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NO

Differentiation Equations

Chapter 1:-

- function   1. $y = f(x)$: fn of one variable

dependent variable   independent variable

so, we can find y', y'', $\frac{dy}{dx}$ ---

2. $y = f(x, t)$: fn of two variables

dependent   independent

so, we use $\frac{dy}{dx}$, $\frac{dy}{dt}$, $\frac{d^2y}{dx^2}$, $\frac{d^2y}{dt\,dx} = \frac{d\left(\frac{dy}{dx}\right)}{dt}$

* Definition :- A Differential Equ (D.E) is an equ which contains derivatives of one or more dependent variables with respect to one or more independent variables

⟹ Classfiction of D.E :-

① By type :- O.D.E — y', y'', y''', $\frac{dy}{dx}$

P.D.E :- $\frac{dy}{dx}$, $\frac{dy}{dt}$, $\frac{d^2y}{dx\,dt}$

⟹

NO.

(2) By order :- the order of the D.E

" " " of the highest derivative

(ordinary or partial) in the D.E

ex:- 1. $\sin x \frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^7 = e^x$

O.D.E of order 3

2. $y^{(4)}(x) + 5\ln\left(\frac{dy}{dx}\right) = \sin x$

O.D.E of order 4

(3) By Linearity :- a linear O.D.E of order n has the form:-

$$a_n(x)\, y^{(n)}(x) + a_{n-1}\, y^{(n-1)}(x) + \cdots\, a_1(x)\, y'(x) + a_0(x)y(x) = f(x)$$

⇒ the O.D.E is linear if

1. the power of all derivative is 1
2. the coefficient of all " are fn of x only (or constant)
3. No derivatives multiples / composition functions

ex:- 1. $x^3 \frac{d^2y}{dx^2} + e^x \frac{dy}{dx} = x$

linear.

2. $y'(x)\,(y''(x))^2 + y(x) = x$

not linear.

3. $\sqrt{y} - y'(x) = x$

not linear

ex:- 1. $y^2 \frac{dx}{dy} + yx = x$

$\Rightarrow y^2 \frac{dx}{dy} + \cancel{x}(y-1) = 0$

Linear in x and not for y

2. $(x - xy) \frac{dy}{dx} = y^2$

not linear in y

$\frac{dx}{dy} = \frac{x - xy}{y^2} \Rightarrow y^2 \frac{dx}{dy} - x(y-1) = 0$

linear for x

Type of solutions:-

1. explicit solution
- $y(x) = \dots$ fn
- substitute in the DE
- find y, y', y''
- satisfy the DE

2. implicit
- equation in x & y
- substitute in the DE
- use Implicit diff to find y, y', y''
- satisfy the DE

ex: show that $y(x) = \cos x$ is a solution explicit to the DE $y'' + y = 0$?

$y = \cos x \quad y' = -\sin x \quad y'' = -\cos x$

$-\cos x + \cos x = 0 \Rightarrow$ explicit sol ✓

Ex:- show that $y^2 - x^3 + 8 = 0$ is an implicit solution to the D E

$y \frac{dy}{dx} - \frac{3}{2} x^2 = 6$ in the interval $(2, \infty)$

$$y^2 - x^3 + 8 = 0$$

$$2y \frac{dy}{dx} - 3x^2 = 0 \Rightarrow \frac{dy}{dx} = \frac{3x^2}{2y}$$

نعوضها بالمعادلة الأصلية

* Chapter 2:- solving first order O.D.E

- $\frac{dy}{dx} = f(x, y)$
- $a_1(x) y'(x) + a_0(x) y(x) = f(x)$
- $(N(x,y)) dy + (M(x,y)) dx = 0$ ex:- $(y^2 - xy) dy - (x^2 - y) dx = 0$

- Method to solve order O.D.E:-

1. linear $a_1(x) \frac{dy}{dx} + a_x \, y(x) = f(x)$
2. non-linear * $\frac{dy}{dx} = F(x, y)$

   * $N(x,y) dy + M(x,y) dx = 0$

* Method 1:- separable O.D.E

- Given $dy/dx = F(x, y)$
- If $F(x,y) = h(x) \cdot g(y)$ then the D.E is said to be separable.

NO.

Ex:- The D.E $\frac{dy}{dx} = \frac{y^2 x - x}{y e^x - y}$?

$$dy/dx = \frac{x(y^2-1)}{y(e^x-1)} = \left(\frac{x}{e^x-1}\right)\cdot\left(\frac{(y^2-1)}{y}\right)$$

$$= h(x) \cdot g(y)$$

- So $\Rightarrow$ $dy/dx = h(x) \cdot g(y)$

$$\int \frac{dy}{g(y)} = \int h(x)\, dx$$

$$G(y) + C_1 = H(x) + C_2$$

$$G(y) = H(x) + C$$

ex:- solve the D.E (or find the general solution)

1. $x\, y'(x) - \sqrt{1-y^2} = 0$?

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{x} = \left(\frac{1}{x}\right) \cdot \left(\sqrt{1-y^2}\right) \text{ is separable}$$

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{x} \Longrightarrow \sin^{-1}(y) = \ln|x| + c$$

$y = y(x) = \sin(\ln|x| + c)$ is an explicit sol

2. $xy' - \sqrt{1-y^2} = 0 \qquad y(1) = 1/2 \ ?$

نفس خطوات حل السؤال الي قبله

$y(x) = \sin^{-1}(\ln x + c)$

$x=1,\ y=\frac{1}{2} \Rightarrow \frac{1}{2} = \sin c \qquad \Rightarrow \qquad c = \pi/6$

$y(x) = \sin\left(\ln|x| + \frac{\pi}{6}\right)$: unique solution

3. $(xy - 3y + x - 3)\,dy + (xy + 2y + x + 2)\,dx = 0$

$$\frac{dy}{dx} = \frac{-(xy + 2y + x + 2)}{(xy - 3y + x - 3)}$$

$$= \frac{-(y(x+2) + x + 2)}{(y(x-3) + (x-3))} \Rightarrow \frac{-\cancel{(y+1)}(x+2)}{\cancel{(y+1)}(x-3)}$$

$$\int dy = -\int \frac{x+2}{x-3}$$

$$y = -\int \left(1 + \frac{5}{x-3}\right) dx$$

$$y = -(x + 5\ln|x-3|) + c$$

$$y(x) = -x - 5\ln|x-3| + c$$

explicit sol

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ex:- solve :- $(xy - 2y - 3x + 6)\,dy - (xy - 4x + 2y - 8)\,dx = 0$?

$y(x-2) - 3(x-2)$ $\qquad$ $x(y-4) + 2(y-4)$

$(y-3)(x-2)$ $\qquad$ $(x+2)(y-4)$

$$\int \frac{dy}{dx} = \int \frac{(y-4)(x+2)}{(y-3)(x-2)} \Rightarrow \ldots$$

note:- $M(x,y)\,dx + N(x,y)\,dy = 0$

If $M(x,y) = f_1(x) \cdot g_1(y)$ and $N(x,y) = f_2(x) \cdot g_2(y)$ then the D.E is separable

* H.w solve:- 1. $\frac{dy}{dx} = (1+y^2) \cdot \tan^{-1} x$ $\qquad$ $y(0) = \sqrt{3}$

2. $y' = \frac{x - e^{-x}}{y - e^{-y}}$

Sol:- 1. $\int \frac{dy}{1+y^2} = \int dx \tan^{-1} x$ is separable

$\tan^{-1} y = x\tan^{-1} x - \frac{1}{2}\ln|1+x^2| + C$

$\frac{\pi}{6} = 0 - 0 + C \Rightarrow C = \pi/6$

$\int dx \tan^{-1} x$

$v = \tan^{-1} x \Rightarrow dv = \frac{1}{1+x^2}\,dx$

$du = dx \Rightarrow u = x$

$x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx$

$x\tan^{-1} x - \frac{1}{2}\ln|1+x^2| + C$

$y = \tan\left(x\tan^{-1} x - \frac{1}{2}\ln|1+x^2| + \frac{\pi}{6}\right)$ explicit

2. $\frac{dy}{dx} = (x - e^{-x}) \cdot \left(\frac{1}{y - e^{-y}}\right)$ is separable

$\int y - e^{-y}\,dy = \int x - e^{-x}\,dx \Rightarrow \frac{y^2}{2} + e^{-y} = \frac{x^2}{2} + e^{-x} + C$

implicit

* Method 2 :- Homogeneous D.E متجانس

If the D.E $\frac{dy}{dx} = F(x,y)$ is not separable, then check if $F(x,y) = G\left(\frac{y}{x}\right)$

$\frac{dy}{dx} = \frac{xy - y^2}{x^2} \Rightarrow \frac{dy}{dx} = \left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2$ مثال توضيحي

let $u = \frac{y}{x} \rightarrow U - U^2$

1. $u = \frac{y}{x}$
2. $xu = y$
3. diff $\Rightarrow x\frac{du}{dx} + u = \frac{dy}{dx}$

Now substitute in the D.E $\Rightarrow \frac{dy}{dx} = G\left(\frac{y}{x}\right)$

$x\frac{du}{dx} + u = G(u) \Rightarrow \frac{du}{dx} = \frac{G(u) - U}{x}$

is separable now solve

then replace u by $\frac{y}{x}$

ex:- 1. $(x-2y)\,dy - (x+y)\,dx = 0$?

not sep

hom?

$\div x \Rightarrow \left(1 - \frac{2y}{x}\right)dy - \left(1 + \frac{y}{x}\right)dx = 0$

$\frac{dy}{dx} = \frac{\left(1 + \frac{y}{x}\right)}{\left(1 - \frac{2y}{x}\right)} \Rightarrow$ let $u = \frac{y}{x}$

ex:- 1. $(y-4x)\,dx-(x-y)\,dy=0$

is not sep

$$\frac{dy}{dx}=\frac{y-4x}{x-y}=\frac{\frac{y}{x}-4}{1-\frac{y}{x}}$$

$$u=\frac{y}{x}\Rightarrow xu=y\overset{\text{diff}}{\Rightarrow}\frac{x\,du}{dx}+u=\frac{dy}{dx}$$

$$\frac{x\,du}{dx}+u=\frac{u-4}{1-u}\qquad \text{is sep}$$

$$\frac{x\,du}{dx}=\frac{u-4}{1-u}-u\Rightarrow\frac{du}{dx}=\frac{u^2-4}{1-u}\cdot\frac{1}{x}$$

$$\int\frac{1-u}{u^2-4}\qquad \cancel{\int\frac{u^2-4}{1-u}\,du}=\int\frac{1}{x}\,dx$$

$$=\ln|x|+C$$

$$=\int\frac{A}{u-2}+\frac{B}{u+2}$$

$$A(u-2)+B(u+2)=1-u$$

$$A=\frac{-1}{4}\qquad B=\frac{-3}{4}$$

$$=\frac{-1}{4}\ln|u-2|-\frac{3}{4}\ln|u+2|$$

$$\frac{-1}{4}\left(\ln\left|\frac{y}{x}-2\right|\left|\frac{y}{x}+2\right|^3\right)=\ln|x|+C$$

2. $x\,dy-y(\ln(y)-\ln(x)+1)\,dx=0$??

$$x\,dy-y\left(\ln\left(\frac{y}{x}\right)+1\right)dx=0$$

$$dy-\frac{y}{x}\left(\ln\left(\frac{y}{x}\right)+1\right)dx=0$$

NO.

$$\frac{dy}{dx} = \frac{y}{x}\left(\ln\left(\frac{y}{x}\right)+1\right)$$

$$u = \frac{y}{x} \Rightarrow xu = y \xRightarrow{div} x\frac{du}{dx} + u = \frac{dy}{dx}$$

$$x\frac{du}{dx} + u = u\ln(u) + u$$

$$x\frac{du}{dx} = \frac{u\ln u}{x}$$

$$\int \frac{du}{u\ln u} = \int \frac{1}{x}\,dx$$

$w = \ln u$

$dw = \frac{du}{u}$

$\int \frac{1}{w}\,dw$

$= \ln|w| \Rightarrow \ln(\ln u)$

Ans :- $\ln|\ln|\frac{y}{x}|| = \ln(x) + C$

H.w :- 1 $3xy\,dy - (x^2 - y^2)\,dx = 0$

2 $xy\,dy - (x^2 + x\sqrt{x^2+y^2})\,dx = 0$

* Method 3 :- Almost separable :-

- If $\frac{dy}{dx} = F(x,y)$ is not separable, then check if:-

$$F(x,y) = G(ax+by) \text{ or } G(ax+by+C)$$

and we can transform the D.E into separable by the substitute let $u = ax + by\ (+C)$

$$\frac{du}{dx} = a + b\frac{dy}{dx}$$

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ex: find the general sol:-

$y\frac{dy}{dx} = y^2 - 8xy + 16x^2$?

- Not sep
- Not hom
- is almost sep

$(y-4x)^2$

$dy/dx = (y-4x)^2$

let $u = y-4x$

$\frac{du}{dx} = \frac{dy}{dx} - 4 \Rightarrow \frac{du}{dx} + 4 = \frac{dy}{du}$

$\frac{du}{dx} = u^2 - 4$ is separable

$\int \frac{du}{(u^2-4)} = \int dx$

تحليل

$A\ln|u+2| + B\ln(u-2) = x + c$

$\int u' = \int u^2 - 4$

$u = \frac{u^3}{3} - 4u + c$

$(y-4x) = \frac{(y-4x)^3}{4} - 4(y-4x) + c$

2.) $\frac{dy}{dx} = (x+y)\left[1 + \ln(x+y)\right]$ $u = x+y$

3.) $\frac{3dy}{dx} = (2x+3y-1) + \frac{4}{(2x+3y-1)^3}$ $u = 2x+3y-1$ or $2x+3y$

4.) $\frac{dy}{dx} = \frac{2y}{x} + \cos\left(\frac{y}{x^2}\right)$ $u = \frac{y}{x^2}$

5.) $x^3 dy + y(4x^2 - \sin x)\,dx = 0$, use the substitution $u = x^4 y$.

هاي sep جاهزة بس لازم تكامل فبتعلم الفرضية اجباري

* Method 4:- Linear 1st order O.D.E
* General form:-

$$a_1(x)\, y'(x) + a_0\, y(x) = f(x)$$

* Standard form:-

$$y'(x) + p(x)\, y(x) = g(x)$$

where $p(x) = \dfrac{a_0(x)}{a_1(x)}$ $\qquad g(x) = \dfrac{f(x)}{a_1(x)}$

$$y'(x) + p(x)\, y(x) = g(x) \quad --- \quad (*)$$

⇒ multiply $(*)$ by a non zero (positive) function $\mu(x)$

- $\mu(x)\, y'(x) + \underline{\mu(x)\, p(x)}\, y(x) = \mu(x)\, g(x)$

if $\mu'(x) = \mu(x)\, p(x)$

then we get:-

$$(\mu(x)\, y(x))' = \mu(x)\, g(x)$$

$$\mu(x)\, y(x) = \int \mu(x)\, g(x)$$

قاعدة

$$\boxed{y(x) = \frac{1}{\mu(x)} \left[\int \mu(x)\, g(x)\, dx + C \right]}$$

* Now:- to find $\mu(x)$:-

$$\int \frac{\mu'(x)}{\mu(x)} = \int \frac{\cancel{\mu(x)}\, p(x)}{\cancel{\mu(x)}} \Rightarrow \ln|\mu(x)| = \int p(x)\, dx$$

$$\mu(x) = e^{\int p(x)\, dx} \qquad \text{The integration factor}$$

Ex:- Find the general solution of the D.E:-

1. $(x+y+1)\, dx - dy = 0$

- not sep $\quad$ - not hom $\quad$ - almost ✓ $(u = y+x+1)$
- linear in y ✓

sol:-
$$\frac{dy}{dx}(x+y+1)-\frac{dy}{dx}=0$$

$$\frac{dy}{dx}-y=x+1$$

$P(x)=-1$

$g(x)=(x+1)$

$\mu_{(x)}=e^{\int -1 dx}=e^{-x}$

تكامل بالأجزاء

$$y(x)=\frac{1}{e^{-x}}\left[\int \underline{e^{-x}(x+1)}\,dx+C\right]$$

$$=e^{x}\left[-e^{-x}(x+1)-e^{-x}+C\right]$$

$$=-(x+1)-1+Ce^{x}$$

$$\boxed{y=-x-2+Ce^{x}}$$

* extra for 1 if $y(0)=1$ find $y(2)$

$y=1$ / $x=0$

$1=-2+C \Rightarrow C=3$

$y=-x-2+3e^{x}$

$$\boxed{y(2)=-4+3e^{2}}$$

2. $x\,dy+(2y-x-1)\,dx=0$?

- not sep. hom. almost

- linear in y ✓ ... H.W

3.) $y\,dx + (2x - y - 1)\,dy = 0$

- not linear in y ✗ - linear in x ✓

$$y\frac{dx}{dy} + 2x = y + 1$$

$$\frac{dx}{dy} + \frac{2}{y}x = \frac{y+1}{y}$$

$$\mu_{(y)} = e^{\int \frac{2}{y}dy} = e^{2\ln y} = y^2$$

$$x = \frac{1}{y^2}\left[\int y^2 \frac{(y+1)}{y}\,dy + C\right]$$

$$x = \frac{1}{y^2}\left[\frac{y^3}{3} + \frac{y^2}{2} + C\right]$$

$$x = \frac{y}{3} + \frac{1}{2} + \frac{C}{y^2}$$

4.) $(e^{2y} - 3x)\,dy - dx = 0$ H.w

5.) $(1 + x^2)\,dy - x(3 + 3x^2 - y)\,dx = 0$ H.w

6.) $x\sec^2 y\,dy + \frac{1}{x}(\tan y - 1)\,dx = 0$?

- is sep ✓
- not homo, almost, linear in x or y ✗

* use the substitution $w = \tan y$ to transform the D.E into

$w = \tan y$

$$\frac{dw}{dx} = \sec^2(y)\cdot\frac{dy}{dx}$$

$$x \cdot \sec^2 y \cdot \frac{1}{\sec^2 y} \cdot \frac{dw}{dx} + \frac{1}{x}(w - 1) = 0$$

⟹

$X \frac{dw}{dx} + \frac{w}{x} = \frac{1}{x}$ linear in w

- standard form $\frac{dw}{dx} + \frac{w}{x^2} = \frac{1}{x^2}$

$\mu_{(x)} = e^{\int \frac{1}{x^2} dx} \Rightarrow e^{\frac{-1}{x}}$

$w = e^{\frac{1}{x}} [\int e^{\frac{-1}{x}} (\frac{1}{x^2}) dx + C]$

let $u = \frac{-1}{x}$

$\tan(y) = \ldots$

Method 5 :- Bernoulli D.E

$\frac{dy}{dx} + P(x) y = g(x) y^n$ $n \neq 0 \neq 1$

- If $y' + P(x) y^1 = g(x) y^n$ then this D.E is not linear in y $(n \neq 0, 1)$ and it is called Bernoulli in y.

- we can be transformed into linear O.D.E in a new variable u where $\boxed{u = y^{1-n}}$

- $y' + P(x) y = g(x) y^n$

Step 1 :- $\div y^n \Rightarrow \frac{1}{y^n} y' + P(x) y^{1-n} = g(x)$

Step 2 :- $u = y^{1-n} \Rightarrow du/dx = (1-n) y^{-n} \, dy/dx$

$\frac{dy}{dx} = \frac{1}{1-n} \cdot \frac{1}{y^{-n}} \cdot \frac{du}{dx}$

3. subst $\frac{1}{y^n}\left(\frac{1}{y^{-n}}\cdot\frac{1}{(1-n)}\cdot\frac{du}{dx}\right)+P(x)\,u=g(x)$

حفظ بلا اشتقاق

$$\boxed{\frac{du}{dx}+(1-n)P(x)\,u=(1-n)g(x)}$$ is linear in u

solve

$$\cancel{\mu_{(x)}=e^{\int(1-n)}}\qquad \mu_{(x)}=e^{\int(1-n)P(x)\,dx}$$

$$u(x)=\frac{1}{\mu_{(x)}}\left[\int \mu_{(x)}(1-n)g(x)\,dx+C\right]$$

$$y^{1-n}=\text{-----}$$

ex: find the general sol. of

1. $\frac{dy}{dx}=\frac{2y}{x}-x^2y^2$

$$\frac{dy}{dx}-\frac{2y}{x}=-x^2y^2 \qquad \boxed{n=2}$$

in Bernoulli in y, with $n=2 \Rightarrow$ let $u=y^{1-2}=y^{-1}$

$$\frac{du}{dx}+(-1)\left(\frac{-2}{x}\right)u=(-1)(-x^2)$$

$$u'+\frac{2u}{x}=x^2$$

$$\mu=e^{\int 2/x\,dx}=e^{2\ln x}=x^2$$

$$u=\frac{1}{x^2}\left[\int x^2(x^2)\,dx+C\right] \Longrightarrow$$

$$u = \frac{1}{x^2}\left[\frac{x^5}{5} + C\right]$$

$$y^{-1} = \frac{x^3}{5} + \frac{C}{x^2}$$

$$y = \frac{5x^2}{x^5 + 5C}$$

2. $(x^2y^2 + y^5)\,dy - x\,dx = 0$

not Bern in y

$$x\frac{dx}{dy} = x^2y^2 + y^5$$

$$\frac{dx}{dy} - xy^2 = y^5 x^{-1} \qquad (n = -1)$$

$$u = x^{1-n} = x^{1--1} = x^2$$

$$\frac{du}{dy} + (2)(-y^2)u = 2y^5$$

$$\mu(y) = e^{\int -2y^2\,dy} = e^{-\frac{2}{3}y^3}$$

$$u(y) = e^{\frac{2}{3}y^3}\left[\int e^{-\frac{2}{3}y^3}\cdot 2y^5\,dy + C\right]$$ تعويض ثم أجزاء

H.w use the subst $v = y - x$ to transform the D.E $x\,dy - (x^4(y-x)^2 + y)\,dx = 0$ into a new $\boxed{\text{برنولي}}$ D.E

ans:- $$\frac{1}{y-x} = \frac{-x^4}{5} + \frac{C}{x}$$

Method 6 :- Exact O.D.E

Given $F(x,y)$:-

- the (total differential) of F is $dF(x,y) = \frac{dF}{dx}dx + \frac{dF}{dy}dy$

if $F(x,y) = c \Longrightarrow d(F(x,y) = c)$

$$dF(x,y) = dc$$

$$\underbrace{\frac{dF}{dx}}_{M} dx + \underbrace{\frac{dF}{dy}}_{N} dy = 0$$

- we get $M(x,y)dx + N(x,y)dy = 0$ and this D.E is called exact D.E

because $\frac{dF}{dx} = M$ and $\frac{dF}{dy} = N$

- if :- $\frac{dM}{dy} = \frac{d}{dy}\left(\frac{dF}{dx}\right) = \frac{d^2F}{dy\,dx}$

$\frac{dN}{dx} = \frac{d}{dx}\left(\frac{dF}{dy}\right) = \frac{d^2F}{dx\,dy}$

so, if $\frac{dM}{dy} = \frac{dN}{dx}$ then the D.E is exact and we can find $F(x,y)$

- Summary :-

1. Given $M(x,y)dx + N(x,y)dy = 0$

Test if exact , $\frac{dM}{dy} = \frac{dN}{dx}$

- to find the solution $F(x,y) = c$:-

1. use $\frac{dF}{dx} = M(x,y) \longrightarrow F(x,y) = \int M(x,y)dx + g(y)$

or

2. use $\frac{dF}{dx} = N(x,y) \longrightarrow F(x,y) = \int N(x,y)\,dy + h(x)$

NO.

ex:- solve $(2xy^2+1)dx + 2x^2y\,dy = 0$

$\underbrace{\qquad}_{M}$ $\underbrace{\qquad}_{N}$

test :- $\frac{dM}{dy} = 4xy$

$\therefore$ exact

$\frac{dN}{dx} = 4xy$

= we find $F(x,y) = C$:-

$F(x,y) = \int 2xy^2 + 1\,dx + g(y)$

$= x^2y^2 + x + g(y)$

$F(x,y) = \int 2x^2y\,dy + h(x)$

$= x^2y^2 + h(x)$

$h(x) = x \qquad g(y) = 0$

$F(x,y) = C$ is :-

$x^2y^2 + x = C$

ex: solve :- $(1+ye^x+xye^x)\frac{dx}{dy} = (-xe^x-2)$

linear form y

$$\underbrace{(1+ye^x+xye^x)}_{M}dx + \underbrace{(xe^x+2)}_{N}dy = 0$$

$$\frac{dM}{dy} = e^x + xe^x \qquad \frac{dN}{dy} = e^x + xe^x$$

exact

$F(x,y) = c$

$F(x,y) = \int (xe^x+2)dy + h(x)$ بنفاضل الشكل مع الـ dy

$F(x,y) = xe^x y + 2y + h(x)$

- to find $h(x)$

$M(x,y) = \frac{dF}{dx} = xe^x y + e^x y + 0 + h'(x)$

$\cancel{1}+\cancel{xe^x y}+\cancel{e^x y} = \cancel{xe^x y}+\cancel{e^x y}+h'(x) \Rightarrow h'(x) = 1$

$h(x) = x$

$F(x,y) = c$

$$\boxed{xye^x + 2y + x = c}$$

Method 7:- Non exact O.D.E (special Integration factor)

- $\frac{dM}{dy} = M_y$ - $\frac{dN}{dx} = N_x$

- If the D.E $M(x,y)dx + N(x,y)dy = 0$ ~~then~~ is not exact then sometimes we can transform this equation into an exact equation by multiplying the D.E by some integration factor (I.F) of a form:-

1. $\mu(x)$ = function of x only (or constant)
2. $\mu(y)$: " " y " (or ")
3. $\mu(z)$: " " (x,y) (or ")

test 1:- if $\frac{M_y - N_x}{N(x,y)}$ is a fn of x only or constant then

the I.F $\mu(x) = e^{\int \frac{M_y - N_x}{N(x,y)} dx}$

or:-

test 2:- If $\frac{N_x - M_y}{M(x,y)}$ is a fn of y only (or const) then the I.F

$$\mu(y) = e^{\int \frac{N_x - M_y}{M(x,y)}}$$

Note:- if test 1 and 2 fail, then the I.F $\mu(x,y)$ will be given in the problem

ex:- Find the general sol of 1. $(2x^2 + y)dx + (x^2y - x)dy = 0$

$$M_y = 1 \neq N_x = 2xy - 1 \quad \text{not exact}$$

test 1:- $\frac{M_y - N_x}{N(x,y)} = \frac{1 - 2xy + 1}{x^2y - x} = \frac{2(1-xy)}{x(xy-1)} = \frac{-2}{x}$ fn of x only

$$\mu(x) = e^{\int \frac{-2}{x}} = e^{-2\ln x} = x^{-2} = \frac{1}{x^2}$$

Multiply by $\frac{1}{x^2}$

$$\Rightarrow \left(2+\frac{y}{x^2}\right)dx + \left(y-\frac{1}{x}\right)dy = 0 \quad , \boxed{x \neq 0}$$

is exact

$F(x,y) = c$

$$F(x,y) = \int \left(2+\frac{y}{x^2}\right)dx + g(y)$$

$$= \boxed{2x - \frac{y}{x} + g(y)}$$

$$F(x,y) = \cancel{\frac{y^2}{2} - \frac{y}{x}} \int \left(y-\frac{1}{x}\right)dy + h(x)$$

$$= \boxed{\frac{y^2}{2} - \frac{y}{x} + h(x)}$$

$h(x) = 2x \qquad g(y) = \frac{y^2}{2}$

$F(x,y) = c$

$$2x - \frac{y}{x} + \frac{y^2}{2} = c$$

$\therefore$ The sol to the non exact D.E is :-

$$2x - \frac{y}{x} + \frac{y^2}{2} = c \quad \text{and} \quad x=0 \ (\text{or } y=0)$$

- يجب أنه لا ننسى عند نهاية الحل إضافة الحل الذي حذفناه عند الضرب بـ I.F (إنه لم يكن محذوف بالأساس)

- يجب حذف الحل الذي أضفناه عند الضرب بـ I.F (إنه لم يكن موجوداً من الأساس)
يعني بقدر أعوضنا أولاً"

H.w :- Find the general sol of :-

$$\left(5xe^{-y}+2\cos(3x)\right)y' + 5e^{-y} = \cancel{\ \ } \ 3\sin(3x) \ ?$$

* Note: 1. If the D.E is separable then it is also exact

2. Given $M(x,y)dx + N(x,y)dy = 0$ If it is separable, then its best for you to solve it as exact

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Or:- $f_1(x)\, g_1(y)\,dx + f_2(x)\, g_2(y) = 0$

now ~~we~~ $\div g_1(y)$ and $\div f_2(x)$

$$\int \frac{f_1(x)}{f_2(x)}\,dx + \int \frac{g_2(y)}{g_1(y)} = 0$$

ex:- let $\mu(x,y) = x^n y^m$ be an I.F of

$(2y^2 - 6xy)dx + (3xy - 4x^2)dy = 0$

1. Find n and m 2. use part (a) to find the general sol.

sol:- 1. Multiply by $x^n y^m$:-

$$(2x^n y^{m+2} - 6x^{n+1} y^{m+1})dx + (3x^{n+1} y^{m+1} - 4x^{n+2} y^m)dy = 0$$

$$\frac{d}{dy} = \frac{d}{dx}$$

$$2(m+2)x^n y^{(m+1)} - 6(m+1)x^{n+1} y^m = 3(n+1)x^n y^{m+1} - 4(n+2)x^{n+1} y^m =$$

$2(m+2) = 3(n+1)$

$-6(m+1) = -4(n+2)$ .... $n=1$, $m=1$

2. we get $\mu(x,y) = x^1 y^1$

$$\underset{exact}{F(x,y)} = C \equiv \underset{nonexact}{F(x,y)} = C$$

H.W. 1. if $\mu(x,y) = \frac{1}{x^2+y^2}$ is an I.F of $y\,dx - (y^2+x^2+x)dy = 0$ if $y(1) = 1$, find $y(\frac{2}{\pi})$

check answer $y + \tan^{-1}(\frac{y}{x}) = c$ ...

2. Q27/Page 62 (book):- Find $M(x,y)$ such that the D.E

$$M(x,y)dx + (\sin x \cos y - xy - e^{-y})dy = 0$$

NO.

H.W :- 1. Find the value of (b) which make the D.E exact

$(e^x \sin y + b x^2 y^2)\,dx + (e^x \cos y + x^3 y)\,dy = 0$

2. Find the form of an exact D.F whose implicit sol is given by:-

$x^3 y^3 + 6y = C$

$F = x^3 y^3 + 6y$

$F = \int M\,dx$

$x^3 y^3 + 6y = \int M\,dx$ بنشتقه بالنسبة لـ x

$3x^2 y^3 = M$

$x^3 y^3 + 6y = \int N\,dy$ نشتقه بالنسبة لـ y

$3y^2 x^3 + 6 = N$

$$(3x^2 y^3)\,dx + (3y^2 x^3 + 6)\,dy = 0$$

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Chapter (4)

· Solving 2^{nd} order linear O.D.E

General form:-

$$a_2(x)\,y''(x) + a_1(x)\,y'(x) + a_0(x)\,y(x) = f(x)$$ (same for linear in X)

Standard form:-

$$y''(x) + p(x)\,y'(x) + q(x)\,y = g(x)$$

$a_2 \neq 0 \Longrightarrow$ (a_1, a_0 قد يساوي صفر)

Notes:- 1. If $F(x) = 0$ then the D.E is said to be Homogeneous

〃 $F(x) \neq 0$ 〃 〃 〃 〃 non- 〃

2. IF $a_2(x), a_1(x), a_0(x)$ are constants then the D.E. is called constant coefficient D.E.

- $ay'' + by' + cy = 0$ or $f(x)$

⇒ Part(I):- solving 2^{nd} order linear Homog. O.D.E

- we have 3 methods

1. D.E. with constant coefficient

$$ay'' + by' + cy = 0$$

2. Cauchy-Euler D.E

$$ax^2y'' + bxy' + cy = 0$$

3. Reduction of order

$$y''(x) + p(x)\,y'(x) + q(x)\,y(x) = 0$$

given that the first solution $y_1(x)$ is given

الاقتران الأول

Method 1:- 2^{nd} order linear O.D.E (Homog) with constant Coefficient

$ay'' + by' + cy = 0$

the D.E has two solution :- $y_1(x)$ & $y_2(x)$

*The general solution to the D.E is:-

$y(x) = c_1 y_1(x) + c_2 y_2(x)$

- we find C_1 and C_2 using the initial conditions:-

$y(x_0) = y_0 \Rightarrow y'(x_0) = y_1$ (I.C)

⇒ The suggested solution to is ~~$y(x) = e^{rx}$~~ $ay'' + by' + cy = 0$

is $y(x) = e^{rx}$

Now to find r

$y = e^{rx}$ $\qquad y' = re^{rx}$ $\qquad y'' = r^2 e^{rx}$

$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0 \qquad e^{rx} \neq 0$

$ar^2 + br + c = 0$ characteristic equation

or auxillary equation

* we have 3 cases:-

Case$_1$:- Two distinct real roots :-

If $b^2 - 4ac > 0$

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$r_1 = \frac{-b - \sqrt{b^2-4ac}}{2a} \qquad r_2 = \frac{-b + \sqrt{b^2-4ac}}{2a}$$

$y_1 = e^{r_1 x} \qquad y_2 = e^{r_2 x}$

general sol:- $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

NO.

ex:- Find the general sol:-

1. $y'' + 5y' + 6y = 0$, $y(0) = 1$ $y'(0) = -1$

Charact eq:-

$$r^2 + 5r + 6 = 0$$

$$(r+2)(r+3) = 0$$

$$r_1 = -3 \Rightarrow y_1(x) = e^{-3x}$$

$$r_2 = -2 \Rightarrow y_2(x) = e^{-2x}$$

$y(0)=1 \Rightarrow c_1 + c_2 = 1$

$y'(0) = -1 \Longrightarrow y' = -3c_1 e^{-3x} - 2c_2 e^{-2x}$

$$y'(0) = -3c_1 - 2c_2 = -1$$

$c_2 = 2$ $c_1 = -1$

$$y(x) = -e^{-3x} + 2e^{-2x}$$

Case 2:- Repeted roots

IF $b^2 - 4ac = 0$

$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $\sqrt{b^2-4ac} = 0$ $\Rightarrow r_2 = r_1 = r = \frac{-b}{2a}$

$$y_1 = e^{rx} = e^{\frac{-b}{2a}x}$$

$\Rightarrow y_2 = x e^{rx}$

general sol :-

$y = c_1 r^{ex} + c_2 x r^{ex}$

ex solve :- 1. $y'' + 4y' + 4 = 0$

$$r^2 + 4r + 4 = 0$$

$$r = -2$$

$$y_1 = e^{-2x} \qquad y_2 = x e^{-2x}$$

$$y = c_1 e^{-2x} + c_2 x e^{-2x} \quad \#$$

2. $y'''(x) + 4y'' + 3y' = 0$

$$r^3 + 4r^2 + 3r = 0$$

$$r(r^2 + 4r + 3) = 0$$

$$r(r+3)(r+1) = 0$$

$r_1 = 0 \Rightarrow y_1 = e^{0x} = 1$

$r_2 = -3 \Rightarrow y_2 = e^{-3x}$

$r_3 = -1 \Rightarrow y_3 = e^{-x}$

the general sol

$$y = c_1 + c_2 e^{-3x} + c_3 e^{-x}$$

Case 3:- Complex Roots

IF $b^2 - 4ac < 0$

$$r = \frac{-b \pm \sqrt{\boxed{b^2 - 4ac}}}{2a}$$ negative

$$r = \frac{-b \pm \sqrt{(-1)(4ac - b^2)}}{2a} \Rightarrow r = \frac{-b}{2a} \pm \sqrt{-1} \cdot \frac{\sqrt{4ac - b^2}}{2a}$$

$\sqrt{-1}$ /// i = the imaginary number

$$r = \underbrace{\frac{-b}{2a}}_{\alpha} \pm i \underbrace{\frac{\sqrt{4ac - b^2}}{2a}}_{\beta}$$

then:-

$$y_1 = e^{\alpha x}\cos(\beta x)$$

$$y_2 = e^{\alpha x}\sin(\beta x)$$

General sol:-

$$y(x) = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$$

ex:- solve

1. $y'' + 4y' + 7y = 0$

$r^2 + 4r + 7 = 0$

$$r = \frac{-4 \pm \sqrt{16 - 4(1)(7)}}{2(1)} = \frac{-4 \pm \sqrt{-12}}{2} = -2 \pm \frac{\sqrt{-12}}{2}$$

$$r = \underset{\alpha}{-2} \pm i \underset{\beta}{\sqrt{3}}$$

$y_1 = e^{-2x}\cos(\sqrt{3}\,x)$ $\qquad$ $y_2 = e^{-2x}\sin(\sqrt{3}\,x)$

general sol:- $\quad y = C_1 e^{-2x}\cos\sqrt{3}x + C_2 e^{-2x}\sin\sqrt{3}x$

2. $y''' - 2y'' - 5y' + 6y = 0$

$r^3 - 2r^2 - 5r + 6 = 0$ $\qquad$ $r = 1$

$(r-1)$ is a factor

$(r-1)(r^2 - r - 6) = 0$

$(r-1)(r-3)(r+2) = 0 \Rightarrow r = 1, 3, -2$

$y_1 = e^{x}$ $\quad$ $y_2 = e^{3x}$ $\quad$ $y_3 = e^{-2x}$

$$y = c_1 e^{x} + c_2 e^{3x} + c_3 e^{-2x}$$

$$\begin{array}{r|l} & r^2 - r - 6 \\ \hline r-1 & r^3 - 2r^2 - 5r + 6 \\ & -r^3 + r^2 \\ \hline & -r^2 - 5r + 6 \\ & +r^2 - r \\ \hline & -6r + 6 \\ & +6r - 6 \\ \hline & \text{zero} \end{array}$$

3. $y''' - y'' - 4y' + 4y = 0$ H.w

4. $y^{(4)}(x) + 2y''(x) + y(x) = 0$

$r^4 + 2r^2 + 1 = 0$

$(r^2+1)(r^2+1) = 0$

$r^2+1=0 \Rightarrow r^2 = -1 \Rightarrow r = \pm\sqrt{-1} = \pm i$

$r = 0 \pm 1\, i$

$\alpha = 0, \ \beta = 1$

$r^2+1=0$ ... repeated

$y_1 = e^{0x} \cdot \cos x = \cos x$ $\qquad$ $y_3 = x \cdot e^{0x} \cos x = x \cos x$

$y_2 = e^{0x} \sin x = \sin x$ $\qquad$ $y_4 = x \cdot e^{0x} \sin x = x \sin x$

ex: If $y = xe^{3x}$ is a solution to the D.E $y'' + by' + cy = 0$

Find b, c ?

$r_1 = r_2 = 3$

$(r-3)^2 = 0$

$r^2 - 6r + 9 = 0$

$b = -6$ $\qquad$ $c = 9$ $\quad (y'' - 6y' + 9y = 0)$

H.w: If $y = e^{3x} \sin(2x)$ is a sol to the D.E, $y'' + by' + cy = 0$

find b, c ? $\boxed{\text{H.w}}$

ex: If $p(r) = r^3(r^2+3r+2)^2(r^2-4)(r^2+9)^3 = 0$ is the char eq of some n^{th} order linear O.D.E with the constant coeff

1. find n ?
2. Find the general sol :-

Sol:- $n = 3+4+2+6 = 15$

$r^3 = 0 \quad r_1 = r_2 = r_3 = 0 \Rightarrow y_1 = 1 \Rightarrow y_2 = x \Rightarrow y_3 = x^2$

$(r^2 + 3r + 2)^2 = 0 \Rightarrow (r+2)^2 (r+1)^2 = 0 \quad r_4 = r_5 = -2 \Rightarrow y_4 = e^{-2x} \quad y_5 = x e^{-2x}$

$r_6 = r_7 = -1 \Rightarrow y_6 = e^{-x} \quad y_7 = x e^{-x}$

⊛ Method ②:- Cauchy-Euler D.E (of 2^{nd} order) / Homog

General form:- $a x^2 y''(x) + b x y'(x) + c y(x) = 0$

$a \neq 0 \quad x \neq 0$

the suggested sol:-

$y(x) = x^r$

r to be found:-

$y = x^r \Rightarrow y' = r x^{r-1} \quad y'' = r(r-1) x^{r-2}$

substi:-

$a x^2 r(r-1) x^{r-2} + b x r x^{r-1} + c x^r = 0$

$a r(r-1) x^r + b r x^r + c x^r = 0$

$0 \neq x^r [a r^2 - a r + b r + c] = 0$

$a r^2 + (b-a) r + c = 0$ char. eq

Case 1:- Two distinct real roots:

$$r = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4ac}}{2a}$$

$y_1 = x^{r_1} \quad y_2 = x^{r_2}$

Case 2:- Repeated Roots

$$r = \frac{-(b-a) \pm 0}{2a} = r_1 = r_2$$

$$y_1(x) = x^{\frac{-(b-a)}{2a}}$$

$$y_2 = \ln(x) \cdot x^{\frac{-(b-a)}{2a}}$$

note:-

	const	cauchy
-	$y = e^{rx} = (e^x)^r$	$y = x^r$
repeat -	بنضرب بـ x	بنضرب بـ $\ln x$

- Case (3):- complex roots

$$r = \frac{-(b-a)}{2a} \mp i \frac{\sqrt{4ac-(b-a)^2}}{2a}$$

α β

$$y_1 = x^{\alpha} \cos(\beta \ln x)$$
$$y_2 = x^{\alpha} \sin(\beta \ln x)$$

ex: solve:-

1. $x^2 y''(x) + 6x y'(x) + 6y(x) = 0$?

$r^2 + \cancel{5}r + \cancel{6} = 0$ $(r+3)(r+2) = 0$ $r = -2, -3$

$y_2 = x^{-2}$ $y_1 = x^{-3}$

$$y = c_1 x^{-3} + c_2 x^{-2}$$

2. $x^2 y''(x) + 5xy' + 4y = 0$

$r^2 + 4r + 4 = 0$

$(r+2)(r+2) = 0 \qquad r = r_1 = r_2 = -2$

$y_1 = x^{-2}$

$y_2 = \ln x \cdot x^{-2}$

3) $x^2 y''(x) + 3xy'(x) + 4y = 0$

$r^2 + 2r + 4 = 0$

$r = \frac{-2 \mp \sqrt{(4) - 4(1)(4)}}{2}$

$r = -1 \pm i \frac{\sqrt{12}}{2}$

$= -1 \pm i\sqrt{3} \implies y_1 = x^{-1} \cos(\sqrt{3} \ln x)$

$y_2 = x^{-1} \sin(\sqrt{3} \ln x)$

H.w:- 1. $y'' + \frac{1}{x^2} y = 0$

$x^2 y'' + y = 0$

$r^2 - r + 1 = 0$

$r = \frac{1}{2} \mp i \frac{\sqrt{4-1}}{2} \implies r = \frac{1}{2} \mp i \frac{\sqrt{3}}{2}$

$\alpha = \frac{1}{2} \qquad \beta = \frac{\sqrt{3}}{2}$

$y_1 = x^{\frac{1}{2}} \cos\left(\frac{\sqrt{3}}{2} \ln x\right) \qquad y_2 = x^{\frac{1}{2}} \sin\left(\frac{\sqrt{3}}{2} \ln x\right)$

$y = c_1 x^{\frac{1}{2}} \cos\left(\frac{\sqrt{3}}{2} \ln x\right) + c_2 x^{\frac{1}{2}} \sin\left(\frac{\sqrt{3}}{2} \ln x\right)$

NO.

* Method 3 :- Reduction of order Method :-

- We need some indroduction

* Matrices :- المصفوفات

$$\underset{2\times 2}{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

صف ← 2×2 → عمود

* The determinant of A

$$\det(A) = |A|$$

$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= ad - bc$$

* if $$A_{3\times 3} = \begin{bmatrix} \overset{+}{a} & \overset{-}{b} & \overset{+}{c} \\ \overset{-}{d} & \overset{+}{e} & \overset{-}{f} \\ \overset{+}{g} & \overset{-}{h} & \overset{+}{i} \end{bmatrix}$$

det(A) = along first row :-

$$= (+a)\begin{vmatrix} e & f \\ h & i \end{vmatrix} - (b)\begin{vmatrix} d & f \\ g & i \end{vmatrix} + (c)\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

along 3^{rd} column :-

$$= +c\begin{vmatrix} d & e \\ g & h \end{vmatrix} - f\begin{vmatrix} a & b \\ g & h \end{vmatrix} + i\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

* بنوقف الصف أو العمود الي يحتوي على أصفار أكثر

* Wronskian :-

- Given two fn $f(x)$, $g(x)$ (cont an interval I) then:-
  the wronskian of $f(x)$ and $g(x)$ is defined by:-

$$W(f(x), g(x))(x) := \begin{bmatrix} f(x) & g(x) \\ f' & g' \end{bmatrix}$$

$$= fg' - f'g \equiv \text{function of } x \text{ or constant} \neq 0$$

- $$w[f(x), g(x), h(x)](x) = \begin{bmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{bmatrix}_{3\times 3}$$

ex: find:

1. $w[e^x, x]$

$$= \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix}$$

$$= e^x - e^x x$$

$$= e^x(1-x) \boxed{\neq 0 \text{ for all } x \neq 1}$$

* note: $W(x) \neq 0$ for all $x \neq 1$ then e^x and x are independent on $\mathbb{R} \setminus \{1\}$

ex:- find:- 1. $w[e^{5x}, 3e^{5x}](x) = ?$

$$\begin{vmatrix} e^{5x} & 3e^{5x} \\ 5e^{5x} & 15e^{5x} \end{vmatrix} = 15e^{10x} - 15e^{10x} = 0$$

dependent

2. $w\,[\sin^2(x),\ 1,\ \cos 2x]$

$\frac{1}{2}(1-\cos 2x)$

$= \frac{1}{2}(1) + (-\frac{1}{2})\cos 2x$

so $w = 0$

are linearly dependent $w(x)=0$ for all $x \in \mathbb{R}$

* If y_1 and y_2 are solution to $\boxed{y'' + P(x)y' + q(x)y = 0}$

with $\boxed{w[y_1, y_2](x) \neq 0}$ on an interval I then:-

we say that $S = \{y_1, y_2\}$ is a Fundamental solution set

* Abel's Theorem:-

- If $y_1(x)$ and $y_2(x)$ are two solution (usually unknown) of the D.E $y'' + P(x)y' + q\,y(x) = 0$ then

$$W[y_1(x), y_2(x)] = C e^{-\int P(x)dx}$$

where C is a constant depends on y_1 and y_2:-

1. if y_1 and y_2 are dep then $C = 0$
2. if y_1 and y_2 are indep $C \neq 0$

* Note:- that we have two methods to find $w[y_1, y_2](x)$

1. $w[y_1, y_2](x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1 y_2' - y_2 y_1'$

2. $W[y_1, y_2](x) = Ce^{-\int P(x)\,dx}$

- Now Given that $y_1(x)$ is a solution known to the D.E

$$y''(x) + P(x)y'(x) + q(x)y(x) = 0$$

- we need to find $y_2(x)$ (a second Linearly indep sol for the D.E)

- To do this:-

we have $\underbrace{y_1 y_2' - y_2 y_1'}_{\text{method 1}} = \underbrace{Ce^{-\int P(x)\,dx}}_{\text{method 2}}$

is a linear first order O.D.E in $y_2(x)$

- solve for $y_2(x)$:-

$$y_2'(x) - \frac{y_1'(x)}{y_1}y_2(x) = C\,\frac{e^{-\int P(x)\,dx}}{y_1}$$

$$\mu(x) = e^{\int \frac{-y_1'(x)}{y_1(x)}dx} = e^{-\ln y_1(x)} = \frac{1}{y_1(x)}$$

$$\rightarrow y_2 = y_1(x)\left[\int \frac{1}{y_1(x)} C \cdot \frac{e^{-\int P(x)\,dx}}{y_1(x)}\,dx + C_0\right]$$

$$= y_1\left[C\underbrace{\int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx}_{v(x)} + C_0\right]$$

$$\boxed{y_2(x) = C\,y_1(x)\,v(x) + C_0\,y_1(x)}$$

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but:- the general sol is :-

$$y(x) = c_1 y_1 + c_2 y_2$$

$$= c_1 y_1(x) + c_2 (c y_1 v(x) + c_0 y_1(x))$$

$$= c_1 y_1(x) + c_2 c\, y_1(x)\, v(x) + c_2 c_0\, y_1(x)$$

$$= \underbrace{(c_1 + c_0)}_{c_1'} y_1(x) + \underbrace{c_2 c}_{c_2'} (y_1(x)\, v(x))$$

$$\boxed{y(x) = c_1' y_1(x) + c_2' y_2}$$ $\therefore$ so we can assume that $c_0 = 0$ & $c = 1$

$$\boxed{\therefore\ y_2(x) = y_1(x) \cdot v(x)}$$

Note:- There are two methods to find $v(x)$:-

1. $$v(x) = \int \frac{W[y_1, y_2](x)}{y_1^2(x)}\, dx$$

2. $$v(x) = \int \frac{e^{-\int p(x) dx}}{y_1^2(x)}\, dx$$

ex let $y_1(x) = e^x$ be a solution to the D.E:-

$$x y'' - (x+1) y' + y = 0$$

Find a second linear independent sol?

c) $y_2(x) = y_1(x)\, v(x) = e^{x}\, v(x)$

$$v(x) = \int \frac{e^{-\int -\frac{x+1}{x}dx}}{e^{2x}}\, dx$$

$$= \int \frac{e^{\int 1+\frac{1}{x}dx}}{e^{2x}}\, dx = \int \frac{e^{x}\cdot x}{e^{2x}}\, dx$$

u	du
x	e^{-x}
1	$-e^{-x}$
0	e^{-x}

$$= \int x e^{-x}\, dx = -x e^{-x} - e^{-x}$$

$$y_2 = e^{x}\left[-e^{-x}\cdot x - e^{-x}\right] = -(x+1)$$

* ملاحظة مهمة :- يمكن لما أكتب ال general وتكون معي قيمة y_2 بالسالب
يمكن نعوض السالب مع قيمة (C) وتكون في الخيارات و بتكون هي الإجابة الصح

$$y = c_1 e^{x} + c_2 (x+1)$$

H.W :- Ex :- If $y(x) = x$ is a sol to the D.E $y''(x) + \frac{x^3 P(x)}{2} y'(x) + q\, y(x) = 0$

1. Find the general sol if the wronskin of the two soluation is x^2.
2. Find $P(x)$

1. $y_2 = y_1 \cdot v \Rightarrow v = \int \frac{W[y_1, y_2]}{y_1^2}\, dx = \int \frac{x^2}{x^2}\, dx = x$

$y_2 = x^2 \xRightarrow{\text{general sol}} y = c_1 y_1 + c_2 y_2 = c_1 x + c_2 x^2$

2. $W[y_1, y_2] = e^{-\int \frac{x^3 P(x)}{2}} \Rightarrow \ln x^2 = -\int x^3 \frac{P(x)}{2}$

$\frac{2x}{x+1}$?? $= -\frac{x^3}{2} \cdot P(x) \Rightarrow P(x) = \frac{-4}{x^3 + x^2}$

$\frac{4}{x^2}$

FIVE APPLE

H.w: If y_1 and y_2 are two lin. indep sol for the D.E

$$x^2 y'' - 2y' + (3+x^2)y = 0$$

Find $w[y_1, y_2](4)$ if $w[y_1, y_2](2) = 3$

a) $3/e$ b) $3e^2$ c) $3\sqrt{e}$

d) $3e^3$ e) $3e\sqrt{e}$ f) $3e$

Ans:- $3\sqrt{e}$

Ex: Why we multiply by "$\ln x$" in a repeated case in homog Cauchy-Euler D.E? let $y_1 = x^{-2}$ be a solution of the D.E $x^2 y''(x) + 5x y'(x) + 4y(x) = 0$, find the general sol (or find 2nd linear indep-

$$y_2 = v(x)\, y_1(x)$$

$$v = \int \frac{e^{-\int \frac{5}{x} dx}}{(x^{-2})^2} dx$$

$$= \int \frac{x^{-5}}{x^{-4}} dx \Rightarrow \int \frac{1}{x} dx = \ln x$$

$$y_2 = \ln x \, (x)^{-2}$$

$$y = c_1 x^{-2} + c_2 (\ln x) x^{-2}$$

Ex: let $y_1(x) = x^r$ be a sol of the D.E $ax^2 y'' + bx y' + cy = 0$ find the general sol?

$\Rightarrow$

Sol:-

$$v = \int \frac{e^{-5\frac{b}{4x}}\,dx}{(x^r)^2}$$

$$= \int \frac{x^{-\frac{b}{a}}}{x^{2r}} = \int x^{-\frac{b}{a}} \cdot x^{\cancel{+}(1-\frac{b-a}{a})} dx$$

$$= \int x^{-\frac{b}{a}} \cdot x^{\frac{b}{a}} \cdot x^{-1} dx$$

$$= \int x^{-1} dx = \ln x$$

$$y_2 = x^r \ln x$$

Part (II):- Non-Homogeneous D.E

- general form:-

$$a_2(x) y''(x) + a_1(x) y'(x) + a_0(x) y(x) = f(x) \neq 0$$

- Standard form:-

$$y''(x) + p(x) y'(x) + q(x) y(x) = g(x)$$

Recall $L[y] = y''(x) + p(x) y'(x) + q\, y(x)$

linear Diff operator

ex:- let $L[y] = y'' + 5y' + 6y = 0$ Find:-

1. $L[e^{-2x}] =$ zero (homog)
2. $L[e^{-3x}] =$ zero (homog)
3. $L[e^{4x}] \neq 0$

$$L[e^{4x}] = 16e^{4x} + 5(4e^{4x}) + 6(e^{4x}) = 42e^{4x} \neq 0$$

e^{4x} is a sol to the (non-homog) D.E. $y'' + 5y' + 6y = 42e^{4x}$

(general sol:-) $y(x) = \underbrace{c_1 y_1 + c_2 y_2}_{y_h(x)\ \text{homog sol}} + \underbrace{e^{4x}}_{y_p(x)\ \text{particular sol}}$

- How to find $y_p(x)$? we have two method

1. Variation of parameters method
2. Undetermined coefficient method

1) <u>Method one</u>:- Variation of parameters method

* يجب كتابة المعادلة على standard form

Given $y'' + p(x) y' + q\, y = g(x)$

step 1:- find $y_h(x) = c_1 y_1 + c_2 y_2$

step 2:- find $y_p(x) = y_1 u_1 + y_2 u_2$

$$\boxed{u_1(x) = \int \frac{-g(x)\, y_2(x)}{W[y_1, y_2]}\, dx \qquad u_2(x) = \int \frac{g(x)\, y_1(x)}{W[y_1, y_2]}\, dx}$$

ex:- Find the particular solution of :-

1. $y'' + 5y' + 6y = 42 e^{4x}$?

$y_h(x) = r^2 + 5r + 6 = 0 \Rightarrow r = -2, -3 \Rightarrow y_1 = e^{-2x} \quad y_2 = e^{-3x}$

$$W[y_1, y_2] = \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix} = -3e^{-5x} + 2e^{-5x} = -e^{-5x}$$

$$V_1(x) = \int \frac{-42e^{4x} \cdot e^{-3x}}{-e^{-5x}} = 7e^{6x}\,dx$$

$$V_2(x) = \int \frac{(42e^{4x})\, e^{-2x}}{-e^{-5x}}\,dx = -6e^{7x}$$

$$y_p = 7e^{-2x} \cdot e^{6x} + e^{-3x} \cdot (-6e^{7x}) = e^{4x}$$

General sol $= c_1 e^{-2x} + c_2 e^{-3x} + e^{4x}$

ex- Find the general sol of $3y'' - 6y' + 30y = 15\sin x + e^{x}\tan 3x$?

$$y'' - 2y' + 10y = 5\sin x + \frac{e^{x}}{3}\tan 3X$$

$y_h(x)$ $\quad r^2 - 2r + 10 = 0$

مميز

$$y_1 = e^{x}\cos(3x)$$

$$y_2 = e^{x}\sin(3x)$$

let $y_{p_1}(x)$ be the particular sol of $y'' - 2y' + 10y = 5\sin x$

and:- $y_{p_2}(x)$ 〃 〃 〃 〃 〃 $y'' - 2y' + 10y = \frac{e^{x}}{3}\tan 3x$

then $y_p(x) = y_{p_1}(x) + y_{p_2}(x)$ is the particular sol of

$$y'' - 2y' + 10y = 5\sin(x) + \frac{e^{x}}{3}\tan 3x$$

y_1 $\quad$ y_2

$y_{p_1}(x)$ $\quad$ $y_{p_2}(x)$

$\Longrightarrow$

or given the sol of $y'' - 2y' + 10y = 4\sin(x) + 7e^{x}\tan(3x)$

السؤال ا بقدوبو علقه الـ دالة

$$= \frac{4}{5}(5\sin x) + 21\left(\frac{e^{x}}{3}\tan x\right)$$

$$y_p(x) = \frac{4}{5}y_{p_1}(x) + 21\left(y_{p_2}(x)\right)$$

* Theorem :- Super position principle

let $L[y] = y'' + P(x)y' + q(x)y$

$L[y_1] = g_1$ $\qquad$ $L[y_2] = g_2$

then $L[\alpha_1 y_1 + \alpha_2 y_2] = \alpha_1 g_1 + \alpha_2 g_2$

ex:- let $L[y_1] = 3 - 2x^2$

$L[y_2] = 2x$ and $L[y_3] = 2$

Find the sol to $L[y] = 4x^2 + 6x - 2$?

$$g(x) = 4x^2 + 6x - 2 \equiv \alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3$$

$$4x^2 + 6x - 2 \equiv \alpha_1(3 - 2x^2) + \alpha_2(2x) + \alpha_3(2)$$

$$4x^2 + 6x - 2 \equiv x^2(-2\alpha_1) + x(2\alpha_2) + (3\alpha_1 + 2\alpha_3)$$

$\alpha_1 = -2$ $\qquad$ $\alpha_2 = 3$ $\qquad$ $\alpha_3 = 2$

Method 2: Undetermined Coefficients Method :-

- Given $a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$ or (standard form)

- If $f(x)$ is one of the following forms then we can find the form of the particular sol. $y_p(x)$

form of $f(x)$	Suggested (Trial) form of $y_p(x)$
1. a (const)	$A x^s$ $\quad s = 0, 1, 2 \ldots n$
2. $ax+b$	$(Ax+B)x^s$
3. ax^2+bx+c	$(Ax^2+Bx+C) \cdot x^s$ $\quad s = 0, 1, \ldots n$
4. ae^{bx}	$Ae^{bx} \cdot x^s$
5. $\alpha_1 \sin(bx) + \alpha_2 \cos(bx)$	$(A\sin(bx) + B\cos(bx)) \cdot x^s$

$\cancel{y_1 = e^x}$ $\quad y_2 = e^{-2x}$

$f = 3e^{-2x}$

$y_p = Ae^{-2x} \cdot x^s \Rightarrow$ بنعدل قيمة s بحيث
ما يكونوا اقترانات شبه

$\boxed{s=1}$

$y_1 = e^{3x} \quad y_2 = xe^{3x} \quad f = -e^{3x}$

$y_p = Ae^{3x} \cdot x^s$

$\boxed{s=2}$

$y_1 = \sin 5x \quad y_2 = \cos 5x$

$F_1 = 2\sin 5x$ / $F_2 = 2\sin x + 3\cos x$

$y_{p_1} = (A\sin 5x + B\cos(5x)) \cdot x^{s_1}$

$\boxed{s_1 = 1}$

$y_{p_2} = (C\sin x + D\cos x)\, x^{s_2}$

$s_2 = 0$

NO.

mix 6. $e^{bx}(\sin(\alpha x) + \cos(\alpha x))$	$e^{bx}(A\sin(\alpha x) + B\cos(\alpha x))\, x^{s}$
7. $e^{bx}(ax+c)$	$e^{bx}(Ax+B)\, x^{s}$
8. $e^{bx}(ax^2+cx+d)$	$e^{bx}(Ax^2+Cx+D)\, x^{s}$

ex:- 1. if $f(x) = \overset{(2x+0)}{2x}\sin(3x) - \overset{(0x+1)}{\cos(3x)}$ find y_{p_s}

$$y_p(x) = \big((Ax+B)\sin(3x) + (Cx+D)\cos(3x)\big)\cdot x^{s}$$

2. $f(x) = (2x+1)(\sin 3x - \cos 3x) \equiv (2x+1)\sin 3x - (2x+1)\cos 3x$

$$y_{p(x)} = \big((Ax+B)\sin 3x + (Cx+D)\cos 3x\big)\, x^{s}$$

3. $f(x) = 2e^{3x} x\sin(x) - x^2 e^{3x}\cos x + xe^{x}\sin(x)$

$$y_{p(x)} = e^{3x}\big((A_1x^2+B_1x+C_1)\sin x - (A_2x^2+B_2x+C_2)\cos x\big)\, x^{s_1} +$$

$$+\ e^{x}\big(\underline{(Ax+B)\sin x + (Cx+D)\cos x}\big)\, x^{s_2}$$

* ما بنفع نوقف الـ sin بدون cos والعكس كذلك

* لما يكون أحد sin أو cos مضروب باقترانين فإن ما يحصل عليه ما يحصل على الآخر ولما يكونوا مضروبات باقترانين مختلفات بنوخذ صاحب الدرجة الأعلى ويضربوا في الـX

ex:- Find the general sol of the I.V.P

$y'' + 4y' + 4y = 2e^{x} + 5$, $y(0) = 1$, $y'(0) = 0$ by

using under coefs method:-

$r^2 + 4r + 4 = 0$

$r = -2, -2$ $\quad y_1 = e^{-2x} \quad y_2 = xe^{-2x}$

$y_p = y_{p_1} + y_{p_2}$

$y_{p_1} = Ae^{x}x^{S_1}$ $\quad S_1 = 0$ لا يوجد تكرار

$\boxed{y_{p_1} = Ae^{x}}$

$y_{p_2} = Bx^{S}$ $\quad S_2 = 0 \Rightarrow y_{p_2} = B$

y_{p_1} works:- $\quad y''_{p_1} + 4y'_{p_1} + 4y_{p_1} = 2e^{x}$

$Ae^{x} + 4Ae^{x} + 4Ae^{x} = 2e^{x}$ $\quad A = \frac{2}{9}$

$y_{p_1} = \frac{2}{9}e^{x}$

y_{p_2} :- $\quad y''_{p_2} + 4y'_{p_2} + 4y_{p_2} = 5$

$0 + 0 + 4B = 5 \Rightarrow B = \frac{5}{4} \Rightarrow y_{p_2} = \frac{5}{4}$

$y_p = \frac{2}{9}e^{x} + \frac{5}{4}$

$y_{general} = c_1e^{-2x} + c_2xe^{-2x} + \frac{2}{9}e^{x} + \frac{5}{4}$ $\Rightarrow$ Then find c_1, c_2 باستخدام ا.ق

H.w Find the general sol of $y'' + 4y' + 4y = \underbrace{5xe^{-2x} + e^{-2x}}_{y_{p_1}} + \underbrace{3x - 1}_{y_{p_2}}$

Hw find the form of $y_p(x)$ if :- (Don't find the coefficient)

1. $f(x) = 3\sin^2(x) + \cos^2(x)$

2. $f(x) = 2\sinh(5x) + 5x$

3. $f(x) = \sin(x)\cos(3x)$

1. $f(x) = \frac{3}{2}(1 - \cos 2x) + \frac{1}{2}(1 + \cos 2x) \Rightarrow \underbrace{2}_{y_{p_1}} - \underbrace{\cos 2x}_{y_{p_2}}$

$y_{p_1} = Ax^{s_1}$

$y_{p_2} = -(B\cos 2x + C\sin 2x)x^{s_2}$

$y_p = y_{p_1} + y_{p_2} = Ax^{s_1} - (B\cos 2x + C\sin 2x)x^{s_2}$

2. $f(x) = \cancel{2}\frac{(e^x + e^{-x})}{\cancel{2}} + 5x$

$y_{p_1} = Ae^x x^{s_1}$ $y_{p_2} = Be^{-x}x^{s_2}$ $y_{p_3} = (Cx + D)x^{s_3}$

$y_p = Ae^x x^{s_1} - Be^{-x}x^{s_2} + (Cx + D)x^{s_3}$

3. $f(x) = \frac{1}{2}(\sin(-2x) + \sin 4x) \Rightarrow \underbrace{\frac{-1}{2}\sin 2x}_{y_{p_1}} + \underbrace{\frac{1}{2}\sin 4x}_{y_{p_2}}$

$y_{p_1} = -(A\sin 2x + B\cos 2x)x^{s_1}$ $y_{p_2} = (C\sin 4x + D\cos 4x)x^{s_2}$

$y_p = -(A\sin 2x + B\cos 2x)x^{s_1} + (C\sin 4x + D\cos 4x)x^{s_2}$

NO.

ex:- Find the general sol of $x^2y'' - 2xy' + 2y = 3(\ln x)^2 + x^2 + x - 1$ (using undeter coeff) ?

sol:- let $x = e^t$, $t = \ln x$

charastic $r^2 - 3r + 2 = 0$

مع بعض

$$y''(t) - 3y'(t) + 2y(t) = 3t^2 + e^{2t} + e^t - 1$$

$(r-2)(r-1) = 0$ $\quad y_1 = e^{2t} = x^2 \quad y_2 = e^t = x$

$$y_{p_1} = (At^2 + Bt + C)\, t^{s_1} \qquad s_1 = 0$$

$$y_{p_2} = D e^{2t} t^{s_2} \qquad s_2 = 1$$

$$y_{p_3} = E e^t t^{s_3} \qquad s_3 = 1$$

$$y = c_1 x^2 + c_2 x + A(\ln x)^2 + B(\ln x) + C + Dx^2 \ln x + Ex \ln x$$

THE END

Chapter 7:- The Laplace Transform :-

is a new method to solve I.V.P of linear O.D.Es with initial condition at $x=0$

* Given a function $f(t)$ defined on $[0,\infty)$ then:- the "Laplace transform" of $f(t)$ is a fun $F(s)$ defined by:-

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

* The set of all values of s which make the improper integral converge is the domain of $F(s)$

note:- $\int_0^\infty = \lim_{b\to\infty} \int_0^b$

ex let $f(t)=1$ (constant fun) Find $\mathcal{L}\{1\}$?

$$F(s) = \mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \lim_{b\to\infty} \int_0^b e^{-st}\,dt$$

$$\lim_{b\to\infty} \left(\frac{-1}{s} e^{-st} \Big]_0^b \right) \quad (s \neq 0) = \lim_{b\to\infty} \frac{-1}{s}\left(e^{-sb} - 1\right)$$

$$= \frac{-1}{s}\left(e^{-s(\infty)} - 1\right)$$

$$F(s) = \frac{1}{s} = \begin{cases} \text{div} & \text{if } s<0 \\ \frac{1}{s}, & s>0 \end{cases}$$

2) $f(t) = e^{3t}$?

$$\mathcal{L}\{e^{3t}\} = \int_0^\infty e^{-st} e^{3t}\, dt = \int_0^\infty e^{-(s-3)t}\, dt$$

$$= \lim_{b \to \infty} \int_0^b e^{-(s-3)t}\, dt = \lim_{b \to \infty} \left(\frac{-1}{s-3} e^{-(s-3)t} \Big]_0^b \right)$$

$$= \lim_{b \to \infty} \left(\frac{-1}{s-3} \left[e^{-(s-3)b} - 1 \right] \right) = \frac{-1}{s-3} \left[e^{-(s-3)b\infty} - 1 \right]$$

$$= \begin{cases} \frac{1}{s-3} & , \text{ if } (s-3) > 0 \\ \text{div} & , \text{ if } (s-3) \leq 0 \end{cases}$$

أو بنحط $s \neq 3$ اي احتمال السابق

$$\mathcal{L}\{e^{3t}\} = \frac{1}{s-3} \quad , \quad s > 3 \quad \#$$

	$f(t)$	$F(s)$
	1	$\frac{1}{s}$, $s > 0$
	c	$\frac{c}{s}$, $s > 0$
	e^{at}	$\frac{1}{s-a}$, $s > a$
$n = 0, 1, 2, 3 \ldots$	t^n	$\frac{n!}{s^{n+1}}$
	$\sin(bt)$	$\frac{b}{s^2+b^2}$, $s > 0$
	$\cos(bt)$	$\frac{s}{s^2+b^2}$, $s > 0$

ex:- find $\mathcal{L}\{f(t)\}$

1. $f(t) = \cos^2(3t)$

$= \frac{1}{2}(1 + \cos(6t))$

$\mathcal{L}\{\cos^2(3t)\}$

$= \mathcal{L}\{\frac{1}{2} + \frac{1}{2}\cos(6t)\}$

$= \mathcal{L}\{\frac{1}{2}\} + \mathcal{L}\{\frac{1}{2}\cos 6t\}$

$= \frac{1}{2s} + \frac{1}{2} * \frac{s}{s^2+36}$:- $s > 0$

$(s > 0)$ $(s > 0)$

$f(t) = \sin(\alpha t)\cos(\beta t) = \frac{1}{2}[\sin(\alpha-\beta)t + \sin(\alpha+\beta)t]$

$= \sin(\alpha t)\sin(\beta t) = \frac{1}{2}[\cos(\alpha-\beta)t - \cos(\alpha+\beta)t]$

$= \cos(\alpha t)\cos(\beta t) = \frac{1}{2}[\cos(\alpha-\beta)t + \cos(\alpha+\beta)t]$

* properties of Laplace Transform

1. $\mathcal{L}\{f \pm g\} = \mathcal{L}\{f\} \mp \mathcal{L}\{g\}$

$\mathcal{L}\{c.f\} = c\mathcal{L}\{f\}$

NO

(2) $\mathcal{L}\{e^{at} f(t)\} = \mathcal{L}\{f(t)\} = F(s-a)$ $\boxed{s \to s-a}$

(3) $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$

ex:- 1. $\mathcal{L}\{e^{3t}\cos(2t)\}$

$= \mathcal{L}\{\cos 2t\}_{s \to s-3}$

$= \left(\frac{s}{s^2+4}\right)_{s \to s-3} = \left(\frac{s-3}{(s-3)^2+4}\right)$

2. $\mathcal{L}\{t^6 e^{2t}\}$

$f(t) = e^{2t}$ $\quad f(t) = t^6$

$(-1)^6 \frac{d^6}{ds^6}(1/(s-2))$ $\quad \mathcal{L}\{t^6\}_{s \to s-2} = \frac{6!}{(s-2)^7}$

$\lim_{s \to \infty} F(s) = 0$ because $F(s) \nless 1$

NO.

3. $\mathcal{L}\{t^2 \sin(t)\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right)$

$= \frac{d}{ds}\left(\frac{-2s}{(s^2+1)^2}\right)$

$= \frac{-2(s^2+1)^2 - (-2s)\,2(s^2+1) * 2s}{(s^2+1)^4}$

4. $\mathcal{L}\{t^2 e^{3t} \sin(5t)\}$

وقت زيادة ☹

$\mathcal{L}\{e^{3t}(\underbrace{t^2 \sin 5t}_{f(t)})\}$ $\mathcal{L}\{t^2(\underbrace{e^{3t}\sin(5t)}_{f(t)})\}$

- $\mathcal{L}\{f(t)\} = F(s)$

④ $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

اجزاء

$\equiv \int_0^{\infty} \underbrace{e^{-st}}_{u} \underbrace{f'(t)\,dt}_{dv} \Rightarrow f(t)e^{-sb}\Big]_0^{\infty} - \int -se^{-st} f(t)\,dt$

$\lim_{b\to\infty}[f(b)e^{-sb} - f(0)] + s\int_0^{\infty} e^{-st} f(t)\,dt$ F_s

$\frac{f(\infty)}{e^{s(\infty)}} - f(0) + sF(s) \Rightarrow \boxed{sF(s) - f(0)}$

FIVE APPLE

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(5) $\mathcal{L}\{f''(t)\} = s\,\mathcal{L}\{f'(x)\} - f'(0)$

$$= s\left[sF(s) - f(0)\right] - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

(6) $\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$

- $\mathcal{L}\{f''''(t)\} = s^4F(s) - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0)$

ex:- Find $\mathcal{L}\{\cos(t)\}$ if $\mathcal{L}\{\sin t\} = \dfrac{1}{1+s^2}$?

* (على فرض انو ما بنعرف اللابلاس لل cost)

Sol:-

$$\mathcal{L}\{\cos t\} = \mathcal{L}\{S\sin t\}$$

$$= S\,\mathcal{L}\{\sin t\} - \sin 0$$

$$S \cdot \frac{1}{S^2+1} - 0 \Rightarrow \frac{S}{S^2+1}$$

(7) * $\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \dfrac{1}{S} \cdot F(s) + 0$

$$F(s) = \mathcal{L}\{f(t)\}$$

السؤال = تكامل المعطى

ex: If $\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}$ find:-

1. $\mathcal{L}\{\sqrt{t}\}$ 2. $\mathcal{L}\left\{\sqrt{\frac{2t}{\pi}}\right\}$ H.w 3. $\mathcal{L}\left\{t\sqrt{\frac{3}{\pi t}}\right\}$ H.w

1. $$\sqrt{t} = \frac{1}{2}\left(\int_0^t \frac{1}{\sqrt{x}}\,dx\right)$$

$$\mathcal{L}(\sqrt{t}) = \frac{1}{2}\mathcal{L}\left\{\int_0^t \frac{1}{\sqrt{x}}\,dx\right\}$$

$$= \frac{1}{2}\left(\frac{1}{s}\cdot F(s)\right) = \frac{1}{2\cdot s}\cdot\sqrt{\frac{\pi}{s}}$$

* sec 7.4: Inverse Laplace Transform

Recall:-

Given $f(t)$, then $\mathcal{L}\{f(t)\} = F(s)$

is laplace trans of $f(t)$

Now:- Given $F(s)$,

then $\mathcal{L}^{-1}\{F(s)\} = f(t)$ الاصل

is the Laplace inverse trans... of $F(s)$

ex:- let $F(s) = \frac{2}{s^2+4}$ then $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \sin 2t$

so, we have 4 methods to find $\mathcal{L}^{-1}\{F(s)\}$

1. Direct computation (using the table)

2. completing the square of shifting

NO

3. Partial Fraction

4. using the rule:- $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}$

* why we need to find $\mathcal{L}^{-1}$??

ex:- Solve the I.V.P using Laplace transform:-

$y''(t) - y(t) = -t \qquad y(0)=0 \qquad y'(0)=1$

sol:

Step 1:- take "$\mathcal{L}$" for both sides

$$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = -\mathcal{L}\{t\}$$

$$s^2 y(s) - \cancel{s\,\overset{zero}{y(0)}} - \overset{1}{y'(0)} - y(s) = \frac{-1}{s^2}$$

where: $y(s) = \mathcal{L}\{y(t)\}$

* is the general sol for the I.V.P

step 2:- solve for $y(s)$

$$y(s)(s^2 - 1) = \frac{-1}{s^2} + 1 = \frac{(-1 + s^2)}{s^2}$$

$$y(s) = \frac{1}{s^2}$$

$\Longrightarrow$

Step 3:- $y(t) = \mathcal{L}^{-1}\{ y(s) \}$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{s^2} \right\} = t$$

$y(t) = t$

$\Leftarrow y_p$

$C_1 = C_2 = 0$

Ex: find "$\mathcal{L}^{-1}$" for :-

1. $f(s) = \dfrac{2s+3}{s^2+2}$:-

sol:

$$\frac{2s}{s^2+2} + \frac{3 \times 1}{s^2+2}$$

$$2\mathcal{L}^{-1}\left\{ \frac{s}{s^2+2} \right\} + \frac{3}{\sqrt{2}} \mathcal{L}^{-1}\left\{ \frac{\sqrt{2}}{s^2+2} \right\}$$

* إذا كانت الإشارة سالبة
بتصير $\sinh x$ ، $\cosh x$

$$2\cos(\sqrt{2}t) + \frac{3}{\sqrt{2}}\sin(\sqrt{2}t)$$

2. $F(s) = \dfrac{2}{(s-5)^4} = e^{5t} \mathcal{L}^{-1}\left\{ \dfrac{2}{s^4} \right\}$

$$e^{5t}\mathcal{L}^{-1}\left\{ \frac{2}{3!} \cdot \frac{3!}{s^{3+1}} \right\} = \frac{e^{5t}}{3} \cdot t^3$$

NO.

3) $\mathcal{L}^{-1}\left\{\frac{2s+3}{(s-5)^2+2}\right\}$:-

$e^{5t}\mathcal{L}\left\{\frac{2s+3}{s^2+2}\right\}$ لأنو في s في البسط

$=\mathcal{L}^{-1}\left\{\frac{2(s-5+5)+3}{(s-5)^2+2}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s-5)+13}{(s-5)^2+2}\right\}$

$e^{5t}\mathcal{L}^{-1}\left\{\frac{2s+13}{s^2+2}\right\} = e^{5t}\left(2\cos(\sqrt{2}t)+\frac{13}{\sqrt{2}}\sin(\sqrt{2}t)\right)$

4.) $\mathcal{L}^{-1}\left\{\frac{2s+3}{s^2-10s+27}\right\}$:-

complete the square

$s^2-10s+27 \qquad \pm\left(\frac{-10}{2}\right)^2$

$(s^2-10s+25)+2$

$=\mathcal{L}^{-1}\left\{\frac{2s+3}{(s-5)^2+2}\right\}$:- ——

5) $\mathcal{L}^{-1}\left\{\frac{3-2s}{s^2-4s+3}\right\} \longrightarrow$ complete the square (Hw)

$e^{2x}((\sin x)+\cos x)$

✓ partial fraction

$\frac{3-2s}{(s-3)(s-1)}$

NO.

$\frac{A}{s-1} + \frac{B}{s-3}$ find A, B

$A(s-3) + B(s-1) = \cancel{3s-2}\ 3-2s$

$B = \frac{-3}{2} \quad A = \frac{-1}{2}$

$\mathcal{L}^{-1} = \mathcal{L}^{-1}\left\{\frac{-1/2}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{-3/2}{s-3}\right\}$

$= \frac{-1}{2} e^{t} + \frac{-3}{2} e^{3t}$

6) $\mathcal{L}^{-1}\left\{\frac{3s+1}{s^3-s}\right\}$ ...

$\mathcal{L}^{-1}\left\{\frac{3s+1}{s(s-1)(s+1)}\right\} \Rightarrow \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}$

$3s+1 = A(s-1)(s+1) + B(s)(s+1) + C(s)(s-1)$

$s=1 \quad s=0 \quad s=-1$

$A=-1 \quad B=2 \quad C=-1$

$\mathcal{L}\left\{\frac{3s+1}{s(s-1)(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{-1}{s+1}\right\}$

$-1 + 2e^{t} - e^{-t}$

7. H.w :- find (i) :- $\mathcal{L}^{-1}\left\{\frac{2s+2}{(s^2-1)(s^2+1)}\right\}$

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Ex: $F(s) = \frac{2}{s^3(s-1)}$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^3(s-1)}\right\} \Rightarrow \mathcal{L}^{-1}\left\{\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s-1}\right.$$

$= A(1)+B(t)+C\frac{t^2}{2!}+De^{t}$. to find (A,B,C,D) =

$$As^2(s-1)+B(s)(s-1)+C(s-1)+Ds^3=2$$

$s=0 \Rightarrow C=-2$

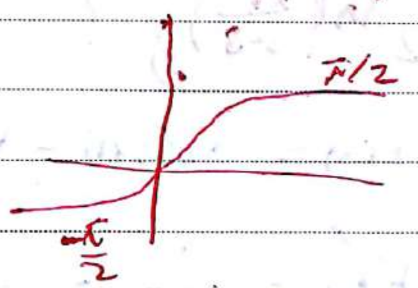
$s=0 \Rightarrow D=2$

sdr $-2-2t-t^2+2e^{t}$

2. H.w $-\mathcal{L}^{-1}\frac{2-3s}{(s-2)(s^2+3)} \longrightarrow \frac{A}{(s-2)}+\frac{Bs+C}{s^2+3}$

pr- $\mathcal{L}^{-1}\frac{s+2}{s(s+3)(s-4)}$

ex. $\mathcal{L}^{-1}\left\{\tan^{-1}\left(\frac{1}{s}\right)\right\}$



$$-\frac{\pi}{2} < \tan^{-1} < \frac{\pi}{2}$$

$$(\tan^{-1}s)' = \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{t^n f(t)\right\} = (-1)^n \frac{dF(s)}{ds}$$ if $n=1$

$$\mathcal{L}\left\{t f(t)\right\} = (-1)F'(s)$$

$$\mathcal{L}\{-t f(t)\} = F'(s)$$

$$= \mathcal{L}^{-1}\{F'(s)\} = -t f(t)$$

1. let $F(s) = \tan^{-1}\left(\frac{1}{s}\right)$ من السؤال

2. $F'(s) = \frac{-1}{s} \cdot \frac{1}{1+\left(\frac{1}{s}\right)^2}$

$$F'(s) = \frac{-1}{s^2+1}$$

$$\mathcal{L}^{-1}\{F'(s)\} = \mathcal{L}^{-1}\left\{\frac{-1}{s^2+1}\right\}$$

$$\therefore -t f(t) = -\sin t$$

$$f(t) = \frac{\sin t}{t}$$

b. $\mathcal{L}^{-1}\left\{\ln\left(\frac{s+1}{s-3}\right)\right\}$?

$$F(s) = \ln(s+1) - \ln(s-3)$$

$$F'(s) = \frac{1}{s+1} - \frac{1}{s-3}$$

$$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$-t f(t) = e^{-t} - e^{-3t} \Rightarrow$$

$$f(t) = \frac{-1}{t}\left(e^{-t} - e^{+3t}\right)$$

H.W 1. $\mathcal{L}^{-1}\left\{ \ln\left(\frac{s^2-1}{s^2+3}\right) \right\}$

2. $\mathcal{L}^{-1}\left\{ \tan^{-1}\left(\frac{1}{s^2}\right) \right\}$

ex: find the laplace transform of the sol to the I.V.P

$$y'' - y' - 2y = e^{2t} \quad , \quad y(0)=0 \quad , \quad y'(0)=1 \quad ?$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{e^{2t}\}$$

$$s^2 Y(s) - s y(0) - s y'(0) - (s Y(s) - y(0)) - 2Y(s) = \frac{1}{s-2}$$

$$Y(s)(s^2 - s - 2) - 1 = \frac{1}{s-2}$$

$$Y(s) = \left(\frac{s-1}{s-2}\right) \cdot \frac{1}{s^2 - s - 2} \Rightarrow \frac{A}{s-2} + \frac{B}{(s-2)^2} + \frac{C}{(s+1)}$$

ex: solve the following I.V.P using laplace transform:

$$y'' - 7y' + 10y = 9\cos(t) - 7\sin(t)$$

$$y(0) = 1 \qquad y'(0) = 0 \qquad ?$$

H.W

sol :-

$$\mathcal{L}\{y''\} - 7\mathcal{L}\{y'\} + 10\mathcal{L}\{y\} = 9\cos t - 7\sin t$$

$$s^2Y(s) - sy(0) - y'(0) - 7sY(s) + 7y(0) + 10Y(s) = \frac{9s-7}{s^2+1}$$

$$\cancel{s^2}\, Y(s)\left(s^2 - 7s + 10\right) = \frac{9s-7}{s^2+1} + s - 7$$

$$Y(s) = \frac{s^3 - 7s^2 + 10s - 14}{s^2 - 7s + 10} \quad \#$$

THE END