Structural Mechanics

Chapter 9

Stresses Transformation

+

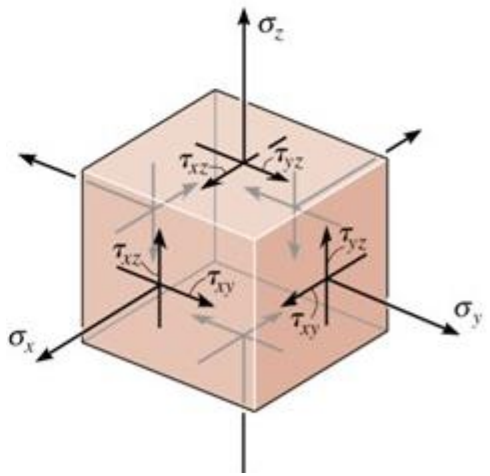
Chapter 11

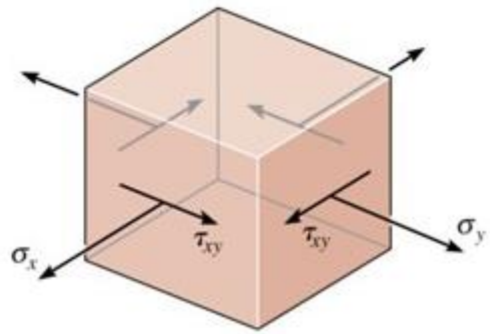
Design of Beams

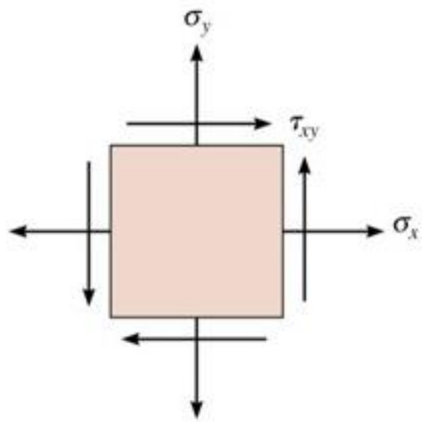
I. Marie

Stress element

The stress element represents a point on or in a structural component which must satisfy equilibrium since the overall structural component is in equilibrium







Plane stress 3D-view

Plane stress 2D-view

3D- general state of stress

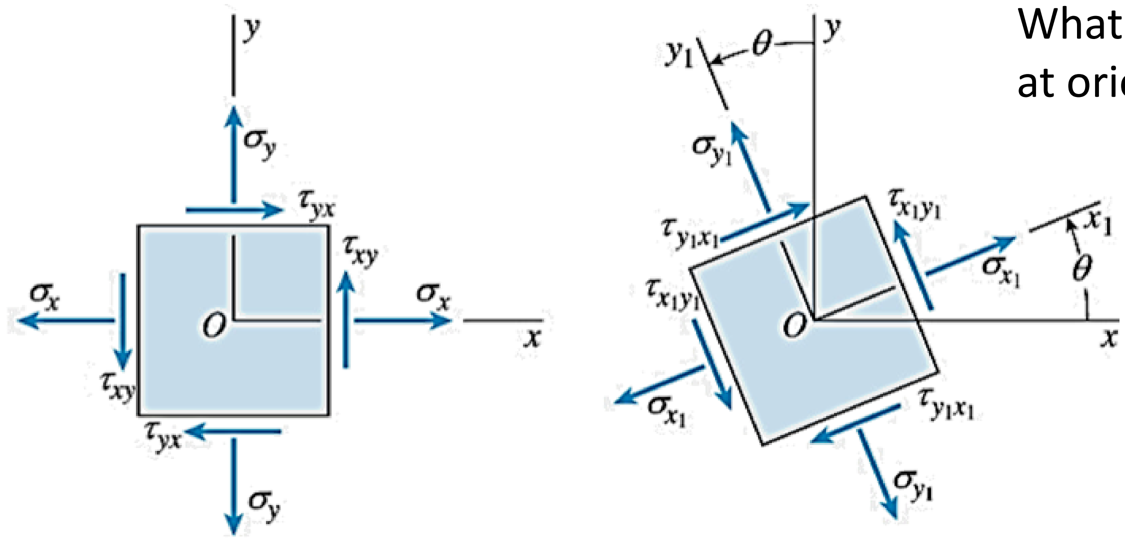
Positive sign convention

A shear stress is positive if it is acting on a positive face and in the positive direction of one of the coordinate axes, or on a negative face and in the negative direction of one of the coordinate axes. A shear stress is negative if it is acting on a negative face and in the positive direction of one of the coordinate axes, or on a positive face and in the negative direction of one of the coordinate axes.

Plane Stress-state of stress in which two faces of the cubic element are free of stress. For the illustrated example, the state of stress is defined by .

$$\sigma_x, \sigma_y, \tau_{xy} \quad \text{and} \quad \sigma_z = \tau_{zx} = \tau_{zy} = 0.$$

9.1 Plane Stress Transformation



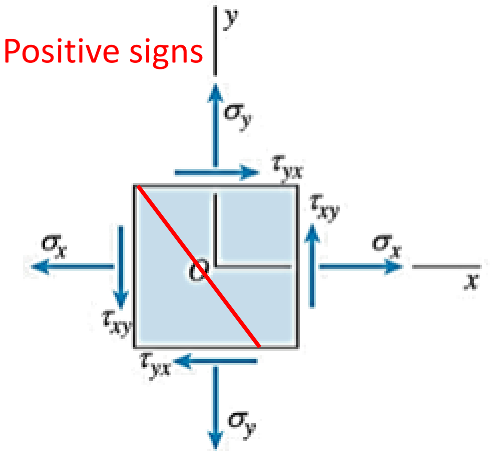
What are the stresses at orientation θ ?

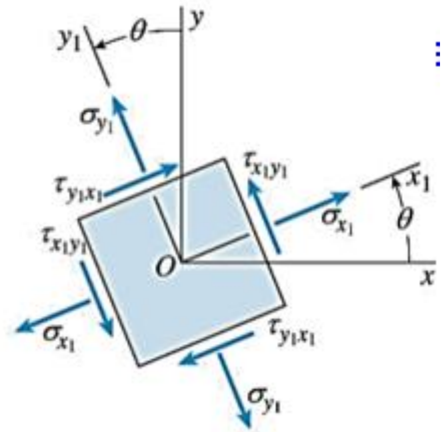
CCW angle is positive

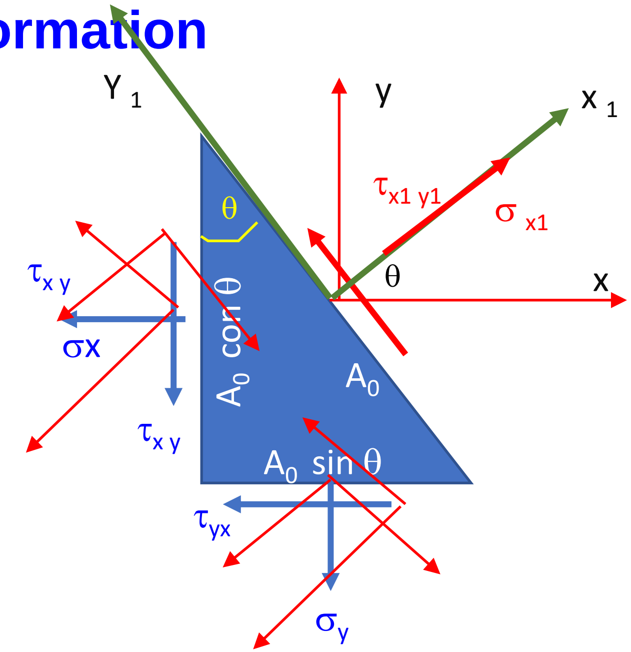
A material may yield or fail at the maximum value of σ or τ. This value may occur at some angle other than θ = 0. (Remember that for uniaxial tension the maximum shear stress occurred when θ = 45 degrees.)

9.2 General equations of Plane Stress Transformation

Stresses on Inclined Sections







$$\sum F_{x_1} = \sigma_{x_1} A_0 - \sigma_x A_0 \cos^2\theta - \tau_{xy} A_0 \sin\theta\cos\theta - \sigma_y A_0 \sin^2\theta - \tau_{yx} A_0 \sin\theta\cos\theta = 0$$

$$\sum F_{y_1} = \tau_{x_1y_1} A_0 + \sigma_x A_0 \sin\theta\cos\theta - \tau_{xy} A_0 \cos^2\theta - \sigma_y A_0 \sin\theta\cos\theta + \tau_{yx} A_0 \sin^2\theta = 0$$

Using $\tau_{xy} = \tau_{yx}$ and simplifying gives:

$$\sigma_{x1} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\,\tau_{xy} \sin\theta\cos\theta$$

$$\tau_{x1y1} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \sin\theta\cos\theta = \frac{\sin 2\theta}{2}$$

$$\boxed{\begin{aligned} \sigma_{x1} &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \tau_{x1y1} &= -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \end{aligned}}$$

$$\boxed{\sigma_{y1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta}$$

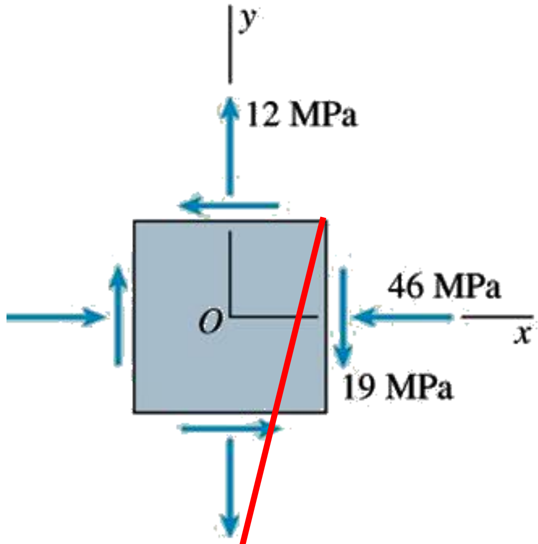
$$\sigma_{x1} + \sigma_{y1} = \sigma_x + \sigma_y$$

$\sigma_x = -46$ MPa

$\sigma_y = 12$ MPa

$\tau_{xy} = -19$ MPa

Determine stresses on the -15° surface



$$\frac{\sigma_x + \sigma_y}{2} = -17 \text{ MPa}$$

$$\frac{\sigma_x - \sigma_y}{2} = -29 \text{ MPa}$$

$$\tau_{xy} = -19 \text{ MPa}$$

$$\theta = -15^\circ$$

$$2\theta = -30^\circ$$

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x1y1} = -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

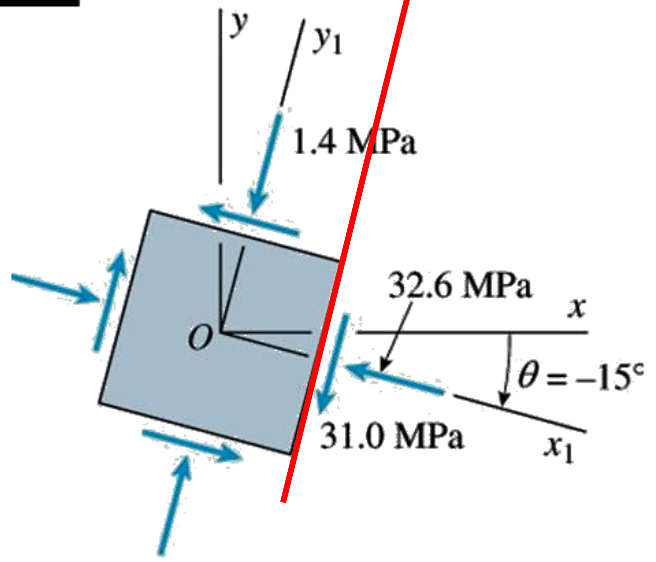
For stresses on the y_1 face, substitute $\theta + 90^\circ$ for θ:

$$\sigma_{y1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$= (-17) + (-29)\cos(-30^\circ) + (-19)\sin(-30^\circ) = -32.6 \text{ MPa}$$

$$\tau_{x_1y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$
$$= -(-29)\sin(-30^\circ) + (-19)\cos(-30^\circ) = -31.0 \text{ MPa}$$

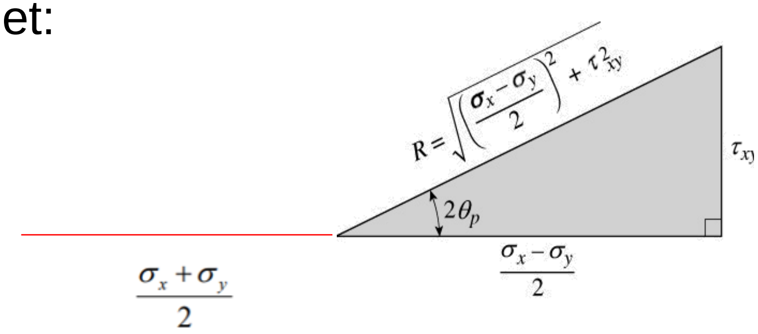
$$\sigma_{y_1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$
$$= (-17) - (-29)\cos(-30^\circ) - (-19)\sin(-30^\circ) = -1.4 \text{ MPa}$$



9.3 Principal Stresses and Maximum Shear Stresses

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

Taking the derivative with respect to θ, we get:



For $\theta = \theta_p$

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta_p + \tau_{xy}\sin 2\theta_p$$

$$= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\left(\frac{\sigma_x - \sigma_y}{2R}\right) + \tau_{xy}\left(\frac{\tau_{xy}}{2R}\right)$$

$$= \frac{\sigma_x + \sigma_y}{2} + R = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

For $\theta = \theta_p + \pi/2$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta_p + \pi) + \tau_{xy}\sin(2\theta_p + \pi)$$

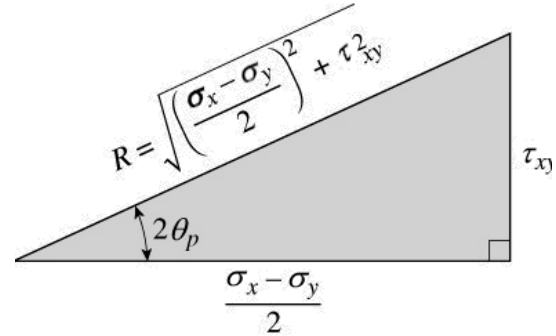
$$= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\left(-\frac{\sigma_x - \sigma_y}{2R}\right) + \tau_{xy}\left(-\frac{\tau_{xy}}{2R}\right)$$

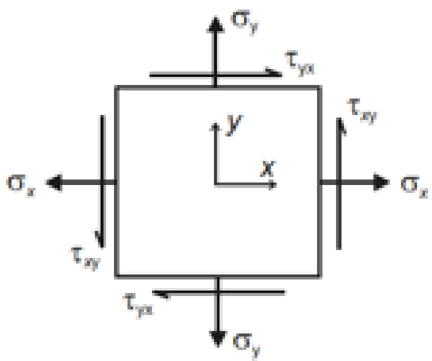
$$= \frac{\sigma_x + \sigma_y}{2} - R = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

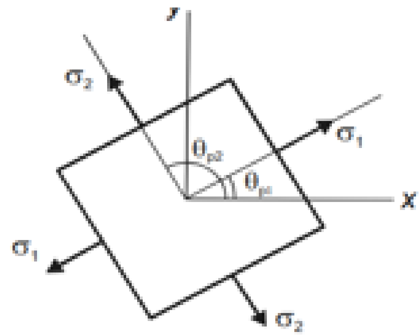
$$\tan 2\theta_p == \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$$

Principal Angle

Compute $\sigma_{ave} = \dfrac{\sigma_x + \sigma_y}{2}$; $\quad R = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$







the shear stresses are zero on the principal planes.

Principal Stresses

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

Principal Angles defining the **Principal Planes**

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Maximum Shear Stress :

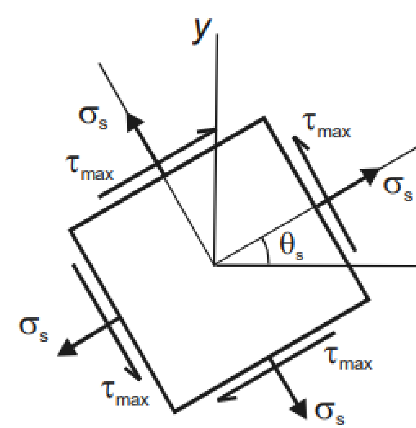
$$\tau_{\max} = R$$

$$\theta_{s_1} = \frac{1}{2}\tan^{-1}\left(-\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right)$$

$$\theta_{s_2} = \theta_{s_1} + 90^\circ$$

$$\sigma_x(\theta = \theta_s) = \sigma_y(\theta = \theta_s) = \sigma_{aver}$$

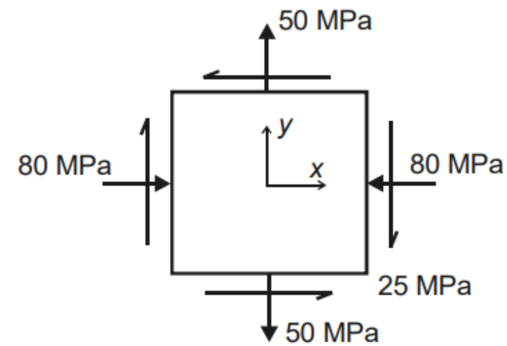
$$\theta_s = \theta_p \pm 45^\circ$$



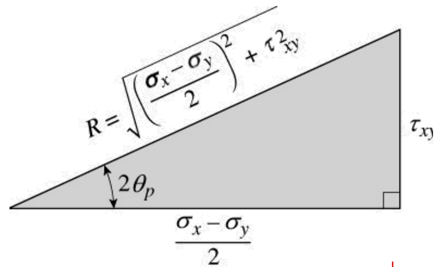
$\sigma_s = \dfrac{\sigma_x + \sigma_y}{2}$

Define the stresses on a stress element, and find the principal stress on an oriented element
What is the maximum sher and its orientation

σ_x = -80 MPa σ_y = 50 MPa
τ_{xy} = -25 MPa



$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

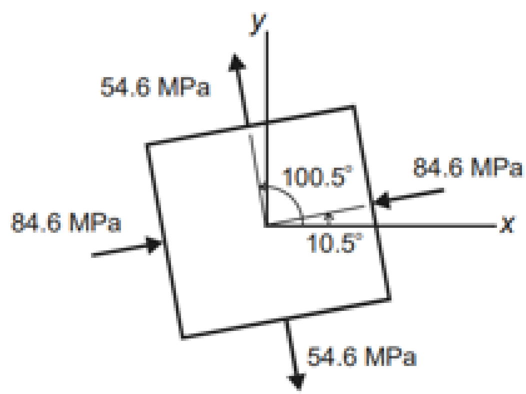


$$\sigma_{1,2} = \frac{-80+50}{2} \pm \sqrt{\left(\frac{-80-50}{2}\right)^2 + (-25)^2} = -15 \pm 69.6$$

$\sigma_1 = 54.6$ MPa $\sigma_2 = -84.6$ MPa

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\tan 2\theta_p = \frac{2(-25)}{-80-50} = 0.3846$$



$$2\theta_p = 21.0° \text{ and } 21.0 + 180°$$

$$\theta_p = 10.5°, 100.5°$$

But we must check which angle goes with which principal stress.

$$\sigma_{x1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{x1} = \frac{-80+50}{2} + \frac{-80-50}{2}\cos 2(10.5°) + (-25)\sin 2(10.5°) = -84.6 \text{ MPa}$$

σ_1 = 54.6 MPa with θ_{p1} = 100.5°
σ_2 = -84.6 MPa with θ_{p2} = 10.5°

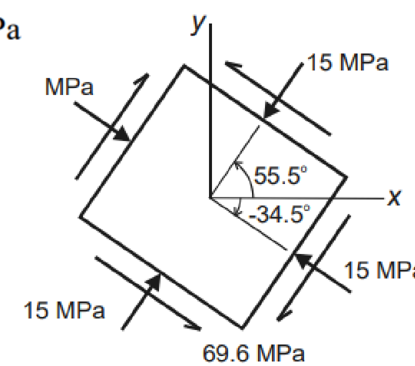
$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}$$

$$\tau_{max} = \sqrt{\left(\frac{-80-50}{2}\right)^2 + (-25)^2} = 69.6 \text{ MPa}$$

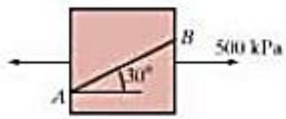
$$\theta_s = \theta_p \pm 45°$$

$$\sigma_s = \frac{\sigma_x + \sigma_y}{2}$$

$$\sigma_s = \frac{-80+50}{2} = -15 \text{ MPa}$$

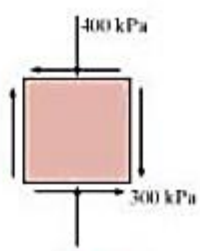


F9–1. Determine the normal stress and shear stress acting on the inclined plane AB. Sketch the result on the sectioned element.



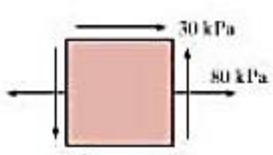
F9–1

F9–2. Determine the equivalent state of stress on an element at the same point oriented 45° clockwise with respect to the element shown.



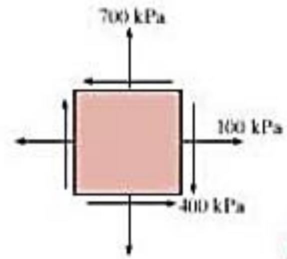
F9–2

F9–3. Determine the equivalent state of stress on an element at the same point that represents the principal stresses at the point. Also, find the corresponding orientation of the element with respect to the element shown.



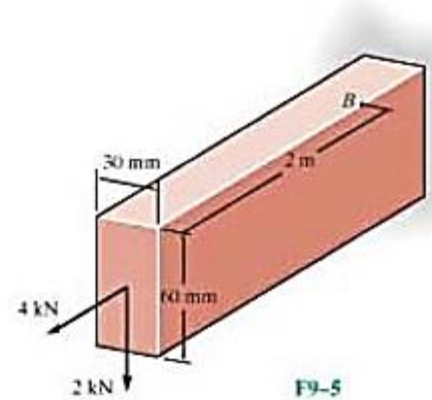
F9–3

F9–4. Determine the equivalent state of stress on an element at the same point that represents the maximum in-plane shear stress at the point.



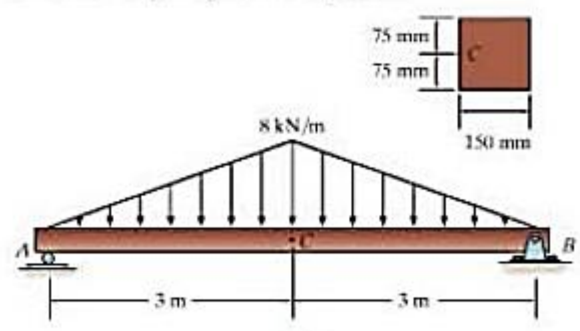
F9–4

F9–5. The beam is subjected to the load at its end. Determine the maximum principal stress at point B.



F9–5

F9–6. The beam is subjected to the loading shown. Determine the principal stress at point C.



F9–6

9.4 Mohr's Circle- Plane stress

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right) + \tau_{x'y'}{}^2 = \left(\frac{\sigma x - \sigma y}{2}\right)^2 + \tau_{xy}{}^2$$

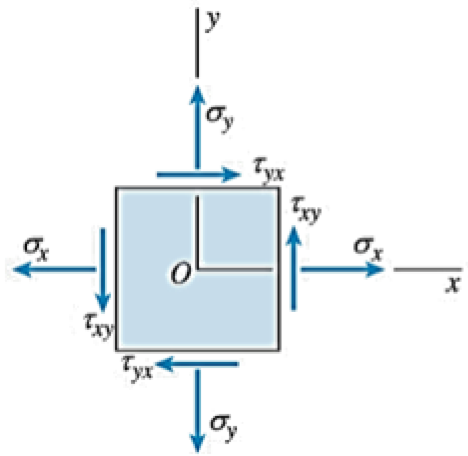
Circle equation

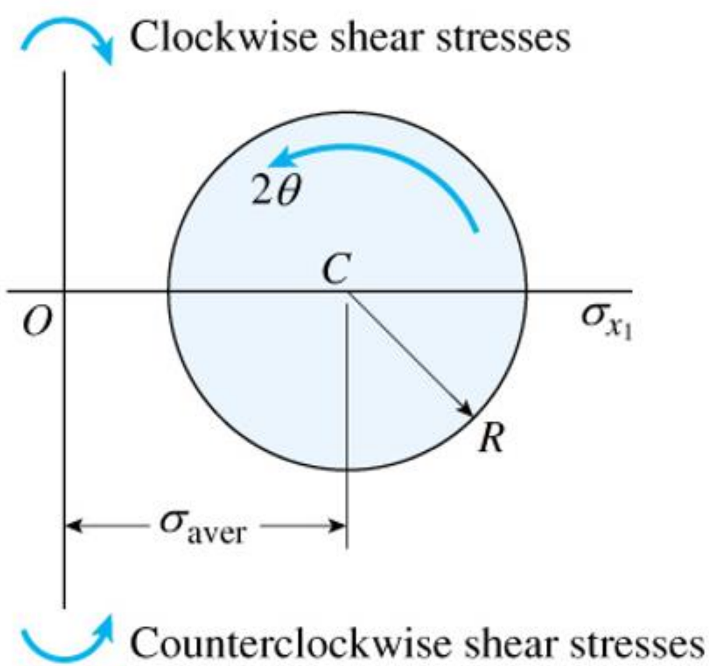
$$(x-n)^2 + (y-k)^2 = R^2$$

$$\boxed{\begin{aligned}\sigma_{x1} &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \tau_{x1y1} &= -\frac{(\sigma_x - \sigma_y)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta\end{aligned}}$$

For stresses on the y_1 face, substitute $\theta + 90^\circ$ for θ:

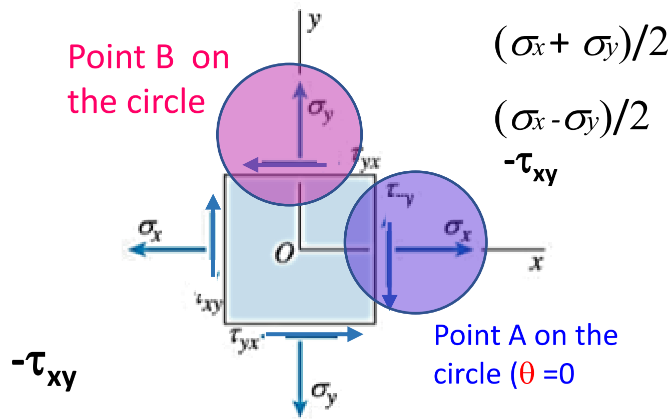
$$\boxed{\sigma_{y1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta}$$

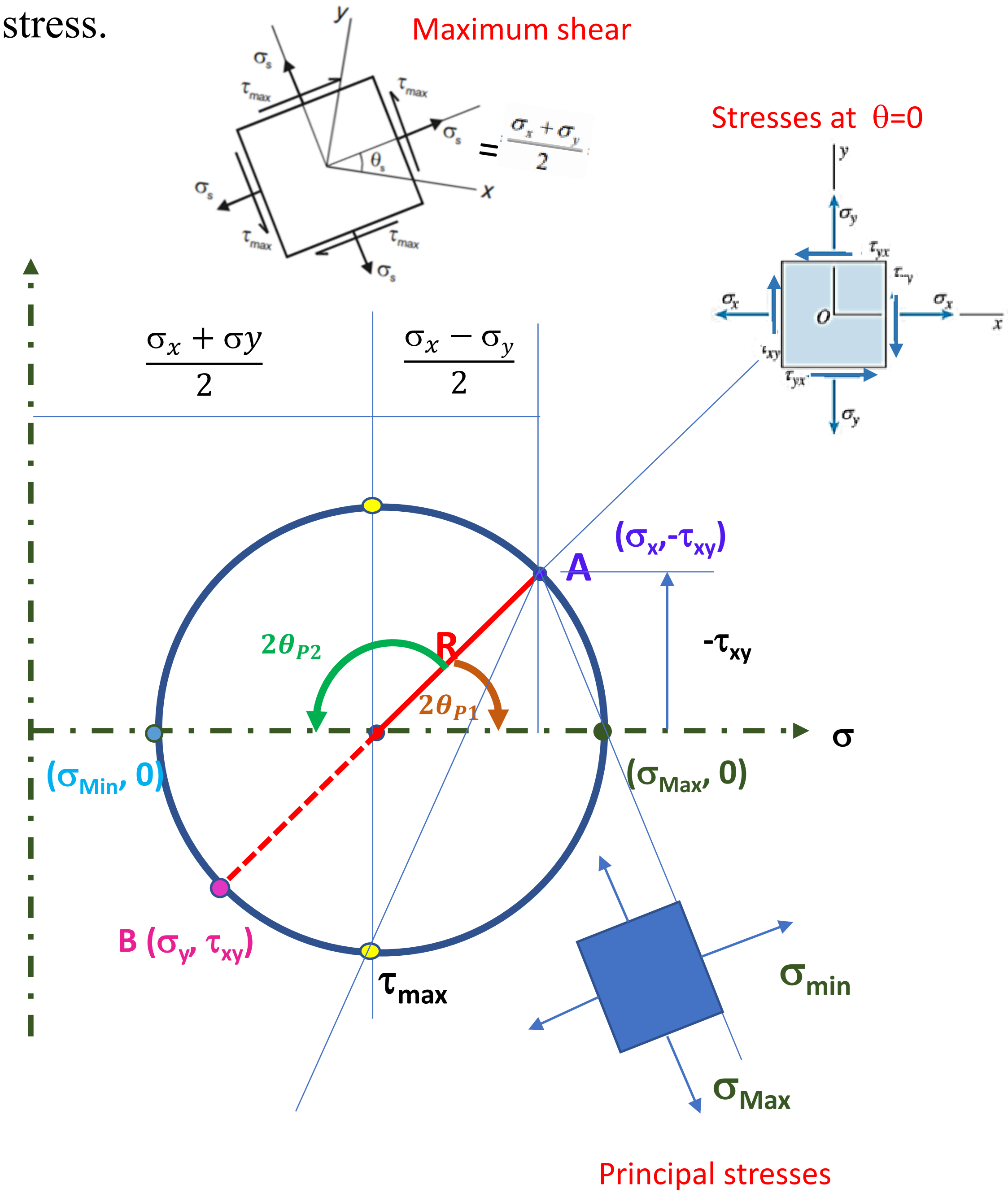




τ_{x1y1} is positive downward and the angle 2θ is positive counterclockwise

Construction of Mohr's circle for plane stress.





$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_{\substack{max \\ min}} = \frac{\sigma_x + \sigma_y}{2} \pm R$$

$$Tan\ 2\theta_p = \frac{\tau_{xy}}{\frac{\sigma_x - \sigma_y}{2}}$$

Using Mohr's circle, find
principal stresses and find the principal angle
Show maximum shear stresses on a stress element
show stresses on the inclined surface shown

$(\sigma_x + \sigma_y)/2 = (-50 + 10)/2 = -20$

$(\sigma_x - \sigma_y)/2 = (-50-10)/2 = -30$

$\tau_{xy} = -40$

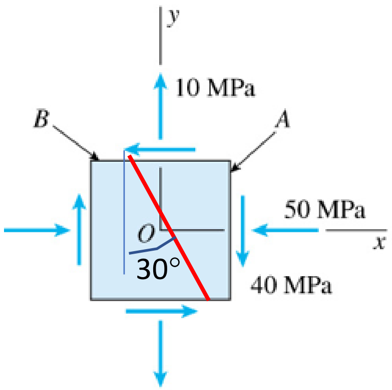
$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 50$$

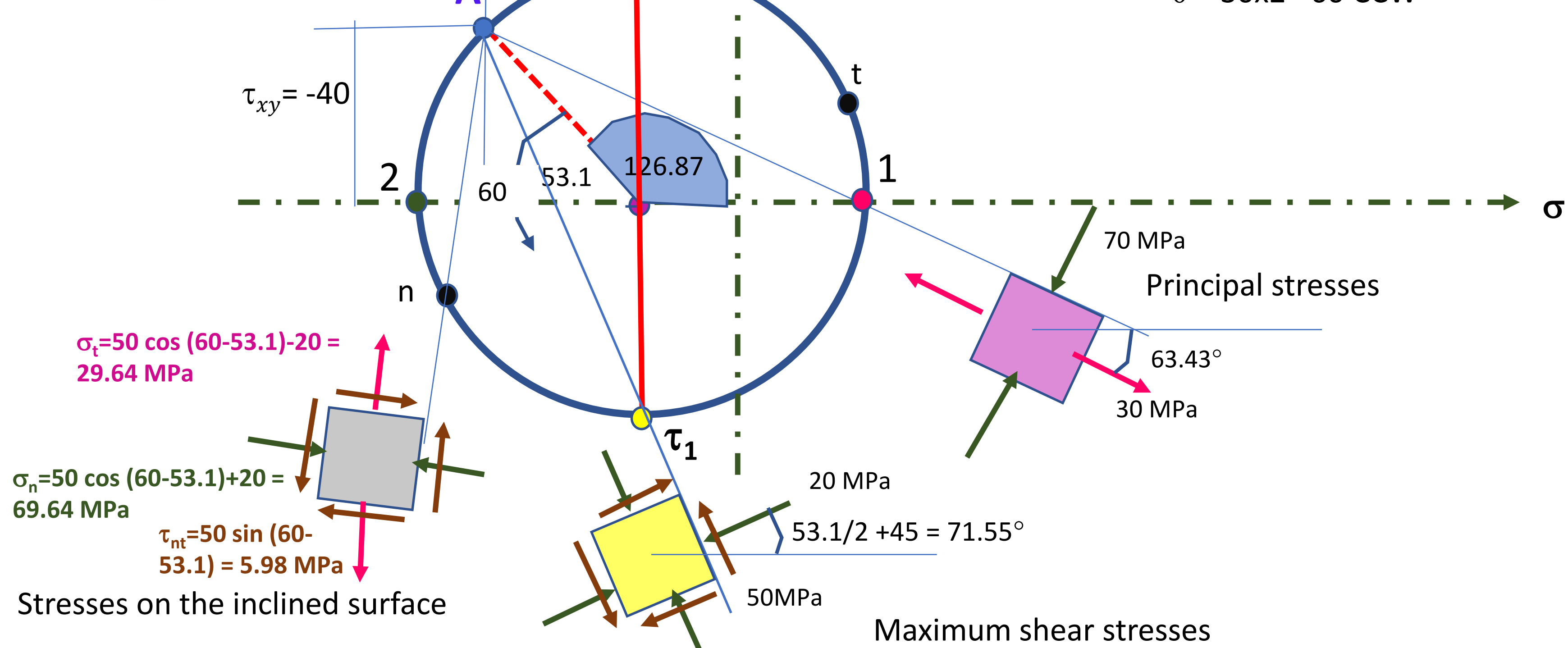
$$Tan\ 2\theta_p = \frac{\tau_{xy}}{\frac{\sigma_x - \sigma_y}{2}}$$

$\theta_1 = 126.87/2 = 63.43°$

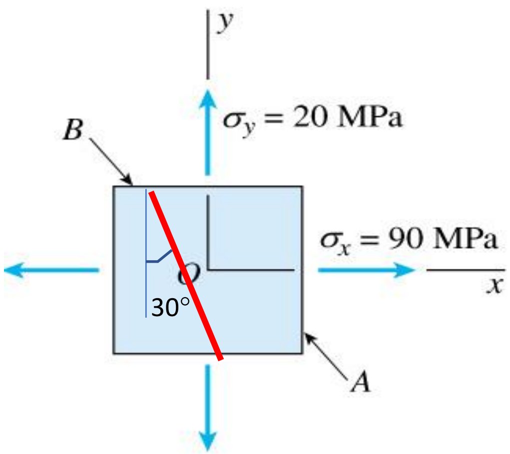
$$\sigma_{\substack{max\\min}} = -20 \pm 50 = \begin{matrix}+30\\-70\end{matrix}$$

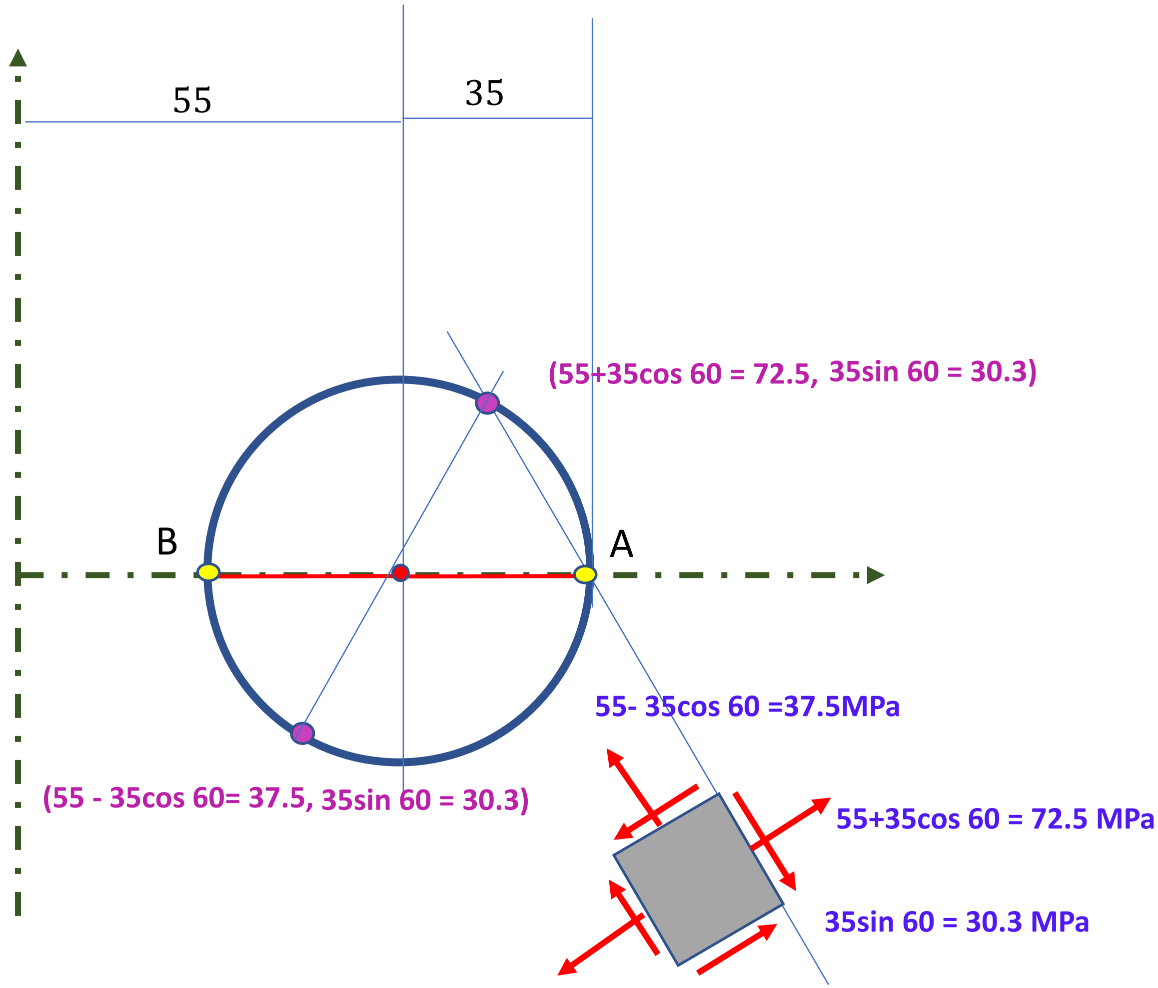
$\theta = 30\text{x}2 = 60$ CCW





Find the stresses on the inclined surface shown . Show the stress element





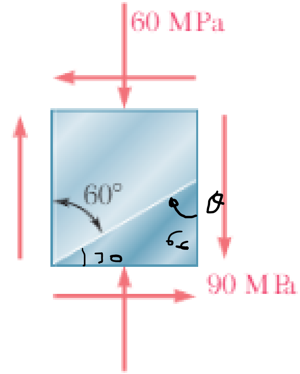
$(\sigma_x + \sigma_y)/2 = (90+20)/2 = 55$

$(\sigma_x - \sigma_y)/2 = (90-20)/2 = 35$

$\tau_{xy} = 0$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 35$$

For the given state of stress, determine , normal and shear stresses exerted on the oblique face.



$\theta = 60^\circ$ c.w $= -60^\circ$: $\underline{2\theta = -120^\circ}$

$\sigma_x = 0$, $\sigma_y = -60$ MPa , $\tau_{xy} = -90$ MPa

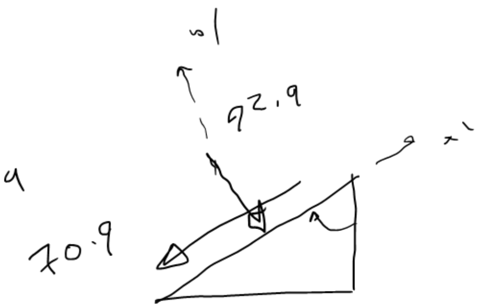
$\frac{\sigma_x + \sigma_y}{2} = -30$

$\frac{\sigma_x - \sigma_y}{2} = 30$

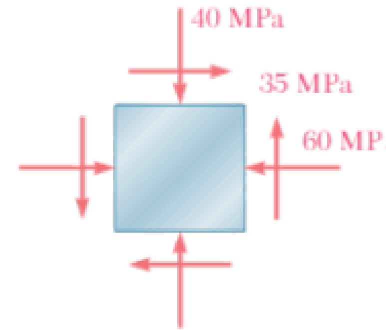
$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$

$= -32.94$ MPa

$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta = -70.9$ MPa



For the given state of stress, determine the principal planes and the principal stresses



$\sigma_x = -60$

$\sigma_y = -40$

$\tau_{xy} = 35$

$\frac{\sigma_x + \sigma_y}{2} = -50$

$\frac{\sigma_x - \sigma_y}{2} = -10$

$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

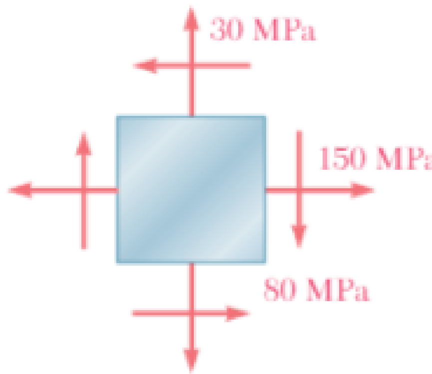
$R = \sqrt{10^2 + 35^2} = 36.4$

$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm R = -50 \pm 36.4 = \begin{cases} -13.6 \text{ MPa} \\ -86.4 \text{ MPa} \end{cases}$

$\tan 2\theta_p = \frac{\tau_{xy}}{\left(\frac{\sigma_x - \sigma_y}{2}\right)} = -\frac{35}{10} = 3.5 \Rightarrow \theta_{p_1} = \underline{-37.1^\circ}$

$\theta_{p_2} = -37.1 + 90 = \underline{52.47^\circ}$

For the given state of stress, determine, the orientation of the maximum in-plane shearing stress, the maximum shearing stress, and the corresponding normal stress



$\sigma_x = 150$

$\sigma_y = 30$

$\tau_{xy} = -80$

$\frac{\sigma_x + \sigma_y}{2} = 90$

$\frac{\sigma_x - \sigma_y}{2} = 60$

$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{60^2 + 80^2} = 100$

$\tau_{max} = R = 100\text{ MPa}$

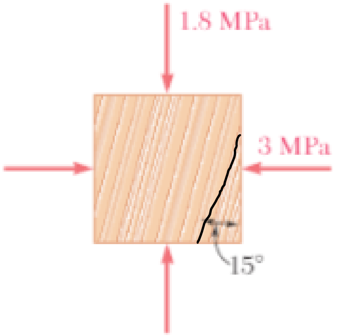
$\tan 2\theta_p = \frac{-\tau_{xy}}{\frac{\sigma_x - \sigma_y}{2}} = \frac{-80}{60} \Longrightarrow \theta_p = -26.56$

$\theta_{s_1} = 45 + \theta_p = 18.4^\circ$ and

$\theta_{s_2} = 18.4 + 90 = 108.4^\circ$

Corresponding Normal Stress $= \frac{\sigma_x + \sigma_y}{2} = 90\text{ MPa}$

The grain of a wooden forms an angle of 15 deg. With the vertical. For the shown state of stress , determine the in-plane shearing stress parallel to the grain and the normal stress perpendicular to the grain.



$\sigma_x = -3\,MPa$

$\sigma_y = -1.8\,MPa$

$\theta = 15^\circ$ C.W $= -15$

$2\theta = -30^\circ$

$\frac{\sigma_x+\sigma_y}{2} = -2.4$

$\frac{\sigma_x-\sigma_y}{2} = -0.6$

$\tau_{xy} = 0$

$$\sigma_n = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$= -2.4 - 0.6\cos(-30) = \underline{-2.92\ MPa}$$

$$\tau = -\frac{\sigma_x-\sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta = -0.6\sin(-30) = \underline{0.3\ MPa}$$

Review and Summary

Transformation of Plane Stress

A state of *plane stress* at a given point Q has nonzero values for σ_x, σ_y, and τ_{xy}. The stress components associated with the element are shown in Fig. 7.66*a*. The equations for the components $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$ associated with that element after being rotated through an angle θ about the z axis (Fig. 7.66*b*) are

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \tag{7.5}$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \tag{7.7}$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \tag{7.6}$$

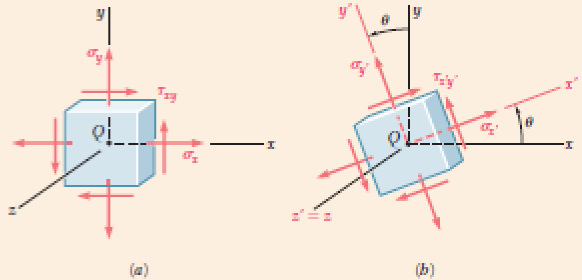


Fig. 7.66 State of plane stress. (*a*) Referred to [*x y z*]. (*b*) Referred to [*x'y'z'*].

The values θ_p of the angle of rotation that correspond to the maximum and minimum values of the normal stress at point Q are

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \tag{7.12}$$

Principal Planes and Principal Stresses

The two values obtained for θ_p are 90° apart (Fig. 7.67) and define the ***principal planes of stress*** at point Q. The corresponding values of the normal stress are called the ***principal stresses*** at Q:

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{7.14}$$

The corresponding shearing stress is zero.

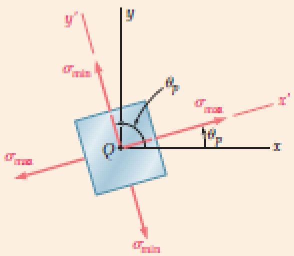


Fig. 7.67 Principal stresses.

Maximum In-Plane Shearing Stress

The angle θ for the largest value of the shearing stress θ_s is found using

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \tag{7.15}$$

The two values obtained for θ_s are 90° apart (Fig. 7.68). However, the planes of maximum shearing stress are at 45° to the principal planes. The maximum value of the shearing stress *in the plane of stress is*

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{7.16}$$

and the corresponding value of the normal stresses is

$$\sigma' = \sigma_{\text{ave}} = \frac{\sigma_x + \sigma_y}{2} \tag{7.17}$$

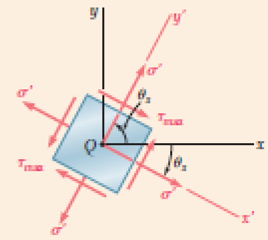


Fig. 7.68 Maximum shearing stress.

Mohr's Circle for Stress

Mohr's circle provides an alternative method for the analysis of the transformation of plane stress based on simple geometric considerations. Given the state of stress shown in the left element in Fig. 7.69*a*, point *X* of

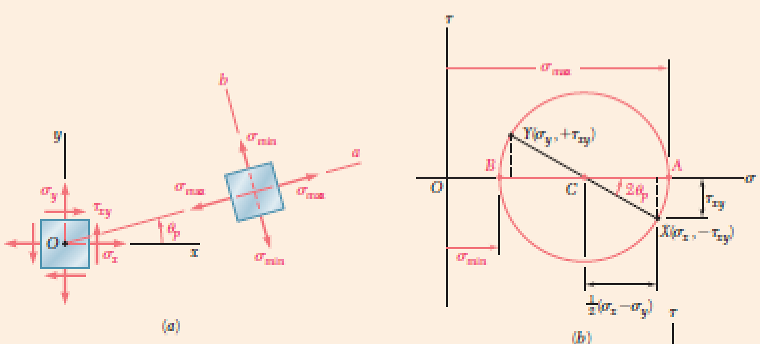


Fig. 7.69 (*a*) Plane stress element, and the orientation of principal planes. (*b*) Corresponding Mohr's circle.

coordinates σ_x, $-\tau_{xy}$ and point *Y* of coordinates σ_y, $+\tau_{xy}$ are plotted in Fig. 7.69*b*. Drawing the circle of diameter *XY* provides Mohr's circle. The abscissas of the points of intersection *A* and *B* of the circle with the horizontal axis represent the principal stresses, and the angle of rotation bringing the diameter *XY* into *AB* is twice the angle θ_p defining the principal planes, as shown in the right element of Fig. 7.69*a*. The diameter *DE* defines the maximum shearing stress and the orientation of the corresponding plane (Fig. 7.70).

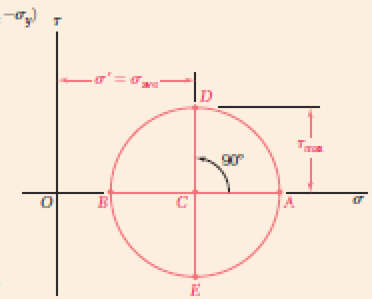


Fig. 7.70 Maximum shearing stress is oriented ±45° from principal directions.

General State of Stress

A *general state of stress* is characterized by six stress components, where the normal stress on a plane of arbitrary orientation can be expressed as a quadratic form of the direction cosines of the normal to that plane. This proves the existence of three *principal axes of stress* and three *principal stresses* at any given point. Rotating a small cubic element about each of the three principal axes was used to draw the corresponding Mohr's circles that yield the values of σ_{max}, σ_{min}, and τ_{max} (Fig. 7.71). In the case of *plane stress* when the x and y axes are selected in the plane of stress, point C coincides with the origin O. If A and B are located on opposite sides of O, the maximum shearing stress is equal to the maximum in-plane shearing stress. If A and B are located on the same side of O, this is not the case. For instance if $\sigma_a > \sigma_b > 0$, the maximum shearing stress is equal to $\frac{1}{2}\sigma_a$ and corresponds to a rotation out of the plane of stress (Fig. 7.72).

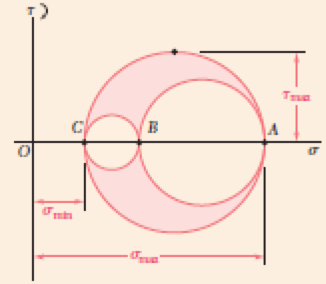


Fig. 7.71 Three-dimensional Mohr's circles for general state of stress.

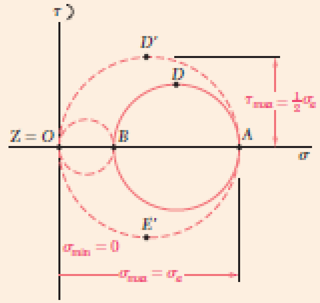


Fig. 7.72 Three-dimensional Mohr's circles for plane stress having two positive principal stresses.

Yield Criteria for Ductile Materials

To predict whether a structural or machine component will fail at some critical point due to yield in the material, the principal stresses σ_a and σ_b at that point for the given loading condition are determined. The point of coordinates σ_a and σ_b is plotted, and if this point falls within a certain area, the component is safe. If it falls outside, the component will fail. The area used with the maximum-shearing-stress criterion is shown in Fig. 7.73, and the area used with the maximum-distortion-energy criterion in Fig. 7.74. Both areas depend upon the value of the yield strength σ_Y of the material.

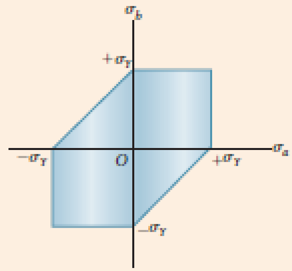


Fig. 7.73 Tresca's hexagon for maximum shearing-stress criterion.

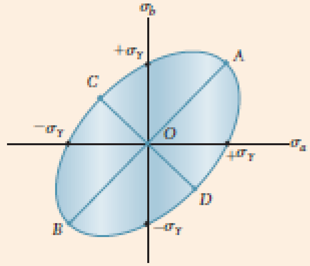


Fig. 7.74 Von Mises surface based on maximum-distortion-energy criterion.

Structural Mechanics

Chapter 11

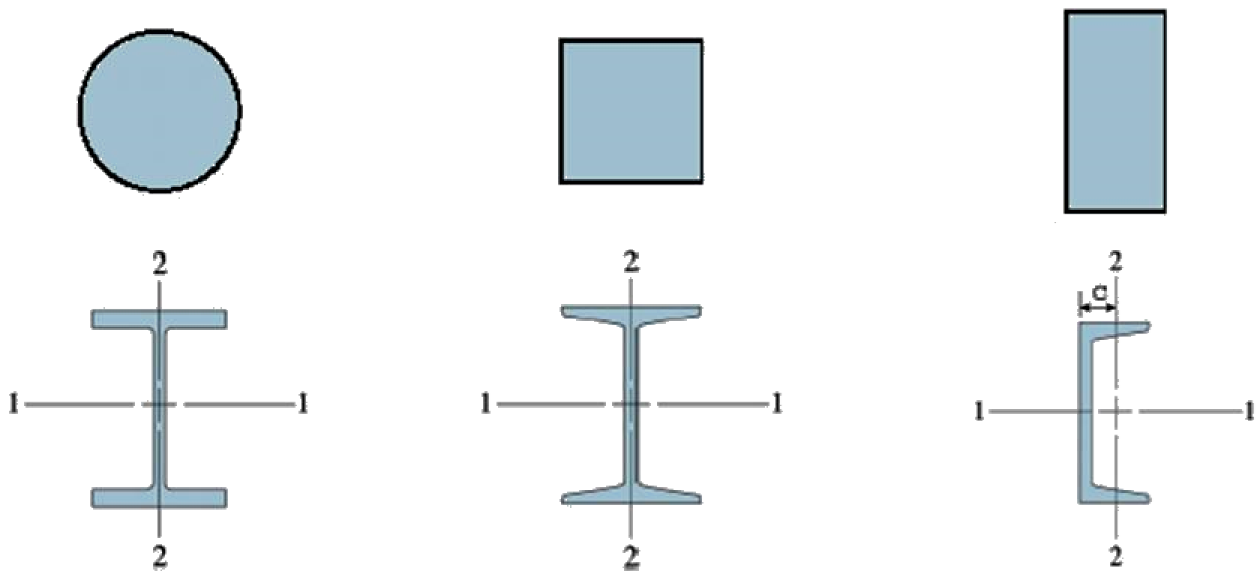
Design of beams

I. Marie

11.2 Prismatic Beam Design

Design of Beams for Bending Stresses

$$\boxed{S = \frac{M_{\max}}{\sigma_{\text{allow}}}}$$

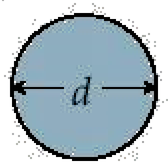


Which cross section is the most efficient one?

Design of Beams for Bending Stresses

$$S = \frac{I}{c} = \frac{M_{\max}}{\sigma_{\text{allow}}}$$

I. Circular Cross Sections

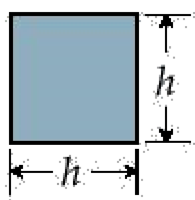


$$Area = \frac{\pi d^2}{4}$$

$$I = \frac{\pi d^3}{64}, \quad c = \frac{d}{2}$$

$$S_{\text{circle}} = \frac{\pi d^3}{32} = 0.0982 d^3$$

II. Square Cross Sections



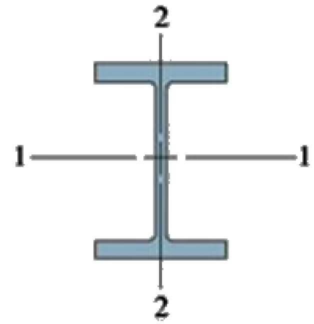
$$Area = h^2$$

$$I = \frac{h^4}{12}, \quad c = \frac{h}{2}$$

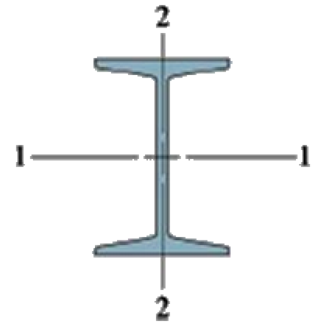
Compare to a circular cross section of identical area

$$Area = h^2 = \frac{\pi d^2}{4} \quad \Rightarrow \quad h = \frac{\sqrt{\pi}\, d}{2} = 0.886 d$$

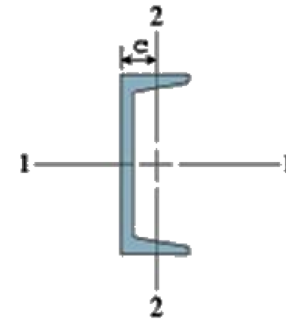
$$S_{\text{square}} = \frac{I}{c} = \frac{h^3}{6} = \frac{1}{6}\left(\frac{\sqrt{\pi}\, d}{2}\right)^3 = 0.116 d^3 = 1.181 S_{\text{circle}}$$



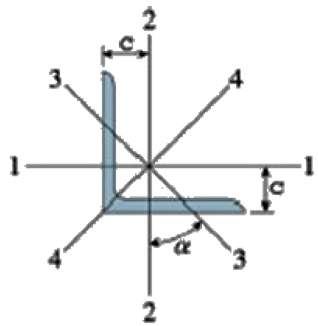
Wide-Flange Sections
(W Shapes)



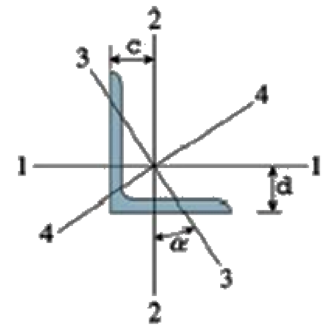
I-Beam Sections
(S Shapes)



Channel Sections
(C Shapes)

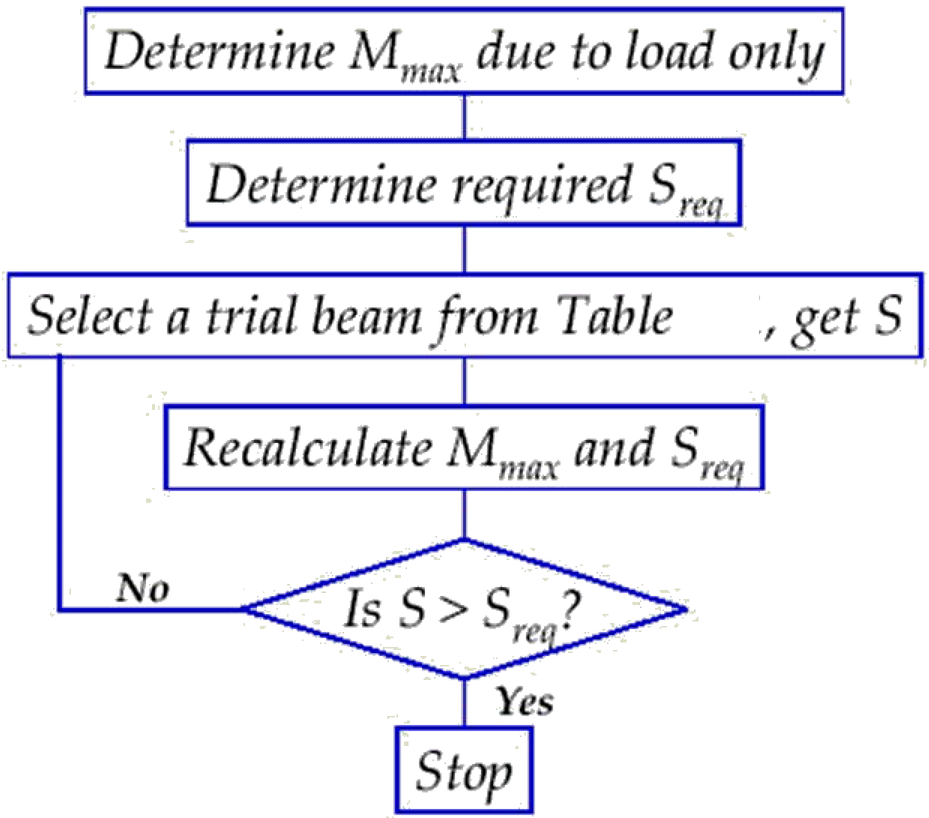


Angle Sections with Equal Legs
(L Shapes)

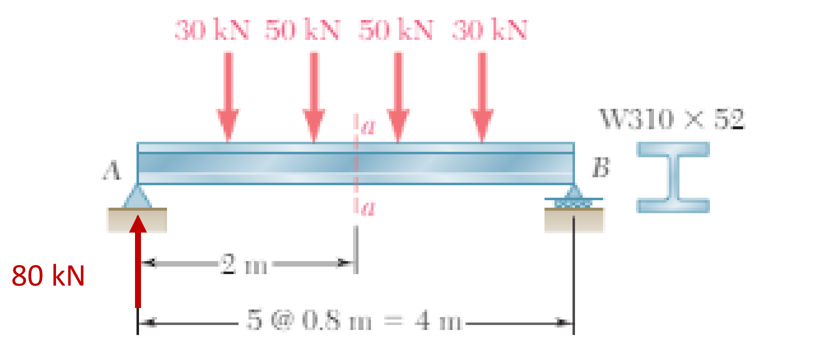


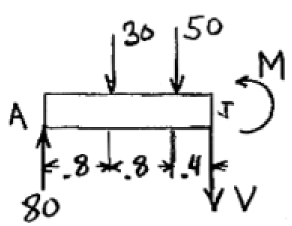
Angle Sections with Unequal Legs
(L Shapes)

Design of Beams for Bending Stresses



determine the maximum normal stress due to bending on section a-a.





$+\circlearrowleft \Sigma M_J = 0$:

$$-(80)(2) + (30)(1.2) + (50)(0.4) + M = 0$$

$$M = 104 \text{ kN} \cdot \text{m} = 104 \times 10^3 \text{ N} \cdot \text{m}$$

For $\quad W310 \times 52, \quad S = 747 \times 10^3 \text{mm}^3$ from table

$$= 747 \times 10^{-6} \text{ m}^3$$

Normal stress: $\quad \sigma = \dfrac{M}{S} = \dfrac{104 \times 10^3}{747 \times 10^{-6}} = 139.2 \times 10^6 \text{ Pa}$

$$\sigma = 139.2 \text{ MPa}$$

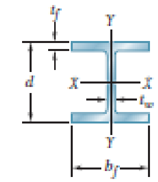
Appendix C Properties of Rolled-Steel Shapes

(SI Units)

Continued from page A19

W Shapes

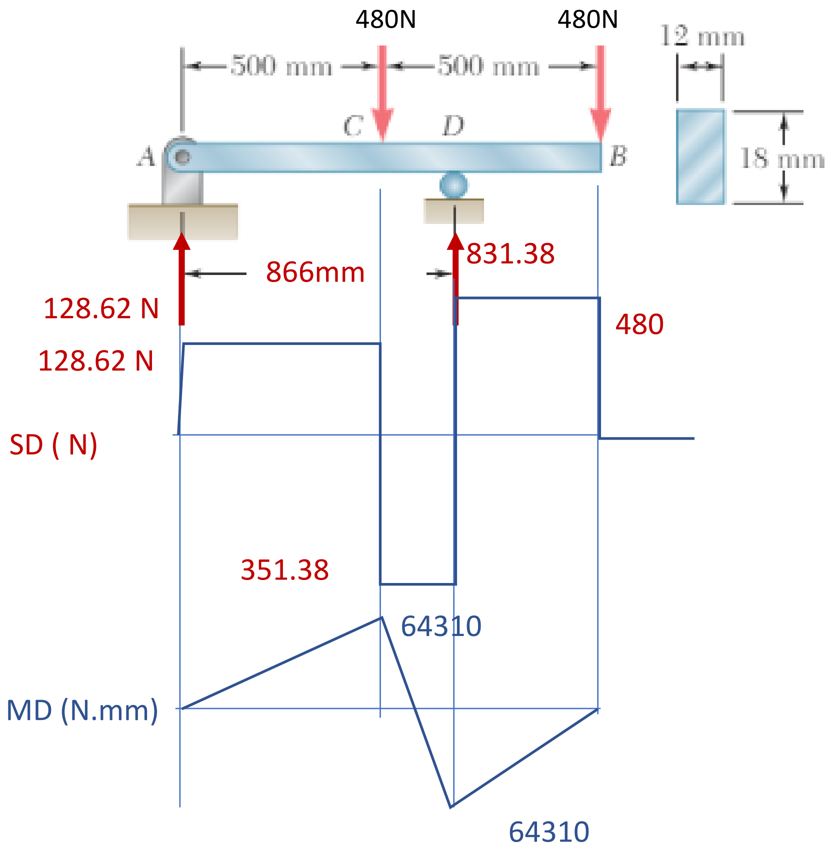
(Wide-Flange Shapes)



Designation*	Area A, mm^2	Depth d, mm	Flange: Width b_f, mm	Flange: Thickness t_f, mm	Web Thickness t_w, mm	Axis X-X: I_x 10^6 mm^4	Axis X-X: S_x 10^3 mm^3	Axis X-X: r_x mm	Axis Y-Y: I_y 10^6 mm^4	Axis Y-Y: S_y 10^3 mm^3	Axis Y-Y: r_y mm
W310 × 143	18200	323	310	22.9	14.0	347	2150	138	112	728	78.5
107	13600	312	305	17.0	10.9	248	1600	135	81.2	531	77.2
74	9420	310	205	16.3	9.40	163	1050	132	23.4	228	49.8
60	7550	302	203	13.1	7.49	128	844	130	18.4	180	49.3
52	6650	318	167	13.2	7.62	119	747	133	10.2	122	39.1
44.5	5670	312	166	11.2	6.60	99.1	633	132	8.45	102	38.6
38.7	4940	310	165	9.65	5.84	84.9	547	131	7.20	87.5	38.4
32.7	4180	312	102	10.8	6.60	64.9	416	125	1.94	37.9	21.5
23.8	3040	305	101	6.73	5.59	42.9	280	119	1.17	23.1	19.6
W250 × 167	21200	290	264	31.8	19.2	298	2060	118	98.2	742	68.1
101	12900	264	257	19.6	11.9	164	1240	113	55.8	433	65.8
80	10200	257	254	15.6	9.4	126	983	111	42.9	338	65.0
67	8580	257	204	15.7	8.89	103	805	110	22.2	218	51.1
58	7420	252	203	13.5	8.00	87.0	690	108	18.7	185	50.3
49.1	6260	247	202	11.0	7.37	71.2	574	106	15.2	151	49.3
44.8	5700	267	148	13.0	7.62	70.8	531	111	6.95	94.2	34.8
32.7	4190	259	146	9.14	6.10	49.1	380	108	4.75	65.1	33.8
28.4	3630	259	102	10.0	6.35	40.1	308	105	1.79	35.1	22.2
22.3	2850	254	102	6.86	5.84	28.7	226	100	1.20	23.8	20.6
W200 × 86	11000	222	209	20.6	13.0	94.9	852	92.7	31.3	300	53.3
71	9100	216	206	17.4	10.2	76.6	708	91.7	25.3	246	52.8
59	7550	210	205	14.2	9.14	60.8	582	89.7	20.4	200	51.8
52	6650	206	204	12.6	7.87	52.9	511	89.2	17.7	174	51.6
46.1	5880	203	203	11.0	7.24	45.8	451	88.1	15.4	152	51.3
41.7	5320	205	166	11.8	7.24	40.8	398	87.6	9.03	109	41.1
35.9	4570	201	165	10.2	6.22	34.4	342	86.9	7.62	92.3	40.9
31.3	3970	210	134	10.2	6.35	31.3	298	88.6	4.07	60.8	32.0
26.6	3390	207	133	8.38	5.84	25.8	249	87.1	3.32	49.8	31.2
22.5	2860	206	102	8.00	6.22	20.0	193	83.6	1.42	27.9	22.3
19.3	2480	203	102	6.48	5.84	16.5	162	81.5	1.14	22.5	21.4
W150 × 37.1	4740	162	154	11.6	8.13	22.2	274	68.6	7.12	91.9	38.6
29.8	3790	157	153	9.27	6.60	17.2	220	67.6	5.54	72.3	38.1
24	3060	160	102	10.3	6.60	13.4	167	66.0	1.84	36.1	24.6
18	2290	153	102	7.11	5.84	9.20	120	63.2	1.24	24.6	23.3
13.5	1730	150	100	5.46	4.32	6.83	91.1	62.7	0.916	18.2	23.0
W130 × 28.1	3590	131	128	10.9	6.86	10.9	167	55.1	3.80	59.5	32.5
23.8	3040	127	127	9.14	6.10	8.91	140	54.1	3.13	49.2	32.0
W100 × 19.3	2470	106	103	8.76	7.11	4.70	89.5	43.7	1.61	31.1	25.4

*A wide-flange shape is designated by the letter W followed by the nominal depth in millimeters and the mass in kilograms per meter.

Determine the maximum normal stress due to bending



$$\sigma = \frac{Mc}{I} = \frac{M}{\frac{I}{c}} = \frac{M}{S}$$

$$S = \frac{bh^3}{12(h/2)} = \frac{bh^2}{6} = \frac{(12)(18)^2}{6} = 648 \text{ mm}^3$$

$$\sigma_{Max} = \frac{M_{max}}{S} = \frac{64310}{648} = \underline{99.2 \text{ MPa}}$$

Design the cross section of the beam, knowing that the grade of timber used has an allowable normal stress of 12 MPa.

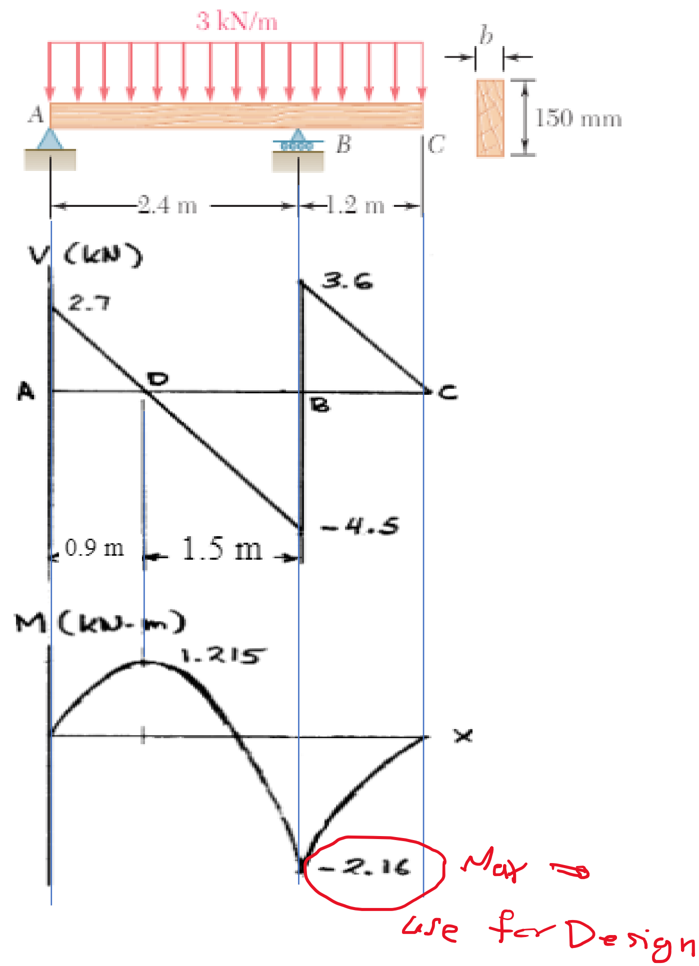
$M_{max} = 2.16 \times 10^6$ N-mm

$\sigma_{all} = 12$ MPa

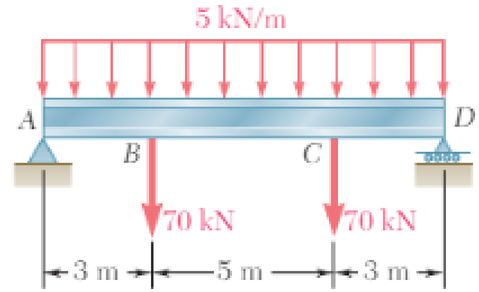
$$\sigma = \frac{M}{S} \Rightarrow 12 = \frac{2.16 \times 10^6}{S} \Rightarrow S = 180 \times 10^3 \text{ mm}^3$$

$$S = \frac{I}{h/2} = \frac{bh^2}{6}$$

$$180 \times 10^3 = \frac{b(150)^2}{6} \Rightarrow \boxed{b = 480 \text{ mm}}$$

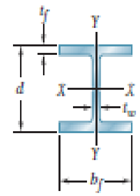


Knowing that the allowable normal stress for the steel used is 160 MPa, select the most economical wide-flange beam to support the loading shown



$$\sigma = \frac{M}{S} = 160$$

$$S = \frac{286 \times 10^6}{160} = 1787 \times 10^3\ mm^3 \Rightarrow$$ Search the table for the nearest (S)



Appendix C Properties of Rolled-Steel Shapes

(SI Units)

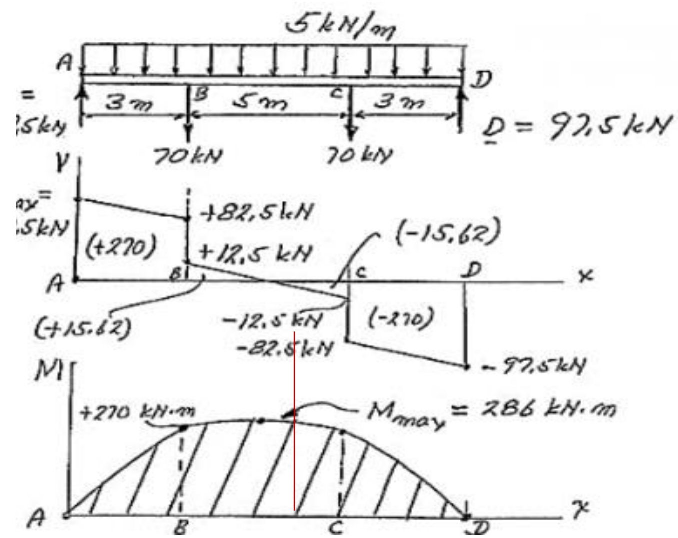
W Shapes

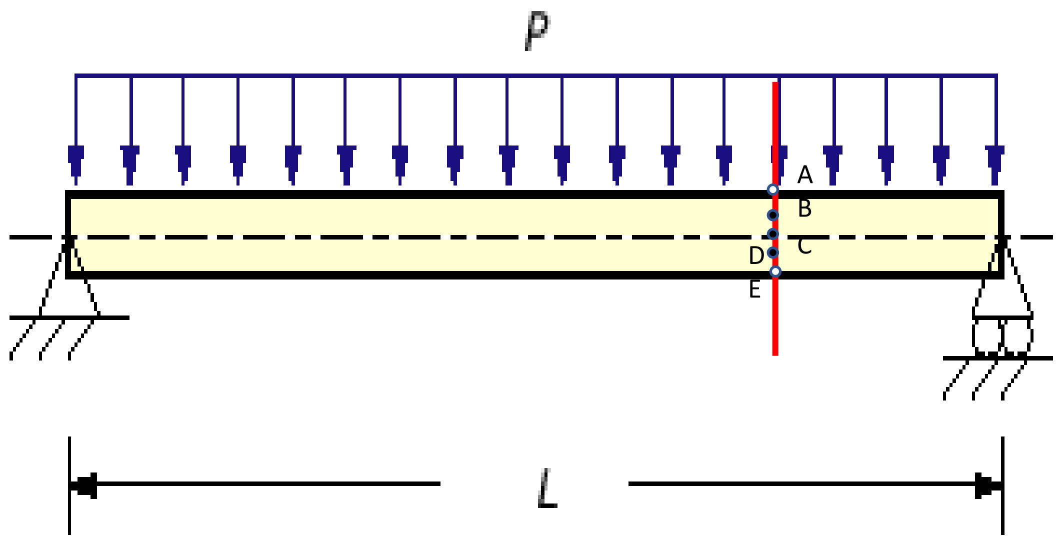
(Wide-Flange Shapes)

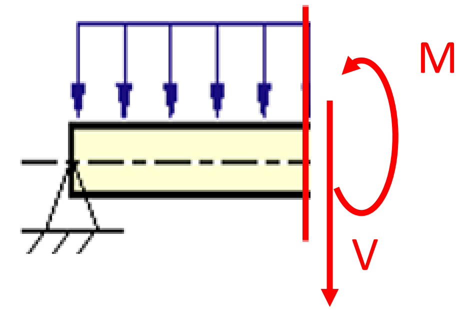
Designation[†]	Area A, mm²	Depth d, mm.	Flange Width b_f, mm	Flange Thickness t_f, mm	Web Thickness t_w mm	Axis X-X I_x 10^6 mm⁴	Axis X-X S_x 10^3 mm³	Axis X-X r_x mm	Axis Y-Y I_y 10^6 mm⁴	Axis Y-Y S_y 10^3 mm³	Axis Y-Y r_y mm
W920 × 449	57300	947	424	42.7	24.0	8780	18500	391	541	2560	97.0
201	25600	904	305	20.1	15.2	3250	7190	356	93.7	618	60.5
W840 × 299	38200	856	399	29.2	18.2	4830	11200	356	312	1560	90.4
176	22400	836	292	18.8	14.0	2460	5880	330	77.8	534	58.9
W760 × 257	32900	772	381	27.2	16.6	3430	8870	323	249	1310	86.9
147	18800	754	267	17.0	13.2	1660	4410	297	53.3	401	53.3
W690 × 217	27800	696	356	24.8	15.4	2360	6780	292	184	1040	81.3
125	16000	678	254	16.3	11.7	1190	3490	272	44.1	347	52.6
W610 × 155	19700	612	325	19.1	12.7	1290	4230	257	108	667	73.9
101	13000	602	228	14.9	10.5	762	2520	243	29.3	257	47.5
W530 × 150	19200	544	312	20.3	12.7	1010	3720	229	103	660	73.4
92	11800	533	209	15.6	10.2	554	2080	217	23.9	229	45.0
66	8390	526	165	11.4	8.89	351	1340	205	8.62	104	32.0
W460 × 158	20100	475	284	23.9	15.0	795	3340	199	91.6	646	67.6
113	14400	462	279	17.3	10.8	554	2390	196	63.3	452	66.3
74	9480	457	191	14.5	9.02	333	1460	187	16.7	175	41.9

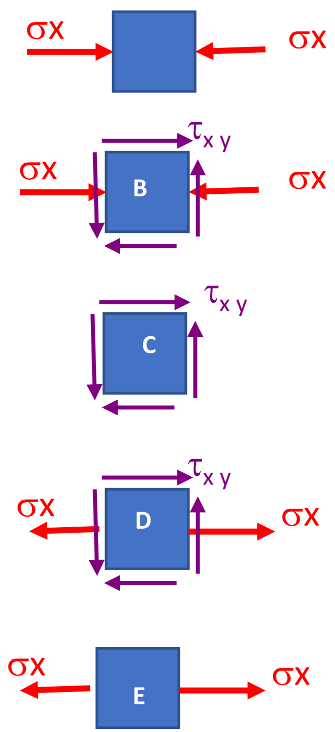
use

W350 X92









$$\sigma = \pm \frac{My}{I}$$

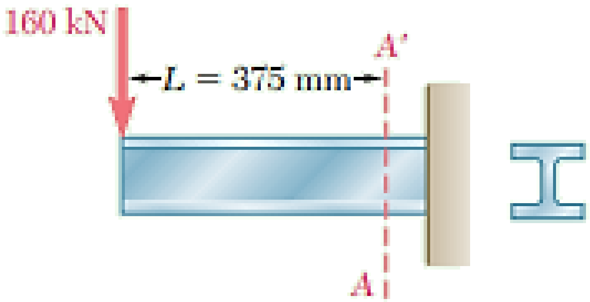
$$\tau_{xy} = \frac{VQ}{Ib} \pm \frac{Tr}{j}$$

Stress elements

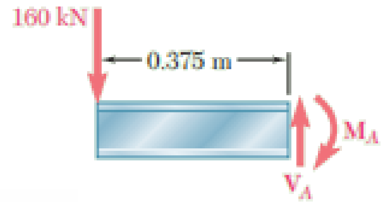
- Indicate the point where stresses are required
- Find internal forces at that point
- Determine the stress at that point (normal and shear stresses)
- Show the stresses on a stress element
- Transform the stresses to determine the maximum stress and show them on an oriented element

PRINCIPAL STRESSES IN A BEAM

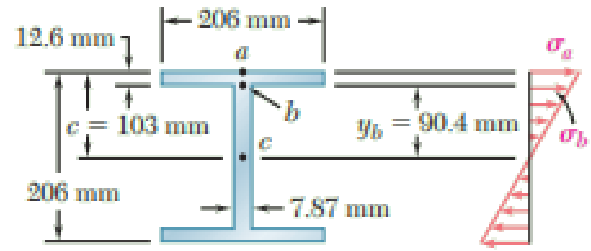
A 160-kN force is applied as shown at the end of a W200 x 52 rolled-steel beam. Neglecting the effect of fillets and of stress concentrations, determine whether the normal stresses in the beam satisfy a design specification that they be equal to or less than 150 MPa at section A–A'.



$$M_A = (160 \text{ kN})(0.375 \text{ m}) = 60 \text{ kN}\cdot\text{m}$$
$$V_A = 160 \text{ kN}$$



Free-body diagram of beam, with section at A–A'



$I = 52.9 \times 10^{-6} \text{m}^4$
$S = 511 \times 10^{-6} \text{m}^3$

Cross-section dimensions and normal stress distribution.

At point a,

$$\sigma_a = \frac{M_A}{S} = \frac{60 \text{ kN}\cdot\text{m}}{511 \times 10^{-6} \text{ m}^3} = 117.4 \text{ MPa}$$

At point b,

$$\sigma_b = \sigma_a \frac{y_b}{c} = (117.4 \text{ MPa})\frac{90.4 \text{ mm}}{103 \text{ mm}} = 103.0 \text{ MPa}$$

At point a,

$$Q = 0 \qquad \tau_a = 0$$

At point b,

$$Q = (206 \times 12.6)(96.7) = 251.0 \times 10^3 \text{ mm}^3 = 251.0 \times 10^{-6} \text{ m}^3$$

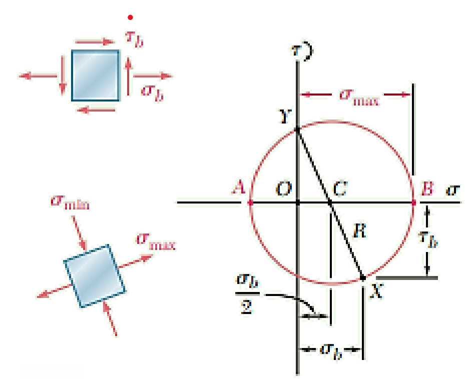
$$\tau_b = \frac{V_A Q}{It} = \frac{(160 \text{ kN})(251.0 \times 10^{-6} \text{ m}^3)}{(52.9 \times 10^{-6} \text{ m}^4)(0.00787 \text{ m})} = 96.5 \text{ MPa}$$

Principal Stress at Point b. The state of stress at point b consists of the normal stress $\sigma_b = 103.0$ MPa and the shearing stress $\tau_b = 96.5$ MPa. Draw Mohr's circle (Fig. 4) and find

$$\sigma_{\max} = \frac{1}{2}\sigma_b + R = \frac{1}{2}\sigma_b + \sqrt{\left(\frac{1}{2}\sigma_b\right)^2 + \tau_b^2}$$

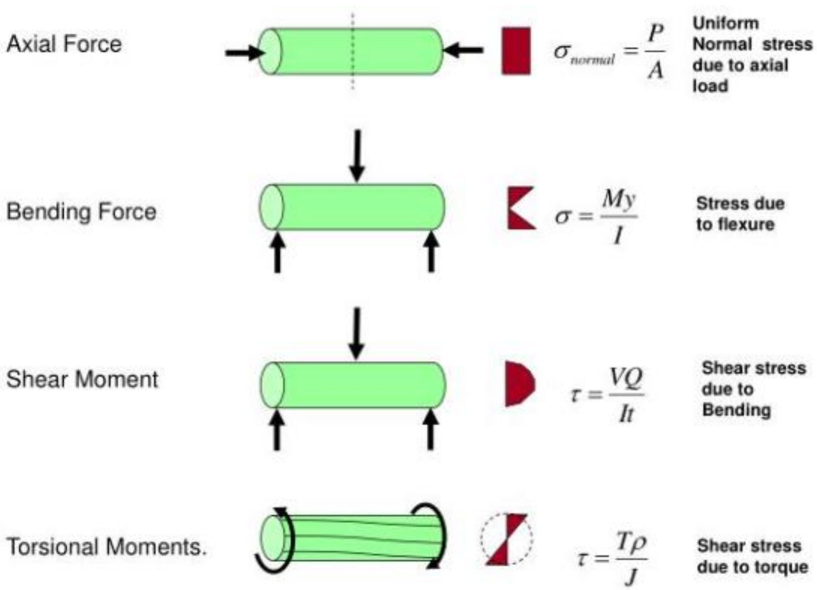
$$= \frac{103.0}{2} + \sqrt{\left(\frac{103.0}{2}\right)^2 + (96.5)^2}$$

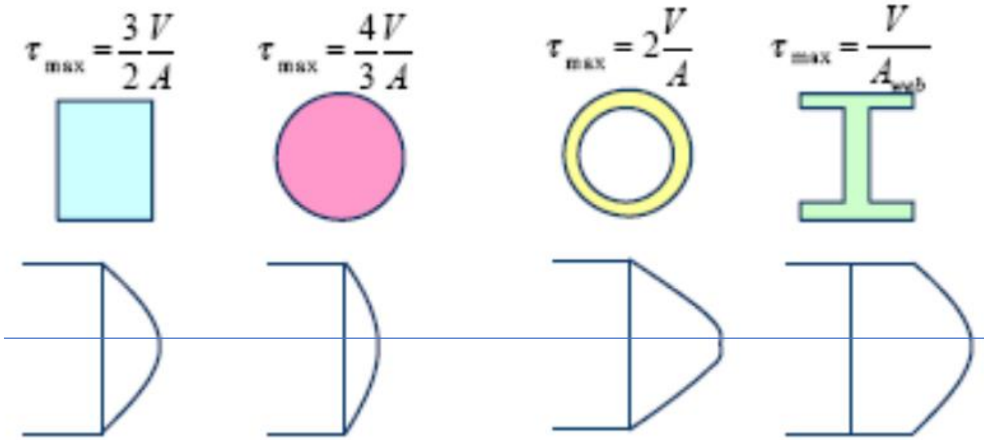
$$\sigma_{\max} = 160.9 \text{ MPa}$$ >150 X



Stress element for coordinate and principal orientations at point b; Mohr's circle for point b.

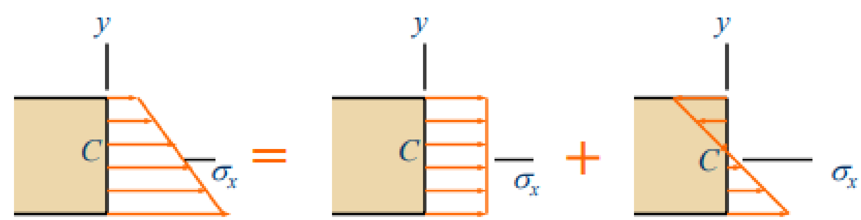
State of stress caused by combined loading



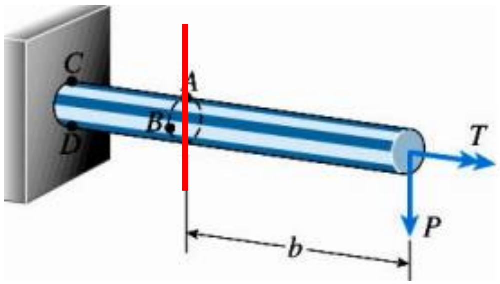


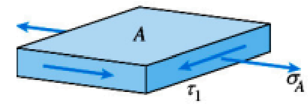
Superposition

- Determine normal and shear stress for each loading . Use superposition principle to determine the resultant normal and shear stresses.
- Show the results as a distribution of stresses acting over the member cross-sectional area
- Represent the results on am element located at the point of interest

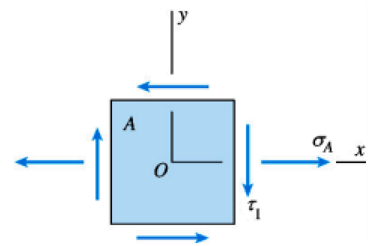


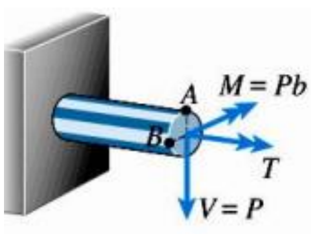
$$\sigma_x = (\sigma_x)_{\text{centric}} + (\sigma_x)_{\text{bending}}$$





$$\sigma = \pm\frac{P}{A} \pm \frac{My}{I}$$

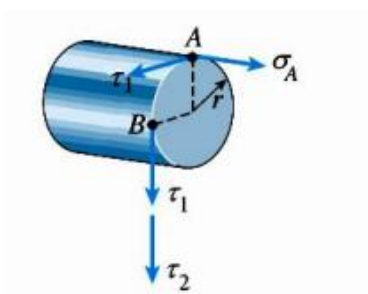


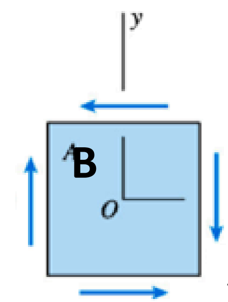


$$\boxed{\tau_{torsion} = \frac{Tr}{I_{Polar}} = \frac{2T}{\pi r^3}}$$

$$\boxed{\sigma_{bending} = \frac{Mr}{I} = \frac{4M}{\pi r^3}}$$

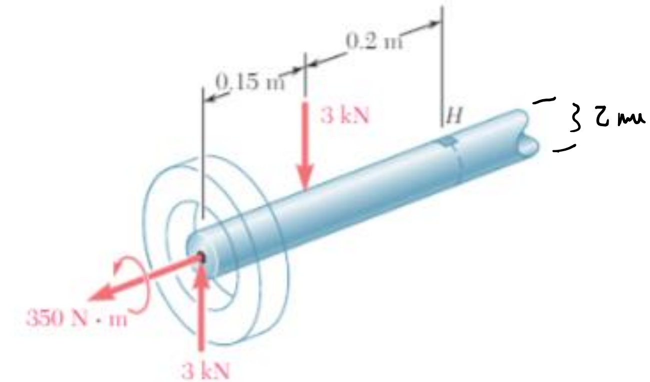
$$\boxed{\tau_{shear} = \frac{VQ}{Ib} = \frac{4V}{3A}}$$





$\tau_1 + \tau_2$

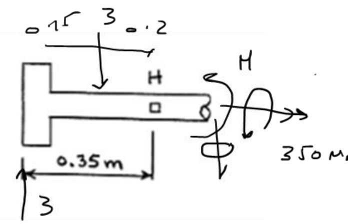
The axle of an automobile is acted upon by the forces and couple shown. Knowing that the diameter of the solid axle is 32 mm, determine (*a*) the principal planes and principal stresses at point *H* located on top of the axle, (*b*) the maximum shearing stress at the same point.



$\sigma = \frac{My}{I}$, $\tau = \frac{Tr}{J}$; $r = 32/2 = 16$

$I = \frac{\pi r^4}{4} = \frac{(3.14)(32/2)^4}{4} = 51445.76 \text{ mm}^4$

$J = \frac{\pi r^4}{2} = \frac{(3.14)(32/2)^4}{2} = 102891.52 \text{ mm}^4$



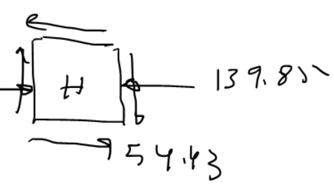
$\circlearrowleft \Sigma M = 0$

$-(3)(0.35) + (3)(0.2) + M = 0$

$M = 0.45 \text{ kN.m}$

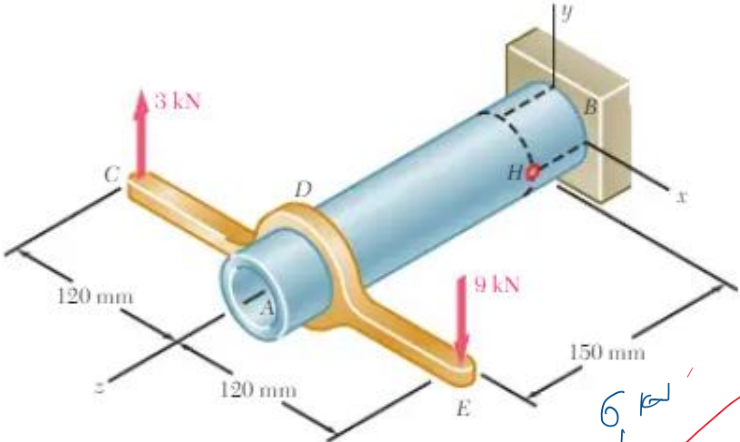
$\sigma = \frac{(0.45 \times 10^6)(16)}{51445.76} = 139.95 \text{ MPa}$

$\tau = \frac{(350000)(16)}{102891.52} = 54.43 \text{ MPa}$



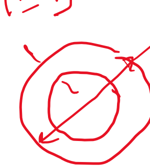
$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm R$

$\tau = R$

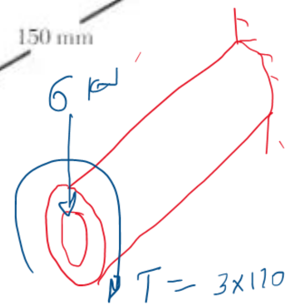


The steel pipe *AB* has a 72-mm outer diameter and a 5-mm wall thickness. Knowing that the arm *CDE* is rigidly attached to the pipe, determine the principal stresses, principal planes, and the maximum shearing stress at point *H*.

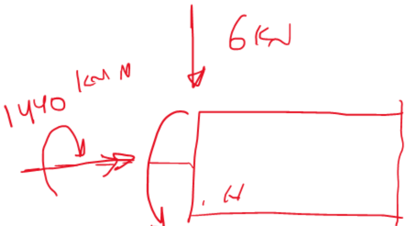
$\sigma_1, \sigma_2, \theta_{p_1} - \tau_{max}$



$d_i = 72 - 10 = 62$ mm



$T = 3\times120 + 9\times120 = 1440$ kN·m



$M = 6\times150 = 900$ kN·mm

$\sigma = \frac{My}{I}$

$\left.\begin{array}{l}\tau_T = \frac{Tr}{J} \\ \tau_v = \frac{VQ}{Ib}\end{array}\right] +$

$I = \frac{\pi(72^4 - 62^4)}{64} = 593835.6$

$J = \frac{\pi(72^4 - 62^4)}{32} = 1187671.2$ mm^4

$Q = \frac{2}{3}(r_o^3 - r_i^3)$

$= \frac{2}{3}(36^3 - 31^3) = 11243.33$ mm^3

Point E on the neutral axis $\Rightarrow \sigma_H = 0$

$\tau_H = \frac{Tr}{J} + \frac{VQ}{Ib} = \frac{(1440\times10^3)(72/2)}{1187671.2} + \frac{(6000)(11243.33)}{(593835.6)(10)}$

$= 55$ MPa

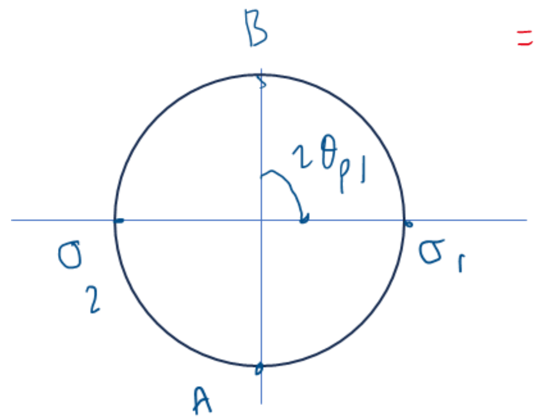
$\frac{\sigma_x + \sigma_y}{2} = 0$, $\frac{\sigma_x - \sigma_y}{2} = 0$

$\tau_{xy} = -55$

$R = 55$

$\sigma_{1,2} = 0 \pm R = \begin{cases} 55 \\ -55 \end{cases}$

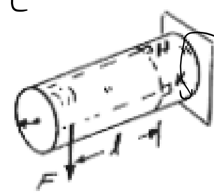
$\theta_{p_1} = 45^\circ$, $\theta_{p_2} = -45^\circ$

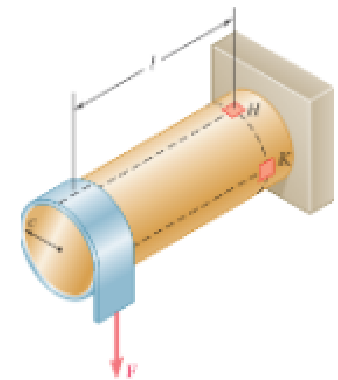


$\tau_{max} = R = 55$ MPa

A thin strap is wrapped around a solid rod of radius c =20 mm as shown. Knowing that l =100 mm and F = 5 kN, determine the normal and shearing stresses at point H, and point K..

$T = [5000](20)\ N\cdot m,\quad M = 5000 \times 100 : N\cdot m$





Bending:

Torsion:

$I = \frac{\pi r^4}{4} = 125600\ m^4$

$J = \frac{\pi r^4}{2} = 251200$

(H)

$\sigma_H = \frac{(5000)(100)(20)}{125600} = \underline{79.6\ MPa}$

$\tau_H = \frac{Tr}{J} = \frac{(5000)(20)(20)}{251200} = \underline{7.96\ MPa}$

(K) on neutral axis

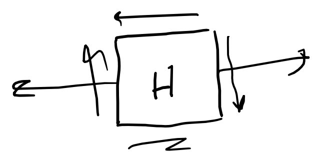
$\underline{\sigma_K = 0}$

$\tau_v = \frac{4}{3}\frac{V}{A} = \frac{4}{3}\left(\frac{5000}{\pi\ 20^2}\right) = 5.3$

$\tau = 7.96$

Torque

$\tau = \tau_v + \tau_T = \underline{13.3\ MPa}$



Structural mechanics

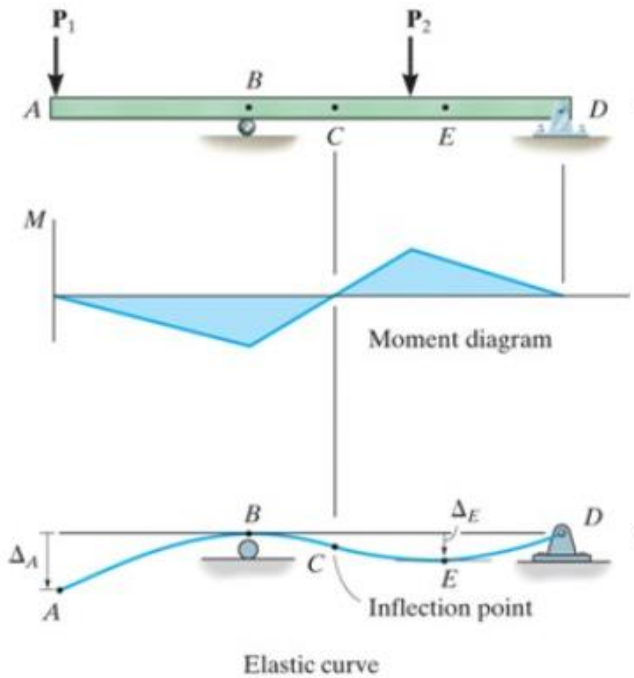
Chapter 12

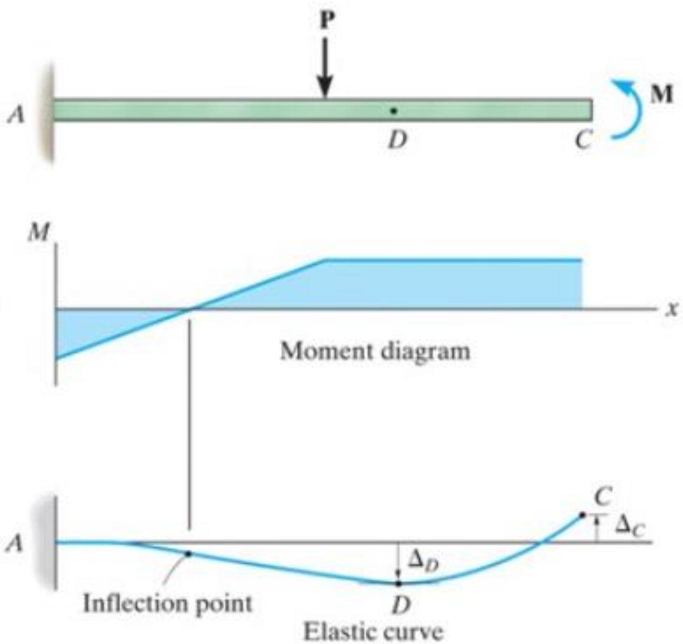
Deflection of Beams

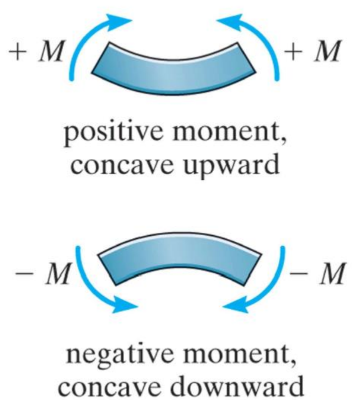
I. Marie

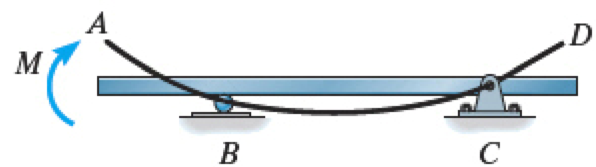
DEFORMATION UNDER TRANSVERSE LOADING

12.1 The Elastic Curve



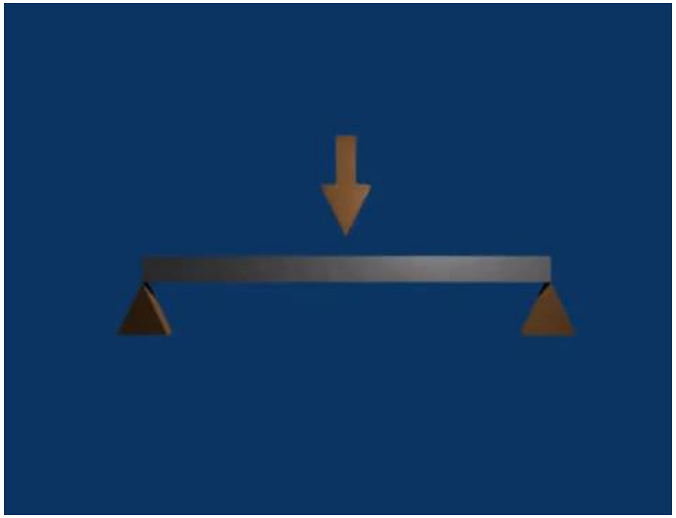






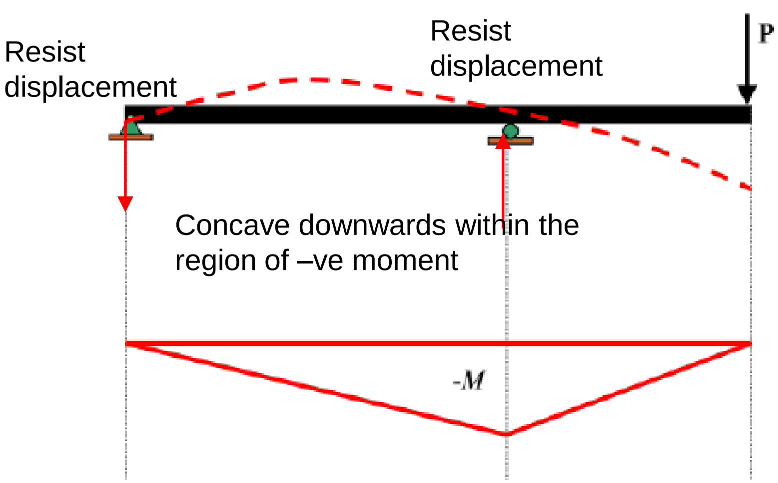
Factors affecting deflection of beams

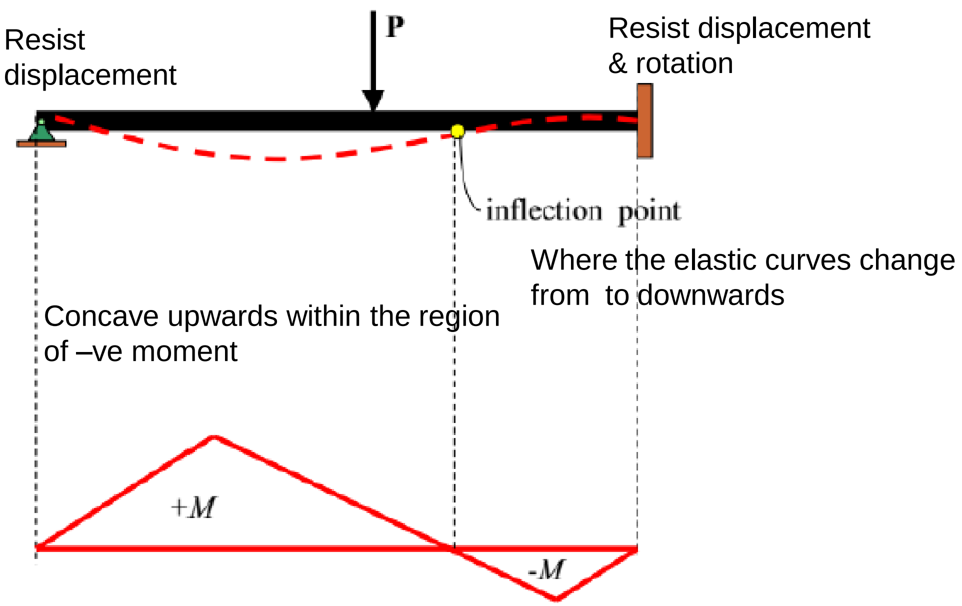
- magnitude and type of loading (bending moment diagram
- Span length of the beam
- Beam supports
- Material properties (E)
- Moment of inertia (shape of beam)



The deflection must be limited in order to provide:

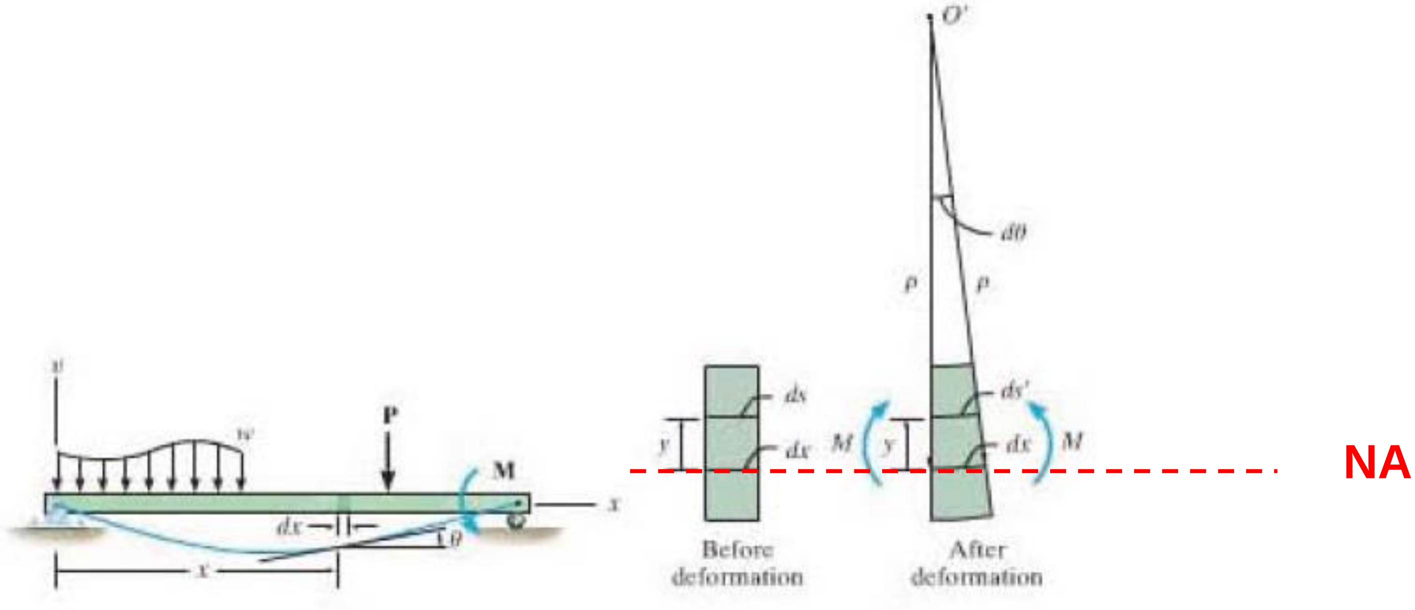
- integrity and stability of a structure or machine
- Prevent cracking of any attached brittle material (concrete or glass)





It is helpful to sketch the deflected shape of the beam when it is loaded to visualize and check the calculated results

Momen



$$dx = \rho \, d\theta$$

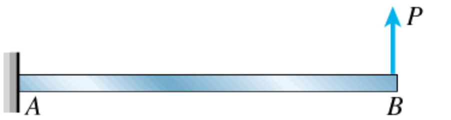
$$ds = (\rho - y)\, d\theta$$

$$\epsilon = \frac{(\rho - y)\, d\theta - \rho \, d\theta}{\rho \, d\theta} \Rightarrow \epsilon = \frac{-y}{\rho} \Rightarrow \frac{1}{\rho} = -\frac{\epsilon}{y} \quad \text{-- } (1)$$

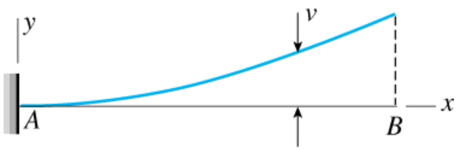
$$\sigma = E\epsilon \; : \quad \frac{My}{I} = E\epsilon \Rightarrow \epsilon = \frac{My}{EI} \Rightarrow \text{sub in } (1)$$

$$\boxed{\frac{1}{\rho} = \frac{M}{EI}}$$

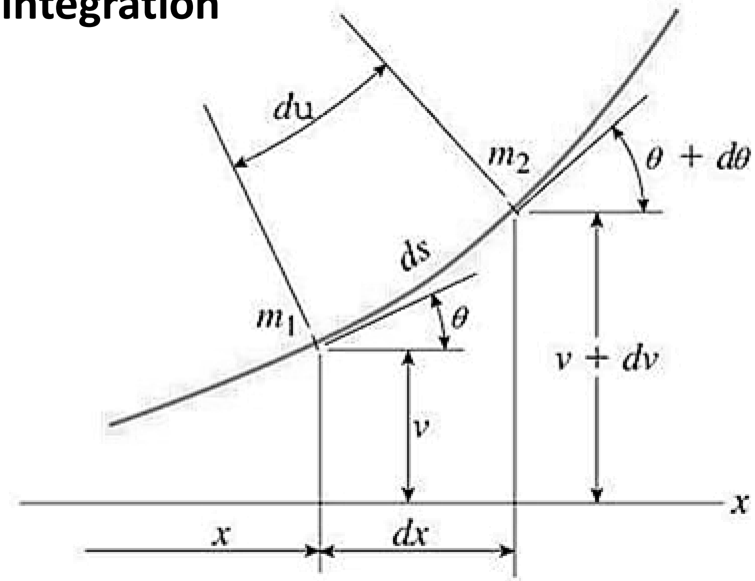
12.2 Slope and displacement by integration



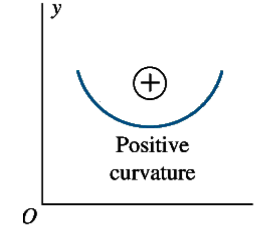
(a)



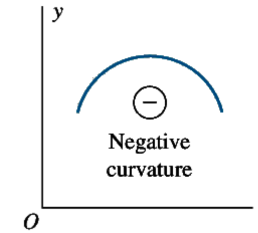
(b)



Deflection curve of a cantilever beam.



(a)



(b)

$$\kappa = \frac{1}{\rho} = \frac{M}{EI}$$

$$\sigma = \frac{My}{I}$$

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{ds}$$

$$ds \approx dx, \theta \approx \frac{dv}{dx}$$

$$\kappa = \frac{1}{\rho} \approx \frac{d^2v}{dx^2}$$

From calculus books

- Exact:

$$\kappa = \frac{1}{\rho} = \frac{d^2v/dx^2}{[1 + \cancel{(dv/dx)^2}]^{3/2}}$$

Non linear second order differential equation

≅ 0

- Small deflections: $\kappa \approx \dfrac{d^2v}{dx^2} = \dfrac{M}{EI}$

Beam Differential Relations

1. Derive the equation for the internal moment at point at distance x.. M(x)
2. Integrate M(x) to find the beam slope, then integrate the second time to find the equation of the elastic curve
3. Apply the boundary conditions to find the constant of integration

$$\frac{dv}{dx} = \theta(x)$$

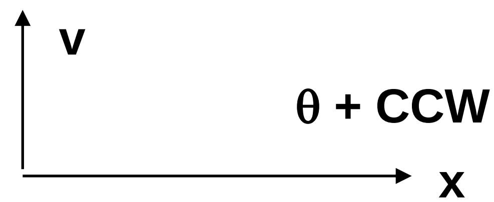
$$\frac{d^2v}{dx^2} = \frac{M(x)}{EI}$$

$$\frac{d^3v}{dx^3} = \frac{V(x)}{EI}$$

$$\frac{d^4v}{dx^4} = -\frac{w(x)}{EI}$$

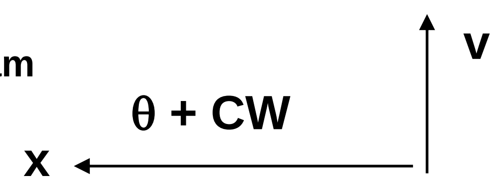
$$\left\{ \begin{aligned} EI\frac{\mathbf{d}v}{\mathbf{d}x} &= \mathbf{EI}\theta = \int M(x)\,\mathbf{d}x + C_1 \\ EIv &= \iint M(x)\,\mathbf{d}x + c_1 x + C_2 \end{aligned} \right.$$

The constant of integration to be found from the boundary conditions



Solution technique

Draw the elastic curve to visualize deflection
Find the internal moment equation at distance x. for each region of the beam
integrate the equations
solve for the integration constants using *boundary* conditions for the beam



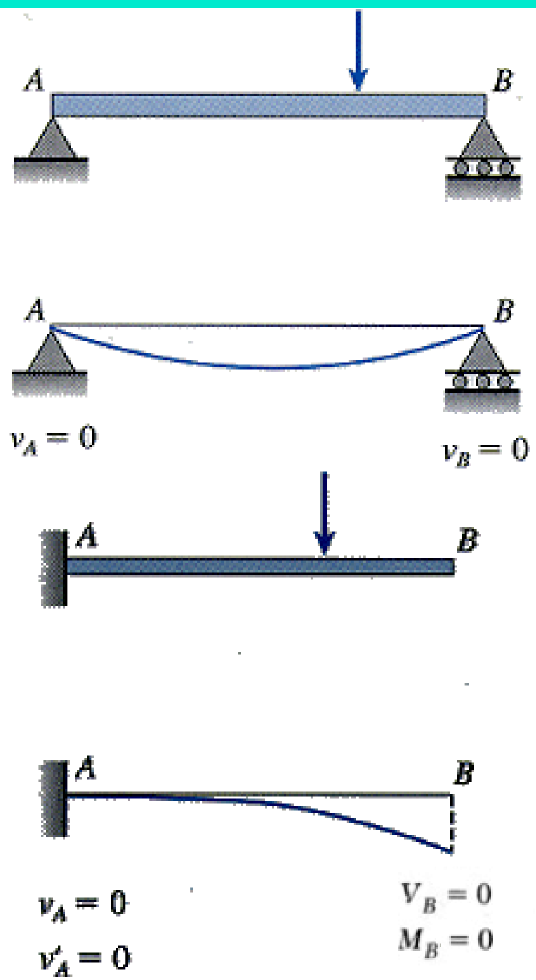
Assumptions

material obeys Hooke's law
slopes of the deflection curve are small
shear deformations are negligible

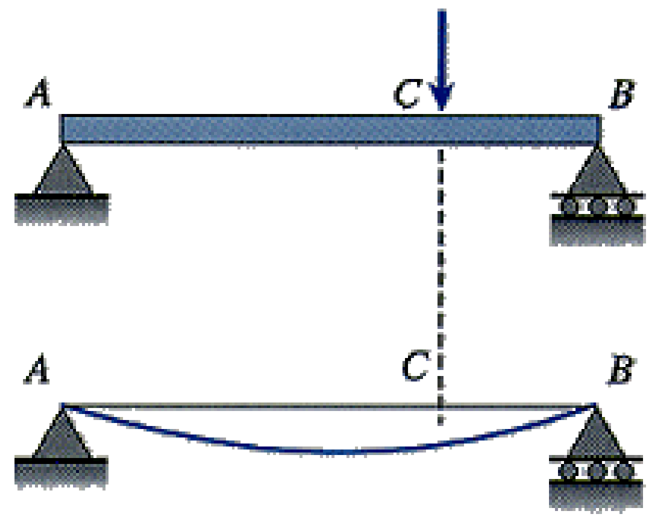
Solution technique

- Divide beam into regions of continuous M equations
- Integrate each region twice to determine beam deflections *v*
- solve for two integration constants per region using :
  - boundary conditions - deflections and slopes at the beam supports
  - continuity conditions - deflections and slopes where regions meet

Boundary condition examples



Continuity and symmetry example



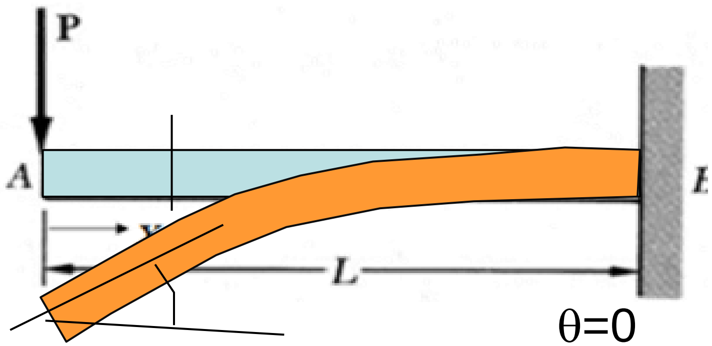
At point C: $(v)_{AC} = (v)_{CB}$

$(v')_{AC} = (v')_{CB}$

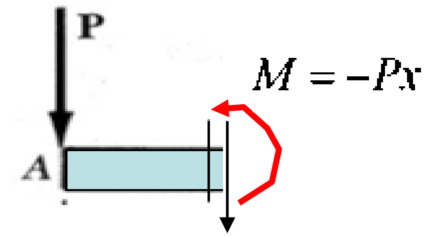
At midpoint: $v' = 0$

Determine the deflection equation and the slope equation for the cantilever shown. What is the maximum deflection?

E= 70 GPa
I= $100x10^6$ mm^4
L=6m
P= 1kN



θ=0
v= 0



$$EI\frac{d^2v}{dx^2} = -Px$$

$$EI\frac{dv}{dx} = -Px2/2 + C_1 \quad \ldots\ldots (1)$$

$$EI\theta = -Px2/2 + PL^2/2$$

$$EIv = -Px3/3 + C_1x + C_2 \quad \ldots\ldots\ (2)$$

@ x=0 … v =v $_{max}$

@ x=L … θ=0 and v= 0

$$v_{max} = -\frac{PL^3}{3EI} \qquad \frac{1000 \times 6000^3}{3 \times 70 \times 10^3 x100x10^6} = 10\text{ mm}$$

C_1= $pL^2/2$
C_2= $-pL^3/3$

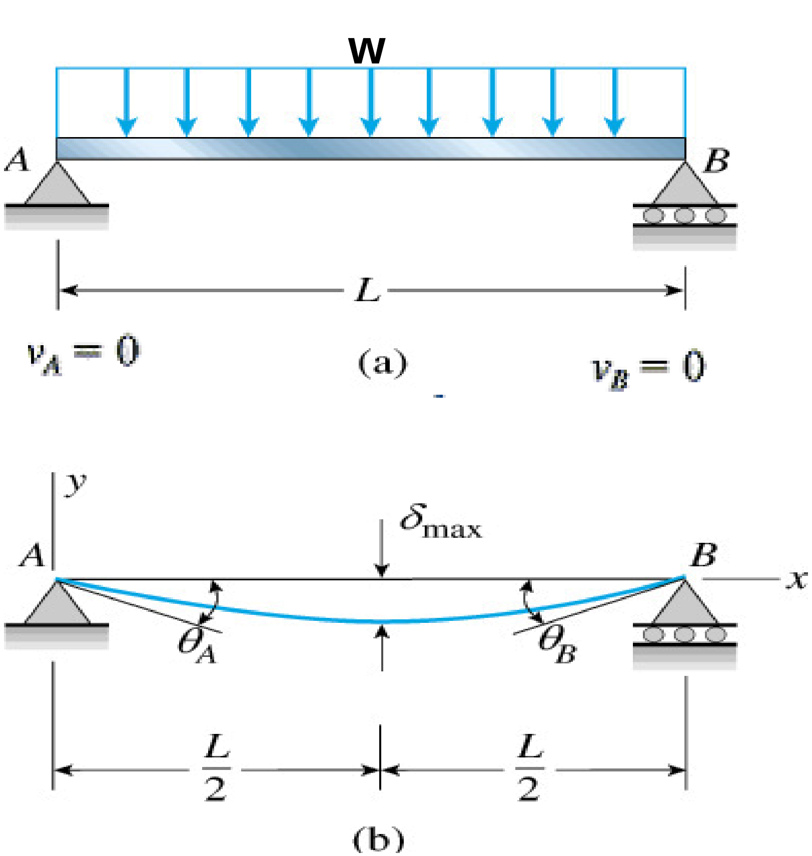
$$EI\theta_{max} = PL^2/2 \qquad \frac{1000 \times 6000^2}{2 \times 70 \times 10^3 x100x10^6} = 2.57x10^{-3}\text{ rad}$$

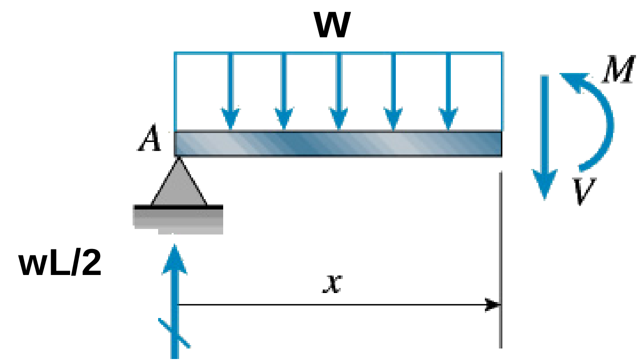
The positive θ
indicates CCW

$$= \frac{2.57x10^{-3}}{3.14} x\ 180 = 0.147\text{ deg}$$

$$v = \frac{P}{6EI}\left(-x^3 + 3L^2x - 2L^3\right)$$

Deflections of a simply supported beam with a uniform load.





$$M = \frac{wLx}{2} - \frac{wx^2}{2}$$

$$EI\frac{d^2v}{dx^2} = M$$

$$EI\theta = \frac{wLx^2}{4} - \frac{wx^3}{6} + C_1$$

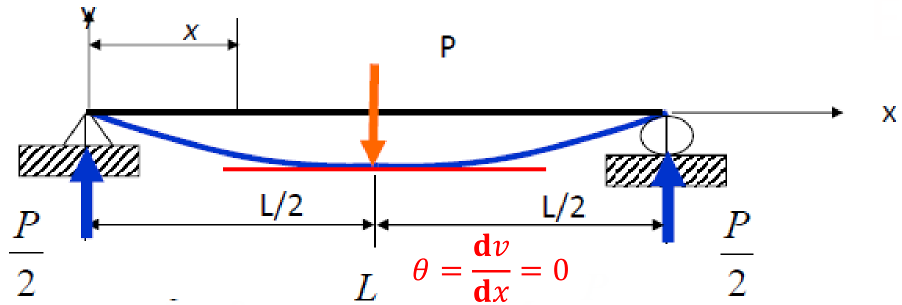
$$EIv = \frac{wLx^3}{12} - \frac{wx^4}{24} + C_1x + \cancelto{0}{C_2}$$

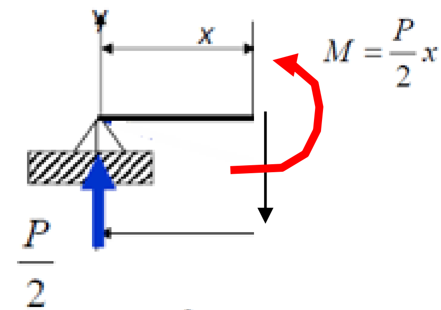
@ x= 0… V=0 ………… C2 = 0

@ x= L …. V=0 ………… C1 =- $\frac{wL^3}{24}$

$$EIv = \frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}$$

Determine the maximum deflection





for $0 < x < \frac{L}{2}$ $\qquad M = \frac{P}{2}x$

$$EI\frac{d^2v}{dx^2} = \frac{P}{2}x$$

$$EI\frac{dv}{dx} = \frac{P}{2}\frac{x^2}{2} + c_1$$

$$EIv(x) = \frac{Px^3}{12} + C_1x + c_2$$

@ x = L /2

$$\Delta_{\max} = \frac{PL^3}{48EI}$$

@x=0 $\qquad \theta_A = -PL^2/16$

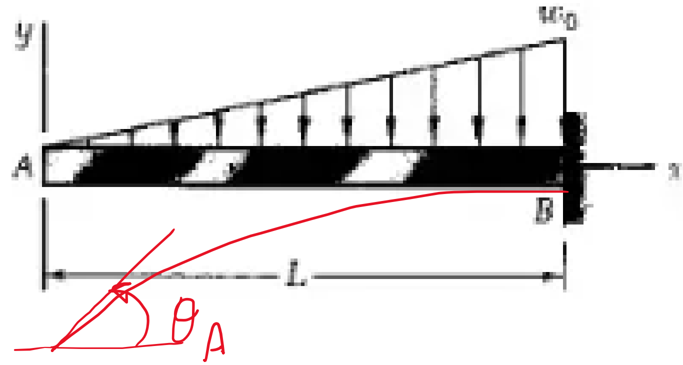
Due to symmetry $\quad @\ x = \frac{L}{2} \quad \frac{dv}{dx} = 0 \longrightarrow EI(0) = \frac{P}{2}\frac{\left(\frac{L}{2}\right)^2}{2} + c_1 \Rightarrow c_1 = -\frac{PL^2}{16}$

$$\therefore \quad EI\frac{dv}{dx} = \frac{P}{4}x^2 - \frac{PL^2}{16}$$

$$EIv = \frac{P}{4}\frac{x^3}{3} - \frac{PL^2}{16}x + c_2$$

@ x = 0 v = 0 $\quad \Rightarrow EI(0) = \frac{P}{4}\frac{(0)^3}{3} - \frac{PL^2}{16}(0) + c_2 \quad \Rightarrow c_2 = 0 \quad \therefore \quad EIv = \frac{P}{12}x^3 - \frac{PL^2}{16}x$

Determine the equation of the elastic curve, the deflection and slope at point A. EI constant



$$EI\frac{d^2v}{dx^2} = M = -\frac{1}{6}\frac{w_0x^3}{L}$$

$$EI\frac{dv}{dx} = -\frac{1}{24}\frac{w_0x^4}{L} + C_1 \qquad (1)$$

$$EIv = -\frac{1}{120}\frac{w_0x^5}{L} + C_1x + C_2 \qquad (2)$$

To find C_1 and C_2

$\left[@\, x = L,\ \frac{dv}{dx} = 0,\ v = 0\right]$

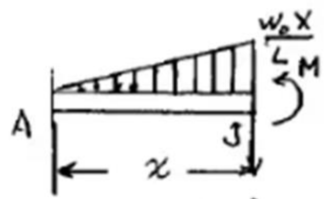
sub in (1)

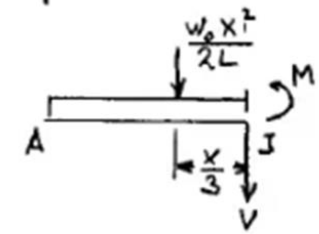
$$\frac{1}{24}\frac{w_0L^4}{L} = C_1 \Rightarrow C_1 = \frac{w_0L^3}{24}$$

sub in (2)

$$0 = -\frac{1}{120}w_0L^4 + \frac{w_0L^4}{24} + C_2 = 0$$

$$C_2 = -\frac{1}{30}w_0L^4$$





$$\Sigma M = 0 \Rightarrow M = -\frac{1}{6}\frac{w_0x^3}{L}$$

elastic curve equation (2)

$$EIv = -\frac{1}{120}\frac{w_0x^5}{L} + \frac{w_0L^3}{24}x - \frac{1}{30}w_0L^4$$

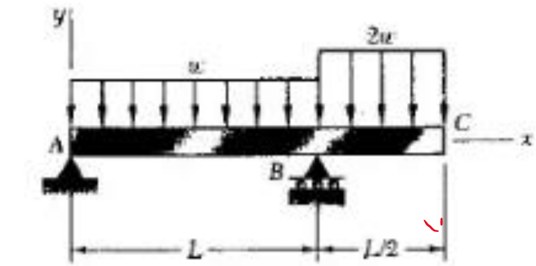
deflection at A $(x = 0)$

$$v_A = -\frac{w_0L^4}{30EI}$$

slope at A

$$\theta_A = w_0L^3/24EI$$

Determine the elastic curve equation for part AB, and find the slope at A and B. EI constant



$[x=0,\ y=0]$ $[x=L,\ y=0]$

$$EI\frac{d^2v}{dx^2} = M = \frac{wL}{4}x - \frac{wx^2}{2}$$

$$EI\frac{dv}{dx} = \frac{wLx^2}{8} - \frac{wx^3}{6} + c_1 \quad —(1)$$

$$EI\,v = \frac{wLx^3}{24} - \frac{wx^4}{24} + c_1x + c_2 \quad —(2)$$

@ $x=0 \Rightarrow v=0$ --- sub in (1) $\longrightarrow$ $c_2 = 0$

@ $x=L \Rightarrow v=0$ -- sub in (1) $\Rightarrow$

$$0 = \frac{wL^4}{24} - \frac{wL^4}{24} + c_1L + 0 = 0 \Rightarrow c_1 = 0$$

Elastic Curve for $0<x<L$

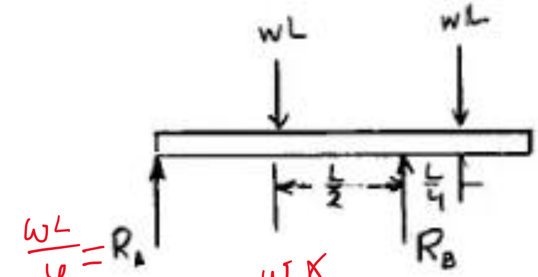
$$v = \frac{w}{24EI}\left(Lx^3 - x^4\right)$$

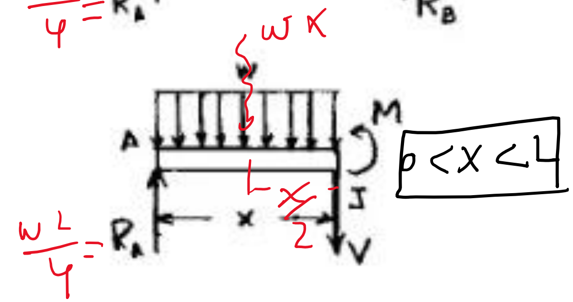
at $x=0 \Rightarrow$ sub in (1)

$\theta_A = 0$

at $x=L$ — sub in (1)

$$\theta_B = \frac{wL^3}{EI\,8} - \frac{wL^3}{6EI} = -\frac{wL^3}{24EI} \quad \text{CW}$$



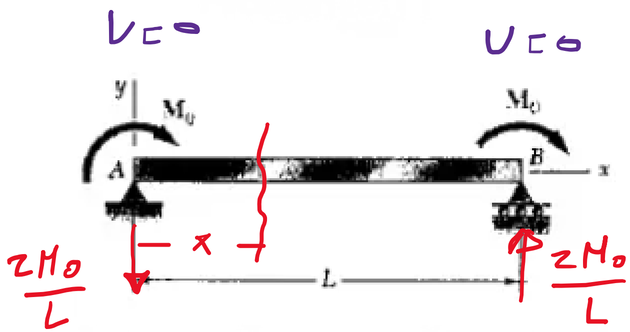


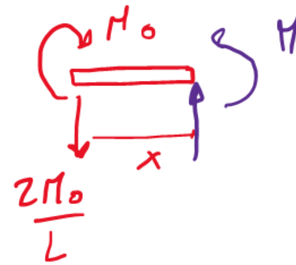
$\Sigma M = 0$

$$-\frac{wL}{4}x + \frac{wx^2}{2} + M = 0$$

$$M = \frac{wL}{4}x - \frac{wx^2}{2}$$

Derive the deflection and slope equation of the beam shown





$$\circlearrowleft^{+}\ \sum M = 0$$

$$-M_0 + \frac{2M_0}{L}x + M = 0$$

$$M = M_0 - \frac{2M_0}{L}x$$

$$EI\frac{d^2v}{dx^2} = M = M_0 - \frac{2M_0}{L}x$$

$$EI\frac{dv}{dx} = M_0x - \frac{M_0}{L}x^2 + C_1$$

$$EIv = \frac{M_0x^2}{2} - \frac{M_0}{3L}x^3 + C_1x + \cancel{C_2}^{\,0}$$

$x = 0 \rightarrow v = 0 \Rightarrow C_2 = 0$

$x = L \rightarrow v = 0$

$$\frac{M_0L^2}{2} - \frac{M_0L^{\cancel{3}\,2}}{3\cancel{L}} + C_1L = 0$$

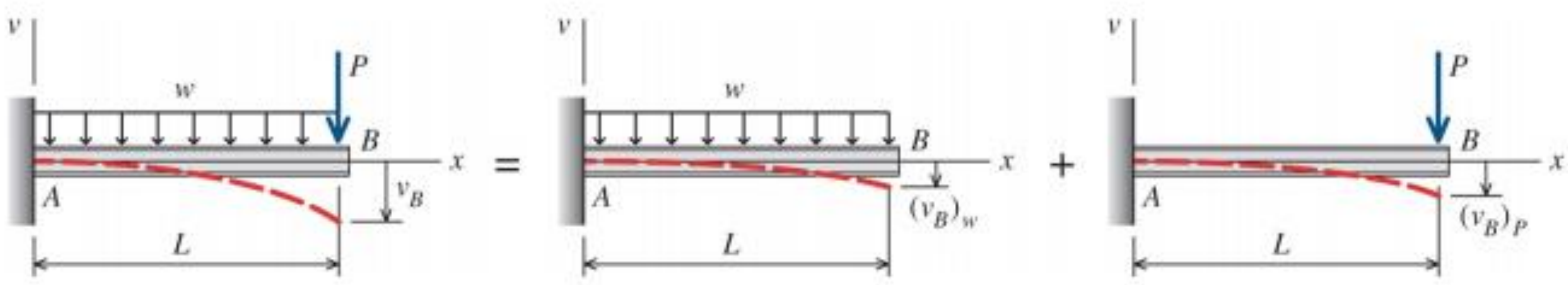
$$C_1 = \frac{M_0L}{3} - \frac{M_0L}{2} = -\frac{M_0L}{6}$$

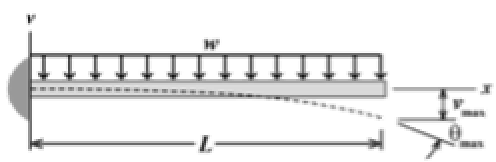
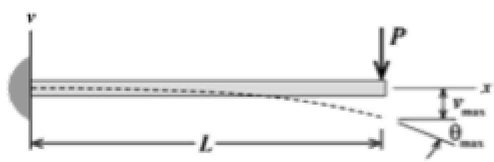
$$EI\frac{dv}{dx} = M_0x - \frac{M_0}{L}x^2 - \frac{M_0L}{6}$$

$$EIv = \frac{M_0x^2}{2} - \frac{M_0x^3}{3L} - \frac{M_0L}{6}x$$

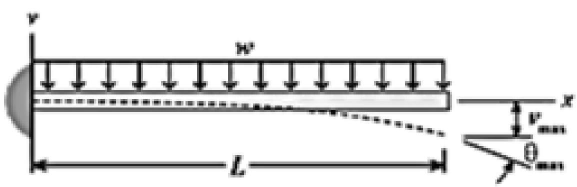
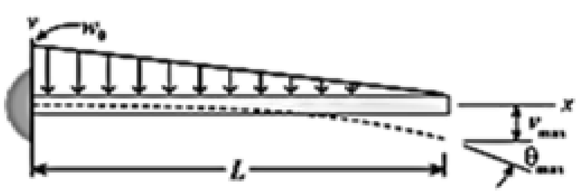
12.5 Method of Superposition (using tables

Find Deflection and slope at point B using tables for deflection of beams

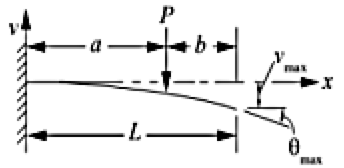
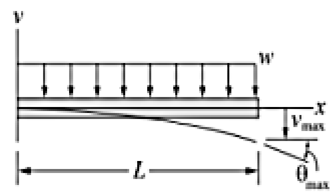
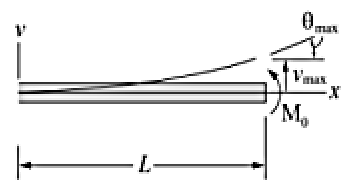
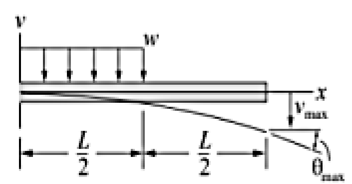
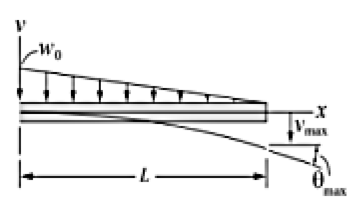


	$\theta_{max} = -\dfrac{wL^3}{6EI}$	$v_{max} = -\dfrac{wL^4}{8EI}$
	$\theta_{max} = -\dfrac{PL^2}{2EI}$	$v_{max} = -\dfrac{PL^3}{3EI}$

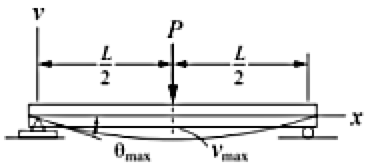
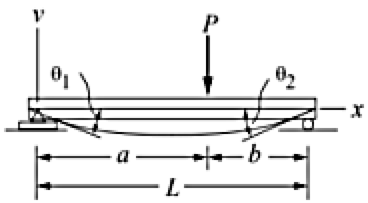
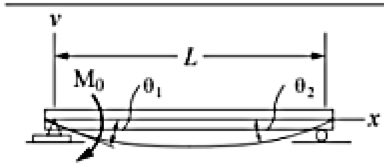
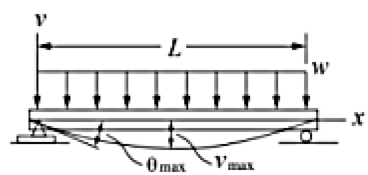
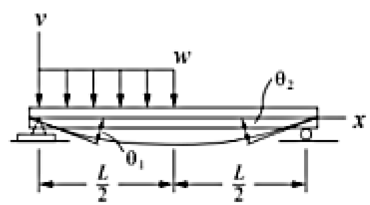
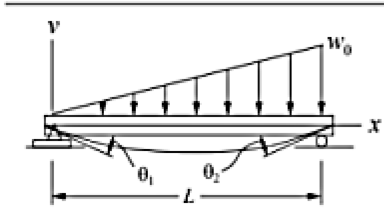
CANTILEVER BEAMS

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = -\dfrac{PL^2}{2EI}$	$v_{max} = -\dfrac{PL^3}{3EI}$	$v = -\dfrac{Px^2}{6EI}(3L - x)$
	$\theta_{max} = -\dfrac{ML}{EI}$	$v_{max} = -\dfrac{ML^2}{2EI}$	$v = -\dfrac{Mx^2}{2EI}$
	$\theta_{max} = -\dfrac{wL^3}{6EI}$	$v_{max} = -\dfrac{wL^4}{8EI}$	$v = -\dfrac{wx^2}{24EI}(6L^2 - 4Lx + x^2)$
	$\theta_{max} = -\dfrac{w_0 L^3}{24EI}$	$v_{max} = -\dfrac{w_0 L^4}{30EI}$	$v = -\dfrac{w_0 x^2}{120LEI}(10L^3 - 10L^2 x + 5Lx^2 - x^3)$

Cantilevered Beam Slopes and Deflections

BEAM	SLOPE	DEFLECTION	ELASTIC CURVE	MAXIMUM MOMENT
	$\theta_{max} = \frac{-Pa^2}{2EI}$	$v_{max} = \frac{-Pa^2}{6EI}(3L - a)$	$v = \frac{-Pa^2}{6EI}(3x - a)$, for $x > a$ $v = \frac{-Px^2}{6EI}(-x + 3a)$, for $x \leq a$	$M_{max}\,(\text{at } x = 0) = Pa$
	$\theta_{max} = \frac{-wL^3}{6EI}$	$v_{max} = \frac{-wL^4}{8EI}$	$v = \frac{-wx^2}{24EI}(x^2 - 4Lx + 6L^2)$	$M_{max}\,(\text{at } x = 0) = \frac{wL^2}{2}$
	$\theta_{max} = \frac{M_0 L}{EI}$	$v_{max} = \frac{M_0 L^2}{2EI}$	$v = \frac{M_0 x^2}{2EI}$	$M_{max}\,(\text{at all } x) = M_0$
	$\theta_{max} = \frac{-wL^3}{48EI}$	$v_{max} = \frac{-7wL^4}{384EI}$	$v = \frac{-wx^2}{24EI}\left(x^2 - 2Lx + \frac{3}{2}L^2\right)$ $0 \leq x \leq L/2$ $v = \frac{-wL^3}{192EI}(4x - L/2)$ $L/2 \leq x \leq L$	$M_{max}\,(\text{at } x = 0) = \frac{wL^2}{8}$
	$\theta_{max} = \frac{-w_0 L^3}{24EI}$	$v_{max} = \frac{-w_0 L^4}{30EI}$	$v = \frac{-w_0 x^2}{120EIL}(10L^3 - 10L^2x + 5Lx^2 - x^3)$	$M_{max}\,(\text{at } x = 0) = \frac{w_0 L^2}{6}$

Simply Supported Beam Slopes and Deflections

BEAM	SLOPE	DEFLECTION	ELASTIC CURVE	MAXIMUM MOMENT
	$\theta_{max} = \frac{-PL^2}{16EI}$	$v_{max} = \frac{-PL^3}{48EI}$	$v = \frac{-Px}{48EI}(3L^2 - 4x^2)$ $0 \le x \le L/2$	M_{max} (at center) $= \frac{PL}{4}$
	$\theta_1 = \frac{-Pab(L+b)}{6EIL}$ $\theta_2 = \frac{Pab(L+a)}{6EIL}$	$v\big\|_{x=a} = \frac{-Pba}{6EIL}(L^2 - b^2 - a^2)$	$v = \frac{-Pbx}{6EIL}(L^2 - b^2 - x^2)$ $0 \le x \le a$	M_{max} (at point of load) $= \frac{Pab}{L}$
	$\theta_1 = \frac{-M_0 L}{3EI}$ $\theta_2 = \frac{M_0 L}{6EI}$	$v_{max} = \frac{-M_0 L^2}{\sqrt{243}EI}$	$v = \frac{-M_0 x}{6EIL}(x^2 - 3Lx + 2L^2)$	M_{max} (at $x = 0$) $= M_0$
	$\theta_{max} = \frac{-wL^3}{24EI}$	$v_{max} = \frac{-5wL^4}{384EI}$	$v = \frac{-wx}{24EI}(x^3 - 2Lx^2 + L^3)$	M_{max} (at center) $= \frac{wL^2}{8}$
	$\theta_1 = \frac{-3wL^3}{128EI}$ $\theta_2 = \frac{7wL^3}{384EI}$	$v\big\|_{x=L/2} = \frac{-5wL^4}{768EI}$ $v_{max} = -0.006563\frac{wL^4}{EI}$ at $x = 0.4598L$	$v = \frac{-wx}{384EI}(16x^3 - 24Lx^2 + 9L^3)$ $0 \le x \le L/2$ $v = \frac{-wL}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3)$ $L/2 \le x < L$	$M_{max}\left(\text{at } x = \frac{3}{8}l\right) = \frac{9}{128}wL^2$
	$\theta_1 = \frac{-7w_0L^3}{360EI}$ $\theta_2 = \frac{w_0L^3}{45EI}$	$v_{max} = -0.00652\frac{w_0L^4}{EI}$ at $x = 0.5193L$	$v = \frac{-w_0 x}{360EIL}(3x^4 - 10L^2x^2 + 7L^4)$	$M_{max}\left(\text{at } x = \frac{L}{\sqrt{3}}\right) = \frac{w_0L^2}{9\sqrt{3}}$

In the following problems assume that the flexural rigidity *EI* of each beam is constant.

9.1 through 9.4 For the loading shown, determine (*a*) the equation of the elastic curve for the cantilever beam *AB*, (*b*) the deflection at the free end, (*c*) the slope at the free end.

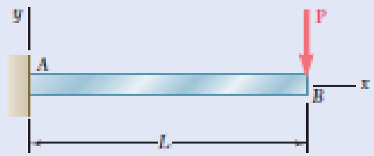


Fig. P9.1

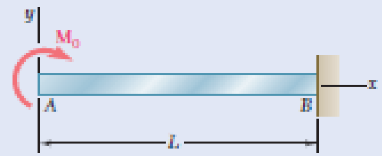


Fig. P9.2

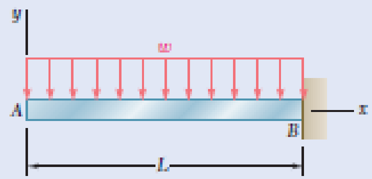


Fig. P9.3

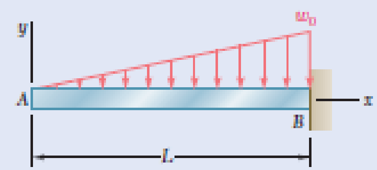


Fig. P9.4

9.5 and 9.6 For the cantilever beam and loading shown, determine (*a*) the equation of the elastic curve for portion *AB* of the beam, (*b*) the deflection at *B*, (*c*) the slope at *B*.

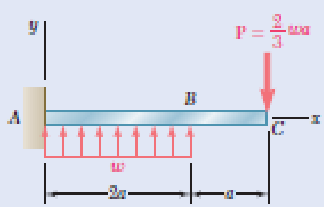


Fig. P9.5

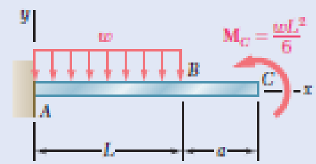


Fig. *P9.6*

9.7 For the beam and loading shown, determine (*a*) the equation of the elastic curve for portion *AB* of the beam, (*b*) the deflection at midspan, (*c*) the slope at *B*.

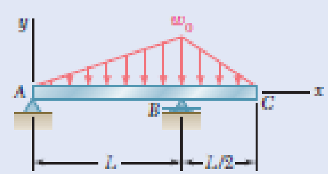
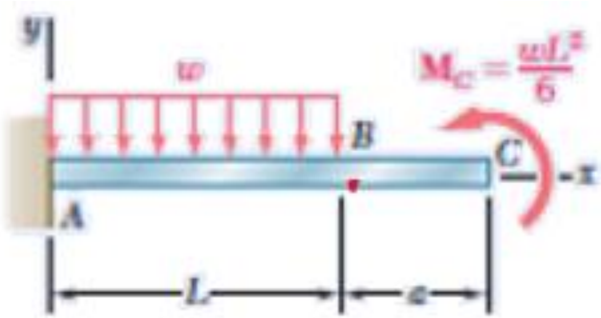
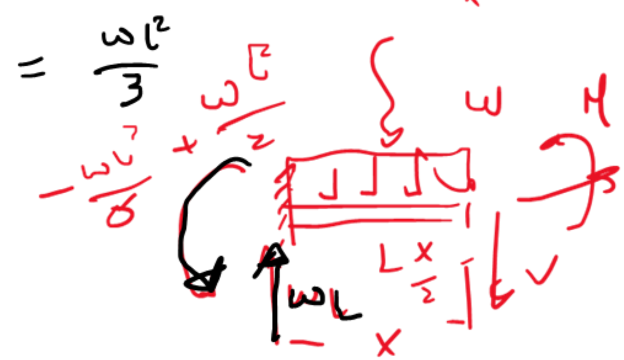


Fig. P9.7

- **Determine the equation of the elastic curve for portion AB of the beam**
- **Determine the deflection at B and the slope at B**



$0 < x < L$



$$(\circlearrowleft +)\ \Sigma M = 0 \Rightarrow \frac{wL^2}{3} - wLx + \frac{wx^2}{2} + M = 0 \Longrightarrow$$

$$M = -\frac{wx^2}{2} + wLx - \frac{wL^2}{3}$$

$$EI\frac{d^2v}{dx^2} = -\frac{wx^2}{2} + wLx - \frac{wL^2}{3}$$

$$EI\frac{dv}{dx} = -\frac{wx^3}{6} + \frac{wLx^2}{2} - \frac{wL^2x}{3} + C_1$$

$$EI\,v = \frac{-wx^4}{24} + \frac{wLx^3}{6} - \frac{wL^2x^2}{6} + C_1x + C_2$$

$x = 0 \Rightarrow$

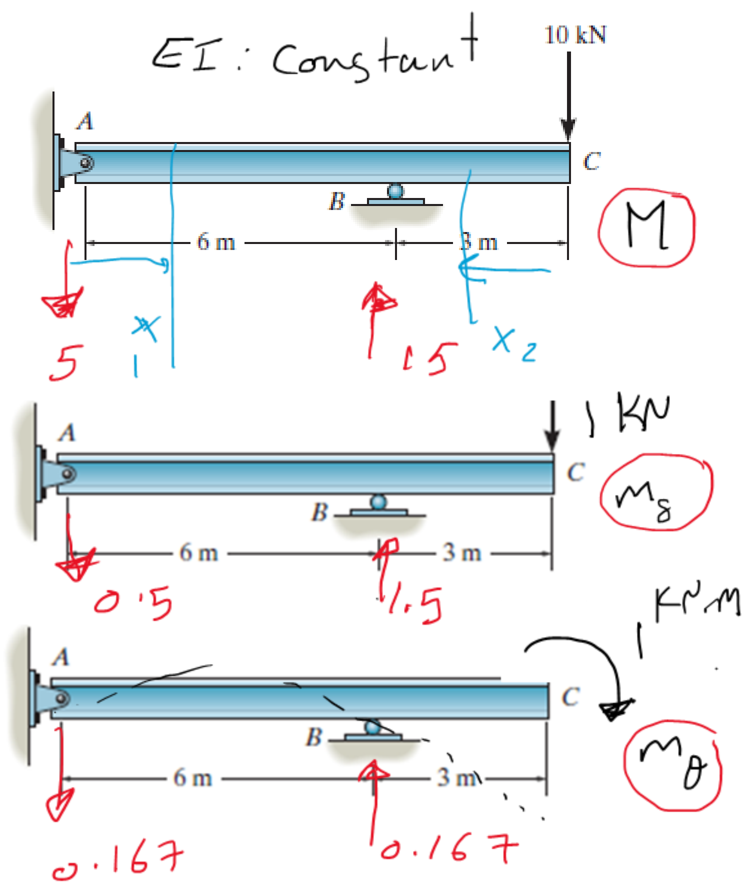
$v = 0,\ \theta = 0$

$\Rightarrow C_2 = 0$

$C_1 = 0$

$$EI\,v_{x=L} = \frac{-wL^4}{24} + \frac{wL}{6} - \frac{wL^4}{6} = \frac{wL^4}{24}$$

Determine the slope and deflection at C

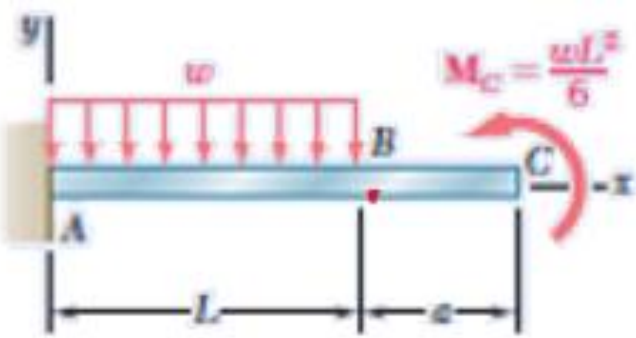


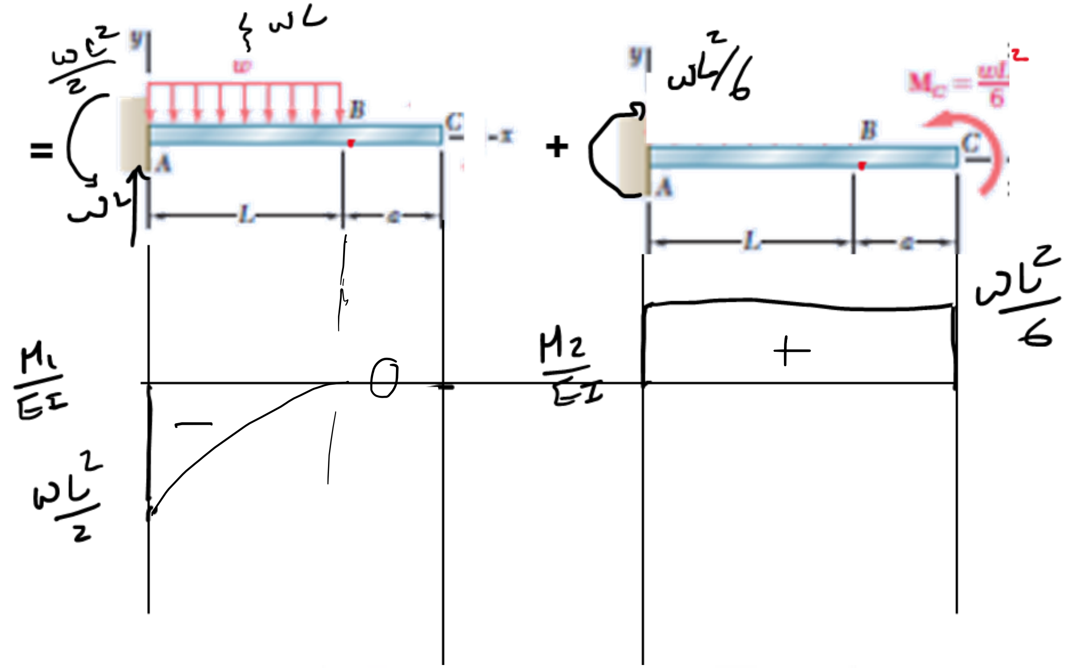
	x	M	m	Mm
δ	$0<x_1<6$	$-5x_1$	$-0.5x_1$	$2.5x_1^2$
	$0<x_2<3$	$-10x_2$	$-x_2$	$10x_2^2$
θ	$0<x_1<6$	$-5x_1$	$-0.167x_1$	$+0.835x_1^2$
	$0<x_2<3$	$-10x_2$	-1	$+10x_2$

$$EI\delta_c = \int Mm = \int_0^6 2.5x_1^2 + \int_0^3 10x_2^2 = 270 \Rightarrow \delta_c = \frac{270\ \text{kN m}^3}{EI}$$

$$EI\theta_c = \int Mm = \int_0^6 0.835x_1^2 + \int_0^3 10x_2 = 105.2 \Rightarrow \theta_c = \frac{105.2\ \text{kN m}^2}{EI}\ \underline{\text{CW}}$$

Find the Deflection at B.

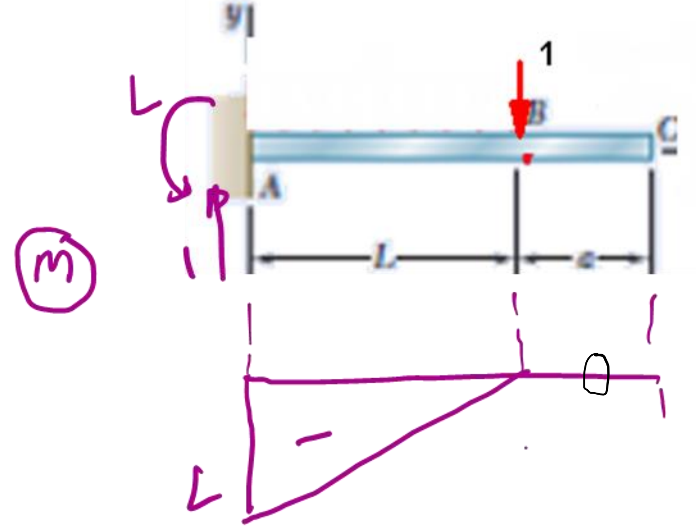




$EI\delta =$

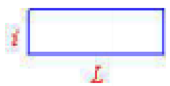
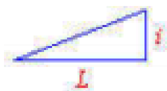
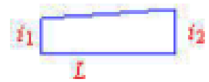
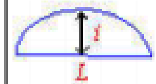
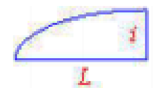
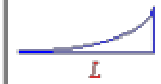
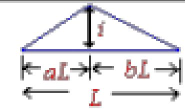
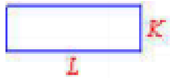
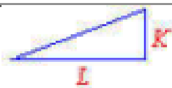
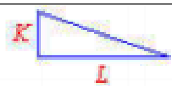
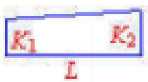
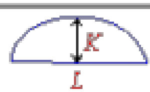
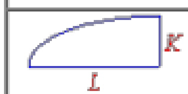
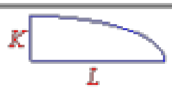
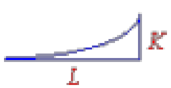
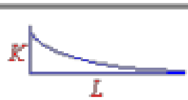
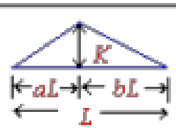
$$-\left(\tfrac{1}{2}\right)(L)\left(\frac{wL^2}{6}\right)L + \frac{1}{4}L \times \frac{wL^2}{2}L =$$

$$\frac{-2wL^4}{2\times 12} + \frac{3wL^4}{8\times 3} = \boxed{\frac{wL^4}{24}}$$



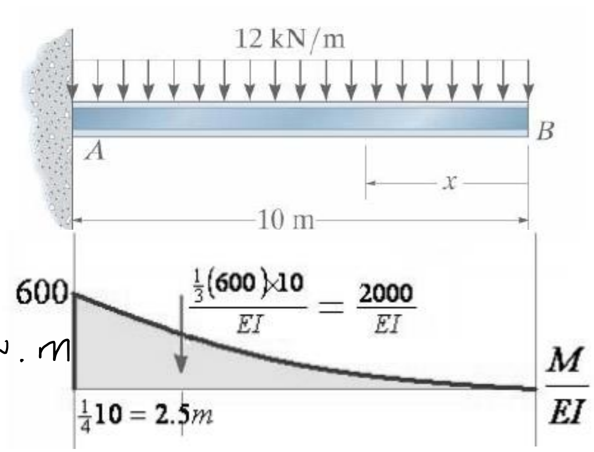
$$1.\Delta = \int_0^L m \frac{M}{EI} dx$$

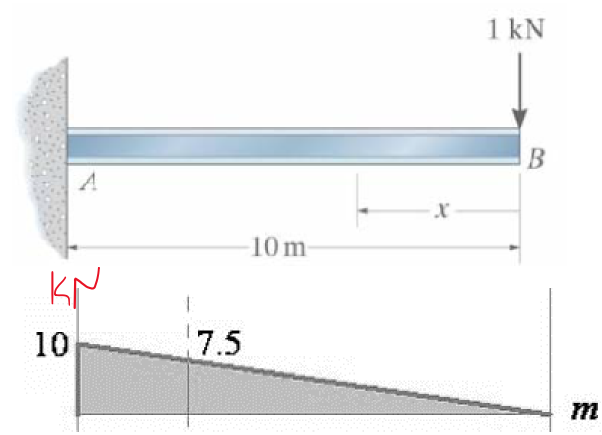
$$1_{(KN.m)}.\theta = \int_0^L \frac{m_\theta M}{EI} dx$$

M_u \ M_p							
	KiL	1/2 KiL	1/2 K (i_1+i_2) L	2/3 KiL	2/3 KiL	1/3 KiL	1/2 KiL
	1/2 KiL	1/3 KiL	1/6 K (i_1+2i_2) L	1/3 KiL	5/12 KiL	1/4 KiL	1/6(1 + a) KiL
	1/2 KiL	1/6 KiL	1/6 K $(2i_1+i_2)$ L	1/3 KiL	1/4 KiL	1/12 KiL	1/6(1 + b) KiL
	1/2 (K_1+K_2) iL	1/6 (K_1+2K_2) iL	1/6 $(2K_1i_1+K_1i_2$ $+K_2i_1+2K_2i_2)$ L	1/3 (K_1+K_2) iL	1/12 $(3K_1+3K_2)$ iL	1/12 (K_1+3K_2) iL	1/6((1 + b) K_1 + (1 + a) K_2)iL
	2/3 KiL	1/3 KiL	1/3 K (i_1+i_2) L	8/15 KiL	7/15 KiL	1/5 KiL	1/3(1 + ab) KiL
	2/3 KiL	5/12 KiL	1/12 K $(3i_1+5i_2)$ L	7/15 KiL	8/15 KiL	3/10 KiL	1/12(5 - b - b^2) KiL
	2/3 KiL	1/4 KiL	1/12 K $(5i_1+3i_2)$ L	7/15 KiL	11/30 KiL	2/15 KiL	1/12(5 - a - a^2) KiL
	1/3 KiL	1/[redacted]4 KiL	1/12 K (i_1+3i_2) L	1/5 KiL	3/10 KiL	1/5 KiL	1/12(1 + a + a^2) KiL
	1/3 KiL	1/12 KiL	1/12 K $(3i_1+i_2)$ L	1/5 KiL	2/15 KiL	1/30 KiL	1/12(1 + b + b^2) KiL
	1/2 KiL	1/6(1+ a) KiL	1/6 KL ((1 + b) i_1 + (1 + a) i_2)	1/3(1 + ab) KiL	1/12(5 - b - b^2) KiL	1/12(1 + a + a^2) KiL	1/3 KiL

Determine the displacement at point B of a steel beam

E = 200 GPa , I = 500(10⁶) mm⁴



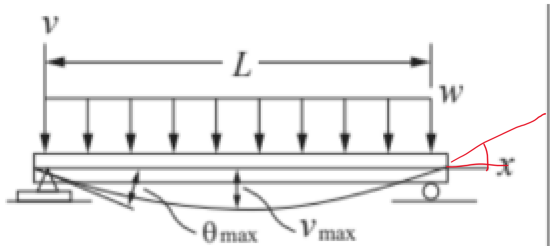


M_u \ M_p	rectangle (i, L)	triangle (i, L)	trapezoid (i_1, i_2, L)	parabola (i, L)	parabola (i, L)	curve (i, L)
rectangle (K, L)	KiL	1/2 KiL	1/2 K ($i_1 + i_2$) L	2/3 KiL	2/3 KiL	1/3 KiL
triangle (K, L)	1/2 KiL	1/3 KiL	1/6 K ($i_1 + 2i_2$) L	1/3 KiL	5/12 KiL	$\boxed{\text{1/4 KiL}}$

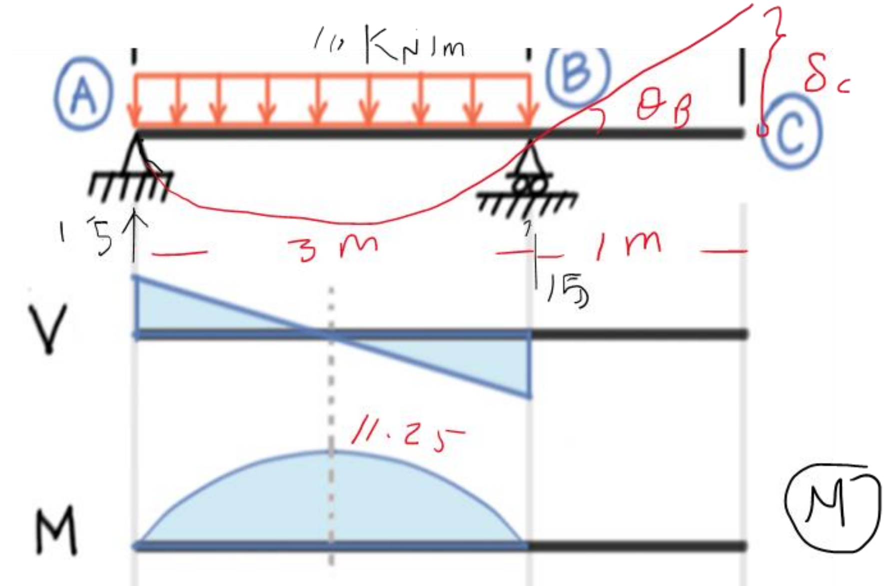
$$EI = (200\times10^{6})(500\times10^{-6}) = 100{,}000$$

$$\delta_B = \left(\frac{1}{4}\right)\frac{(600)(10)(10)}{100000} = 0.15\ m = \underline{150\ mm}$$

For simply-supported beams with unloaded overhangs, the slope and deflection of the overhang is governed by the loads on the back span. $EI = 13000\ kN\cdot m^2$, $w = 10\ kN/m$



$$\theta_{max} = \frac{-wL^3}{24EI}$$



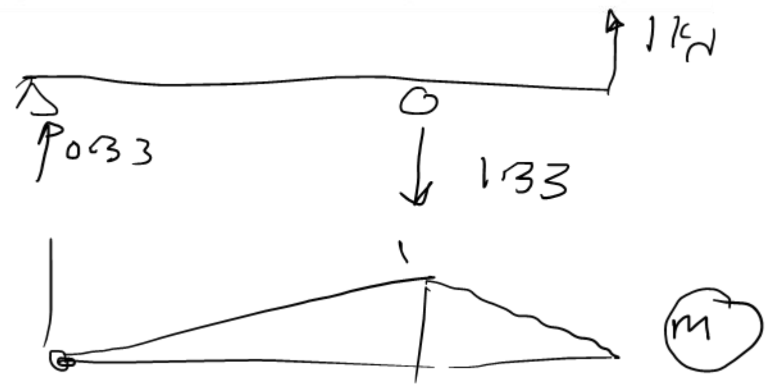
$$\theta_B = \frac{(10)(3)^3}{(24)(13000)} = 8.654\times10^{-4}\ rad$$
$$= 0.05^\circ$$

$$\tan\theta_B = 8.726\times10^{-4} = \frac{\delta_c}{1}$$

$$\delta_c = 8.726\times10^{-4}\ m = 0.87\ mm$$

$\boxed{\text{or}}$

$$\delta_c = \frac{1}{3}(1)(3)(11.25)/13000 = 8.7\times10^{-4}$$



M_u \ M_P	rectangle (l, L)	triangle (l, L)	trapezoid (l_1, l_2, L)	parabola (l, L)
rectangle (K, L)	KlL	1/2 KlL	1/2 K (l_1+l_2) L	2/3 KlL
triangle (K, L)	1/2 KlL	1/3 KlL	1/6 K (l_1+2l_2) L	1/3 KlL

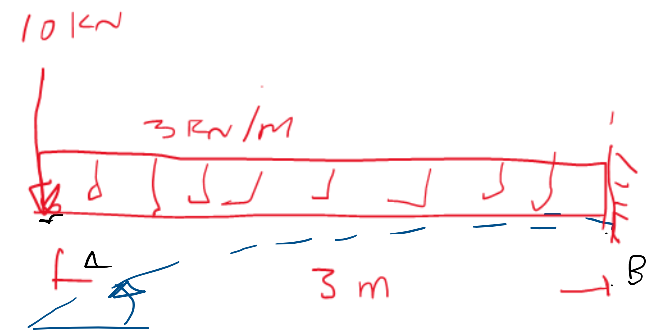
$E = 200\ GPa$

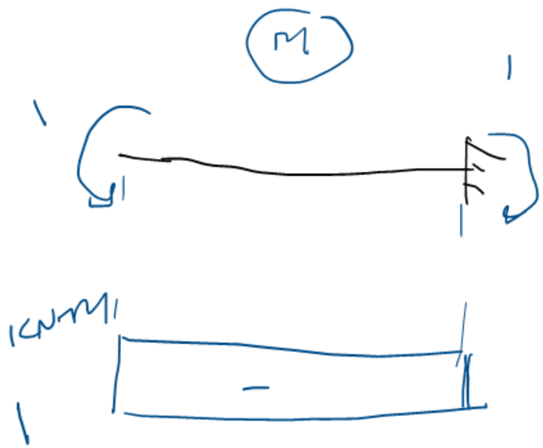
$I = 65 \times 10^{6}\ mm^{4}$

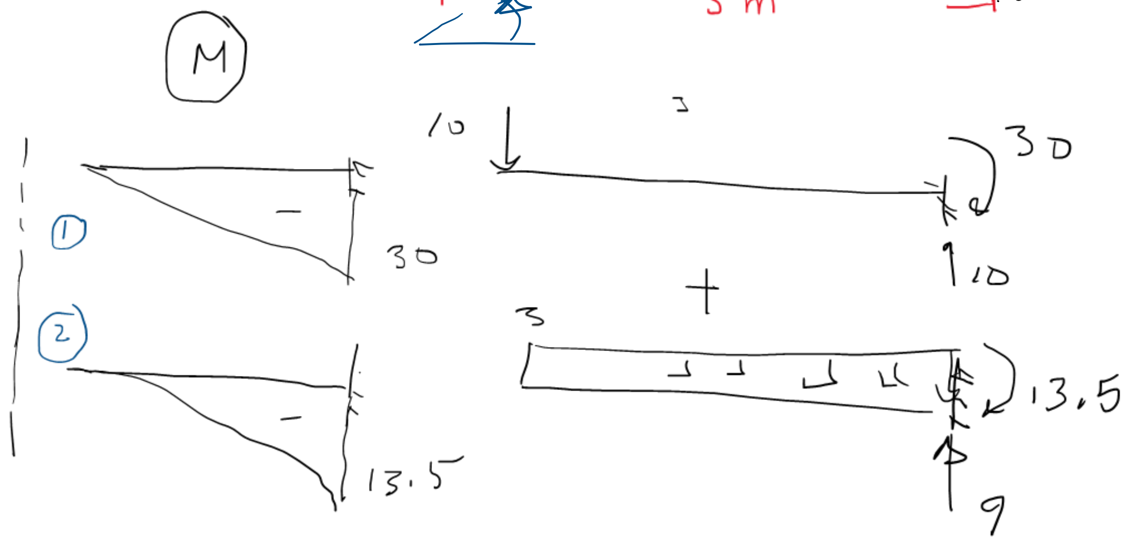
$\theta_A = ?$

$EI = 200 \times 10^{6} \times 65 \times 10^{-6}$

$= 13000$







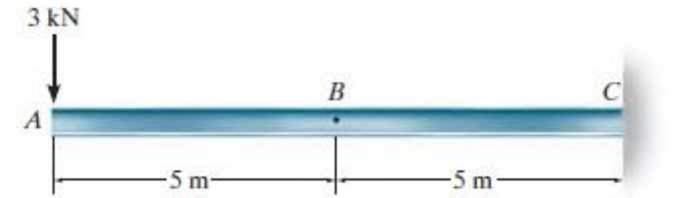
$\theta_A = \theta_1 + \theta_2 =$

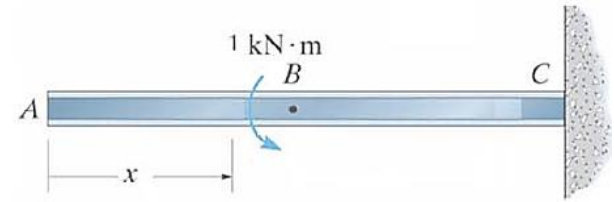
M_u \ M_P	rectangle, i, L	triangle, i, L	trapezoid, i_1, i_2, L	parabola, i, L	parabola, i, L	parabola, i, L
rectangle, K, L	KiL	1/2 KiL	1/2 K $(i_1 + i_2)$ L	2/3 KiL	2/3 KiL	1/3 KiL

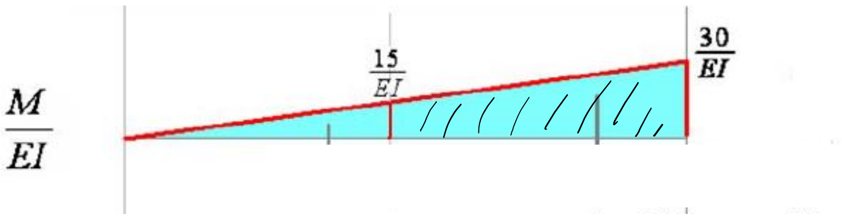
$EI\theta_A = \left(\frac{1}{2}\right)(30)3 + \frac{1}{3}(13.5)(3)$

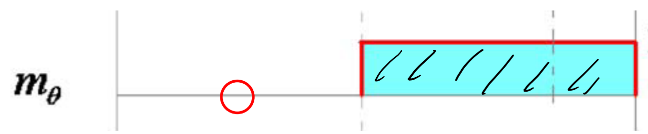
$\theta_A = 58.5/13000 = 4.5 \times 10^{-3}\ rad = 0.258^{\circ}$ c.c.w

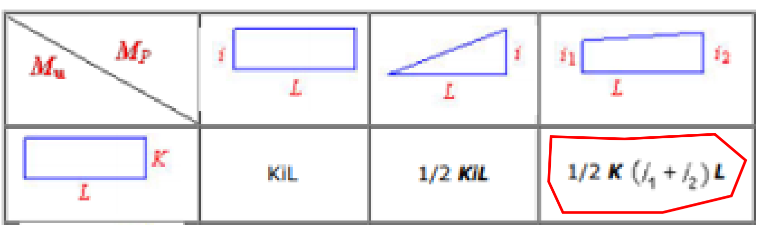
Determine the slope at point *B* of the steel beam shown .
E = 200 GPa, I = $60(10^6)mm^4$.





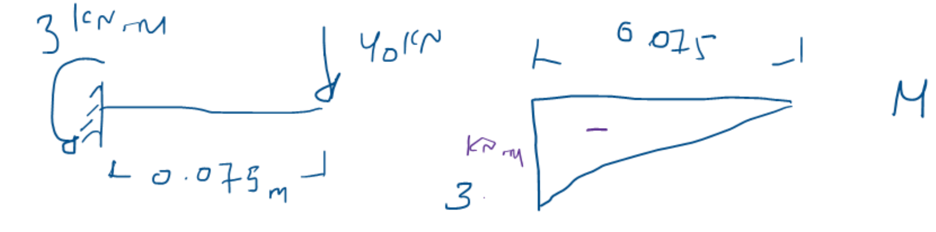


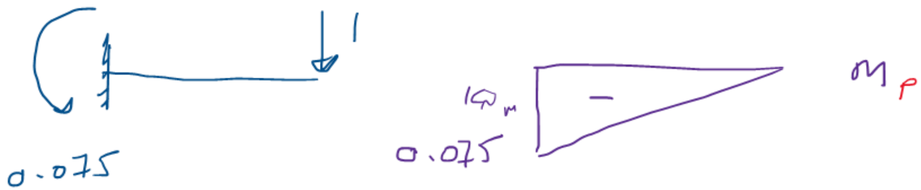


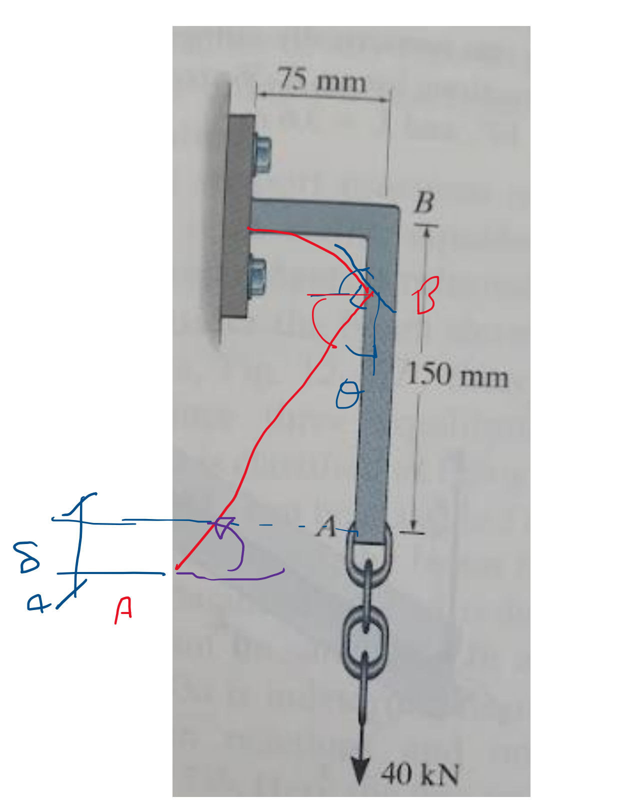


$$\theta_B = (0.5x\text{-}1(-30-15)x5)/(\ 200(10^6\ )\ x60\ x10^{-6}) = 0.00938 \text{ rad}$$

Determine the vertical deflection and slope at the end A of the bracket. Assume that the brackets fixed support at its base and neglect the axial deformation of segment AB . EI is constant

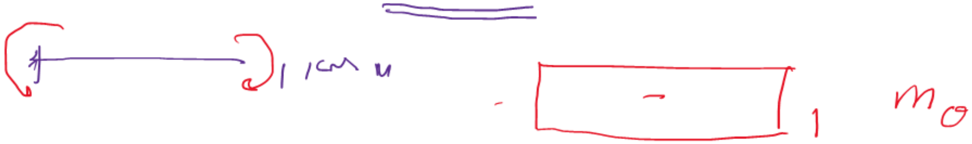




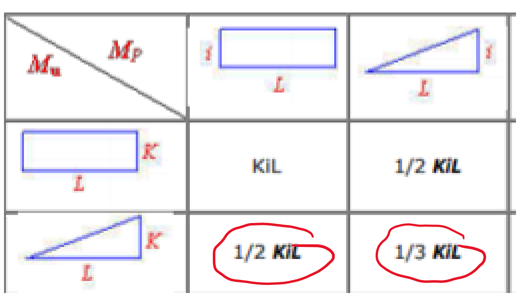


$$\delta_B = \delta_A \Rightarrow \delta_A = \frac{1}{EI}\left(\frac{1}{3}\right)(3\times 0.075\times 0.075)$$

$$= \frac{-5.625\times 10^{-3}}{EI}$$

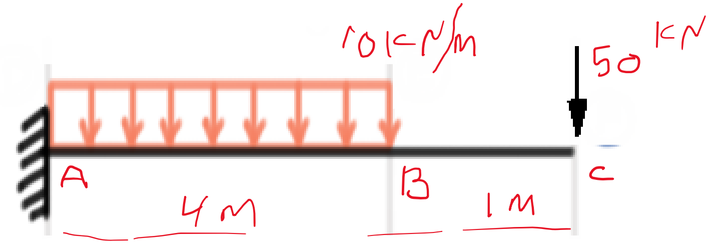


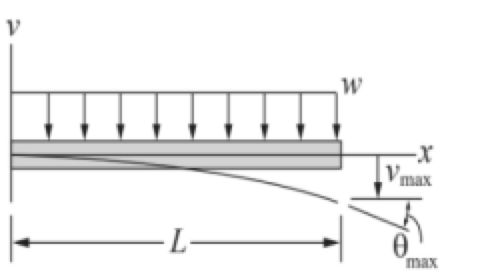
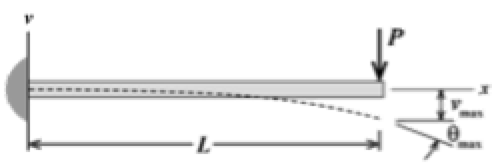
$$\theta_B = \frac{1}{EI}\left(\frac{1}{2}\right)(3)(1)(0.075) = \frac{0.1125}{EI}$$

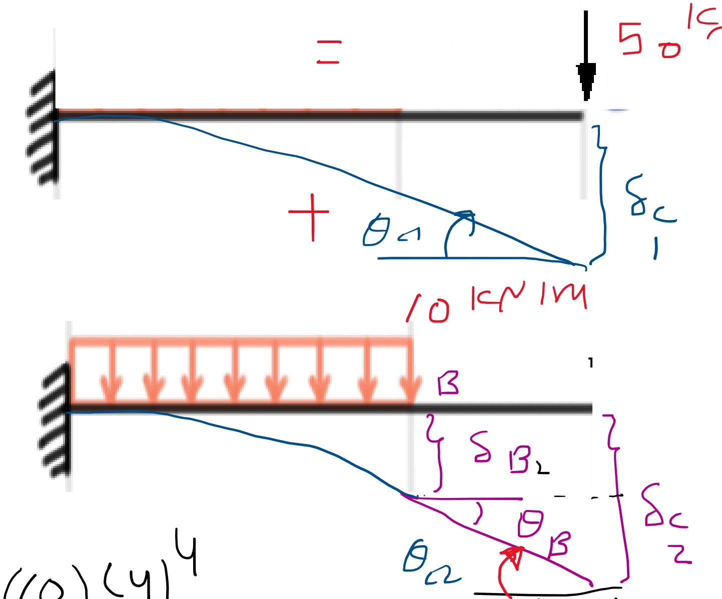


Determine the deflection at C and the slope at C.

$E = 200\ GPa$, $I = 500 \times 10^{-6}\ m^4$



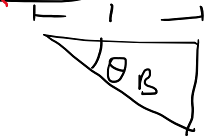
	$\theta_{max} = \dfrac{-wL^3}{6EI}$	$v_{max} = \dfrac{-wL^4}{8EI}$
	$\theta_{max} = -\dfrac{PL^2}{2EI}$	$v_{max} = -\dfrac{PL^3}{3EI}$



$$\delta_C = \delta_{C_1} + \delta_{C_2} = -\frac{50 \times 5^3}{(3)(200 \times 10^6)(500 \times 10^{-6})} - \frac{(10)(4)^4}{(8)(200 \times 10^6 \times 500 \times 10^{-6})}$$

$$-\operatorname{Tan}\left(\frac{(10)(4)^3}{(6) \times 200 \times 10^6 \times 500 \times 10^{-6}}\right) = -0.021 - 3.2 \times 10^{-3} - 1.06 \times 10^{-3}$$

$$= -0.024\ m = \underline{24.3\ mm}$$



$\operatorname{Tan} \theta_B = \frac{y}{1}$

$y = \operatorname{Tan} \theta_B$

$$\theta_C = \theta_{c_1} + \theta_{c_2} = -\frac{10 \times 4^3}{(6)(200 \times 10^6 \times 500 \times 10^{-6})} - \frac{(50)(5)^2}{2 \times 200 \times 10^6 \times 500 \times 10^{-6}}$$

$$-1.07 \times 10^{-3} - 6.25 \times 10^{-3} = 7.32 \times 10^{-3}\ rad = \underline{0.42^\circ\ c.w}$$

Structural mechanics

Chapter 13

Buckling Columns

I. Marie

Columns:

A column is a long vertical slender bar or vertical member, subjected to an axial compressive load and fixed rigidly at both ends.

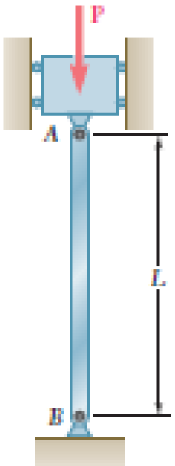
A strut is a slender bar or a member in any position other than vertical, subjected to a compressive load and fixed rigidly or hinged or pin jointed at one or both the ends.

types of column failure

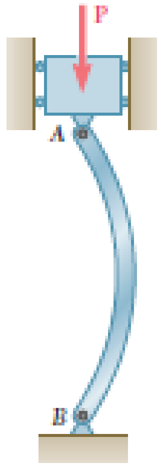
Crushing failure

Buckling failure

Buckling is a failure is due to lateral deflection of the column. The load at which the column just buckles is called buckling load or crippling load or critical load. This type of failure occurs in long column.



Pinned-ended axially loaded column.



Buckled pin-ended column.

Design of columns

Select cross section area to satisfy :

1. **Allowable stresses**

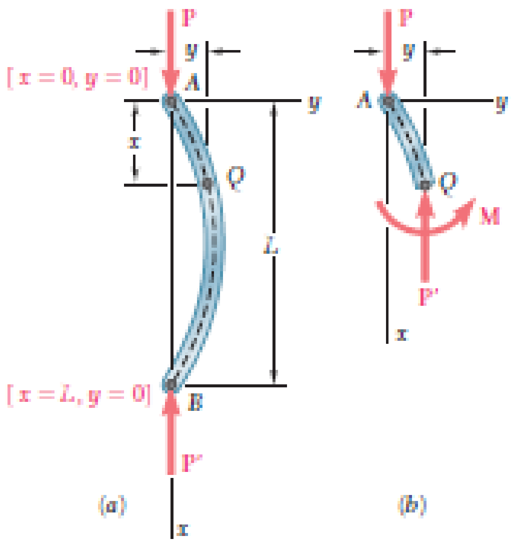
$$\sigma = \frac{p}{A} \le \sigma_{all}$$

2. **Deformation**

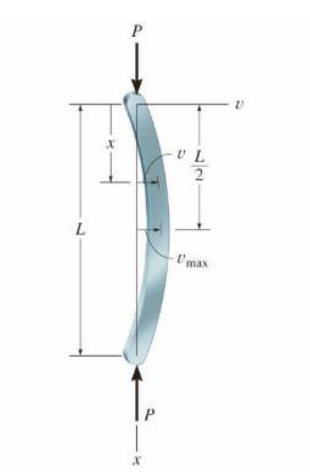
$$\delta = \frac{PL}{EA} \le \delta_{spec}$$

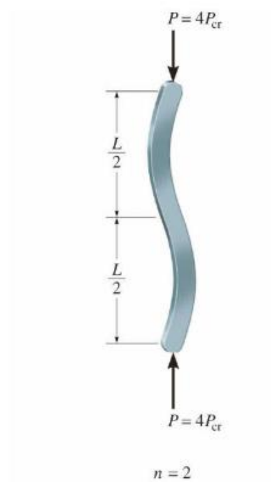
However, it may be unstable under loading and buckles

13.2 Euler's Formula for Pin-Ended Columns



Free-body diagrams of (a) buckled column and (b) portion AQ.





$$\frac{d^2 y}{dx^2} = \frac{M}{EI} = -\frac{P}{EI} y$$

$$\frac{d^2 y}{dx^2} + \frac{P}{EI} y = 0$$

Homogeneous second order , linear, deferential equation with constant coefficients.

Solution:

$$v = C_1 \sin\left(\sqrt{\frac{P}{EI}} x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}} x\right)$$

C_1 and C_2 = constants of integration. υ=0 at x=0 so C_2=0.

Which leaves:

$$C_1 \sin\left(\sqrt{\frac{P}{EI}} x\right)$$

$$\sin\left(\sqrt{\frac{P}{EI}} x\right) = 0$$

Which is satisfied if:

$$\left(\sqrt{\frac{P}{EI}} L\right) = n\pi$$

$$P = \frac{n^2 \pi^2 EI}{L^2} \qquad n = 1, 2, 3, \ldots\ldots$$

Smallest value of P??

Solution with assumed configuration can only be obtained if

$$P > P_{cr} = \frac{\pi^2 EI}{L^2}$$

It is important to realize that a column will buckle about the principal axis of the cross section having the least moment of inertia (the weakest axis)

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

where

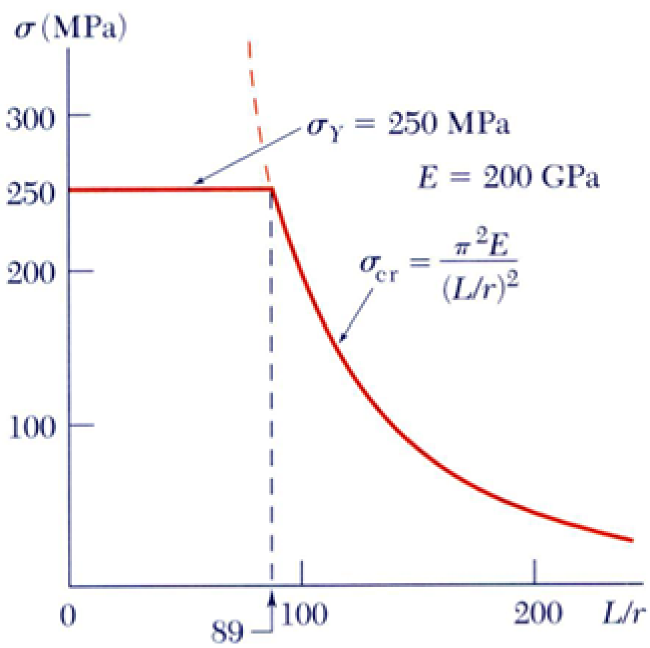
P_{cr} = critical or maximum axial load on the column just before it begins to buckle. This load must *not* cause the stress in the column to exceed the proportional limit

E = modulus of elasticity for the material

I = *least* moment of inertia for the column's cross-sectional area

L = unsupported length of the column, whose ends are pinned

The stress corresponding to the critical load is the *critical stress* $\boldsymbol{\sigma}_{cr}$.
setting $I=Ar^2$, where A is the cross-sectional, area and r its radius of gyration



- The value of stress corresponding to the critical load,

$$P > P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$\sigma = \frac{P}{A} > \sigma_{cr} = \frac{P_{cr}}{A}$$

$$\sigma_{cr} = \frac{\pi^2 E\left(Ar^2\right)}{L^2 A}$$

$$= \frac{\pi^2 E}{(L/r)^2} = \textit{critical stress}$$

$$\frac{L}{r} = \textit{slenderness ratio}$$

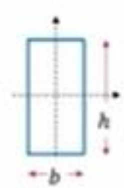
Slenderness ratio [1] $\lambda = \frac{L_e}{r}$

L_e effictive length of the column

r Least radius of gyration $= \sqrt{\frac{I_{min}}{A}}$

I_{min} Min. of I_x and I_y

A Area of cross section



λ

$\lambda < 32$	$32 < \lambda > 120$	$\lambda > 120$
Short column	Intermediate column	Long column

λ

$\lambda < 32$

Short column

$32 < \lambda > 120$

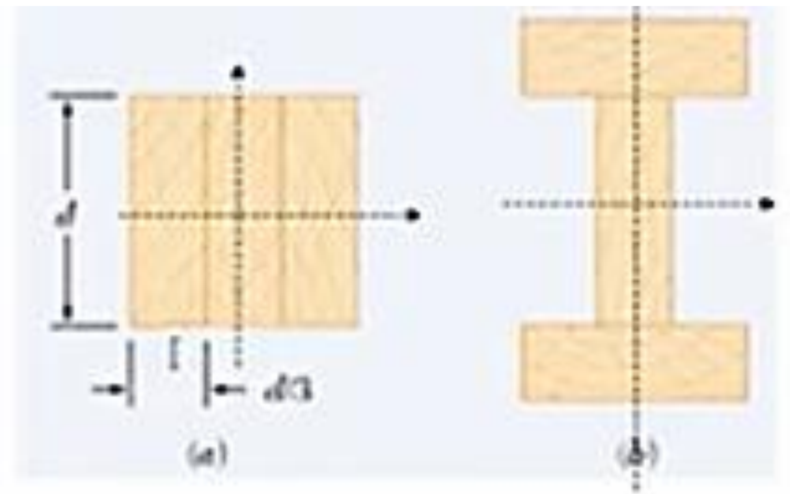
Intermediate column

$\lambda > 120$

Long column

A column of effective length L can be made by gluing together identical planks in either of the arrangements shown. Determine the ration of the critical load using the arrangement (a) to the critical load using the arrangement (b).

$$P_{cr} = \frac{\pi^2 E I}{L_e^2}$$



a) $$I = \frac{d \times d^3}{12} = \frac{d^4}{12}$$

$$P_{cr} = \frac{\pi^2 E d^4}{12 L^2} = \frac{0.0833\, \pi^2 E d^4}{L^2}$$

b) Min I is $I_y = 2\frac{d \times d^3}{12 \times 3} + \frac{d\, d^3}{12 \times 3^3} = 0.0586\, d^4$

$$P_{cr} = \frac{\pi^2 E (0.0586 d^4)}{L^2}$$

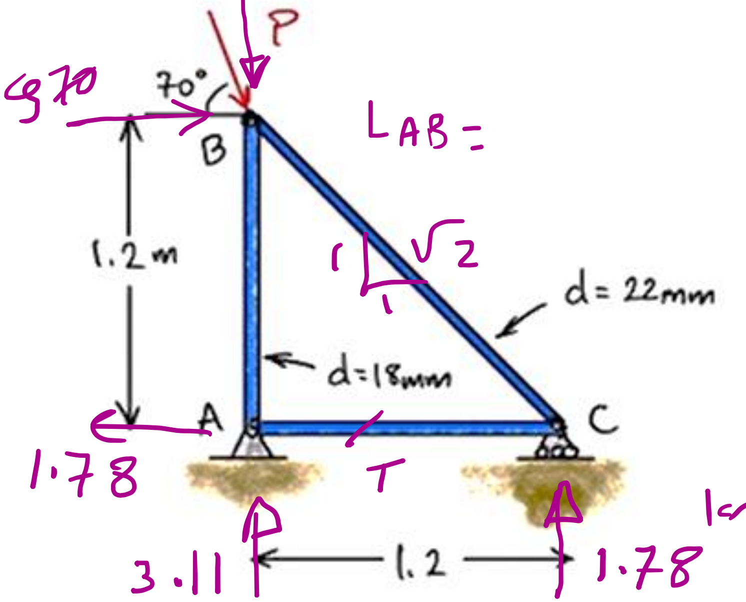
$$\frac{P_{cr(a)}}{P_{cr(b)}} = \frac{0.0833}{0.0586} = \underline{1.42}$$

P= 5.2kN, E = 200GPa
What is the factor of safety against buckling ?

$5.2 \sin 70 = 4.89 \text{ kN}$

Factor of safety $$n = \frac{p_{Cr}}{p}$$

$1.78^{\text{kN}} = 5.2 \cos 70$



We should check all members under compression and take the lowest (n)

Solve the truss for internal forces in all members and indicate if in T or C

$$P_{cr} = \frac{\pi^2 EI}{Le^2}$$

Joint A $\Rightarrow F_{AB} = 3.11^{\text{kN}}$ (C)

Joint C $\Rightarrow F_{BC} = 2.52$ kN (C)

$$P_{cr\,AB} = \frac{\pi^2 \times (200 \times 10^3)(\pi\, 9^4/4)}{(1.2 \times 1000)^2} = 7063.6\text{N}$$

$$P_{cr\,BC} = \frac{\pi^2 \times (200 \times 10^3)(\pi\, 11^4/4)}{(1.2\sqrt{2} \times 1000)^2} = 7881.3\text{N}$$

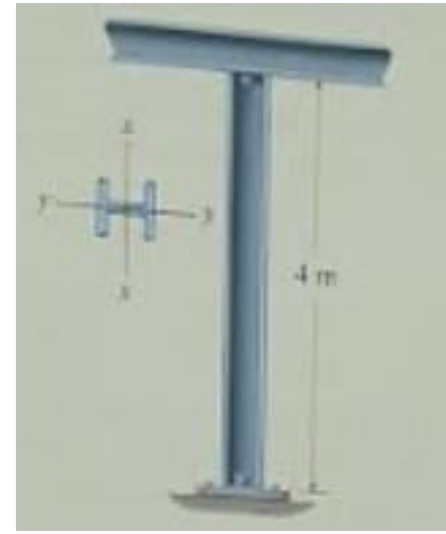
smaller

$n_{AB} = 7.064/3.11 = \boxed{2.27}$

$$n_{BC} = \frac{7.8813}{2.52} = 3.13$$

The A36- steel W200x46 member is to be used as a pin connected column., Determine the largest axial load it can support before it either begins to buckle or the steel yield

$A = 5890\ mm^2$



A36 → $E = 200\ GPa$, $\sigma_y = 250\ MPa$

W200x46 → $I_x = 45.5 \times 10^6\ mm^4$, $I_y = 15.3 \times 10^6\ mm^4$

$I_y < I_x \Rightarrow$ Buckling will occur about the y-axis

$$P_{cr} = \frac{\pi^2 E I}{L^2} = \frac{\pi^2 (200 \times 10^3)(15.3 \times 10^6)}{(4000)^2} = 1887.6\ kN$$

Compressive stress $\sigma_{cr} = \frac{P_{cr}}{A} = \frac{1887.6 \times 10^3}{5890}$

$$= 320.5\ MPa$$

$\sigma_{cr} = 320.5 > \sigma_y = 250 \Rightarrow$ use σ_y to determine P

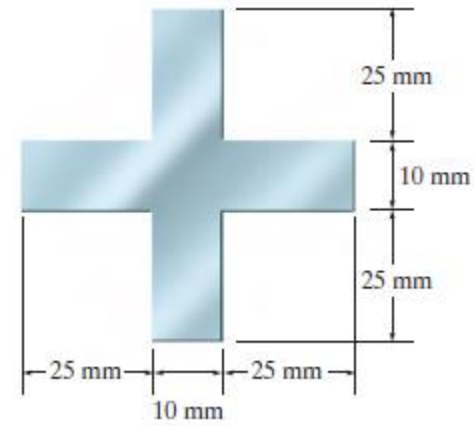
$$\sigma_y = \frac{P}{A} = 250 = \frac{P}{5890} \Rightarrow P = 1472.5\ kN$$

* A factor of safty should be applied

An A-36 steel column has a length of 4m and is pinned at both ends. Determine the critical load.

A-36 steel $\Rightarrow$ $E = 200\ \text{GPa}$

$\sigma_y = 250\ \text{MPa}$



Section Properties

$$A = (10)(60) + (50)(10) = 1100\ \text{mm}^2$$

$$I_x = I_y = \frac{(10)(60)^3}{12} + \frac{(50)(10)^3}{12} = 184167\ \text{mm}^4$$

pinned ends $\Rightarrow$ $L_e = l$

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{(\pi^2)(200\times10^3)(184167)}{(4000)^2} = 22720\ \text{N} = \underline{22.72\ \text{kN}} \leftarrow$$

$$\text{Critical stress} = \frac{P_{cr}}{A} = \frac{22720}{1100} = 20.66\ \text{MPa} < \sigma_y \Leftarrow \underline{\text{o.k}}$$

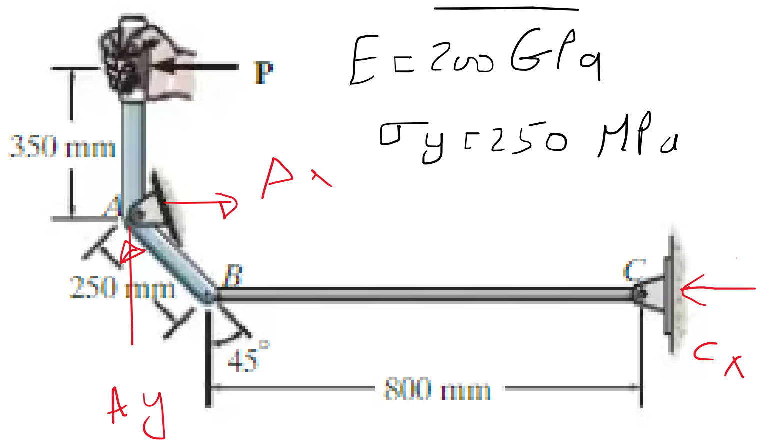
Then $\underline{P_{cr} = 22.72\ \text{kN}}$

Determine the maximum force P that can be applied to the handle so that the A-36 steel control rod BC does not buckle. The rod has a diameter of 25 mm

A-36

$E = 200\ GPa$

$\sigma_y = 250\ MPa$



Support Reaction:

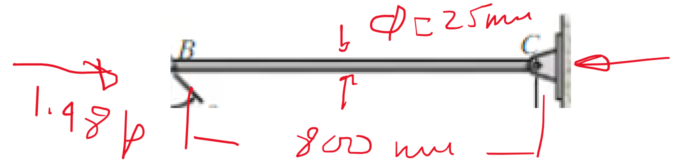
$\circlearrowleft + \sum M_A = 0 \Rightarrow 350P + 250 \sin 45\ C_x = 0$

$C_x = 1.98P$

section Properties:

$A = \pi d^2/4 = \frac{(\pi)(25)^2}{4} = 490.625\ mm^2$

$I_y = I_x = \frac{\pi r^4}{4} = \frac{\pi (12.5)^4}{4} = 19174.8\ mm^4$



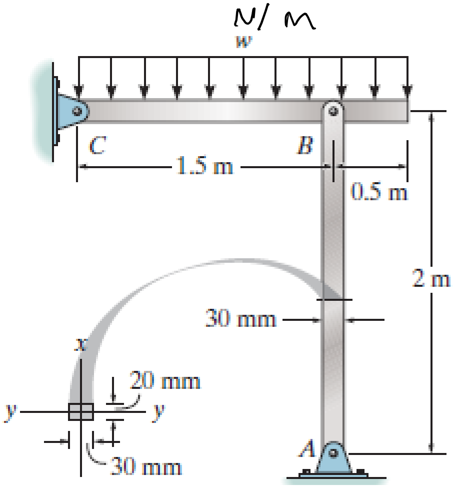
$C_x = \frac{\pi^2 E I}{L_e^2} = \frac{\pi^2 \times 200 \times 10^3 \times 19174.8}{(800)^2} = 1.98P \Rightarrow$

$P = 29.9\ kN$

check stress $= \frac{C_x}{A} = \frac{1.98 \times 29.9 \times 10^3}{490.625} = 120.7\ MPa < \sigma_y$ — o.k

$\therefore P = 29.3\ kN$

Determine the maximum allowable intensity w of the distributed load that can be applied to member BC without causing member AB to buckle. Assume that AB is made of steel and is pinned at its ends for x–x axis buckling and fixed at its ends for y–y axis buckling. use a factor of safety with respect to buckling is 3.



$\sigma_y = 360\ MPa$, $E = 200\ GPa$

$I_x = \dfrac{(20)(30)^3}{12} = 45000\ mm^4$

$I_y = \dfrac{(30)(20)^3}{12} = 20,000\ mm^4$

For y-axis

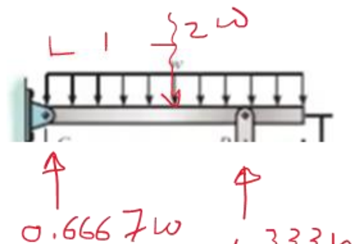
$L_e = 0.5$

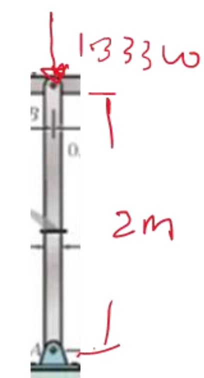
for x-axis

$P_{cr} = F.S.\,P = (3)(1.333\,w) = 4w$, (Pin-Pin $\Rightarrow L_e = L$)

$P_{cr} = \dfrac{\pi^2 E I}{L^2} \Rightarrow 4w = \dfrac{\pi^2 \times (200 \times 10^3)(45000)}{(2000)^2} \Rightarrow$

$w = 5552\ N/m = 5.55\ kN/m$





check critical stress

$\sigma_{cr} = \dfrac{P_{cr}}{A} = \dfrac{4(5552)}{20 \times 30} = 37\ MPa < \sigma_y$ ok

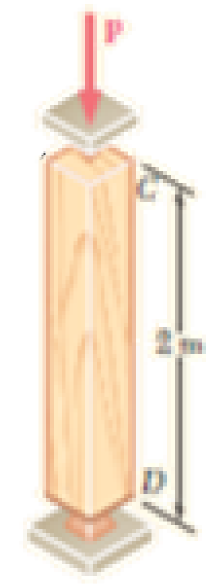
$w = 5.55\ kN/m$

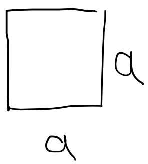
2 m long pin-ended column, square cross section, E=13 GPa, σ_{all} =12 MPa. Factor of safety 2.5 to calculate Euler's critical load for buckling. Determine the size of the cross section if the column is to support :

a: 100 kN load

b. 200 kN load

$$P > P_{cr} = \frac{\pi^2 EI}{L^2}$$





a. for 100 kN load

$P_{cr} = F.S \times P = 2.5 \times 100 = 250$ kN

using Euler's formulae.

$$I = \frac{P_{cr} L^2}{\pi^2 E} = \frac{(250)(2)^2}{\pi^2 \times 13 \times 10^6} = 7.8 \times 10^{-6}\ m^4$$

$$I = \frac{a a^3}{12} = 7.8 \times 10^{-6} \Rightarrow a = 98.3 \approx 100\ mm$$

check stress

$$\sigma = \frac{P}{A} = \frac{100}{(0.1)^2} = 10\ MPa < \sigma_{all}\ \text{--- o.k}$$

$\Rightarrow$ section 100×100 is acceptable.

b. for 200 kN load $\Rightarrow P_{cr} = F.S \times 200 = 2.5 \times 200 = 500$ kN

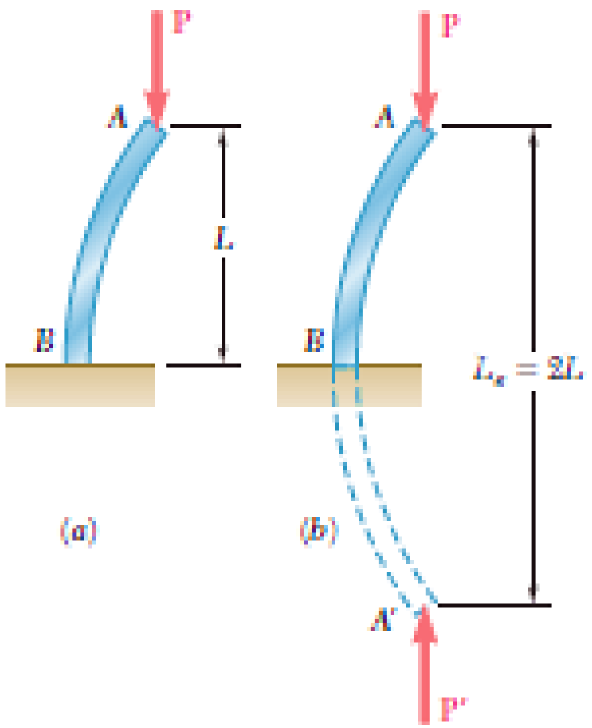
$$I = \frac{P_{cr} L^2}{\pi^2 E} = \frac{(500)(2)^2}{\pi^2 \times 13 \times 10^6} = 1.56 \times 10^{-5}\ m^4 = 15.6 \times 10^{-6}\ m^4$$

$$I = \frac{a^4}{12} \Rightarrow a = 116.95\ mm$$

$$\sigma = \frac{P}{A} = \frac{200000}{(116.95)^2} = 14.62\ MPa > \sigma_{all}$$ $\Rightarrow$ the dimension is not acceptable

$$\therefore \sigma = 12 = \frac{P}{A} = \frac{200000}{a^2} \Rightarrow a = 129.1\ mm = 130\ mm$$

13.3 Columns having various types of supports Euler's Formula



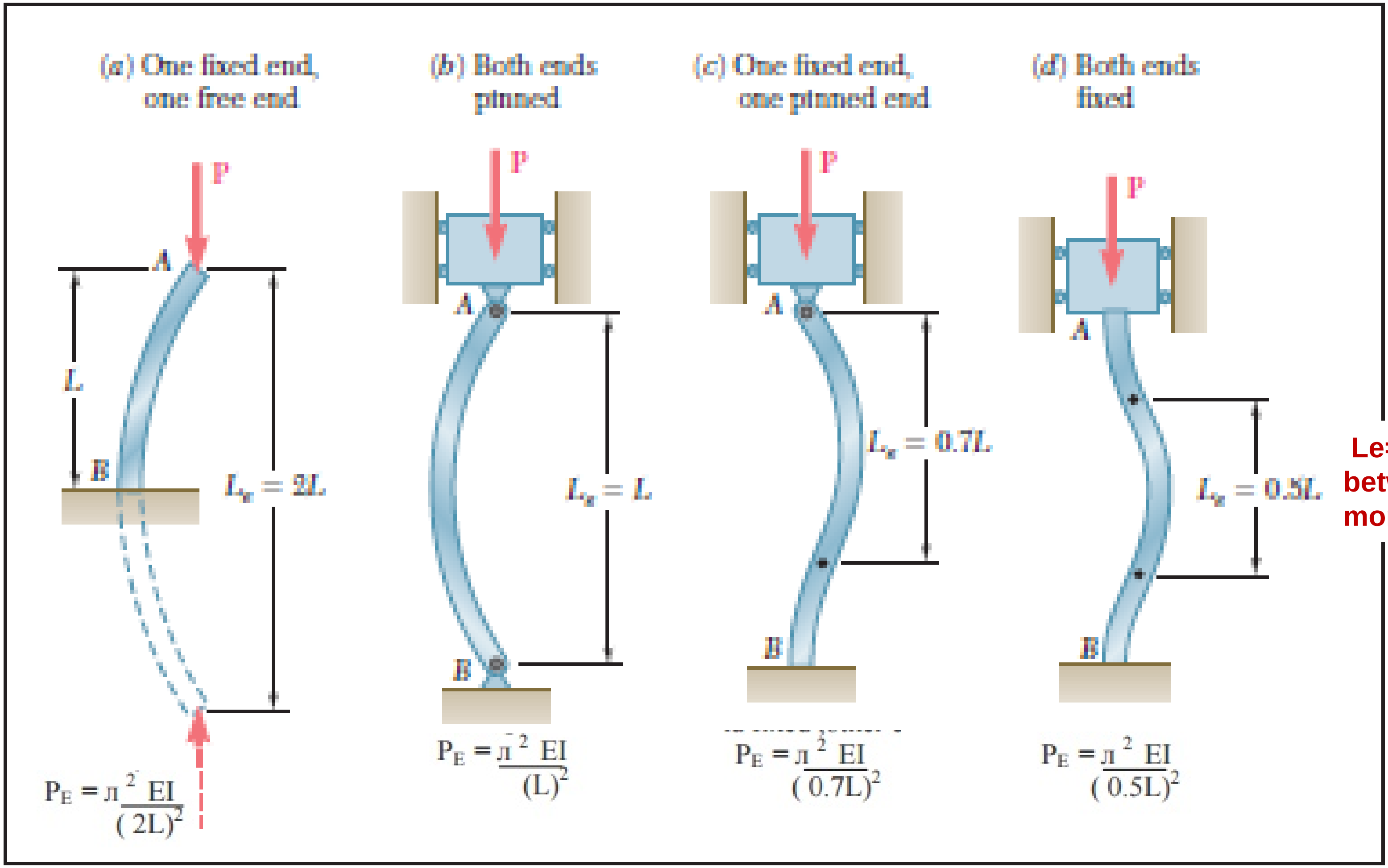
Effective length of a fixed-free column of length L is equivalent to a pin-ended column of length $2L$.

- A column with one fixed and one free end, will behave as the upper-half of a pin-connected column.
- The critical loading is calculated from Euler's formula,

$$P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2}$$

$$L_e = 2L = \text{equivalent length}$$



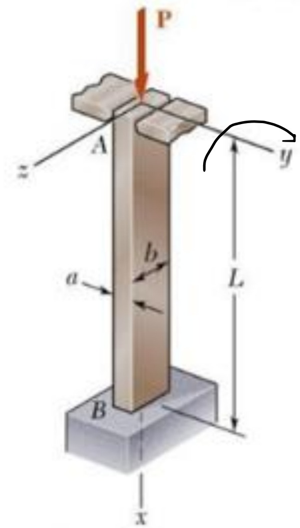
Le= Distance between zero moments

Fig. 10.18 Effective length of column for various end conditions.

End conditions: Strength of the column depends upon the end conditions also.

An aluminum column of length L and rectangular cross-section has a fixed end at B and supports a centric load at A. Two smooth and round fixed plates restrain end A from moving in one of the vertical planes of symmetry but allow it to move in the other plane.

Design the most efficient cross section for the column if a= 0.35b



$L = 0.5$ m

$E = 70$ GPa

$P = 20$ kN

$FS = 2.5$

$$\sigma_{cr} = \frac{\pi^2 E(Ar^2)}{L^2 A} = \frac{\pi^2 E}{(L/r)^2} = \text{critical stress}$$

$$\frac{L}{r} = \text{slenderness ratio}$$

$L_e = 2L$

$$\frac{L_e}{r} = \frac{2L}{r_y}$$

$$r_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{ab^3}{12ab}} = \frac{b}{\sqrt{12}}$$

$$\frac{L_e}{r_y} = \frac{2L}{b/\sqrt{12}} = \frac{(2)(0.5)}{b/\sqrt{12}} = \frac{3.464}{b}$$

$P_{cr} = F.S\, P = (20)(2.5) = 50$ kN

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{50000}{(0.35b)(b)}$$

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2} = \frac{\pi^2(70\times10^3)}{(3.464/b)^2}$$

$\Rightarrow$

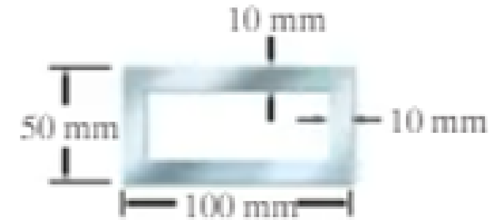
$b = 39.7$ mm

$a = 0.35b = 13.9$ mm

An A-36 steel column has length of 5m and is fixed at both ends. If the cross-sectional area has the dimensions shown, determine the critical load

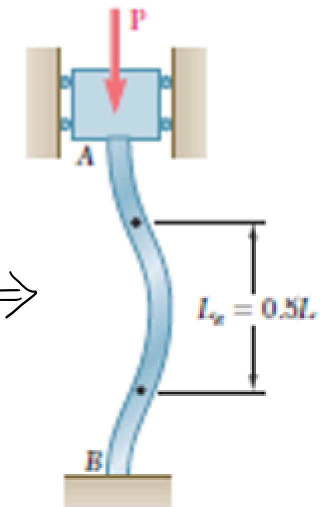
A36 ⇒ from table ⇒ $\sigma_{yield} = 250\ MPa$

$E = 200\ GPa$



$$I = \frac{1}{12}(0.1)(0.05^3) - \frac{1}{12}(0.08)(0.03^3) = 0.86167\,(10^{-6})\ \mathrm{m}^4$$

$$A = (0.1)(0.05) - (0.08)(0.03) = 2.6(10^{-3})\ \mathrm{m}^2$$



$$P_{Crt} = \frac{\Pi^2 EI}{L_e^2} = \frac{\pi^2 \times \mathbf{200} \times 10^6 \times \mathbf{0.86167(10^{-6})}}{(0\cdot 5x5)^2} = \mathbf{272.138\ kN}$$

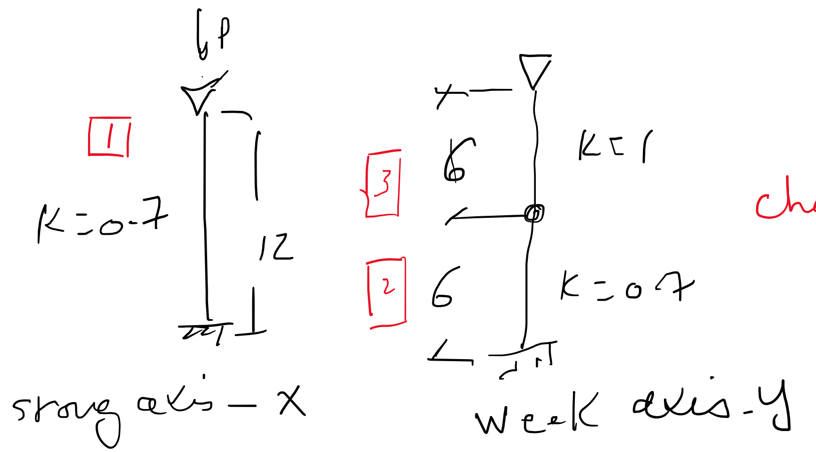
$$\sigma_{Crt} = \frac{P_{cr}}{A} = \frac{272.138 \times 10^3}{2.6 \times 10^{3}} = 104.67\ MPa < \sigma_y \Rightarrow$$

∴ Euler's formula is Valid

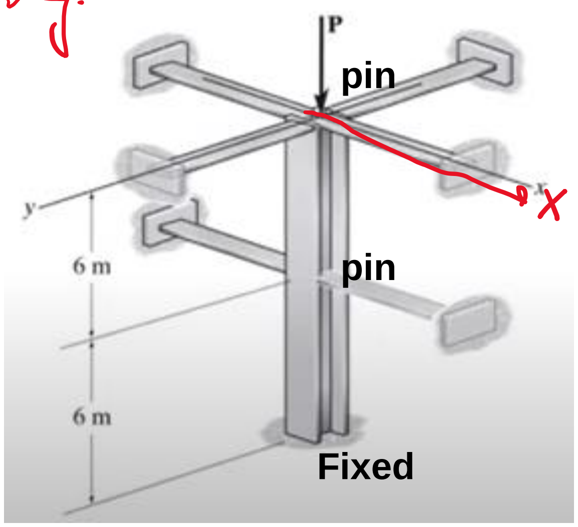
and $P_{crt} = \underline{272.138\ kN}$

The A36 steel column can be considered pinned at its top and fixed at the bottom and braced against weak axis bending at the mid-hight, Determine **the maximum allowable force P that the column can support without buckling.** Apply a F.S = 2 against buckling. Take A= 7.4×10^{-3} m^2 Ix= 87.3 $\times10^6$ mm^4 and Iy=18.8×10^6 mm^4

Take lowest P that cause Buckling.



check the 3-Cases



① strong axis (x-axis)

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} = \frac{\pi^2(200\times10^3)(87.3\times10^6)}{(0.7\times12000)^2} = 2440 \text{ kN}$$

Week axis y-axis

Part 3 ⇒ $P_{cr} = \dfrac{\pi^2(200\times10^3)(18.8\times10^6)}{(6000)^2} = \boxed{1029 \text{ kN}}$ *

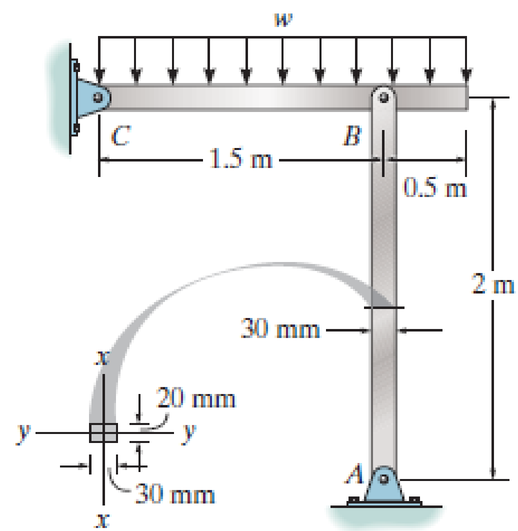
Part 2 (week)

$$P_{cr} = \frac{\pi^2(200\times10^3)(18.6\times10^6)}{(0.7\times6000)^2} = 2079 \text{ kN}$$

Take smaller

$$P = \frac{P_{cr}}{F.S} = \frac{1029}{2} = \boxed{514.5 \text{ kN}}$$

*13–28. Determine if the frame can support a load of $w = 6$ kN/m if the factor of safety with respect to buckling of member AB is 3. Assume that AB is made of steel and is pinned at its ends for x–x axis buckling and fixed at its ends for y–y axis buckling. $E_{st} = 200$ GPa, $\sigma_Y = 360$ MPa.



Check $x-x$ axis buckling:

$$I_x = \frac{1}{12}(0.02)(0.03)^3 = 45.0(10^{-9})\text{ m}^4$$

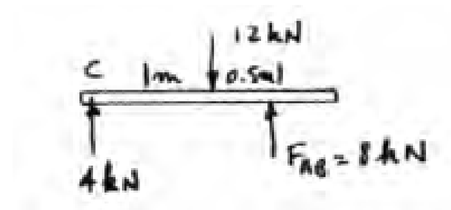
$K = 1.0 \qquad L = 2$ m

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2} = \frac{\pi^2(200)(10^9)(45.0)(10^{-9})}{((1.0)(2))^2}$$

$$P_{cr} = 22.2 \text{ kN}$$

$$\circlearrowleft + \Sigma M_C = 0; \qquad F_{AB}(1.5) - 6(2)(1) = 0$$

$$F_{AB} = 8 \text{ kN}$$

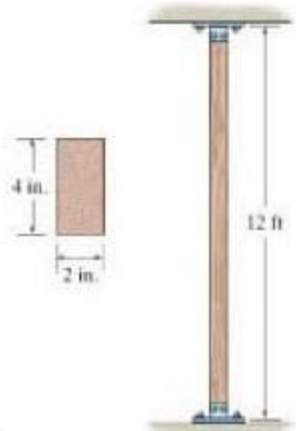


$$P_{\text{req'd}} = 8(3) = 24 \text{ kN} > 22.2 \text{ kN}$$

No, AB will fail.

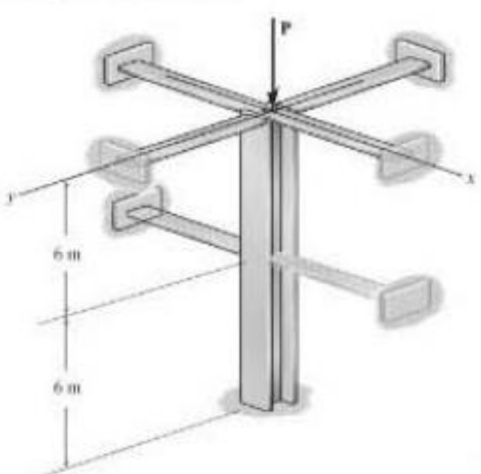
F13–1. A 50-in.-long rod is made from a 1-in.-diameter steel rod. Determine the critical buckling load if the ends are fixed supported. $E = 29(10^3)$ ksi, $\sigma_Y = 36$ ksi.

F13–2. A 12-ft wooden rectangular column has the dimensions shown. Determine the critical load if the ends are assumed to be pin-connected. $E = 1.6(10^3)$ ksi. Yielding does not occur.



F13–2

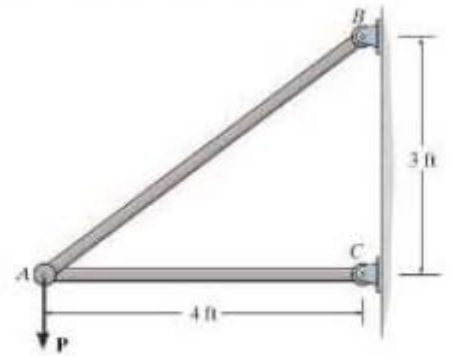
F13–3. The A-36 steel column can be considered pinned at its top and bottom and braced against its weak axis at the mid-height. Determine the maximum allowable force **P** that the column can support without buckling. Apply a F.S. = 2 against buckling. Take $A = 7.4(10^{-3})\ \text{m}^2$, $I_x = 87.3(10^{-6})\ \text{m}^4$, and $I_y = 18.8(10^{-6})\ \text{m}^4$.



F13–3

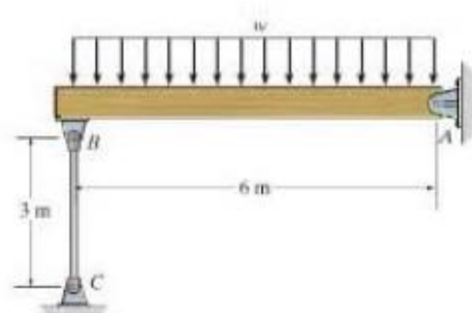
F13–4. A steel pipe is fixed supported at its ends. If it is 5 m long and has an outer diameter of 50 mm and a thickness of 10 mm, determine the maximum axial load P that it can carry without buckling. $E_{st} = 200$ GPa, $\sigma_Y = 250$ MPa.

F13–5. Determine the maximum force **P** that can be supported by the assembly without causing member AC to buckle. The member is made of A-36 steel and has a diameter of 2 in. Take F.S. = 2 against buckling.



F13–5

F13–6. The A-36 steel rod BC has a diameter of 50 mm and is used as a strut to support the beam. Determine the maximum intensity w of the uniform distributed load that can be applied to the beam without causing the strut to buckle. Take F.S. = 2 against buckling.



F13–6

Average Mechanical Properties of Typical Engineering Materials[a]
(SI Units)

Materials		Density ρ (Mg/m^3)	Modulus of Elasticity E (GPa)	Modulus of Rigidity G (GPa)	Yield Strength (MPa) σ_Y			Ultimate Strength (MPa) σ_u			% Elongation in 50 mm specimen	Poisson's Ratio ν	Coef. of Therm. Expansion α $(10^{-6})/°C$
					Tens.	Comp.[b]	Shear	Tens.	Comp.[b]	Shear			
Metallic													
Aluminum Wrought Alloys	2014-T6	2.79	73.1	27	414	414	172	469	469	290	10	0.35	23
	6061-T6	2.71	68.9	26	255	255	131	290	290	186	12	0.35	24
Cast Iron Alloys	Gray ASTM 20	7.19	67.0	27	–	–	–	179	669	–	0.6	0.28	12
	Malleable ASTM A-197	7.28	172	68	–	–	–	276	572	–	5	0.28	12
Copper Alloys	Red Brass C83400	8.74	101	37	70.0	70.0	–	241	241	–	35	0.35	18
	Bronze C86100	8.83	103	38	345	345	–	655	655	–	20	0.34	17
Magnesium Alloy	[Am 1004-T61]	1.83	44.7	18	152	152	–	276	276	152	1	0.30	26
Steel Alloys	Structural A36	7.85	200	75	250	250	–	400	400	–	30	0.32	12
	Stainless 304	7.86	193	75	207	207	–	517	517	–	40	0.27	17
	Tool L2	8.16	200	75	703	703	–	800	800		22	0.32	12
Titanium Alloy	[Ti-6Al-4V]	4.43	120	44	924	924	–	1,000	1,000	–	16	0.36	9.4
Nonmetallic													
Concrete	Low Strength	2.38	22.1	–	–	–	12	–	–	–	–	0.15	11
	High Strength	2.38	29.0	–	–	–	38	–	–	–	–	0.15	11
Plastic Reinforced	Kevlar 49	1.45	131	–	–	–	–	717	483	20.3	2.8	0.34	–
	30% Glass	1.45	72.4	–	–	–	–	90	131	–	–	0.34	–
Wood Select Structural Grade	Douglas Fir	0.47	13.1	–	–	–	–	2.1[c]	26[d]	6.2[d]	–	0.29[e]	–
	White Spruce	3.60	9.65	–	–	–	–	2.5[c]	36[d]	6.7[d]	–	0.31[e]	–

[a] Specific values may vary for a particular material due to alloy or mineral composition, mechanical working of the specimen, or heat treatment. For a more exact value reference books for the material should be consulted.

[b] The yield and ultimate strengths for ductile materials can be assumed equal for both tension and compression.

[c] Measured perpendicular to the grain.

[d] Measured parallel to the grain.

[e] Deformation measured perpendicular to the grain when the load is applied along the grain.

Wide-Flange Sections or W Shapes SI Units

Designation	Area A	Depth d	Web thickness t_w	Flange width b_f	Flange thickness t_f	x–x axis I	x–x axis S	x–x axis r	y–y axis I	y–y axis S	y–y axis r
mm × kg/m	mm^2	mm	mm	mm	mm	10^6 mm^4	10^3 mm^3	mm	10^6 mm^4	10^3 mm^3	mm
W610 × 155	19 800	611	12.70	324.0	19.0	1 290	4 220	255	108	667	73.9
W610 × 140	17 900	617	13.10	230.0	22.2	1 120	3 630	250	45.1	392	50.2
W610 × 125	15 900	612	11.90	229.0	19.6	985	3 220	249	39.3	343	49.7
W610 × 113	14 400	608	11.20	228.0	17.3	875	2 880	247	34.3	301	48.8
W610 × 101	12 900	603	10.50	228.0	14.9	764	2 530	243	29.5	259	47.8
W610 × 92	11 800	603	10.90	179.0	15.0	646	2 140	234	14.4	161	34.9
W610 × 82	10 500	599	10.00	178.0	12.8	560	1 870	231	12.1	136	33.9
W460 × 97	12 300	466	11.40	193.0	19.0	445	1 910	190	22.8	236	43.1
W460 × 89	11 400	463	10.50	192.0	17.7	410	1 770	190	20.9	218	42.8
W460 × 82	10 400	460	9.91	191.0	16.0	370	1 610	189	18.6	195	42.3
W460 × 74	9 460	457	9.02	190.0	14.5	333	1 460	188	16.6	175	41.9
W460 × 68	8 730	459	9.14	154.0	15.4	297	1 290	184	9.41	122	32.8
W460 × 60	7 590	455	8.00	153.0	13.3	255	1 120	183	7.96	104	32.4
W460 × 52	6 640	450	7.62	152.0	10.8	212	942	179	6.34	83.4	30.9
W410 × 85	10 800	417	10.90	181.0	18.2	315	1 510	171	18.0	199	40.8
W410 × 74	9 510	413	9.65	180.0	16.0	275	1 330	170	15.6	173	40.5
W410 × 67	8 560	410	8.76	179.0	14.4	245	1 200	169	13.8	154	40.2
W410 × 53	6 820	403	7.49	177.0	10.9	186	923	165	10.1	114	38.5
W410 × 46	5 890	403	6.99	140.0	11.2	156	774	163	5.14	73.4	29.5
W410 × 39	4 960	399	6.35	140.0	8.8	126	632	159	4.02	57.4	28.5
W360 × 79	10 100	354	9.40	205.0	16.8	227	1 280	150	24.2	236	48.9
W360 × 64	8 150	347	7.75	203.0	13.5	179	1 030	148	18.8	185	48.0
W360 × 57	7 200	358	7.87	172.0	13.1	160	894	149	11.1	129	39.3
W360 × 51	6 450	355	7.24	171.0	11.6	141	794	148	9.68	113	38.7
W360 × 45	5 710	352	6.86	171.0	9.8	121	688	146	8.16	95.4	37.8
W360 × 39	4 960	353	6.48	128.0	10.7	102	578	143	3.75	58.6	27.5
W360 × 33	4 190	349	5.84	127.0	8.5	82.9	475	141	2.91	45.8	26.4