Review
Matrix Analysis

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Basic Concepts

- The Finite Element Method **(FEM)** or Finite Element Analysis **(FEA)** is a numerical method for solving problems of engineering and mathematical physics.

- For problems involving complicated geometries, loadings, and material properties, it is generally not possible to obtain analytical mathematical solutions.

Basic Concepts

- Analytical solutions are those given by a mathematical expression that yields the values of the desired unknown quantities at any location in a body. These analytical solutions generally require the solution of ordinary or partial differential equations, which are not usually obtainable.
- Hence we need to rely on numerical methods, such as the finite element method, for acceptable solutions. The finite element formulation results in a system of simultaneous algebraic equations for solution, rather than requiring the solution of differential equations.

Basic Concepts

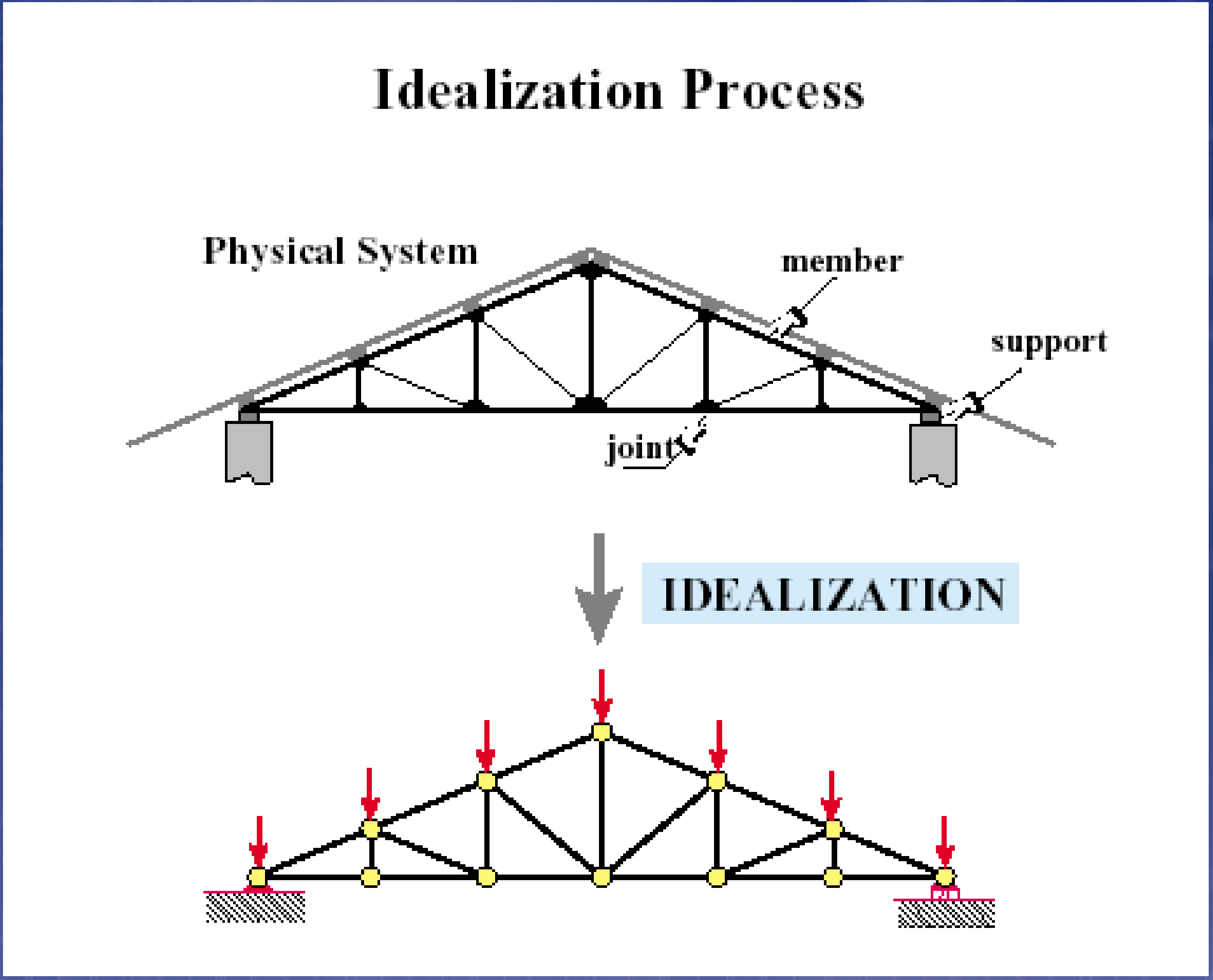
- The Finite Element Method is based on the idea of building a complicated object with simple blocks, or dividing a complicated objects into small manageable pieces.

- **<u>Discretization:</u>** modeling a body by dividing it into an equivalent system of smaller bodies or units (finite elements) interconnected at points common to two or more elements (nodal points or nodes).

Basic Concepts

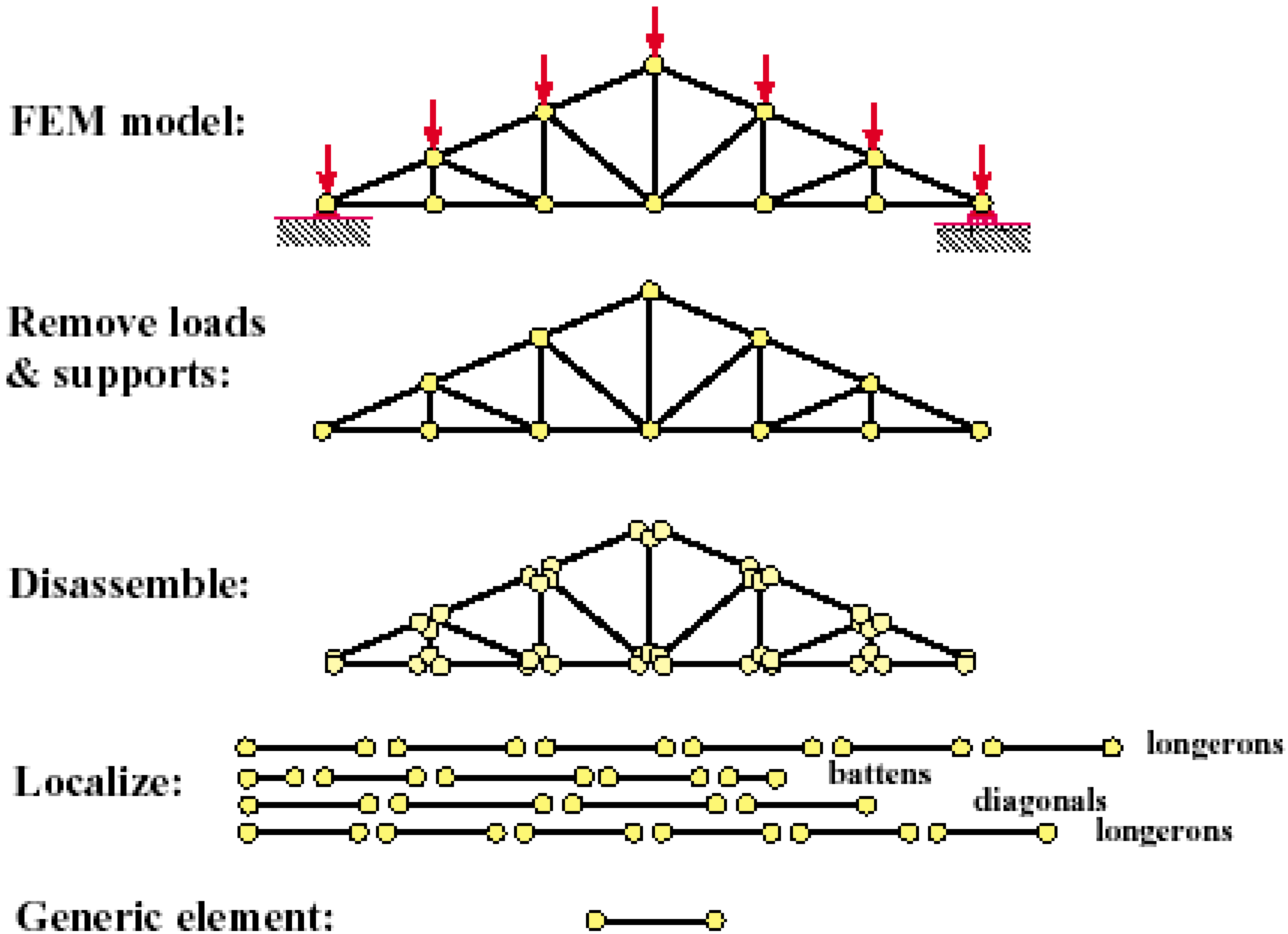
- In the finite element method, instead of solving the problem for the entire body in one operation, we formulate the equations for each finite element and combine them to obtain the solution of the whole body.
- Briefly, the solution for structural problems typically refers to determining the displacement at each node and the stresses within each element making up the structure that is subjected to applied loads.

Basic Concepts

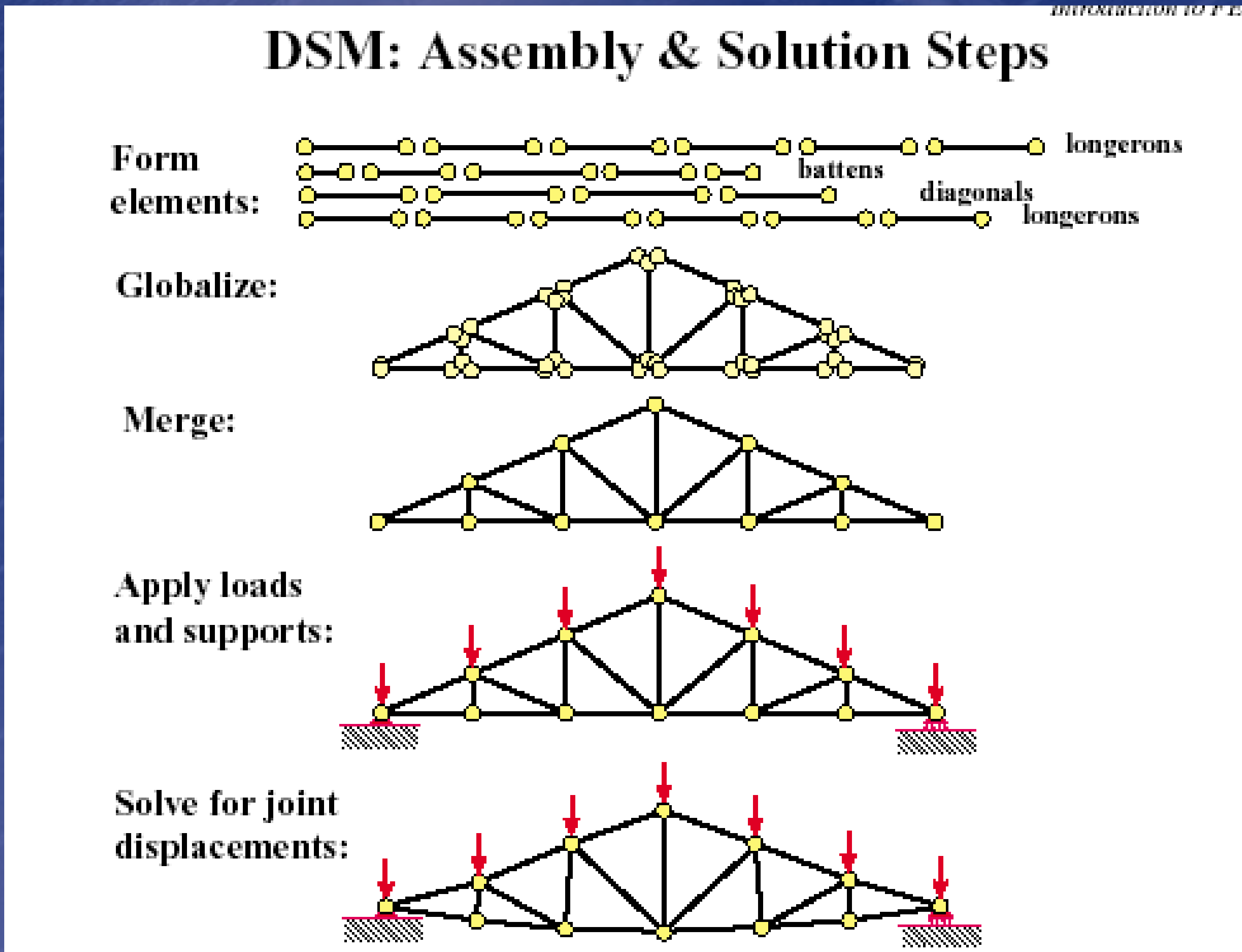


Basic Concepts

DSM: Breakdown Steps

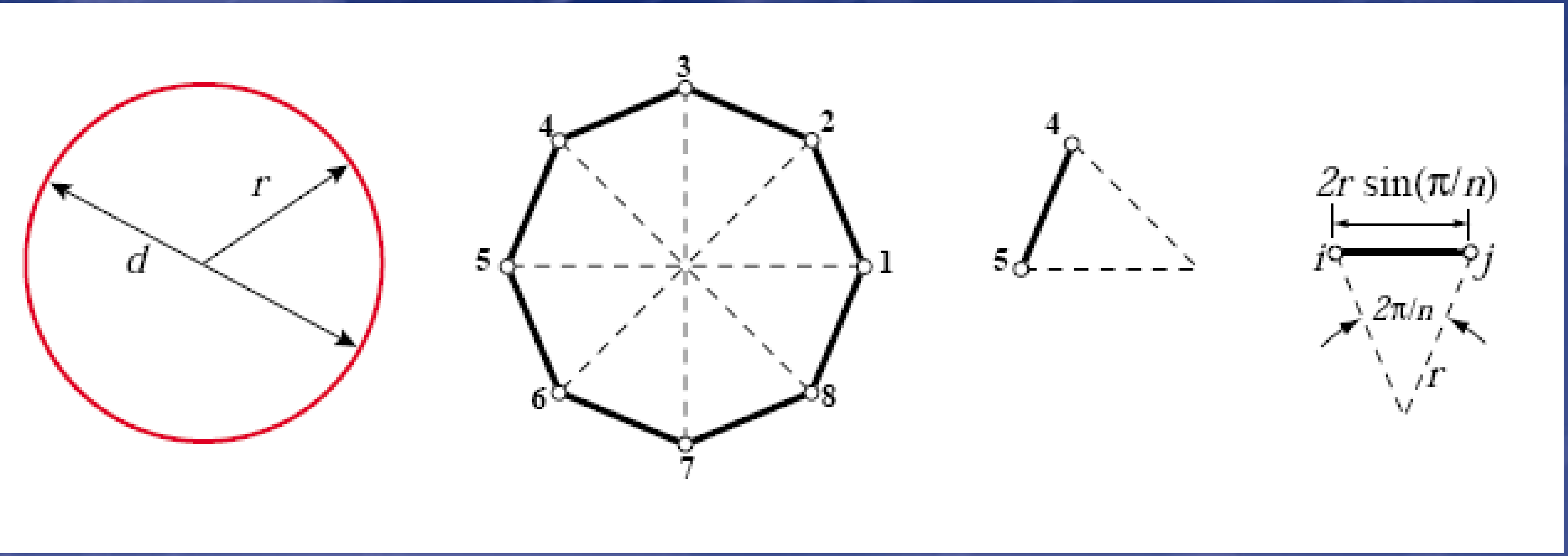


Basic Concepts



Basic Concepts

- The solutions from the finite element method (numerical method) will yield an approximate results rather than exact.



Basic Concepts

- **Solving Using computer:**
- Defining the finite element model, inputs the information into the computer (nodal coordinates, the manner in which elements are connected, the material properties of the elements, the applied loads, boundary conditions, or constraints)
- The computer then uses this information to generate and solve the equations necessary to carry out the analysis.

Basic Concepts

- ***Available Commercial FEM Software Packages:***

- SAP
- ETAB
- STAAD PRO
- SAFE
- ANSYS
- ADINA
- ABAQUS

General Steps of the Finite Element Method

- Two general direct approaches:

  1- Force or Flexibility method.

  2- Displacement or Stiffness method.

General Steps of the Finite Element Method

- *Force or Flexibility method :*

- It uses internal forces as the unknowns of the problem.

- To obtain the governing equations, first the equilibrium equations are used. Then necessary additional equations are found by introducing compatibility equations.

- The result is a set of algebraic equations for determining the redundant or unknown forces.

General Steps of the Finite Element Method

- *Displacement or Stiffness method :*
- It assumes the displacements of the nodes as the unknowns of the problem.
- The governing equations are expressed in terms of nodal displacements using the equations of equilibrium.
- The result is a set of algebraic equations for determining the unknown displacements.

General Steps of the Finite Element Method

- **Steps of FEM:**
- **Step 1:** Discretize and Select the Element Types
- Step 1 involves dividing the body into an equivalent system of finite elements with associated nodes and choosing the most appropriate element type to model most closely the actual physical behavior.
- The elements must be made small enough to give usable results and yet large enough to reduce computational effort.

General Steps of the Finite Element Method

- **Step 1:** - continued-
- The discretized body or mesh is often created with mesh-generation programs or preprocessor programs available to the user.
- The choice of elements used in a finite element analysis depends on the physical makeup of the body under actual loading conditions and on how close to the actual behavior the analyst wants the results to be.

General Steps of the Finite Element Method

- **Step 1:** - continued-

- Types of Elements:

  1- Line elements:

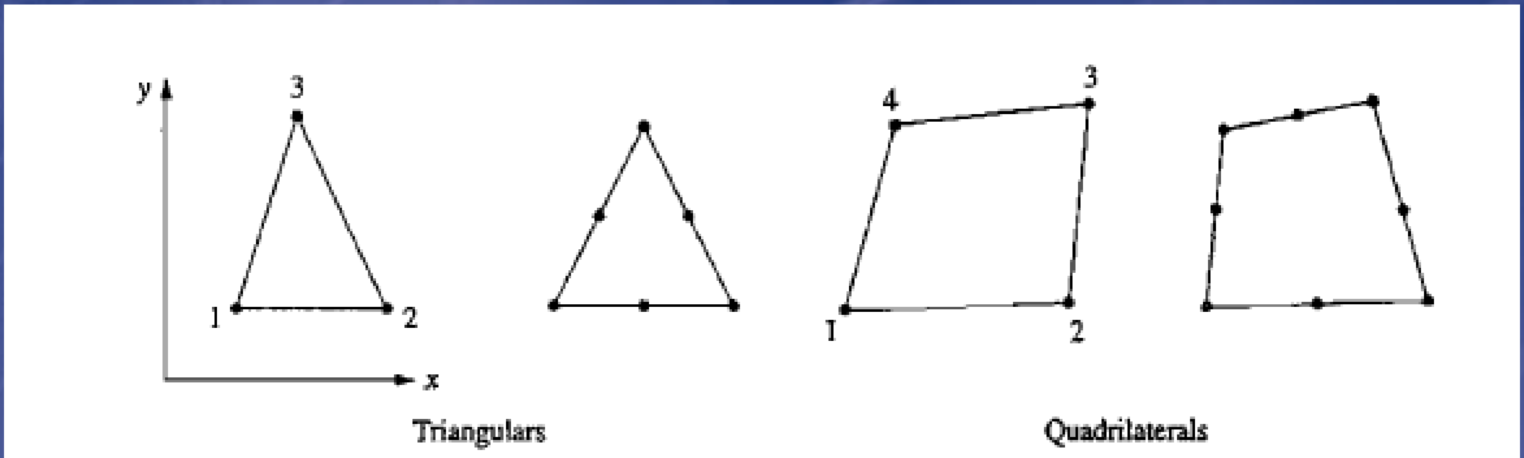
- Consist of bar (or Truss) and beam elements.
- These elements are often used to model trusses and frame structures.

General Steps of the Finite Element Method

- **Step 1:** - continued-

2- Two-dimensional (or plane) elements:

- They are triangular or quadrilateral elements.
- These elements are often used to model a wide range of engineering problems

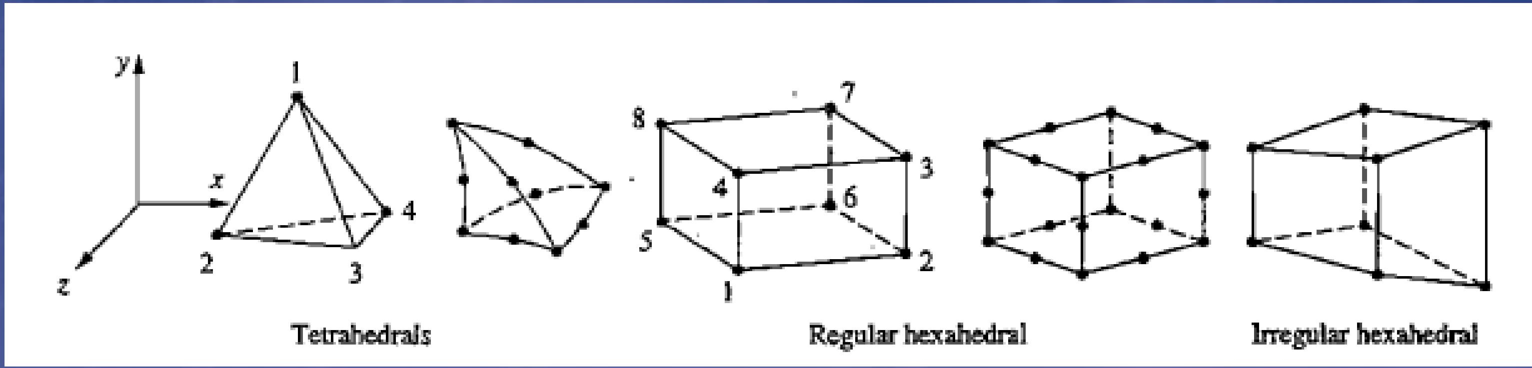


General Steps of the Finite Element Method

- **Step 1:** - continued-

  3- Three-dimensional elements:

- They are tetrahedral and hexahedral (or brick) elements.
- These elements are often used to perform a three-dimensional stress analysis.



General Steps of the Finite Element Method

- **Step 2:** Select a Displacement Function

- **Step 3:** Define the Strain/Displacement and Stress/Strain Relationships

- **Step 4:** Derive the Element Stiffness Matrix and Equations

General Steps of the Finite Element Method

- **Step 5:** Assemble the Element Equations to Obtain the Global or Total Equations and Introduce Boundary Conditions

- In this step the individual element nodal equilibrium equations generated in step 4 are assembled into the global nodal equilibrium equations (Direct Stiffness Method)

$$\{F\} = [K]\{d\}$$

General Steps of the Finite Element Method

- **Step 6:** Solve for the Unknown Degrees of Freedom (or Generalized Displacements)

- Equation modified to account for the boundary conditions, is a set of simultaneous algebraic equations that can be written in expanded matrix form.

- These equations can be solved for by using an elimination method (such as Gauss's method) or an iterative method (such as the Gauss-Seidel method).

General Steps of the Finite Element Method

- **Step 7:** Solve for the Element Strains and Stresses
- The solution from step 6 is called primary unknowns (i.e. displacements).
- Important secondary quantities (unknowns) of strain and stress (or moment and shear force) can be obtained in this step because they can be directly expressed in terms of the displacements determined in step 6.

General Steps of the Finite Element Method

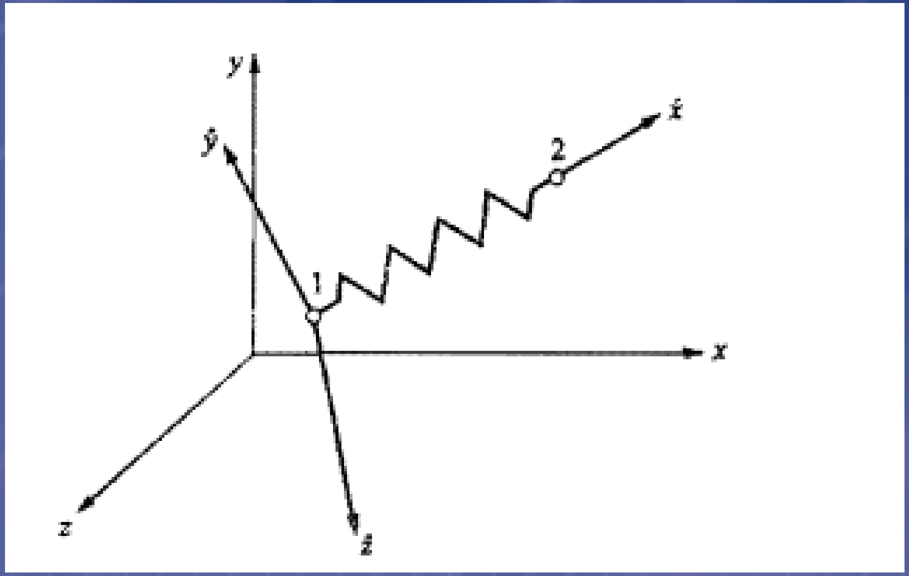
- **Step 8:** Interpret the Results
- The final goal is to interpret and analyze the results for use in the design/analysis process.
- Determination of locations in the structure where large deformations and large stresses occur is generally important in making design/analysis decisions.

Definition of the Stiffness Matrix

- For an element, a stiffness matrix $\hat{\underline{k}}$ is a matrix such that:

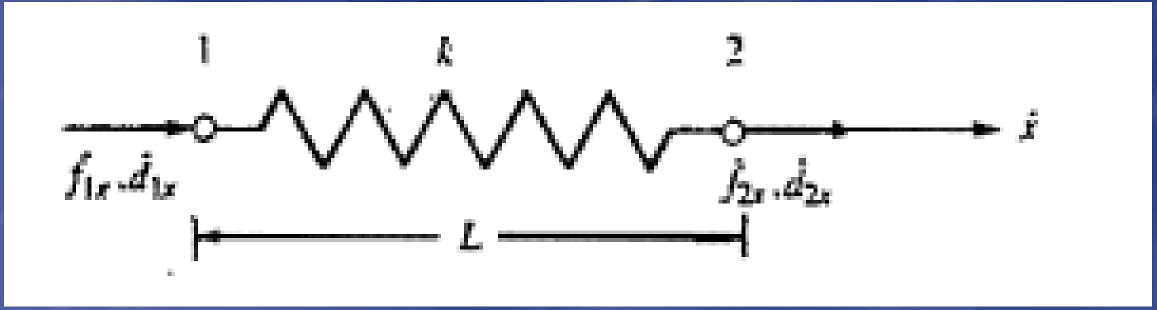
$$\hat{\underline{f}} = \hat{\underline{k}}\hat{\underline{d}}$$

- where $\hat{\underline{k}}$ relates local-coordinate $(\hat{x}, \hat{y}, \hat{z})$ nodal displacement $\hat{\underline{d}}$ to local forces $\hat{\underline{f}}$ of a single element.



Derivation of the Stiffness Matrix for a Spring Element

- Using the direct equilibrium approach, we will now derive the stiffness matrix for a one-dimensional linear spring-that is, a spring that obeys Hooke's law and resists forces only in the direction of the spring.



Derivation of the Stiffness Matrix for a Spring Element

- Nodes of the spring are points 1 and 2 that are located at the ends of the element.

- Local nodal forces are $\hat{f}_{1x}$ and $\hat{f}_{2x}$ for the spring element associated with the local axis $\hat{x}$.

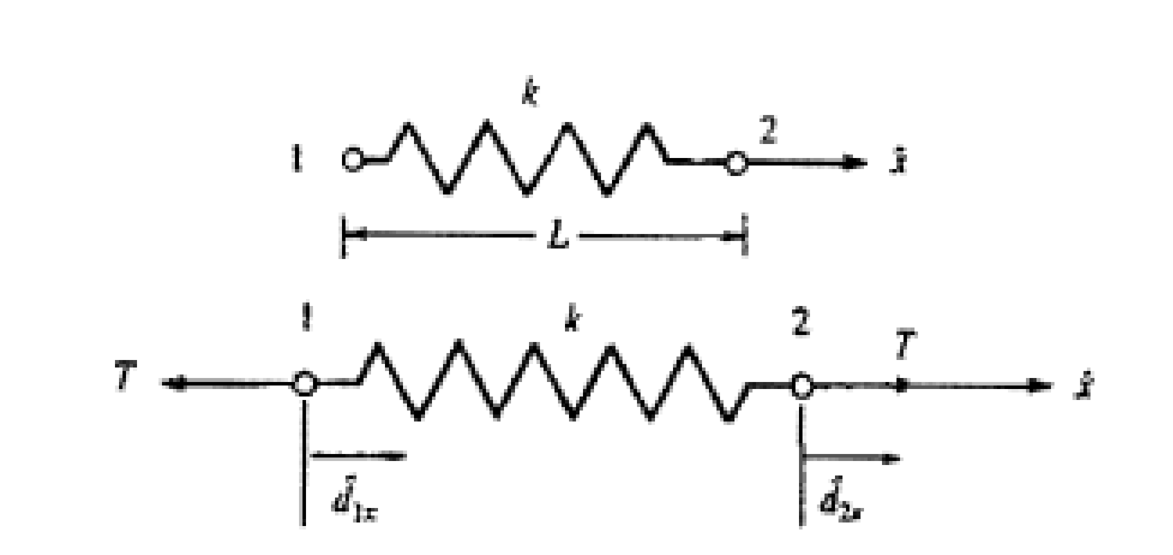
- The local nodal displacements are $\hat{d}_{1x}$ and $\hat{d}_{2x}$ for the spring element. These nodal displacements are called the *degrees of freedom* at each node.

- The symbol *k* is called the *spring constant* or *stiffness* of the spring.

Derivation of the Stiffness Matrix for a Spring Element

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

Select the Element Type



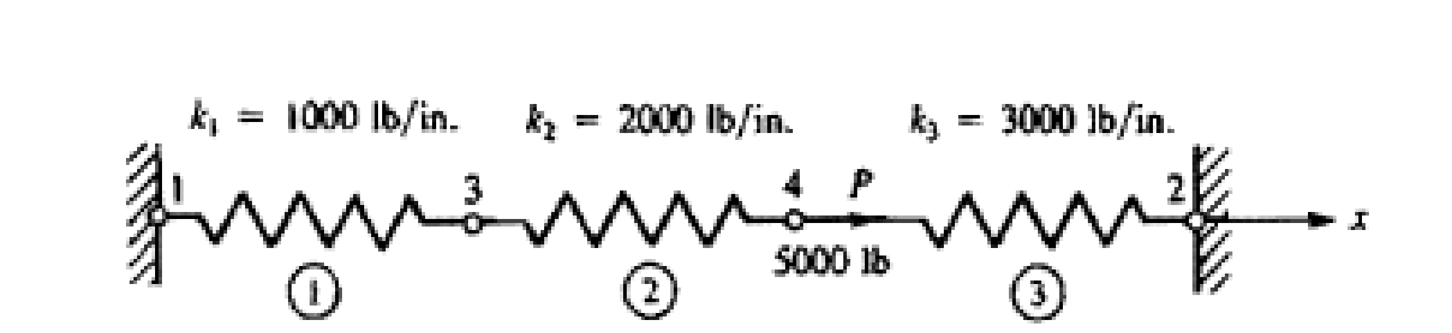
Derivation of the Stiffness Matrix for a Spring Element

- Local stiffness matrix

$$\hat{\underline{k}} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

Examples

- Example 1:



- Required:
- (a) The Global Stiffness Matrix.
- (b) The displacements of nodes 3 and 4.
- (c) The reactions.
- (d) The forces in each spring.

Example 1:

- (a)

$$\begin{array}{cc} 1 & 3 \\ \end{array}$$

$$\underline{k}^{(1)} = \begin{array}{c} \begin{array}{cc} 1 & 3 \end{array} \\ \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 3 \end{array} \qquad \underline{k}^{(2)} = \begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix} \end{array} \begin{array}{c} \\ 3 \\ 4 \end{array}$$

$$\underline{k}^{(3)} = \begin{array}{c} \begin{array}{cc} 4 & 2 \end{array} \\ \begin{bmatrix} 3000 & -3000 \\ -3000 & 3000 \end{bmatrix} \end{array} \begin{array}{c} \\ 4 \\ 2 \end{array}$$

$$\underline{K} = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 1000 + 2000 & -2000 \\ 0 & -3000 & -2000 & 2000 + 3000 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Example 1:

- (b)

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 3000 & -2000 \\ 0 & -3000 & -2000 & 5000 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \\ d_{4x} \end{Bmatrix}$$

- Applying the homogeneous boundary conditions $d_{1x} = 0$ and $d_{2x} = 0$

$$\begin{Bmatrix} 0 \\ 5000 \end{Bmatrix} = \begin{bmatrix} 3000 & -2000 \\ -2000 & 5000 \end{bmatrix} \begin{Bmatrix} d_{3x} \\ d_{4x} \end{Bmatrix}$$

$$d_{3x} = \frac{10}{11} \text{ in.} \qquad d_{4x} = \frac{15}{11} \text{ in.}$$

Example 1:

- (c)

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 3000 & -2000 \\ 0 & -3000 & -2000 & 5000 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \frac{10}{11} \\ \frac{15}{11} \end{Bmatrix} .$$

$$F_{1x} = \frac{-10,000}{11}\ \text{lb} \qquad F_{2x} = \frac{-45,000}{11}\ \text{lb} \qquad F_{3x} = 0$$

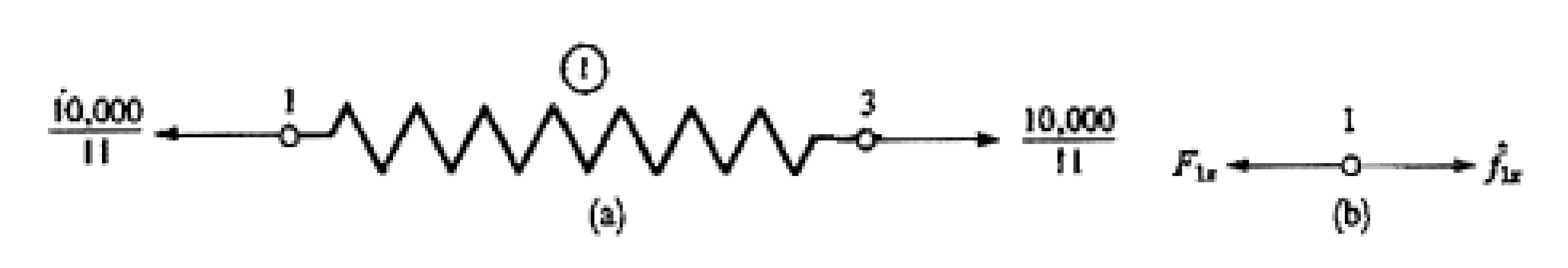
$$F_{4x} = \frac{55,000}{11}\ \text{lb}$$

Example 1:

- (d) Next we use local element Equation to obtain the forces in each element.
- <u>Element 1:</u>

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0 \\ \frac{10}{11} \end{Bmatrix}$$

$$\hat{f}_{1x} = \frac{-10{,}000}{11}\ \text{lb} \qquad \hat{f}_{3x} = \frac{10{,}000}{11}\ \text{lb}$$

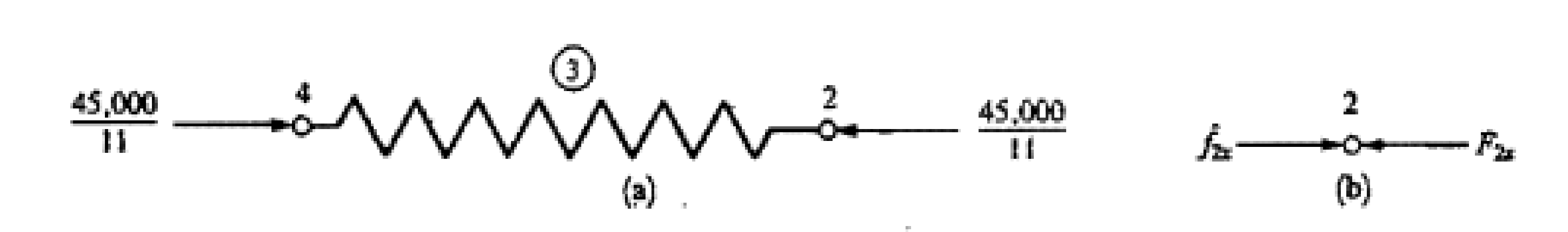


Example 1:

- (d) Element 3:

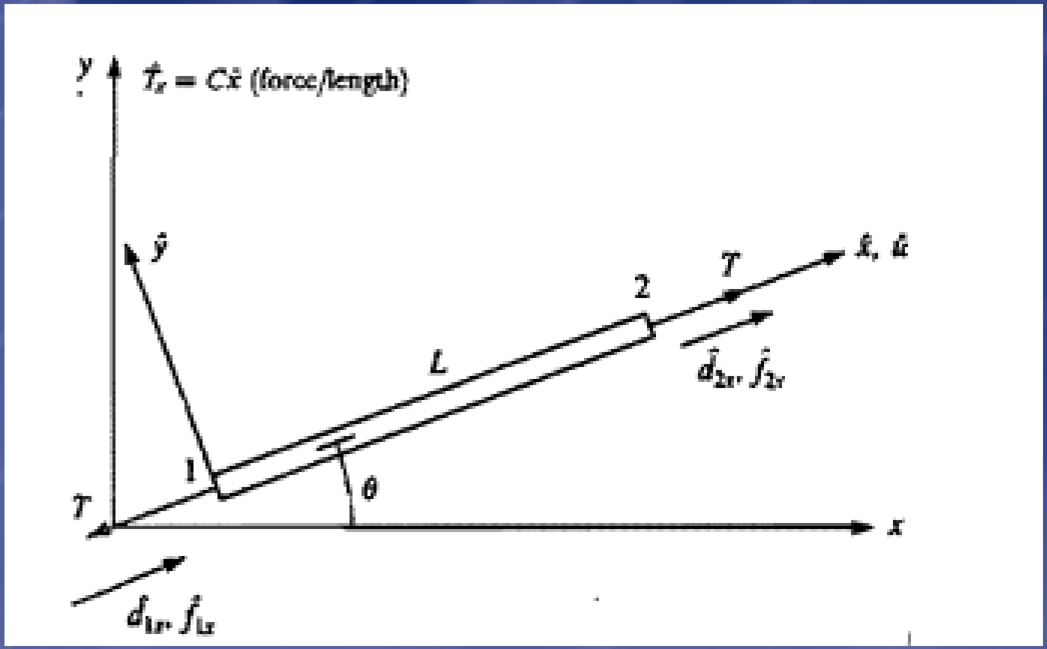
$$\begin{Bmatrix} \hat{f}_{4x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} 3000 & -3000 \\ -3000 & 3000 \end{bmatrix} \begin{Bmatrix} \frac{15}{11} \\ 0 \end{Bmatrix}$$

$$\hat{f}_{4x} = \frac{45{,}000}{11} \text{ lb} \qquad \hat{f}_{2x} = \frac{-45{,}000}{11} \text{ lb}$$



Derivation of the Stiffness Matrix for a Bar Element in local Coordinates

- We will now consider the derivation of the stiffness matrix for the linear-e1astic, constant cross-sectional area (prismatic) bar element.



Derivation of the Stiffness Matrix for a Bar Element in local Coordinates

- The bar element is assumed to have:
- Constant cross-sectional area A, modulus of elasticity E, and initial length L.
- The nodal degrees of freedom are local axial displacements (longitudinal displacements directed along the length of the bar) represented by d_{1x} and d_{2x} at the ends of the element.

Derivation of the Stiffness Matrix for a Bar Element in local Coordinates

- The following assumptions are used in deriving the bar element stiffness matrix:

1- The bar cannot sustain shear force or bending moment, that is, $\hat{f}_{1y} = 0$, $\hat{f}_{2y} = 0$, $m_1 = 0$ and $m_2 = 0$.

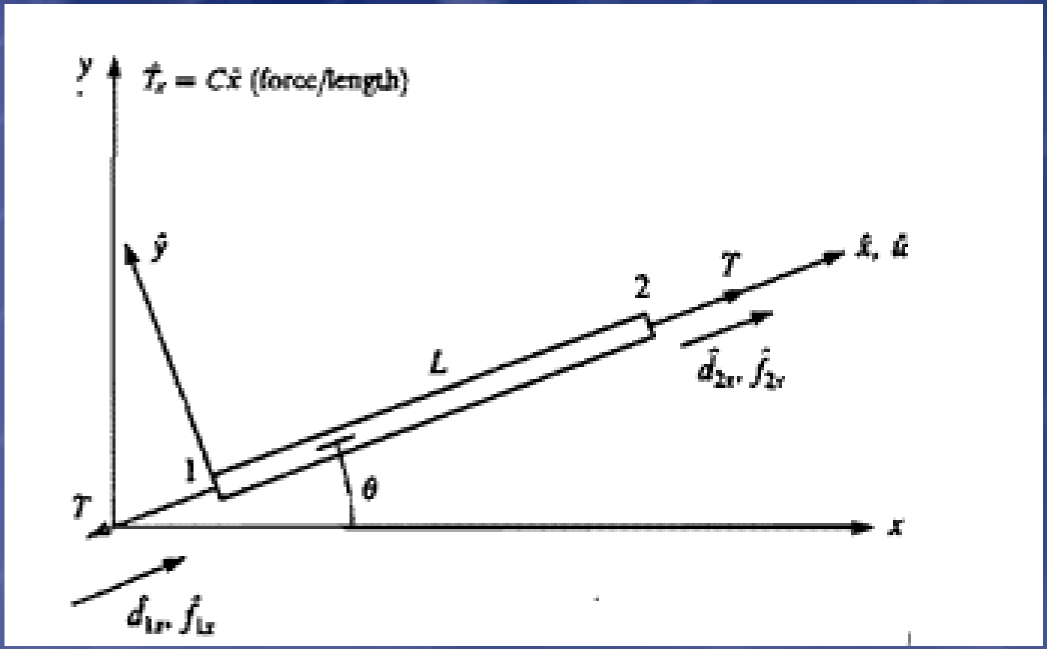
2- Any effect of transverse displacement is ignored.

3- Hooke's law applies; that is, axial stress σ_x is related to axial strain ε_x by $\sigma_x = E\varepsilon_x$

4- No intermediate applied loads.

Derivation of the Stiffness Matrix for a Bar Element in local Coordinates

Select the Element Type

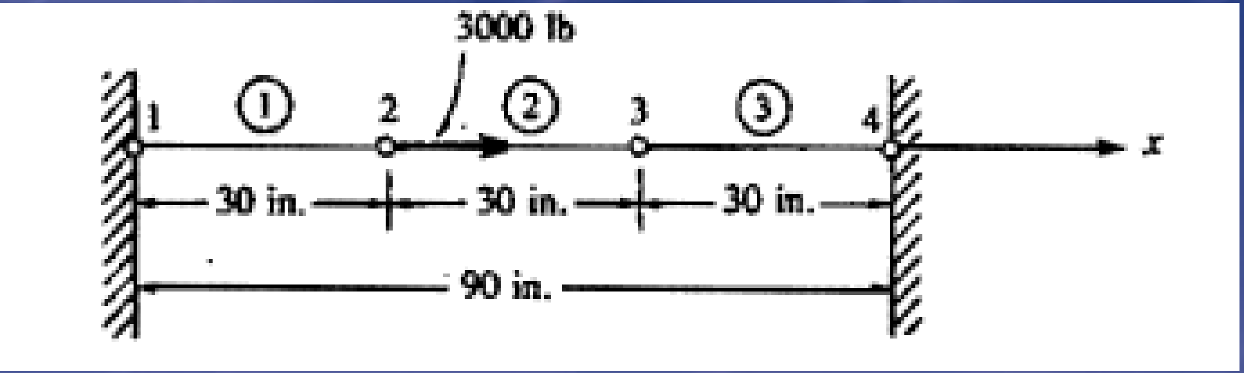


Derivation of the Stiffness Matrix for a Bar Element in local Coordinates

$$\underline{\hat{k}} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Example

- <u>Given:</u>
- A force of 3000 lb is applied in the x direction at node 2. The length of each element is 30 in. Let $E = 30 \times 10^6$ psi and $A = 1$ in^2 for elements 1 and 2, and let $E = 15 \times 10^6$ psi and $A = 2$ in^2 for element 3. Nodes 1 and 4 are fixed.



Example

- Required:

- (a) The global stiffness matrix.
- (b) The displacements of nodes 2 and 3.
- (c) The reactions at nodes 1 and 4.

Example

- (a)

$$\underline{k}^{(1)} = \underline{k}^{(2)} = \frac{(1)(30 \times 10^6)}{30}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{array}{c} \begin{array}{cc} 1 & 2^{(1)} \\ 2 & 3^{(2)} \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}}$$

$$\underline{k}^{(3)} = \frac{(2)(15 \times 10^6)}{30}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}}$$

$$\underline{K} = 10^6 \begin{array}{c} \begin{array}{cccc} d_{1x} & d_{2x} & d_{3x} & d_{4x} \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1+1 & -1 & 0 \\ 0 & -1 & 1+1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \frac{\text{lb}}{\text{in.}}$$

Example

- (b)

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = 10^6 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \\ d_{4x} \end{Bmatrix}$$

$$d_{1x} = 0 \qquad d_{4x} = 0$$

$$\begin{Bmatrix} 3000 \\ 0 \end{Bmatrix} = 10^6 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix}$$

$$d_{2x} = 0.002 \text{ in.} \qquad d_{3x} = 0.001 \text{ in.}$$

Example

- (c) Back-substituting , we obtain the global nodal forces, which include the reactions at nodes 1 and 4, as follows:

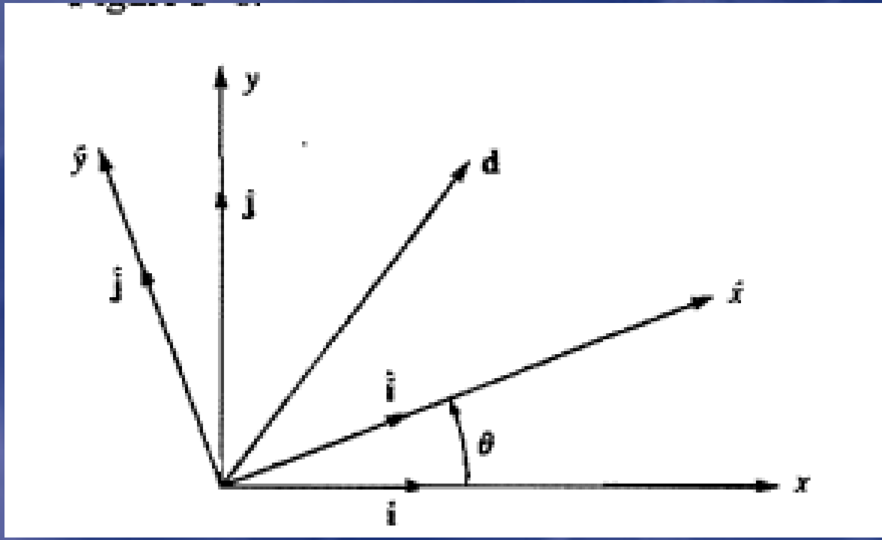
$$F_{1x} = 10^6(d_{1x} - d_{2x}) = 10^6(0 - 0.002) = -2000 \text{ lb}$$

$$F_{2x} = 10^6(-d_{1x} + 2d_{2x} - d_{3x}) = 10^6[0 + 2(0.002) - 0.001] = 3000 \text{ lb}$$

$$F_{3x} = 10^6(-d_{2x} + 2d_{3x} - d_{4x}) = 10^6[-0.002 + 2(0.001) - 0] = 0$$

$$F_{4x} = 10^6(-d_{3x} + d_{4x}) = 10^6(-0.001 + 0) = -1000 \text{ lb}$$

Transformation of Vectors in Two Dimensions



$$\mathbf{d} = d_x\mathbf{i} + d_y\mathbf{j} = \hat{d}_x\hat{\mathbf{i}} + \hat{d}_y\hat{\mathbf{j}}$$

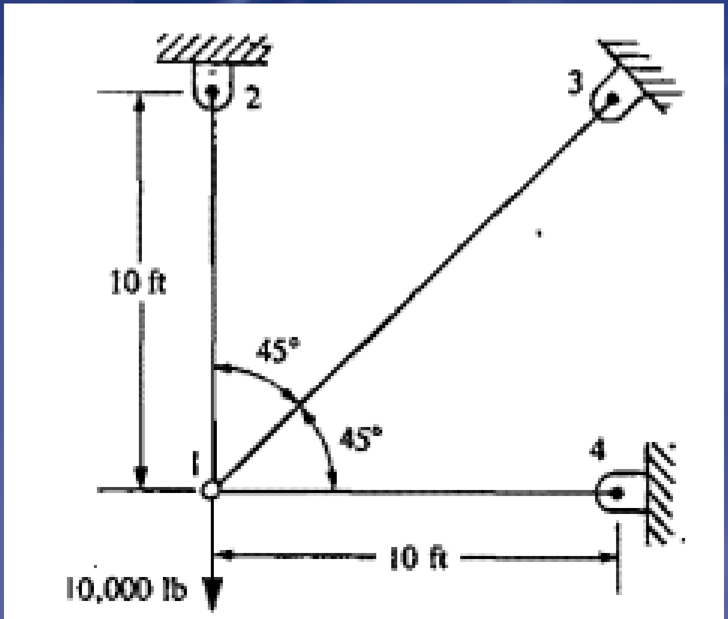
- where i and j are unit vectors in the *x* and *y* global directions and $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors in the $\hat{x}$ and $\hat{y}$ local directions.

Global Stiffness Matrix

$$\underset{\sim}{k} = \frac{AE}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ & S^2 & -CS & -S^2 \\ & & C^2 & CS \\ \text{Symmetry} & & & S^2 \end{bmatrix}$$

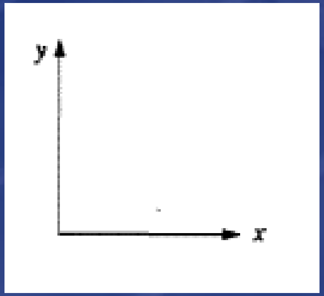
Examples

- *Example 1:* For the plane truss composed of the three elements shown in Figure subjected to a downward force of 10,000 lb applied at node 1, determine the x and y displacements at node 1 and the stresses in each element. Let $E = 30 \times 10^6$ psi and $A = 2$ in^2 for all elements.



Examples

- *Example 1:* -Continued-
- The angle θ is positive when measured counter-clockwise from positive x to $\hat{x}$



Element	$\theta°$	C	S	C^2	S^2	CS
1	90°	0	1	0	1	0
2	45°	$\sqrt{2}/2$	$\sqrt{2}/2$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
3	0°	1	0	1	0	0

Examples

- *Example 1:* -Continued-

$$\underline{k}^{(1)} = \frac{(30\times 10^6)(2)}{120} \begin{array}{c} \begin{array}{cccc} d_{1x} & d_{1y} & d_{2x} & d_{2y} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

$$\underline{k} = \frac{AE}{L} \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ & S^2 & -CS & -S^2 \\ & & C^2 & CS \\ \text{Symmetry} & & & S^2 \end{bmatrix}$$

$$\underline{k}^{(2)} = \frac{(30\times 10^6)(2)}{120\times\sqrt{2}} \begin{array}{c} \begin{array}{cccc} d_{1x} & d_{1y} & d_{3x} & d_{3y} \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \end{array}$$

$$\underline{k}^{(3)} = \frac{(30\times 10^6)(2)}{120} \begin{array}{c} \begin{array}{cccc} d_{1x} & d_{1y} & d_{4x} & d_{4y} \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Examples

- *Example 1:* -Continued-

$$
\underline{K} = (500{,}000)
\begin{array}{c}
\begin{array}{cccccccc} d_{1x} & d_{1y} & d_{2x} & d_{2y} & d_{3x} & d_{3y} & d_{4x} & d_{4y} \end{array} \\
\begin{bmatrix}
1.354 & 0.354 & 0 & 0 & -0.354 & -0.354 & -1 & 0 \\
0.354 & 1.354 & 0 & -1 & -0.354 & -0.354 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\
-0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

Examples

- *Example 1:* -Continued-

$$\begin{Bmatrix} 0 \\ -10{,}000 \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{Bmatrix} = (500{,}000) \begin{bmatrix} 1.354 & 0.354 & 0 & 0 & -0.354 & -0.354 & -1 & 0 \\ 0.354 & 1.354 & 0 & -1 & -0.354 & -0.354 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\times \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} = 0 \\ d_{2y} = 0 \\ d_{3x} = 0 \\ d_{3y} = 0 \\ d_{4x} = 0 \\ d_{4y} = 0 \end{Bmatrix}$$

Examples

- *Example 1:* -Continued-

$$\begin{Bmatrix} 0 \\ -10{,}000 \end{Bmatrix} = (500{,}000)\begin{bmatrix} 1.354 & 0.354 \\ 0.354 & 1.354 \end{bmatrix}\begin{Bmatrix} d_{1x} \\ d_{1y} \end{Bmatrix}$$

$$d_{1x} = 0.414 \times 10^{-2} \text{ in.} \qquad d_{1y} = -1.59 \times 10^{-2} \text{ in.}$$

- The minus sign in the d_{1y} result indicates that the displacement component in the *y* direction at node 1 is in the direction opposite that of the positive *y* direction based on the assumed global coordinates, that is, a downward displacement occurs at node 1.

Examples

- *Example 1:* -Continued-

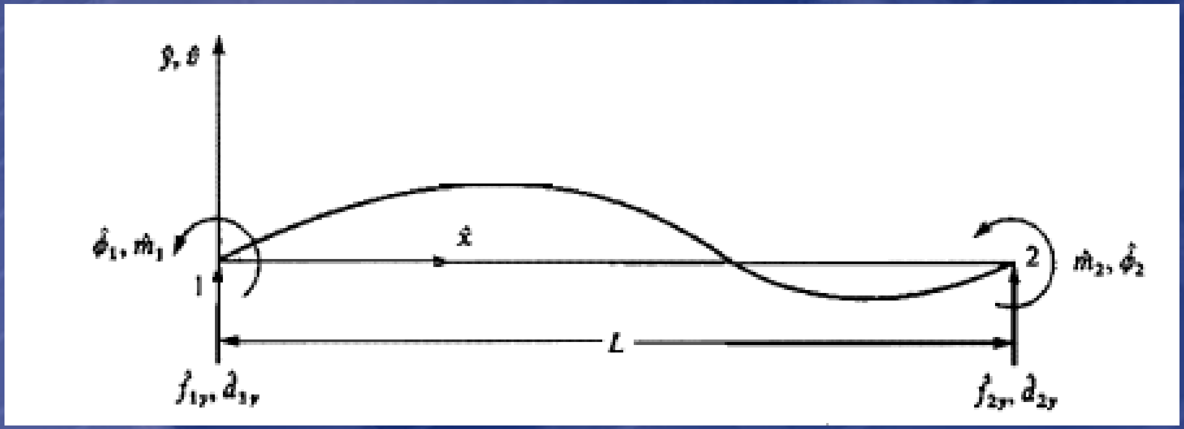
$$\sigma^{(1)} = \frac{30 \times 10^6}{120}\begin{bmatrix} 0 & -1 & 0 & 1 \end{bmatrix}\begin{Bmatrix} d_{1x} = 0.414 \times 10^{-2} \\ d_{1y} = -1.59 \times 10^{-2} \\ d_{2x} = 0 \\ d_{2y} = 0 \end{Bmatrix} = 3965 \text{ psi}$$

$$\sigma^{(2)} = \frac{30 \times 10^6}{120\sqrt{2}}\begin{bmatrix} \frac{-\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{Bmatrix} d_{1x} = 0.414 \times 10^{-2} \\ d_{1y} = -1.59 \times 10^{-2} \\ d_{3x} = 0 \\ d_{3y} = 0 \end{Bmatrix}$$

$$= 1471 \text{ psi}$$

$$\sigma^{(3)} = \frac{30 \times 10^6}{120}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{Bmatrix} d_{1x} = 0.414 \times 10^{-2} \\ d_{1y} = -1.59 \times 10^{-2} \\ d_{4x} = 0 \\ d_{4y} = 0 \end{Bmatrix} = -1035 \text{ psi}$$

Derivation of the Stiffness Matrix for a Beam



- At all nodes, the following sign conventions are used:

1. Moments are positive in the counterclockwise direction.
2. Rotations are positive in the counterclockwise direction.
3. Forces are positive in the positive y direction.
4. Displacements are positive in the positive y direction.

Derivation of the Stiffness Matrix for a Beam

Step 4: -Continued-

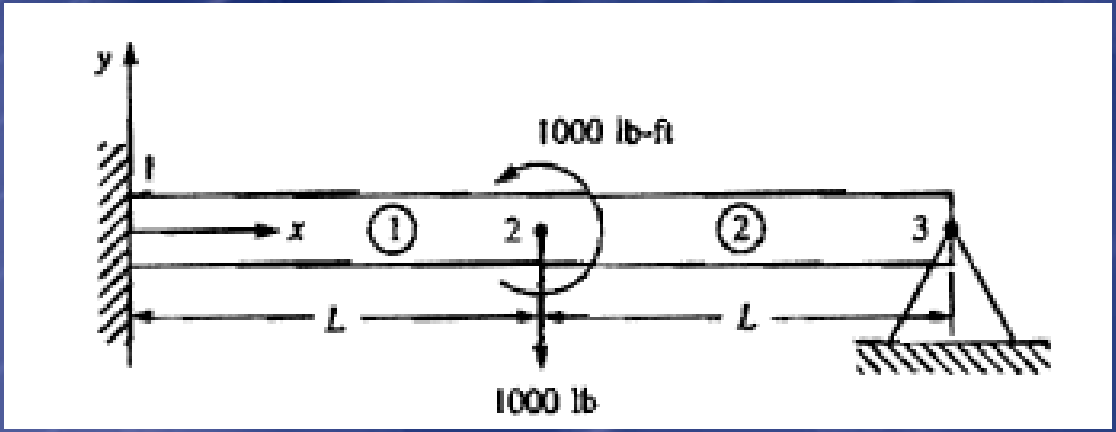
$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix}$$

$$\hat{\underline{k}} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

- It is assumed that the beam is long and slender; that is, the length, *L* , to depth, *h* , dimension ratio of the beam is large.

EXAMPLES

- Example 1:



- EI = Constant

EXAMPLES

- Example 1:-Continued-

$$\begin{array}{cccccc} & & d_{1y} & \phi_1 & d_{2y} & \phi_2 \\ \underline{k}^{(1)} = \dfrac{EI}{L^3} & \left[\begin{array}{c}\\ \\ \\ \\ \end{array}\right. & \begin{array}{c}12\\6L\\-12\\6L\end{array} & \begin{array}{c}6L\\4L^2\\-6L\\2L^2\end{array} & \begin{array}{c}-12\\-6L\\12\\-6L\end{array} & \left.\begin{array}{c}6L\\2L^2\\-6L\\4L^2\end{array}\right] \end{array}$$

$$\begin{array}{cccccc} & & d_{2y} & \phi_2 & d_{3y} & \phi_3 \\ \underline{k}^{(2)} = \dfrac{EI}{L^3} & \left[\begin{array}{c}\\ \\ \\ \\ \end{array}\right. & \begin{array}{c}12\\6L\\-12\\6L\end{array} & \begin{array}{c}6L\\4L^2\\-6L\\2L^2\end{array} & \begin{array}{c}-12\\-6L\\12\\-6L\end{array} & \left.\begin{array}{c}6L\\2L^2\\-6L\\4L^2\end{array}\right] \end{array}$$

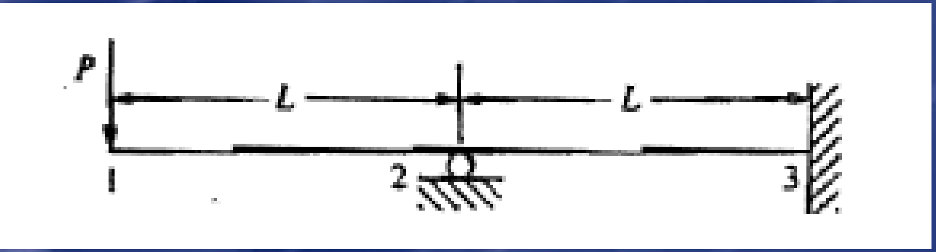
$$\phi_1 = 0 \qquad d_{1y} = 0 \qquad d_{3y} = 0$$

$$\begin{Bmatrix} -1000 \\ 1000 \\ 0 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 24 & 0 & 6L \\ 0 & 8L^2 & 2L^2 \\ 6L & 2L^2 & 4L^2 \end{bmatrix} \begin{Bmatrix} d_{2y} \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 12+12 & -6L+6L & -12 & 6L \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix}$$

EXAMPLES

- Example 2:



- EI = Constant
- Roller at 2

EXAMPLES

- Example 2: - Continued-

$$
\underline{K} = \frac{EI}{L^3}
\begin{array}{c}
\begin{array}{cccccc} d_{1y} & \phi_1 & d_{2y} & \phi_2 & d_{3y} & \phi_3 \end{array} \\
\begin{bmatrix}
12 & 6L & -12 & 6L & 0 & 0 \\
 & 4L^2 & -6L & 2L^2 & 0 & 0 \\
 & & 12+12 & -6L+6L & -12 & 6L \\
 & & & 4L^2+4L^2 & -6L & 2L^2 \\
 & & & & 12 & -6L \\
\text{Symmetry} & & & & & 4L^2
\end{bmatrix}
\end{array}
$$

$$
\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix}
= \frac{EI}{L^3}
\begin{bmatrix}
12 & 6L & -12 & 6L & 0 & 0 \\
6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\
-12 & -6L & 24 & 0 & -12 & 6L \\
6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\
0 & 0 & -12 & -6L & 12 & -6L \\
0 & 0 & 6L & 2L^2 & -6L & 4L^2
\end{bmatrix}
\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix}
$$

$$
d_{2y} = 0 \qquad d_{3y} = 0 \qquad \phi_3 = 0
$$

EXAMPLES

- Example 2: - Continued-

$$\begin{Bmatrix} -P \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & 6L \\ 6L & 4L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix} \begin{Bmatrix} d_{1y} \\ \phi_1 \\ \phi_2 \end{Bmatrix}$$

$$d_{1y} = -\frac{7PL^3}{12EI}$$

$$\phi_1 = \frac{3PL^2}{4EI} \qquad \phi_2 = \frac{PL^2}{4EI}$$

- where the minus sign indicates that the displacement of node 1 is downward and the positive signs indicate counterclockwise rotations at nodes 1 and 2.

EXAMPLES

- Example 2: - Continued-

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -\frac{7PL^3}{12EI} \\ \frac{3PL^2}{4EI} \\ 0 \\ \frac{PL^2}{4EI} \\ 0 \\ 0 \end{Bmatrix}$$

$$F_{1y} = -P \qquad M_1 = 0 \qquad F_{2y} = \tfrac{5}{2}P$$

$$M_2 = 0 \qquad F_{3y} = -\tfrac{3}{2}P \qquad M_3 = \tfrac{1}{2}PL$$

EXAMPLES

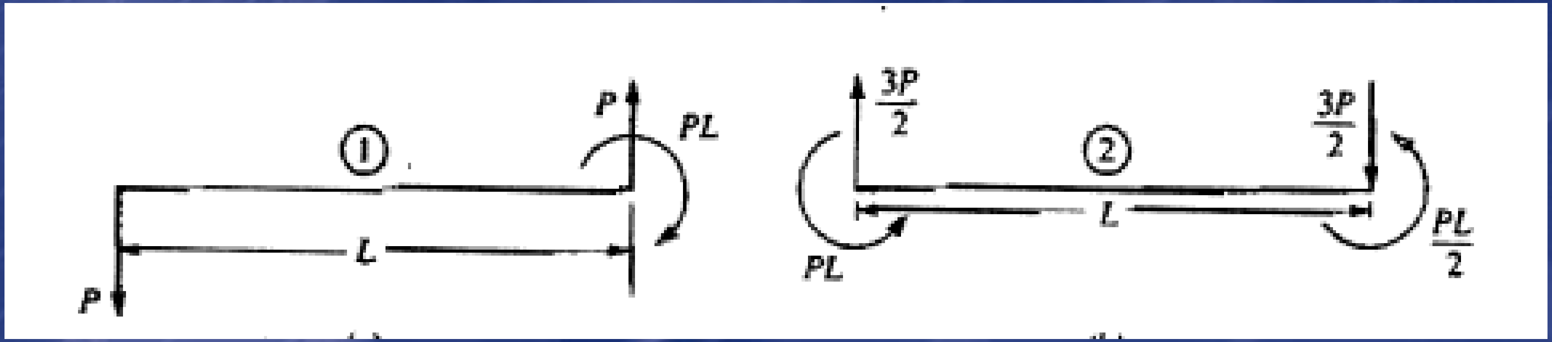
- Example 2: - Continued-
- Local Nodal Forces for Element 1:

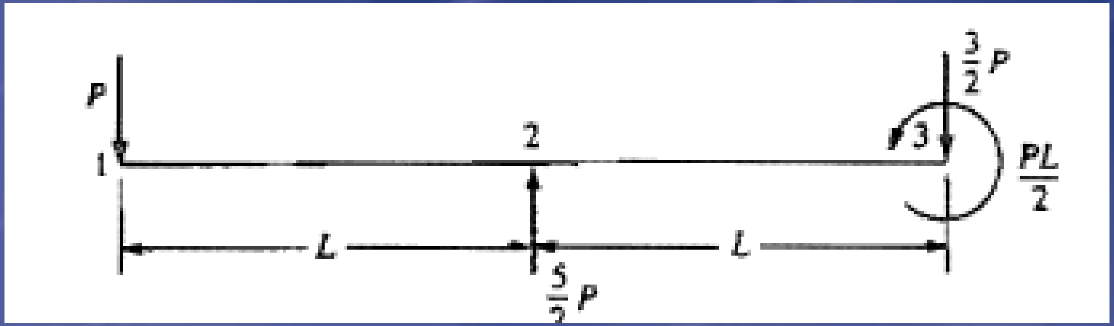
$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -\frac{7PL^3}{12EI} \\ \frac{3PL^2}{4EI} \\ 0 \\ \frac{PL^2}{4EI} \end{Bmatrix}$$

$$\hat{f}_{1y} = -P \qquad \hat{m}_1 = 0 \qquad \hat{f}_{2y} = P \qquad \hat{m}_2 = -PL$$

EXAMPLES

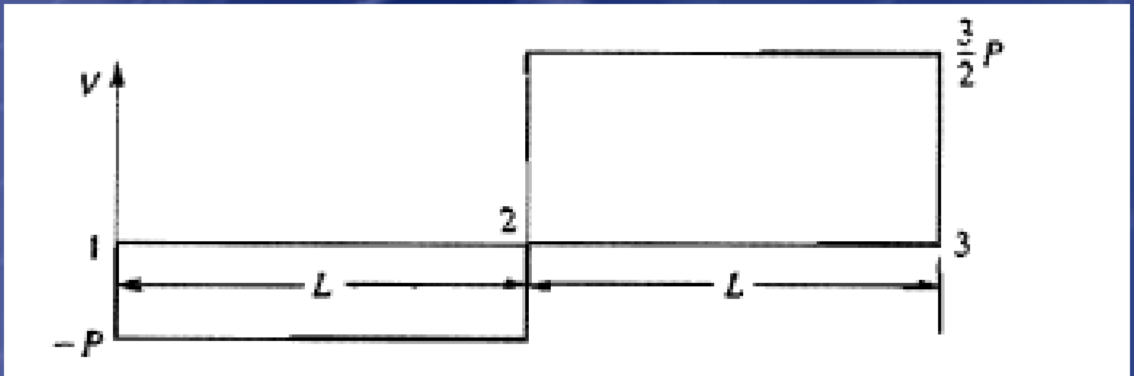
- Example 2: - Continued-

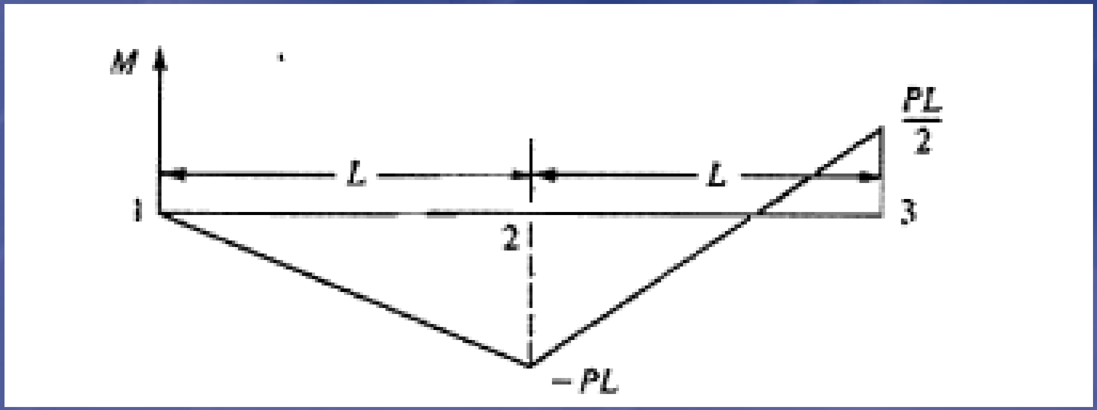




EXAMPLES

- Example 2: - Continued-
- Using the beam sign conventions:





Two-Dimensional Arbitrarily Oriented Beam Element

- Global stiffness matrix for a beam element that includes axial force, shear force, and bending moment effects is as follows:

$$\underline{k} = \frac{E}{L} \times$$

$$\begin{bmatrix} AC^2+\frac{12I}{L^2}S^2 & \left(A-\frac{12I}{L^2}\right)CS & -\frac{6I}{L}S & -\left(AC^2+\frac{12I}{L^2}S^2\right) & -\left(A-\frac{12I}{L^2}\right)CS & -\frac{6I}{L}S \\ & AS^2+\frac{12I}{L^2}C^2 & \frac{6I}{L}C & -\left(A-\frac{12I}{L^2}\right)CS & -\left(AS^2+\frac{12I}{L^2}C^2\right) & \frac{6I}{L}C \\ & & 4I & \frac{6I}{L}S & -\frac{6I}{L}C & 2I \\ & & & AC^2+\frac{12I}{L^2}S^2 & \left(A-\frac{12I}{L^2}\right)CS & \frac{6I}{L}S \\ & & & & AS^2+\frac{12I}{L^2}C^2 & -\frac{6I}{L}C \\ \text{Symmetry} & & & & & 4I \end{bmatrix}$$