New 2018

Civilittee

اللجنة الأكاديمية للهندسة المدنية

تلخيص

معادلات تفاضلية عادية

مجدولين سمارة

* $\sec(x) = \dfrac{1}{\cos(x)}$

$\csc(x) = \dfrac{1}{\sin(x)}$

$\tan(x) = \dfrac{\sin(x)}{\cos(x)}$

$\cot(x) = \dfrac{\cos(x)}{\sin(x)}$

- same for $\sinh(x), \cosh(x) \ldots$

* $\sin(-x) = -\sin(x)$

$\tan(-x) = -\tan(x)$

$\cos(-x) = \cos(x)$

- same for $\sinh(-x), \cosh(-x), \ldots$

* $\sin^2(x) + \cos^2(x) = 1$

$\cosh^2(x) - \sinh^2(x) = 1$

$\sec^2(x) = 1 + \tan^2(x)$

$\csc^2(x) = 1 + \cot^2(x)$.

* $\sin(2x) = 2\sin(x)\cos(x)$

$\cos(2x) = $
- $\cos^2(x) - \sin^2(x)$
- $2\cos^2(x) - 1$
- $1 - 2\sin^2(x)$.

$\sinh(2x) = 2\sinh(x)\cosh(x)$.

* $\cosh(x) = \dfrac{e^{x} + e^{-x}}{2}$

$\sinh(x) = \dfrac{e^{x} - e^{-x}}{2}$

نفس الاشارة

* $\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$.

عكس الاشارة :

$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$.

* $\sin(a)\cos(b) = \dfrac{\sin(a+b)}{2} + \dfrac{\sin(a-b)}{2}$

$\cos(a)\sin(b) = \dfrac{\sin(a+b)}{2} - \dfrac{\sin(a-b)}{2}$

$\cos(a)\cos(b) = \dfrac{\cos(a+b)}{2} + \dfrac{\cos(a-b)}{2}$

$\sin(a)\sin(b) = \dfrac{\cos(a+b)}{2} - \dfrac{\cos(a-b)}{2}$

- same for $\sinh(x), \cosh(x) \ldots$ -

* ملاحظة :

$\dfrac{d}{dx}\left(\tan^{-1}(x)\right) = \left(\dfrac{1}{x^2+1} \cdot \text{مشتقة الزاوية}\right)$

- المشتقات المطلوبة في المادة ...

- تلخيص دوق 1 سكند + فاينل - مجدولين سماره .

أنواع المعادلات :

★ طريقة الحل :
نستبدل كل y بـ r أس رتبة المشتقة .

Zero

1) Homogeneous → $y'' + p(x)y' + q(x)y = 0$

عندما نكرنه بدلالة r $\quad = r^2 + p(x)\,r^1 + q(x)\,r^0 = 0$

تسمى auxillary eq $\quad = r^2 + p(x)\,r + q(x) = 0$

★ طريقة الحل :
نفس الـ homog
لكن نستخدم فرضا لـ g(x) سيشرح في وقته .

2) non-homogeneous → $y'' + p(x)y' + q(x)y = g(x)$ ثابت / متغير بدلالة x .

$= r^2 + p(x)\,r + q(x) = \dots\dots$

not-Zero

3) caushy → $a x^2 y'' + b x y' + c y = \square$ 0 / g(x) .

كيف نميزها caushy . → 2 أس x ، 2 مشتقة — 1 أس x ، 1 مشتقة — لا يوجد مشتقة وأس الـ x = 0 .

الحل سيكون بدلالة t

★ طريقة الحل $\xrightarrow[(t = \ln x)]{\text{تحول إلى homog}}$ $a y'' + (b-a) y' + c y = \square$

وعند التكرار نقرب بـ t

حفظ ★ قانون التحويل من caushy إلى homog .

★ Examples: (للتمييز بين أنواع المعادلات)

1) $y'' + \sin(x)\, y' + y = 0$ → homog.

2) $x^2 y'' + y' + x y = \dfrac{1}{2x}$ → non-homog. (الطرف الذي بعد المساوي ليس صفر)

3) $2x^2 y'' + 5x y' + 6y = 0$ → caushy.

4) $x(x^2 y'' + 5 x y' + y) = 0$ → $x^3 y'' + 5x^2 y' + x y = 0$ (ليست caushy)

استراتيجية الحل

- r diff roots → $y = e^{rx}$ ★ جذور مختلفة
- r repeted roots → $y_1 = e^{rx}$, $y_2 = x e^{rx}$ ★ جذور مكررة
- complex roots → $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ → $e^{\alpha x} \sin(\beta x)$, $e^{\alpha x}\cos(\beta x)$.
  المميز سالب $= b^2 - 4ac < 0$
  $r = \alpha \pm \beta i$

Homog:

① Solve: $y'' - 5y' + 6y = 0$

sol → $r^2 - 5r + 6 = 0$

$(r-3)(r-2) = 0$

$\therefore r_1 = 3 ,\ r_2 = 2$ (diffrent roots)

$\therefore y_1 = e^{3x} ,\ y_2 = e^{2x}$

First sol. second sol.

general sol. or complementry sol. → $y_c = c_1 e^{3x} + c_2 e^{2x}$

لاحظ

الحل العام للـ homog ظهر فيه e والـ r هي أس الـ e.

② $y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0$

$r_1 = 1 ,\ r_2 = 1$ (repeted roots)

$\therefore y_1 = e^{1x} ,\ y_2 = x e^{1x}$

لأنه تكرر

$y_c = c_1 e^{x} + c_2 x e^{x}$.

③ Find the general sol. of:

$y'' + 2y' + 3y = 0$

sol $\boxed{1} r^2 + \boxed{2} r + \boxed{3} = 0$ ($a = 1$, $b = 2$, $c = 3$)

المميز

$\Delta = b^2 - 4ac$

$= 4 - 4 \cdot 1 \cdot 3$

$= 4 - 12 = -8$

بما أنه المميز سالب إذا أكو complex root

$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-2 \pm \sqrt{-8}}{2 \cdot 1}$

$\sqrt{-8} = \sqrt{-1} \cdot \sqrt{8} = i \cdot \sqrt{8}$

$= \frac{-2}{2} \pm \frac{\sqrt{8}\, i}{2} = \underbrace{-1}_{\alpha} \pm \underbrace{\frac{\sqrt{8}}{2}}_{\beta} i$

$\therefore y_1 = e^{\alpha x} \sin(\beta x) = e^{-1x} \sin\left(\frac{\sqrt{8}}{2} x\right)$

$y_2 = e^{\alpha x} \cos(\beta x) = e^{-1x} \cos\left(\frac{\sqrt{8}}{2} x\right)$.

$\therefore y_c = c_1 e^{-x} \sin\left(\frac{\sqrt{8}}{2} x\right) + c_2 e^{-x} \cos\left(\frac{\sqrt{8}}{2} x\right)$

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* Find the general sol. or/How many sol. we have?

$(r+5)^2 (r^2 - 9)^2 (r^2 + 3) = 0$

sol:

* $(r+5)^2 = 0 \to (r+5)(r+5) = 0 \therefore r_1 = -5 ,\ r_2 = -5 \to y_1 = e^{-5x} ,\ y_2 = x e^{-5x}$

* $(r^2 - 9)^2 = 0 \to (r^2 - 9)(r^2 - 9) = 0$

$\Rightarrow (r-3)(r+3)\ (r-3)(r+3) = 0$

$r_3 = 3 \quad r_4 = -3 \quad r_5 = 3 \quad r_6 = -3$

$y_3 = e^{3x} ,\ y_4 = e^{-3x}$

$y_5 = x e^{3x} \quad y_6 = x e^{-3x}$

* $(r^2 + 3) = 0 \to r^2 = -3 \to r = \pm\sqrt{-3} \to r = \pm\sqrt{3}\, i \ \therefore \alpha = 0 ,\ \beta = \sqrt{3}$

$y_7 = e^{0x} \cos(\sqrt{3} x) ,\ y_8 = e^{0x} \sin(\sqrt{3} x)$.

* general sol: $y_c = c_1 e^{-5x} + c_2 x e^{-5x} + \cdots + c_8 \sin(\sqrt{3} x)$.

we have 8 sol.s.

يعني بدو الي بدلالة r.

* Find the auxillary eq of the DE:

① $y_c = c_1 e^{x} + c_2 x e^{x} + c_3 e^{2x}$ ← اصلها homog لانه ال general sol. ظهر فيه $\underline{\underline{e}}$. وال r هي أس ال e.

$r = 1$, $r = 1$, $r = 2$

$(r-1)(r-1)(r-2) = 0$.

$= r^3 - 3r + 2 = 0$ ← the auxillary eq.

$= y''' - 3y' + 2y = 0$.

② $y = e^{2x} \cos(3x)$. ← فيها cos اذا الجذور complex وr على صورة $\alpha \pm \beta i$

$\alpha = 2$, $\beta = 3$.

$\therefore r = \alpha \pm \beta i$

$\therefore r = 2 \pm 3i$

$\Rightarrow r - 2 = \pm 3i$ ← نربع الطرفين لكي نتخلص من i وال $-1 = (i)^2$

$\therefore (r-2)^2 = (\pm 3i)^2$

$\therefore r^2 - 4r + 4 = -9$
$+9 \quad +9$

$\therefore r^2 - 4r + 13 = 0$.

$y'' - 4y' + 13y = 0$ #

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* Caushy :-

تكون بدلالة t عند التكرار نضرب ب t

solve: ① $x^2 y'' - 2 x y' - 4 y = 0$ ⇓ $\boxed{t = \ln x}$

(a, b)

عادة الها Caushy نحولها ل homog عن طريق القانون →

$a y'' + (b-a) y' + c y = 0$.

$\Rightarrow 1 y'' + (-2-1) y' + -4 y = 0$.

$y'' - 3y' - 4y = 0$.

$r^2 - 3r - 4 = 0$

$(r-4)(r+1) = 0$

$r_1 = 4$, $r_2 = -1$ (diff rout).

$y_1 = e^{4t}$, $y_2 = e^{-1t}$ (but $t = \ln x$)

$\therefore y_1 = e^{4 \ln x}$, $y_2 = e^{-1 \ln x}$

$y_1 = e^{\ln x^4}$, $y_2 = e^{\ln x^{-1}}$

$y_1 = x^4$, $y_2 = x^{-1}$ ⟶ $y_c = c_1 \underline{x}^4 + c_2 \underline{\underline{x}}^{-1}$

لاحظ
الحل العام للcaushy ظهر فيه x وال r اس ال x.

② $x^2y'' + xy' + 16y = 0$ (caushy)

$t = \ln x$

عالقانونه $\longrightarrow$ $ay'' + (b-a)y' + cy = 0$

$y'' + (1-1)y' + 16y = 0$

$y'' + 16y = 0$

$r^2 + 16 = 0$

$r^2 = -16$

$r = \pm\sqrt{-16}$

$r = \pm 4$

$\therefore \alpha = 0 , \beta = 4$

$\therefore y_1 = e^{0t}\cos(4t) \xrightarrow[t=\ln x]{\text{but}} = \cos(4\ln x)$

$y_2 = e^{0t}\sin(4t) = \sin(4\ln x)$

$\therefore y_c = c_1\cos(\ln x^4) + c_2\sin(\ln x^4)$

③ $y_c = c_1 x^{\boxed{-1}\,r_1} + c_2 x^{\boxed{-4}\,r_2}$ (caushy)

لانه فيها $x^{\square}$

Sol

$r_1 = -1$, $r_2 = -4$

$\therefore (r+1)(r+4) = 0$

$\therefore \underset{a}{\boxed{\ }}r^2 + \underset{b-a}{\boxed{5}}r + \underset{c}{\boxed{4}} = 0$

$y'' + 5y' + 4y = 0$

شكل ال caushy $*$ $\boxed{ax^2y'' + bxy' + cy = 0}$

$1x^2y'' + 6xy' + 4y = 0$.

بما انها caushy
فهي اصلاً على صورة
$ay'' + (b-a)y' + cy$

$\therefore b - a = 5$

$b - 1 = 5$

$\therefore \underline{b = 6}$

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$*$ Wronskian:

قانون 1

$$w(y_1, y_2)(x) = \underbrace{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}_{\text{det}} = y_1 y_2' - y_2 y_1'$$

**نستخدم هذا القانون اذا كانه معطيني y_1 و y_2.

**اذا كانت det $\neq 0$ فهنو
Linearly indep.

أمّا اذا كانت det $= 0$ فهنس
Linearly dep.

$*$ عندما يكون فوق القطر الرئيسي او تحته اصفار او كلاهما فإنه ال det هو حاصل ضرب اعداد القطر الرئيسي :

مثلا

$$6\times4\times1 = 24 = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} \quad \text{او} \quad 4 = 2\times1\times2 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

$$* \; w(y_1, y_2, y_3)(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

موقع y_1 الصف الاول والعمود الاول

الصف الاول ← العمود الاول ← $1+1$; العمود الثاني ← صف اول ← $1+2$

$$= y_1(-1)^{1+1}\begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} + y_2(-1)^{1+2}\begin{vmatrix} y_1' & y_3' \\ y_1'' & y_3'' \end{vmatrix} + y_3(-1)^{1+3}\begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix}$$

ما تبقى عندما نشطب الصف والعمود الذي يحوي y_1

$$= (-1)^2 y_1\begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} + (-1)^3 y_2\begin{vmatrix} y_1' & y_3' \\ y_1'' & y_3'' \end{vmatrix} + (-1)^4 y_3\begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix}$$

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* **Ex:** Find $w(\underbrace{\cos(2x)}_{y_1}, \underbrace{\sin(2x)}_{y_2})$ or show that $y_1 = \cos(2x)$ and $y_2 = \sin(2x)$ are Linearly independent ?

sol

$$w(\cos(2x), \sin(2x)) = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix} = 2\cos^2(2x) - -2\sin^2(2x)$$

$$= 2(\cos^2(2x) + \sin^2(2x))$$

$$= 2 * 1 = \underline{2} \ \#$$

$\therefore w(y_1, y_2)(x) = 2 \neq 0 \ \therefore y_1 \ \& \ y_2$ linearly indep.

قانون 2

* $$w(y_1, y_2)(x) = c \cdot e^{-\int p(x)\,dx}$$

معامل y'

** لازم لما اطلق عالقانون يكون معامل $y'' = 1$.

** نستخدم هذا القانون اذا كان معطيني معادلة وبده ال w .

* **Ex:** IF y_1 and y_2 sols For the D.E $[xy'' + (x-1)y' + 3y = 0]$, $x > 0$ Find $w(y_1, y_2)(x)$?

sol

$$\frac{\cancel{x}y''}{\cancel{x}} + \frac{(x-1)y'}{x} + \frac{3y}{x} = \frac{0}{x} \longleftarrow$$

عشان اطلق عالقانون لازم معامل $y'' = 1$

$$\rightarrow y'' + \underbrace{\left(\frac{x-1}{x}\right)}_{P(x)}y' + \frac{3}{x}y = 0$$

$$\therefore w(y_1, y_2)(x) = c \cdot e^{-\int \frac{x-1}{x}dx} \longrightarrow c \cdot e^{-\int 1 - \frac{1}{x}dx} \longrightarrow c \cdot e^{\ln x - x} \longrightarrow c\,x \cdot e^{-x}$$

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* **Ex:** $x^2y'' - 2y' + (3+x)y = 0$, and $w(y_1, y_2)(2) = 3$, Find $w(y_1, y_2)(5)$?

sol $x^2y'' - 2y' + (3+x)y = 0 \quad (\div x^2)$

$$\therefore y'' - \underbrace{\frac{2}{x^2}}_{P(x)}y' + \left(\frac{3+x}{x^2}\right)y = 0.$$

now: $w(y_1, y_2)(x) = c \cdot e^{-\int -\frac{2}{x^2}dx}$

$$= c \cdot e^{-\int -2x^{-2}dx}$$

$$w(y_1, y_2)(x) = c \cdot e^{-\frac{2}{x}}$$

$$w(y_1, y_2)(x) = 3e \cdot e^{-\frac{2}{x}}$$

$$\therefore w(y_1, y_2)(5) = 3 \cdot e^{\frac{5 \times 1}{5 \times 1}} \cdot e^{-\frac{2}{5}}$$

$$= 3e^{\frac{3}{5}} \ \#$$

To find the value of $\underline{(c)}$.

$$w(y_1, y_2)(2) = c \cdot e^{\frac{-2}{2}}$$

$$\frac{3}{e^{-1}} = \frac{c \cdot \cancel{e^{-1}}}{e^{-1}}$$

$$\therefore \boxed{c = 3e^1}$$

* extra ex: IF $w(y_1, y_2)(x) = x e^{-x}$, For $y'' + p(x) y' + \frac{3}{x} y = 0$. Find $p(x)$.

Sol:

$$w(y_1, y_2)(x) = c \cdot e^{-\int p(x)\,dx}$$

$$x e^{-x} = c \cdot e^{-\int p(x)\,dx}$$

نأخذ ln للطرفين حتى نتخلص من e.

$$\ln(x e^{-x}) = \ln(c \cdot e^{-\int p(x)\,dx})$$

من خواص ال ln انه في حالة الضرب يتحول الى جمع

$$\Rightarrow \ln(x) + \cancel{\ln}(e^{-x}) = \ln(c) + \cancel{\ln} e^{-\int p(x)\,dx}$$

$$\ln(x) + (-x) = \ln(c) + -\int p(x)\,dx$$

$$\ln(x) - x = \ln(c) - \int p(x)\,dx$$

نشتق حتى نتخلص من التكامل ونجد قيمة $p(x)$

$$\therefore \left(\int p(x)\,dx\right)' = \left(\underbrace{\ln(c)}_{\text{ثابت}} - \ln(x) + x\right)'$$

$$\therefore p(x) = 0 - \frac{1}{x} + 1$$

$$\therefore \boxed{p(x) = 1 - \frac{1}{x}}$$

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* قانون 3

$$y_2 = y_1 \cdot \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$$

* نستخدم هذا القانون اذا كان معطيني y_1 وبدو y_2 او الحل العام.

* لازم معامل $y'' = 1$.

* Example: if $x^2 y'' + 2x y' - 2y = 0$, $x > 0$, and the first sol. $y_1 = x$, Find the general sol. or Find the second sol. (y_2)?

Sol: بعد القسمة على x^2

$$y'' + \underbrace{\frac{2}{x}}_{p(x)} y' - \frac{2}{x^2} y = 0$$

$$\therefore y_2 = y_1 \cdot \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$$

$$= x \cdot \int \frac{e^{-\int \frac{2}{x}\,dx}}{x^2}\,dx$$

$$= x \cdot \int \frac{e^{-2\ln x}}{x^2}\,dx$$

$$= x \int \frac{\cancel{e}^{\cancel{\ln} x^{-2}}}{x^2}\,dx$$

$$= x \int \frac{x^{-2}}{x^2}\,dx$$

$$\rightarrow y_2 = x \int x^{-4}\,dx$$

$$y_2 = x \cdot \frac{x^{-3}}{-3}$$

$$y_2 = \frac{-1}{3} \cdot x^{-2}$$

اذا طلب y_2 هاي الجواب #

$$\therefore y_c = c_1 \cdot x + c_2 \cdot \frac{-1}{3} x^{-2}$$

ممكن يدمج الثابت مع $\frac{-1}{3}$ فيصبح الجواب

$$y_c = c_1 x + c_2 x^{-2}.$$

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* non - homog.

حفظ
جدول الفرضيات ل $g(x)$:

$g(x)$	y_p particular sol.
ثابت → 3	A
عبارة خطية → x	$Ax+B$
عبارة تربيعية → x^2+2	Ax^2+Bx+C
$3e^{\alpha x}$	$Ae^{\alpha x}$
$xe^{\alpha x}$	$(Ax+B)e^{\alpha x}$
$\sin(\beta x)$	$A\sin(\beta x)+B\cos(\beta x)$
$e^{\alpha x}\cos(\beta x)$	$Ae^{\alpha x}\cos(\beta x)+Be^{\alpha x}\sin(\beta x)$
$xe^{\alpha x}\sin(\beta x)$	$(Ax+B)e^{\alpha x}\sin(\beta x)+(Cx+D)e^{\alpha x}\cos(\beta x)$

عندما تكون المعادلة خطية ننتبه اذا وجد تكرار للصفر في الجزء ال homog

** non - homog : $\underbrace{y''+P(x)y'+q(x)y}_{\text{general sol, or Complementary sol. }(y_c)} = \underbrace{g(x)}_{\text{particular sol. }(y_p)}$

* $y = y_c + y_p$ ← general sol

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* Example: Find the general sol. $[y''-9y = 2x-1]$. ← non-homog

sol:

$y''-9y = 2x-1$ ← خطية

$y''-9y=0$

$r^2-9=0$

$(r-3)(r+3)=0$

$r_1=3,\ r_2=-3$ (diff roots).

$y_1 = e^{3x}$

$y_2 = e^{-3x}$

$y_c = c_1e^{3x}+c_2e^{-3x}$

$y_p = Ax+B$ ← the Form

$y' = A$

$y''=0$

نعوض

$\Rightarrow 0-9(Ax+B) = 2x-1$

$-9Ax-9B = 2x-1$

$\therefore -9A = 2 \rightarrow \boxed{A = \frac{-2}{9}}$

$-9B=-1 \rightarrow \boxed{B=\frac{1}{9}}$

بما انها خطية ننتبه اذا وجد تكرار للصفر في الحل ال homog. لكن الجذور هنا هي 3 و -3 لا يوجد صفر فلا نضرب ب x

$\therefore y_p = \frac{-2}{9}x+\frac{1}{9}$ ← particular sol.

لاحظ عندما تكون المعادلة non-homog فإنه الحل العام لا يظهر في كل الجذور C والذي لم يظهر فيه C هو y_p

general sol → $\therefore y = y_c + y_p$

$\therefore y = \underbrace{c_1e^{3x}+c_2e^{-3x}}_{\text{يحوي C}} + \underbrace{\frac{-2}{9}x+\frac{1}{9}}_{\text{لا يحوي C}}$

*Ex: Find the Form of particular sol.

$$y'' + y = \sin(x)\cos(2x)$$

نستخدم متطابقة - متى ما يصح بإمكاني استخدم الفرض.

$\rightarrow \sin a \cdot \cos b = \frac{\sin(a+b)}{2} + \frac{\sin(a-b)}{2}$

sol:

$y'' + y = 0$

$r^2 + 1 = 0$

$r = \pm i$

$\alpha = 0 \; , \; \beta = 1$

$\therefore y_1 = \cos(x)$

$y_2 = \sin(x)$

$\therefore y_c = c_1 \cos(x) + c_2 \sin(x)$.

$= \frac{\sin(3x)}{2} + \frac{\sin(-x)}{2} = \frac{1}{2}\sin(3x) - \frac{1}{2}\sin(x)$.

$y_p = A\sin(3x) + B\cos(3) + Cx\sin(x) + Dx\cos(x)$

Form

ضربت بـ x بسبب تكرار y_c بـ $\cos(x)$ و $\sin(x)$

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*Ex: Find the general sol. $[x^2 y'' - x y' + y = \ln x]$

homog حولنا إلى حسب القانون $a y'' + (b-a) y' + c y = \text{—}$

sol: $x^2 y'' - x y' + y = \ln(x) \xrightarrow[t = \ln x]{\text{(caushy)}} \boxed{y'' - 2y' + y = t}$ ← خطية

$\therefore y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0$

$r = 1 \; , \; r = 1$

$y_1 = e^t \rightarrow e^{\ln x} \rightarrow x$

$y_2 = t e^t \rightarrow (\ln x) e^{\ln x} \rightarrow x \ln(x)$

$y_c = c_1 x + c_2 x \ln(x)$

$y_p = At + B$ ← خطية ولا يوجد تكرار للصفر لا نضرب بـ t.

$y' = A$

$y'' = 0$

$\therefore 0 - 2A + At + B = t$

$\therefore \boxed{A = 1}$

$-2A + B = 0 \rightarrow -2 + B = 0 \quad \therefore \boxed{B = 2}$

$y_p = t + 2$ but $t = \ln x$

$\therefore y_p = \ln x + 2$

$\therefore y = y_c + y_p = \dots$

---

*Ex: $y''' - 2y'' + y' = x + 2e^{2x} - \cos(2x)$, Find the Form of y_p.

sol: $y''' - 2y'' + y' = 0$

$r^3 - 2r^2 + r = 0$

$r(r^2 - 2r + 1) = 0$

$r = 0 \; , \; r = 1 \; , \; r = 1$

$y_c = c_1 e^{0x} + c_2 e^{x} + c_2 x e^{x}$

$y_p = (Ax + B)x + Ce^{2x} + (D\cos(2x) + E\sin(2x))$

نضرب المعادلة الخطية بـ x. بسبب تكرار الصفر مرة واحدة ولو تكرر مرتين نضرب بـ x^2 وهكذا..

$\therefore y_p = Ax^2 + Bx + \dots$

$\Rightarrow$ **Ex:** $y^{(6)} - 81y'' = 7x + \sin(3x) + e^{-3x}$, Find the form of y_p.

sol $\Rightarrow$ $y^{(6)} - 81y'' = 0$

$r^6 - 81r^2 = 0$

$r^2(r^4 - 81) = 0$

$r^2(r^2 - 9)(r^2 + 9) = 0$

$r^2(r-3)(r+3)(r^2+9) = 0$

$r = 0, 0, 3, -3, \pm 3i$

$y_1 = c_1$ / $y_2 = c_2 x$ / $y_3 = e^{3x}$

$y_4 = e^{-3x}$ / $y_5 = \sin(3x)$

$y_6 = \cos(3x)$.

$y_p = (Ax+B)x^2 + C\sin(3x)x + D\cos(3x)x + Exe^{-3x}$

نضرب المعادلة كلها بـ x^2 لتكرار الصفر مرتين

نضرب بـ x لتكرار $\sin(3x)$ مرة واحدة.

نضرب بـ x لتكرار $\cos(3x)$ مرة واحدة.

نضرب بـ x لتكرار e^{-3x} مرة واحدة

---

* if $y = \underbrace{c_1 x + c_2 x \ln x}_{} + \underbrace{\ln x + 2}_{}$, Find the D.E.

بما انها تحوي c اذاً هاي y_c وبما انه على صورة $x^{\square}$ فهي Caushy.

لا تحوي c اذاً هاي y_p. $\leftarrow$ non-homog.

sol $\Rightarrow$ $r = 1, r = 1$

$(r-1)(r-1) = 0$

$r^2 - 2r + 1 = 0$

$\underset{a}{1}y'' \underset{b-a}{-2}y' + \underset{c}{1}y = 0$

$\therefore 1x^2y'' - xy' + y = 0$

عوض

$y_p = \ln x + 2$

$y' = \frac{1}{x}$

$y'' = \frac{-1}{x^2}$

$\Rightarrow \cancel{x^2} \cdot \frac{-1}{\cancel{x^2}} - \cancel{x} \cdot \frac{1}{\cancel{x}} + \ln x + 2 = g(x)$

$-\cancel{1} - \cancel{1} + \ln x + \cancel{2} = g(x)$

$\therefore g(x) = \ln(x)$. #

$\therefore$ the eq $\rightarrow x^2y'' - xy' + y = \ln x$. #

إذا كانت $g(x)$ ليست من الجدول نستخدم القانون التالي :

$$y_p = v_1 y_1 + v_2 y_2$$

sol : $v_1 = \int \frac{w_1}{w} dx$, $v_2 = \int \frac{w_2}{w} dx$.

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} , \quad w_1 = \begin{vmatrix} 0 & y_2 \\ g(x) & y_2' \end{vmatrix} , \quad w_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x) \end{vmatrix}$$

Ex: Find the general sol. of $y'' + y = \tan(x)$.

$g(x)$ ليست من الجدول ولا يمكن تعديلها $g(x)$ من الجدول فنحل على القانون السابق

sol: $y'' + y = \tan(x)$.

$y'' + y = 0$

$r^2 + 1 = 0 \rightarrow r = \pm i \rightarrow y_1 = \cos(x)$

$\alpha = 0 , \beta = 1$ $\quad y_2 = \sin(x)$.

$\therefore y_c = C_1 \cos(x) + C_2 \sin(x)$

* now to find y_p :

$$w = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = \cos^2(x) + \sin^2(x) = \boxed{1}$$

$$w_1 = \begin{vmatrix} 0 & \sin(x) \\ \tan(x) & \cos(x) \end{vmatrix} = -\sin(x)\tan(x) .$$

$$w_2 = \begin{vmatrix} \cos(x) & 0 \\ -\sin(x) & \tan(x) \end{vmatrix} = \sin(x) .$$

$$\therefore v_1 = \int \frac{-\sin(x)\tan(x)}{1} dx = -\int \frac{\sin^2(x)}{\cos(x)} dx = -\int \frac{1-\cos^2(x)}{\cos(x)} = -\int \sec(x) - \cos(x)\, dx$$

$$= -\ln|\sec(x) + \tan(x)| + \sin(x)$$

$$v_2 = \int \frac{\sin(x)}{1} dx = -\cos(x) .$$

$$\therefore y_p = v_1 y_1 + v_2 y_2$$

$$y_p = (\sin(x) - \ln|\sec(x) + \tan(x)|) \cdot \cos(x) + (-\cos(x)) \cdot \sin(x)$$

$$y = y_c + y_p .$$

لاتنسونا من صالح دعائكم ♡

majdoleen samara.

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* Laplace Transformation:

$$L(f(t)) = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

القانون الذي أوجدو منه القواعد التالية

$f(t) = 1$ const.

Ex: Find $L(1)$

$$L(1) = \int_0^\infty e^{-st} \cdot 1\, dt$$

$$= \lim_{K\to\infty} \int_0^K e^{-st}\, dt$$

$$= \lim_{K\to\infty} \left. \frac{-e^{-st}}{s} \right|_0^K$$

$$= \lim_{K\to\infty} \left(\frac{-e^{-Ks}}{s} + \frac{1}{s} \right)$$

$$= \lim_{K\to\infty} \frac{-e^{-Ks}}{s} + \lim_{K\to\infty} \frac{1}{s}$$

$$= 0 + \frac{1}{s}$$

$$= \frac{1}{s}$$

$$\therefore L(1) = \frac{1}{s}$$

$$L(a) = \frac{a}{s} \quad \#$$

$L^{-1}(F(s)) = f(t)$ $\qquad$ $L(f(t)) = F(s)$

const. $a \longrightarrow \frac{a}{s}$

$t^{a=n} \longrightarrow \frac{a!}{s^{a+1}}$ $\left(\frac{n!}{s^{n+1}}\right)$

$e^{at} \longrightarrow \frac{1}{s-a}$

$\sin(at) \longrightarrow \frac{a}{s^2+a^2}$

$\cos(at) \longrightarrow \frac{s}{s^2+a^2}$

$\sinh(at) \longrightarrow \frac{a}{s^2-a^2}$

$\cosh(at) \longrightarrow \frac{s}{s^2-a^2}$

---

* Examples: , Find:

① $L(2) = \frac{2}{s}$ $\quad$ (a)

② $L(t) = \frac{1!}{s^{1+1}} = \frac{1}{s^2}$ $\quad$ ($a=1$)

③ $L(t^3) = \frac{3!}{s^{3+1}} = \frac{3!}{s^4}$

④ $L(e^{7t}) = \frac{1}{s-7}$

⑤ $L(e^{-2t}) = \frac{1}{s+2}$

⑥ $L(\cos\sqrt{7}t) = \frac{s}{s^2+(\sqrt{7})^2} = \frac{s}{s^2+7}$

⑦ $L(\sinh(-8t)) = \frac{-8}{s^2-64}$

* properties of Laplace:

1. $L(\alpha f(t) \mp \beta g(t)) = \alpha L(f(t)) \mp \beta L(g(t))$

يعني : الـLaplace بتوزع على الجمع والطرح والثابت بخرج خارج الـ"L"

2. $L(e^{at} \cdot f(t)) = L(f(t))\Big|_{s \to s-a}$ → "shifting property"

3. $L(t^n \cdot f(t)) = (-1)^n \dfrac{d^n}{ds^n}\Big[\underbrace{L(f(t))}_{F(s)}\Big]$

---

* Examples:

① (1) $L(3 + 2e^{-7t}) = L(3) + 2L(e^{-7t})$

$$= \frac{3}{s} + 2 \cdot \frac{1}{s+7} = \frac{3}{s} + \frac{2}{s+7}.$$

② (2) $L(e^{2t+3}) = L(e^{2t} \cdot \underset{\text{const.}}{e^{3}}) = e^{3} L(e^{2t})$

$$= e^{3} \cdot \frac{1}{s-2}.$$

③ (5) $L(\sin(2t)\cos(3t)) = L\left(\frac{\sin(-t)}{2} + \frac{\sin(5t)}{2}\right)$

$$= \frac{-1}{2} L(\sin(t)) + \frac{1}{2} L(\sin(5t))$$

$$= \frac{1}{2} \cdot \frac{1}{s^2+1} + \frac{1}{2} \cdot \frac{5}{s^2+25}$$

④ (6) $L(\cos(2t + \frac{\pi}{6})) = L\left(\cos(2t) \cdot \cos(\frac{\pi}{6}) - \sin(2t) \cdot \sin(\frac{\pi}{6})\right)$

$$= L\left(\cos(2t) \cdot \frac{\sqrt{3}}{2} - \sin(2t) \cdot \frac{1}{2}\right)$$

$$= \frac{\sqrt{3}}{2} L(\cos(2t)) - \frac{1}{2} L(\sin(2t))$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{s}{s^2+4} - \frac{1}{2} \cdot \frac{2}{s^2+4} \quad \#$$

5 (7) $\mathcal{L}(\sinh^2(3t)) = \mathcal{L}\left(\left(\frac{e^{3t}-e^{-3t}}{2}\right)^2\right)$

$$= \frac{1}{4}\mathcal{L}\left(e^{6t} - 2e^{3t-3t} + e^{-6t}\right)$$

$$= \frac{1}{4}\mathcal{L}\left(e^{6t} - 2 + e^{-6t}\right)$$

$$= \frac{1}{4}\left(\frac{1}{s-6} - \frac{2}{s} + \frac{1}{s+6}\right) \quad \#$$

shifting

6 (3) $\mathcal{L}(e^{5t}\cdot\cosh(4t)) = \mathcal{L}(\cosh(4t))\Big|_{s\to s-5}$

$$= \frac{s}{s^2-16}\Big|_{s\to s-5} = \frac{(s-5)}{(s-5)^2-16}$$

7 (8) $\mathcal{L}(\cosh(2t)\cdot\sin(3t)) = \mathcal{L}\left(\left(\frac{e^{2t}+e^{-2t}}{2}\right)\cdot\sin(3t)\right)$

$$= \frac{1}{2}\mathcal{L}(e^{2t}\cdot\sin(3t)) + \frac{1}{2}\mathcal{L}(e^{-2t}\cdot\sin(3t))$$

$$= \frac{1}{2}\cdot\mathcal{L}(\sin(3t))\Big|_{s\to s-2} + \frac{1}{2}\mathcal{L}(\sin(3t))\Big|_{s\to s+2}$$

$$= \frac{1}{2}\cdot\frac{3}{s^2+9}\Big|_{s\to s-2} + \frac{1}{2}\cdot\frac{3}{s^2+9}\Big|_{s\to s+2}$$

$1 = t$ أُس

$$= \frac{1}{2}\,\frac{3}{(s-2)^2+9} + \frac{1}{2}\cdot\frac{3}{(s+2)^2+9}.$$

8 (4) $\mathcal{L}(t\cdot\sin(3t)) = (-1)^1\cdot\frac{d^1}{ds^1}\mathcal{L}[\sin(3t)]$

$$= -\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$= -\frac{-3(2s)}{(s^2+9)^2} = \frac{6s}{(s^2+9)^2}$$

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(9) $L(t^2 \cdot e^{3t}) = (-1)^2 \cdot \frac{d^2}{ds^2} L(e^{3t})$

المشتقة الثانية ←

$$= \frac{d^2}{ds^2}\left(\frac{1}{s-3}\right)$$

$$= \frac{2s-6}{(s-3)^4}$$

or $= \frac{2}{(s-3)^3}$ #

$= \frac{1}{s-3}$

المشتقة الأولى

$= \frac{-1}{(s-3)^2}$

المشتقة الثانية

$= \frac{-\,-1 * 2(s-3)}{(s-3)^4}$

$= \frac{2s-6}{(s-3)^4}$

or $= \frac{2\cancel{(s-3)}}{(s-3)^{\cancel{4}}}$

$\boxed{= \frac{2}{(s-3)^3}}$

(10) $L(t \cdot e^{-2t} \cdot \sin(2t)) = L(t\sin(2t))\Big|_{s \to s+2}$

اشتق بعدين الحل shifting. →

$$= (-1)^1 \frac{d}{ds} L(\sin(2t))\Big|_{s \to s+2}$$

$$= -1 \frac{d}{ds}\left(\frac{2}{s^2+4}\right)\Big|_{s \to s+2}$$

$$= \cancel{-}\,\frac{\cancel{-}2 \cdot 2s}{(s^2+4)^2}\Big|_{s \to s+2} = \frac{4s}{(s^2+4)^2}\Big|_{s \to s+2} = \frac{4(s+2)}{((s+2)^2+4)^2}$$

or

الحل shifting بعدين اشتق →

$$= (-1)^1 \frac{d}{ds} L(e^{-2t}\sin(2t))$$

$$= -\frac{d}{ds}\left(\frac{2}{s^2+4}\Big|_{s \to s+2}\right)$$

$$= -\frac{d}{ds}\left(\frac{2}{(s+2)^2+4}\right)$$

$$= \cancel{-}\,\frac{\cancel{-}2 \cdot 2(s+2)}{((s+2)^2+4)^2}$$

$$= \frac{4(s+2)}{((s+2)^2+4)^2} \quad \#$$

* Inverse of Laplace:

$$\mathcal{L}(f(t)) = F(s)$$

$$\rightarrow \mathcal{L}^{-1}\mathcal{L}(f(t)) = \mathcal{L}^{-1}(F(s))$$

$$\rightarrow \boxed{f(t) = \mathcal{L}^{-1}(F(s))}.$$

* Examples:

① $\mathcal{L}^{-1}\left(\frac{2}{s}\right) = 2$

② $\mathcal{L}^{-1}\left(\frac{2}{s^3}\right) = t^2 \rightarrow \boxed{\mathcal{L}^{-1}\left(\frac{a!}{s^{a+1}}\right)}$

$2! = a!$

$2+1 = a+1$

③ $\mathcal{L}^{-1}\left(\frac{3}{s^5}\right) = \frac{3}{4!}\mathcal{L}^{-1}\left(\frac{4!}{s^5}\right) = \frac{3}{4!}\cdot t^4$

④ $\mathcal{L}^{-1}\left(\frac{1}{s+4}\right) = e^{-4t} \rightarrow \boxed{\frac{1}{s-a}}$

⑤ $\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) = \cos(3t)$

$(3)^2$

⑥ $\mathcal{L}^{-1}\left(\frac{1}{s^2+7}\right) = \frac{1}{\sqrt{7}}\mathcal{L}^{-1}\left(\frac{\sqrt{7}}{s^2+7}\right) = \frac{1}{\sqrt{7}}\sin(\sqrt{7}t).$

a

a^2

$a^2 = 7$

$a = \sqrt{7}$

⑦ $\mathcal{L}^{-1}\left(\frac{1}{4s^2-3}\right) = \mathcal{L}^{-1}\left(\frac{1}{4(s^2-\frac{3}{4})}\right) = \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s^2-\frac{3}{4}}\right)$

$a^2 = \frac{3}{4}$

$a = \frac{\sqrt{3}}{2}$

$$= \frac{\frac{1}{4}}{\frac{\sqrt{3}}{2}}\mathcal{L}^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{s^2-\frac{3}{4}}\right)$$

$= \frac{1}{4} \div \frac{\sqrt{3}}{2}$

$= \frac{1}{\cancel{4}_2} \times \frac{\cancel{2}}{\sqrt{3}}$

$= \frac{1}{2\sqrt{3}}.$

$$= \frac{1}{2\sqrt{3}}\sinh\left(\frac{\sqrt{3}}{2}t\right)$$

(6)

2 ⑨ لازم تشوف $s-2$

$$L^{-1}\left(\frac{s}{(s-2)^2+16}\right) = L^{-1}\left(\frac{s-2+2}{(s-2)^2+16}\right) = L^{-1}\left(\frac{s-2}{(s-2)^2+16}\right) + L^{-1}\left(\frac{2}{(s-2)^2+16}\right)$$

$$= e^{2t}\cos(4t) + \frac{1}{2}\sin(4t)$$

1 ⑧
$$L^{-1}\left(\frac{s}{s^2-\sqrt{e}}\right) = \cosh(\sqrt[4]{e}\,t)$$

$$a^2 = e^{\frac{1}{2}}$$

$$a = e^{\frac{1}{4}} \rightarrow a = \sqrt[4]{e}$$

⑩
$$L^{-1}\left(\frac{1}{s^2-2s-3}\right) = L^{-1}\left(\frac{1}{\underbrace{s^2-2s+1}_{(s-1)(s-1)}-1-3}\right) = L^{-1}\left(\frac{1}{(s-1)(s-1)-4}\right)$$

واكمال المربع يكون ب $\pm \left(\frac{\text{معامل } s}{2}\right)^2$

طريقة ① باكمال المربع

$$= \frac{1}{2}L^{-1}\left(\frac{1\times 2}{(s-1)^2-4}\right) = \frac{1}{2}e^{1t}\sinh(2t) = \frac{e^{t}}{2}\cdot\frac{e^{2t}-e^{-2t}}{2}$$

$$= \frac{e^{3t}-e^{-t}}{4}$$

توحيد مقامات

طريقة ② كسور جزئية (partial fraction)

$$\rightarrow L^{-1}\left(\frac{1}{s^2-2s-3}\right) \Rightarrow \frac{1}{(s-3)(s+1)} = \frac{A\,(s+1)}{s-3\,(s+1)} + \frac{B\,(s-3)}{s+1\,(s-3)}$$

$$\rightarrow \quad 1 = A(s+1) + B(s-3)$$

$$s=-1 \rightarrow 1 = A(0) + B(-4) \rightarrow \boxed{B = \frac{-1}{4}}$$

$$s=3 \rightarrow 1 = A(4) + B(0) \rightarrow \boxed{A = \frac{1}{4}}$$

$$\rightarrow L^{-1}\left(\frac{\frac{1}{4}}{s-3} - \frac{\frac{1}{4}}{s+1}\right) = \frac{1}{4}L^{-1}\left(\frac{1}{s-3}\right) + \frac{-1}{4}L^{-1}\left(\frac{1}{s+1}\right)$$

$$= \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t}$$

#

⑪
$$L^{-1}\left(\frac{3}{(s^2+1)(s^2+4)}\right) \xrightarrow{\text{sol 1}} \text{by partial fraction } \frac{3}{(s^2+1)(s^2+4)} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{s^2+4} \cdots$$

كسور جزئية

sol 2 — or

$$\Rightarrow \cancel{3}\cdot\frac{1}{\cancel{-3}}L^{-1}\left(\frac{(s^2+1)-(s^2+4)}{(s^2+1)(s^2+4)}\right)$$

$$s^2+1-s^2-4 = -3$$

$$= \frac{1}{2}\cdot -L^{-1}\left(\frac{\cancel{s^2+1}+2}{\cancel{(s^2+1)}(s^2+4)}\right) + L^{-1}\left(\frac{\cancel{s^2+4}}{(s^2+1)\cancel{(s^2+4)}}\right)$$

$$= \frac{-1}{2}\sin(2t) + \sin(t).$$

* we know: $\mathcal{L}(t^n f(t)) = (-1)^n \frac{d^n}{ds^n}\left[\underbrace{\mathcal{L}(f(t))}_{F(s)}\right]$

when $\underline{n=1} \Rightarrow \mathcal{L}(t\,f(t)) = (-1)\frac{d}{ds}[F(s)]$

نشغل $\mathcal{L}^{-1}$ للطرفين ←

نقسم ع t ← $t\,f(t) = (-1)\,\mathcal{L}^{-1}\frac{d}{ds}[F(s)]$

$$f(t) = \frac{-1}{t}\,\mathcal{L}^{-1}\frac{d}{ds}[F(s)]$$

but $f(t) = \mathcal{L}^{-1}[F(s)]$ $\Rightarrow$ $\boxed{\mathcal{L}^{-1}[F(s)] = \frac{-1}{t}\,\mathcal{L}^{-1}\frac{d}{ds}[F(s)]}$

لما يطلب مني $\mathcal{L}^{-1}$ لاقترانه
مش من الجدول ..

Ex $\boxed{1}$ Find $\mathcal{L}^{-1}\left\{\ln\left(\frac{s^2-9}{s+1}\right)\right\}$

sol: $\mathcal{L}^{-1}\left(\ln(s^2-9) - \ln(s+1)\right)$

$$= \mathcal{L}^{-1}(\ln(s^2-9)) - \mathcal{L}^{-1}(\ln(s+1))$$

$$= \frac{-1}{t}\,\mathcal{L}^{-1}\left(\frac{2s}{s^2-9}\right) - \frac{-1}{t}\,\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$= \frac{-2}{t}\cosh(3t) + \frac{1}{t}e^{-t} \quad \#$$

$\boxed{2}$ $\mathcal{L}^{-1}\left(\tan^{-1}\left(\frac{s}{3}\right) + \frac{\pi}{2}\right)$ $\longrightarrow$ مشتقة $\tan^{-1}x$ = $\left(\text{مشتقة الزاوية}\right)\cdot\frac{1}{1+x^2}$

$$= \frac{-1}{t}\,\mathcal{L}^{-1}\left(\frac{1}{3}\cdot\frac{1}{\frac{s^2}{3^2}+1\times\frac{9}{9}} + 0\right)$$

$$= \frac{-1}{3t}\,\mathcal{L}^{-1}\left(\frac{1\times 9}{s^2+9}\right)$$

$$= \frac{-3}{3\times t}\,\mathcal{L}^{-1}\left(\frac{1\times 3}{s^2+9}\right)$$

$$= \frac{-1}{t}\sin(3t).$$

* IF $L\left(\frac{1}{\sqrt{t}}\right) = \sqrt{\frac{\pi}{s}}$, Find : $L\left(t \cdot \sqrt{\frac{3}{\pi t}}\right)$?

<u>sol</u>

$$L\left(t \cdot \sqrt{\frac{3}{\pi t}}\right) = (-1)^1 \frac{d}{ds} L\left(\underbrace{\sqrt{\frac{3}{\pi}}}_{\text{const.}} \cdot \frac{1}{\sqrt{t}}\right)$$

$$= -\frac{\sqrt{3}}{\sqrt{\pi}} \frac{d}{ds} L\left(\frac{1}{\sqrt{t}}\right).$$

$$= \frac{-\sqrt{3}}{\cancel{\sqrt{\pi}}} \frac{d}{ds}\left(\underbrace{\cancel{\sqrt{\pi}}}_{\text{Const}} \cdot \frac{1}{\sqrt{s}}\right)$$

$$= -\sqrt{3} \frac{d}{ds}\left(\frac{1}{\sqrt{s}}\right)$$

$$= \cancel{-}\sqrt{3}\left[\frac{\cancel{-}1 \cdot \frac{1}{2\sqrt{s}}}{(\sqrt{s})^2}\right]$$

$$= \sqrt{3} \times \frac{1}{2\sqrt{s}} \div s$$

$$= \sqrt{3} \cdot \frac{1}{2\sqrt{s}} \times \frac{1}{s}$$

$$= \frac{\sqrt{3}}{2s\sqrt{s}} \quad \#$$

(8)

eg: If $L^{-1}\left(\ln\left(\frac{s-5}{s+2}\right)\right) = F(t)$, Find $\underbrace{\int_0^{\infty} F(t)\cdot e^{-6t}\,dt}$.

يعني مطلوب إيجاد $L(F(t))$

عندما $\boxed{+6 = s}$

sol take "L" For both side

$$L\{L^{-1}\left(\ln\left(\frac{s-5}{s+2}\right)\right)\} = L(F(t))$$

$$\ln\left(\frac{s-5}{s+2}\right) = L(F(t))$$

$\boxed{s=6} \rightarrow \ln\left(\frac{6-5}{6+2}\right) = L(F(t))$

$\rightarrow \ln\left(\frac{1}{8}\right) = L(F(t))$ #

$\ln(8^{-1}) = L(F(t))$

** Solving I.V.P by laplace transformation.

$$L(y^{n}(t)) = s^{n} L(y(t)) - s^{n-1} y(0) - s^{n-2} y(0) - \dots - y^{n-2}(0) - y^{n-1}(0) .$$

$n=1$

$\Rightarrow L(y^{(1)}(t)) = s^{1} L(y(t)) - y(0)$.

$n=2$ حفظ

$\Rightarrow L(y''(t)) = s^{2} L(y(t)) - s y(0) - y'(0)$

$n=3$

$\Rightarrow L(y'''(t)) = s^{3} L(y(t)) - s^{2} y(0) - s y'(0) - y''(0)$.

* Example: use laplace transformation to solve the I.V.P.

$$y'' - 4y' + 4y = t^{3} e^{2t} \quad , y(0) = 0 \quad , y'(0) = 0$$

$$L(y'') - 4L(y') + 4L(y) = L(t^{3} e^{2t})$$

$$s^{2} L(y) - s y(0) - y'(0) - 4[s L(y) - y(0)] + 4L(y) = \frac{3!}{(s-2)^{4}}$$

$$s^{2} L(y) - \cancel{s y(0)} - \cancel{y'(0)} - 4s L(y) + \cancel{4 y(0)} + 4L(y) = \frac{3!}{(s-2)^{4}}$$

عامل مشترك

$$L(y)(s^{2} - 4s + 4) = \frac{6}{(s-2)^{4}}$$

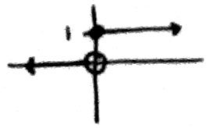
$$\therefore L(y) = \frac{6}{(s-2)^{4}(s^{2}-4s+4)} \Rightarrow L(y) = \frac{6}{(s-2)^{6}} \Rightarrow y = L^{-1}\left(\frac{6}{(s-2)^{6}}\right)$$

$(s-2)(s-2)$

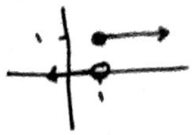
$$\therefore y = 6 \cdot e^{2t} \cdot \frac{t^{5}}{5!}$$ #

The unit step funct.

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$ رسمة $\longrightarrow$



$$u(t-1) = \begin{cases} 0 & t < 1 \\ 1 & 0 \geq t \geq 1 \end{cases} \longrightarrow$$



بصورة عامة

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & 0 > t \geq a \end{cases}$$

*** Rules :**

① $L(u(t-a)) = \dfrac{e^{-as}}{s}$.

② $L(f(t-a)\,u(t-a)) = e^{-as} L(f(t))$.

③ $L(f(t)\,u(t-a)) = e^{-as} L(f(t+a))$.

Ex :

قاعدة 1

1) $L(u(t-\overset{a}{3})) = \dfrac{e^{-3s}}{s}$

2) $L(u(t-\overset{a}{\tfrac{\pi}{2}})) = \dfrac{e^{-\frac{\pi}{2}s}}{s}$

قاعدة 2

3) $L(\underbrace{e^{t-3}}_{f(t-3)} \cdot \underbrace{u(t-\overset{a}{3})}_{u(t-3)}) = e^{-3t} \cdot L(e^{t}) = e^{-3t} \cdot \dfrac{1}{s-1}$

4) $L(\sin(t-\tfrac{\pi}{2}) \cdot u(t-\overset{a}{\tfrac{\pi}{2}})) = e^{-\frac{\pi}{2}s} \cdot L(\sin(t)) = e^{-\frac{\pi}{2}s} \cdot \dfrac{1}{s^2+1}$.

قاعدة 3

5) $L(\sin(t) \cdot u(t-\overset{a}{\tfrac{\pi}{2}})) = e^{-\frac{\pi}{2}s} \cdot L(\sin(t+\tfrac{\pi}{2})) = e^{-\frac{\pi}{2}s} \cdot L(\cos(t)) = e^{-\frac{\pi}{2}s} \cdot \dfrac{s}{s^2+1}$

متطابقة cos(t)

6) $L(t^2 \cdot u(t-\overset{a}{5})) = e^{-5s} L((t+5)^2)$

$= e^{-5s} L(t^2+10t+25)$

$= e^{-5s} \left(L(t^2) + 10L(t) + L(25)\right)$

$= e^{-5s} \left(\dfrac{2}{s^3} + 10 \cdot \dfrac{1}{s^2} + \dfrac{25}{s}\right)$ #

$$7)\ L(\underbrace{e^{3t}}_{\text{shift}} \cdot t \cdot u(t-2)) = L(t \cdot u(t-2))\Big|_{s \to s-3}$$

$$= e^{-2s} \cdot L(t+2)\Big|_{s \to s-3} = e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right)\Big|_{s \to s-3}$$

$$= e^{-2(s-3)}\left(\frac{1}{(s-3)^2} + \frac{2}{(s-3)}\right)$$

$$* \quad f(t) = \begin{cases} \boxed{2} & t<1 \\ 0 & 1<t<2 \\ e^t & t>2 \end{cases} \quad , \text{Find } L(f(t))\ ??$$

$$f(t) = 2 + (0-2)\cdot u(t-1) + (e^t - 0)\cdot u(t-2)$$

$$f(t) = 2 - 2u(t-1) + e^t u(t-2)$$

$$L(f(t)) = L(2) - 2L(u(t-1)) + L(e^t \cdot u(t-2))$$

$$= \frac{2}{s} - 2\frac{e^{-1s}}{s} + e^{-2s}\cdot L(e^{t+2}) \qquad L(e^t \cdot \underbrace{e^2}_{\text{ثابت}})$$

$$= \frac{2}{s} - \frac{2e^{-s}}{s} + e^{-2s}\cdot e^2 \cdot L(e^t)$$

$$= \frac{2}{s} - \frac{2e^{-s}}{s} + e^{-2s+2}\cdot\frac{1}{s-1}$$

$$* \quad f(t) = \begin{cases} t & t<3 \\ 2 & 3<t<5 \\ e^t & t>5 \end{cases} \quad , \text{Find } L(f(t))\ ?$$

$$f(t) = t + (2-t)\,u(t-3) + (e^t - 2)\cdot u(t-5)$$

$$L(f(t)) = L(t) + 2L(u(t-3)) - L(t\cdot u(t-3)) + L(e^t\cdot u(t-5)) - 2L(u(t-5))$$

$$\vdots$$

اكمل :

* $L^{-1}\left(\frac{e^{-as}}{s}\right) = u(t-a)$

* $L^{-1}\left(e^{-as} F(s)\right) = F(t-a) \cdot L^{-1}(F(s))\Big|_{t \to t-a}$

* Ex:

① $L^{-1}\left(\frac{e^{-2s}}{s}\right) = u(t-2)$

② $L^{-1}\left(\frac{e^{-2s}}{s^2+9}\right) = L^{-1}\left(e^{-2s} \cdot \frac{1}{s^2+9}\right)$

$= u(t-2) \cdot \frac{1}{3} L^{-1}\left(\frac{1*3}{s^2+9}\right)\Big|_{t \to t-2}$

$= \frac{1}{3} u(t-2) \cdot \sin(3t)\Big|_{t \to t-2}$

$= \frac{1}{3} u(t-2) \cdot \sin(3(t-2))$.

③ $L^{-1}\left(\frac{e^{-3s}}{(s-4)}\right) = L^{-1}\left(e^{-3s} \cdot \frac{1}{s-4}\right)$

$= u(t-3) \; L^{-1}\left(\frac{1}{s-4}\right)\Big|_{t \to t-3}$

$= u(t-3) \cdot e^{4t}\Big|_{t \to t-3} = u(t-3) \cdot e^{4(t-3)}$.

④ $L^{-1}\left(e^{-4s} \frac{s-2}{(s-2)^2+81}\right)$ e^{2t} sheft

$= u(t-4) \cdot L^{-1}\left(\frac{s-2}{(s-2)^2+81}\right)\Big|_{t \to t-4}$ $a=9$

$= u(t-4) \cdot \left[e^{2t} \cdot \cos(9t)\right]_{t \to t-4}$

$= u(t-4) \cdot e^{2(t-4)} \cdot \cos(9(t-4))$. #