



اللجنة الأكاديمية لقسم الهندسة المدنية

دفتر

تفاضل وتكامل 3 (فيرست)

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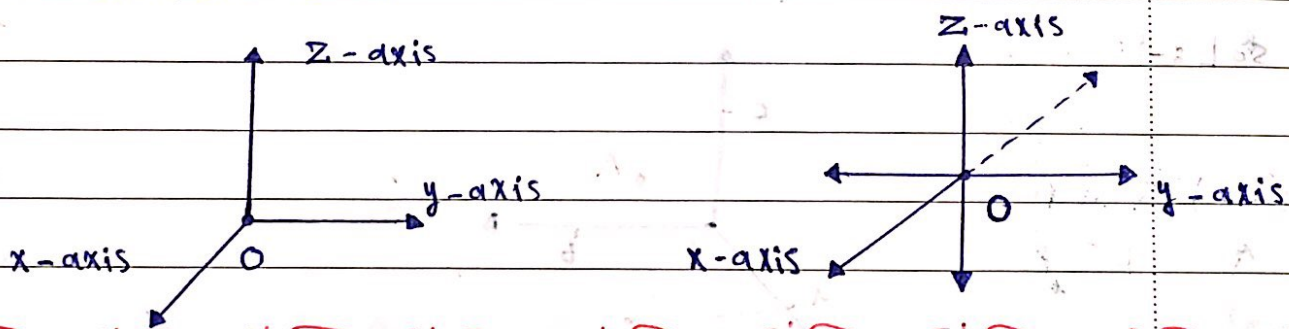


* (CH.12) Vectors and the Geometry of Space

* sec 12.1 Three-dimensional coordinate system.

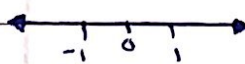
- The coordinate axes in the space are 3 axes are called the x-axis, y-axis, z-axis.

- represented in the space in a way such that any one of them is perpendicular to other 2 axes and intersected (تقاطع) in a point (pt.) called the origin written O.



* Definition (Def) :- The coordinate planes are 3-planes. they are :-
 xy-plane :- contains the x-axis, y-axis.
 xz-plane :- contains the x-axis, z-axis.
 yz-plane :- contains the y-axis, z-axis.

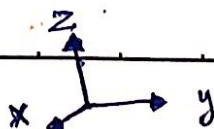
* Def :- (1) $\mathbb{R} = (-\infty, \infty)$



(2) $\mathbb{R} \times \mathbb{R} = \{(a, b) = a, b \in \mathbb{R}\} = \mathbb{R}^2$ (plane)



(3) $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(a, b, c) = a, b, c \in \mathbb{R}\} = \mathbb{R}^3$ space



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*** Remark :-** (ध्यान दें)

The coordinate planes divide the space into 8 parts each part is called octant. The first octant is the octant that contains the positive coordinate axes.

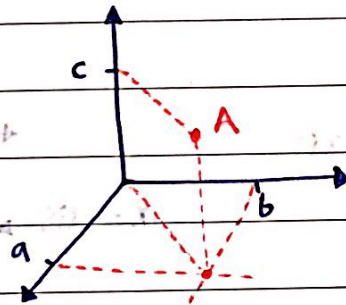
a :- X - coordinate.

b :- y - coordinate.

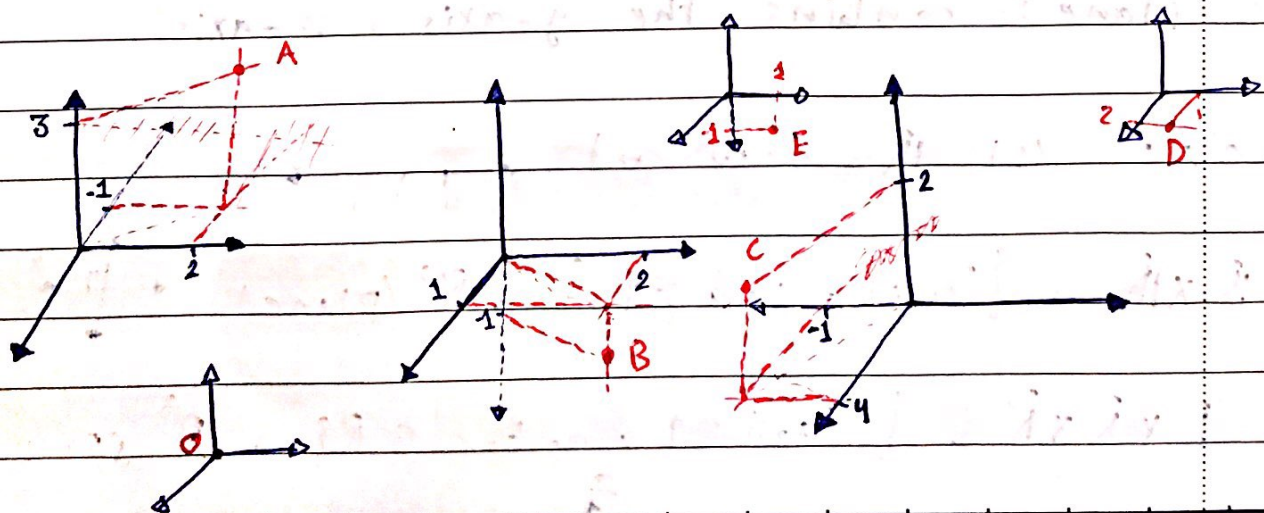
c :- z - coordinate.

E.x) sketch the pt. A (a, b, c), where $a > 0, b > 0, c > 0$

sol :-



E.x) sketch the pts. A(-1, 2, 3), B(1, 2, -1), C(4, -1, 2), D(2, 1, 0), E(0, 1, -1), O(0, 0, 0)



* Remark:- The distance between the pt. $A(a, b, c)$ and the (1) xy -plane is $|c|$
 (2) xz -plane is $|b|$ (3) yz -plane is $|a|$.

* Def:- The equation of a sphere centered at the pt. (a, b, c) of radius r is $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$

E.x) Show that the eq. $2x^2 + 2y^2 + 2z^2 + 8x - 12y + 12 = 0$ is a sphere and find its center and radius.

Sol:-

$$\text{eq.} \div 2 \Rightarrow (x^2 + 4x) + (y^2 - 6y) + z^2 = -6$$

$$(x+2)^2 + (y-3)^2 + z^2 = -6 + 4 + 9 \quad \boxed{7}$$

center $(-2, 3, 0)$, radius $= \sqrt{7}$

* Def:- Let $A(a, b, c)$, $B(d, e, f)$

(1) The distance between A, B is $\text{dist}(A, B) = |AB|$
 $= \sqrt{(d-a)^2 + (e-b)^2 + (f-c)^2}$

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(2) The mid point of line segment with end pt. A, B is $\left(\frac{a+d}{2}, \frac{b+e}{2}, \frac{c+f}{2}\right)$

E.x) Find the eq. of the sphere if one of its diameters has end pts. $A(-3, 2, 1)$, $B(5, -2, 2)$

Sol:-

$$r = \frac{\sqrt{(-3-5)^2 + (2-(-2))^2 + (1-2)^2}}{2} \\ = \frac{1}{2} \sqrt{64 + 16 + 1} \\ = \frac{9}{2}$$

$$\text{center} = \text{mid pt. } (A, B) = \left(\frac{-3+5}{2}, \frac{2+(-2)}{2}, \frac{1+2}{2} \right) \\ = \left(1, 0, \frac{3}{2} \right)$$

The eq. of the sphere is $(x-1)^2 + y^2 + (z-\frac{3}{2})^2 = (\frac{9}{2})^2$

E.x) Find the eq. of the sphere centered at $A(-13, 2, -5)$ and (1) touches the xy -plane. (2) touches the xz -plane. (3) touches the yz -plane.

Sol:-

$$(1) r = \text{dist}(A, xy\text{-plane}) = |-5| = 5$$

$$\text{eq.} \Rightarrow (x+13)^2 + (y-2)^2 + (z+5)^2 = 25$$

$$(2) r = |-2| = 2$$

$$\text{eq.} \Rightarrow (x+13)^2 + (y-2)^2 + (z+5)^2 = 4$$

$$(3) r = |-13| = 13$$

$$\text{eq.} \Rightarrow (x+13)^2 + (y-2)^2 + (z+5)^2 = 169$$

E.X) Find the eq. of the largest possible sphere in the 1st octant centered at the pt. (3, 2, 5)

$$(x-3)^2 + (y-2)^2 + (z-5)^2 = 4$$

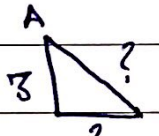
E.X) Find All possible eqs of spheres of radius 3 and touches the 3 coordinate planes.

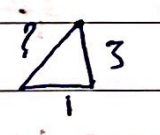
center (3, 3, 3) - (3, -3, 3) - (-3, -3, 3) - (-3, 3, 3)

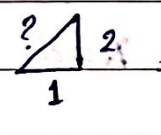
(3, 3, -3) - (3, -3, -3) - (-3, -3, -3) - (-3, 3, -3)

E.X) Find the distance from the pt. A (-1, 2, 3) and the (1) x-axis (2) y-axis (3) z-axis

SOL :-

(1)  $\Rightarrow \sqrt{9+4} = \sqrt{13}$

(2)  $\Rightarrow \sqrt{9+1} = \sqrt{10}$

(3)  $\Rightarrow \sqrt{4+1} = \sqrt{5}$

* Graphs of eqs :-

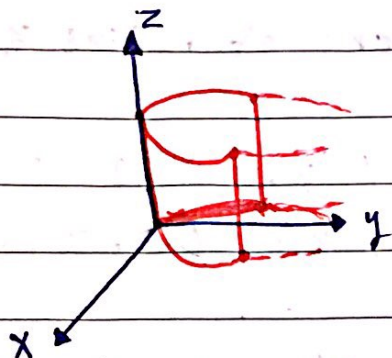
(1) The graph of an eq. in $\mathbb{R}^2$ is a curve

(2) The graph of an eq in $\mathbb{R}^3$ (3D) is a surface.

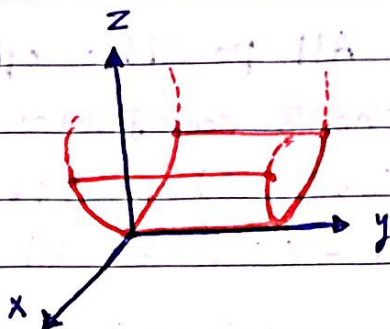
E.x) sketch the graph of the surface whose eqs. are (1) $y = x^2$ (2) $x^2 + z^2 = 4$ (3) $z = x^2$ (4) $z = 0$ (5) $x = 3$

SOL :-

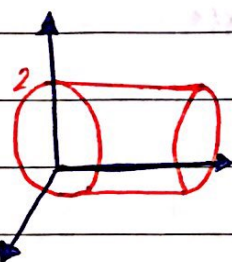
(1)



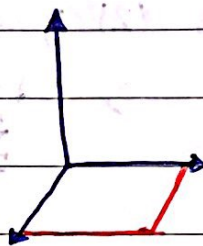
(3)



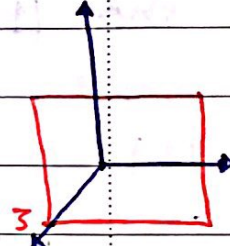
(2)



(4)



(5)



* Remark :- The eq. of the (1) xy -plane is $z = 0$
(2) xz -plane is $y = 0$ (3) yz -plane is $x = 0$

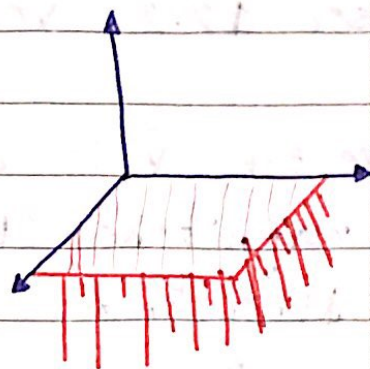
E.x) sketch the region in $\mathbb{R}$ that satisfies the inequality. (مناطق). (مناطق)

(1) $z \leq 0$ (2) $x > -3$ (3) $2 < x^2 + y^2 + z^2 \leq 4$

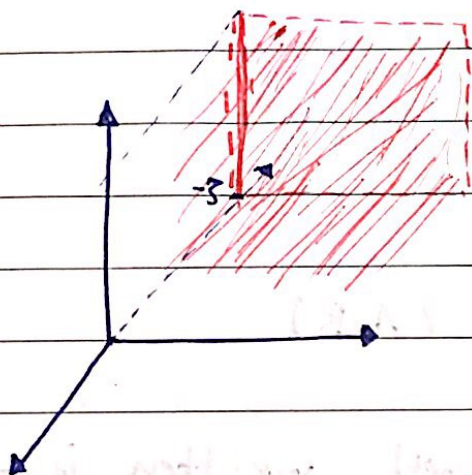
(4) $x^2 + y^2 + z^2 \geq 6z$

SOL :- (1) $z=0$

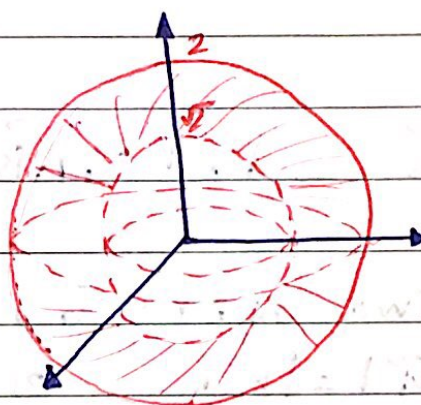
all pts. (x, y, z)
that lies on the xy -plane
or below it.



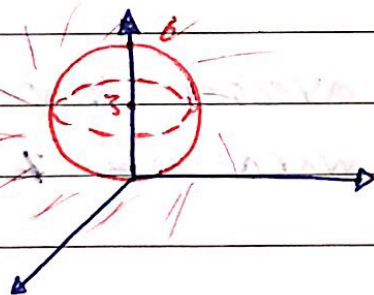
(2)



(3)



(4)



$$x^2 + y^2 + z^2 = 6z$$

$$x^2 + y^2 + (z^2 - 6z + 9) = 9$$

$$x^2 + y^2 + (z - 3)^2 = 9$$

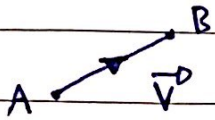
center $(0, 0, 3)$ radius $= 3$

~~$$\frac{a}{c} = \frac{|a|}{|c|} \cos \theta$$~~

~~$$\cos \theta = \frac{a \cdot b}{|a| |b|}$$~~

* sec 12.2 Vectors :-

A vector $\vec{v}$ is a quantity that has
 (1) magnitude (length) written $|\vec{v}|$
 (2) direction

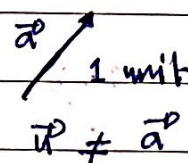
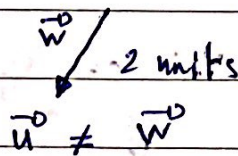
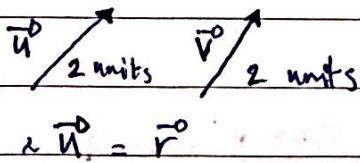


$$\vec{v} = \overrightarrow{AB} \Rightarrow |\vec{v}| = |\overrightarrow{AB}| = \text{dist. (A, B)}$$

* Def :- Two vectors $\vec{u}$, $\vec{r}$ are equal written $\vec{u} = \vec{r}$ if (1) $|\vec{u}| = |\vec{r}|$ (2) they have the same direction

* Remark :- The zero vector $\vec{0}$ is a vector of length 0 but in any direction. In other words $\vec{0} = \overrightarrow{AA} = \overrightarrow{BB}$

* Def :- * E.X)



* Def :- $\vec{v}$ vectors, a scalar (sc)

→ a $\vec{v}$ vector with

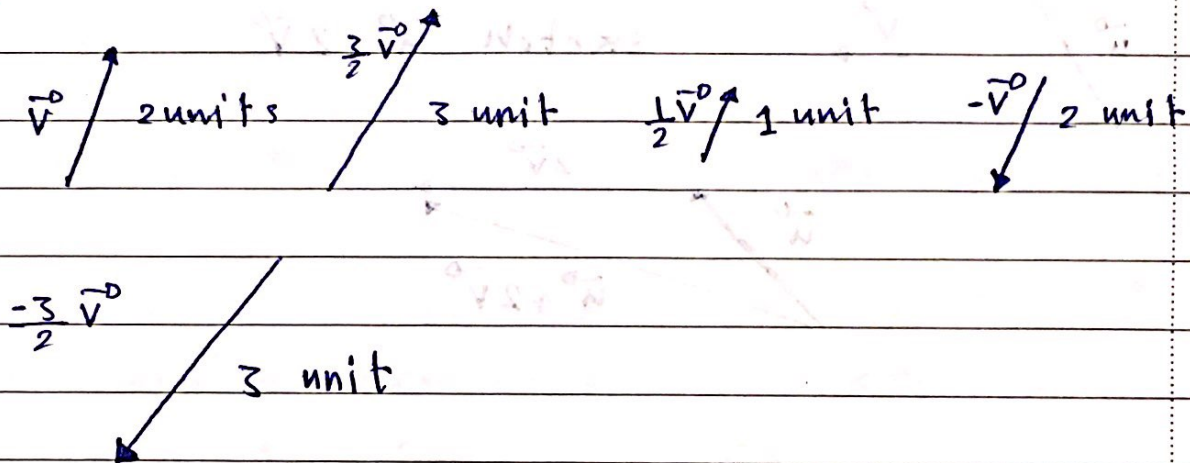
$$(1) |a\vec{v}| = |a||\vec{v}|$$

- (2) * in the same direction of $\vec{v}$ if $a > 0$.
* in the opposite direction of $\vec{v}$ if $a < 0$.

★ Remark :- (1) $-\vec{v}, \vec{v}$ of the same length
 $|\vec{v}| = |-\vec{v}|$ but in opposite direction

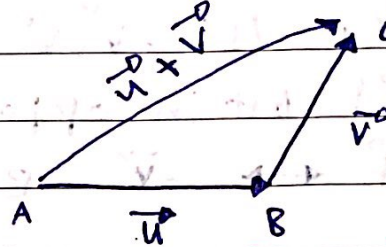
(2) $\vec{v} = \overrightarrow{AB} \longleftrightarrow -\vec{v} = \overrightarrow{BA}$

E.x)

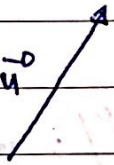


* Def :- Let $\vec{u} = \overrightarrow{AB}$, $\vec{v} = \overrightarrow{BC}$

Then $\vec{u} + \vec{v} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

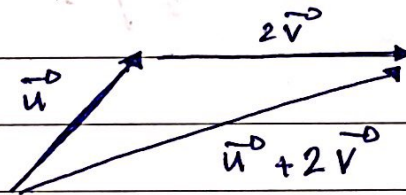


E.x)



Sketch $\vec{u} + 2\vec{v}$

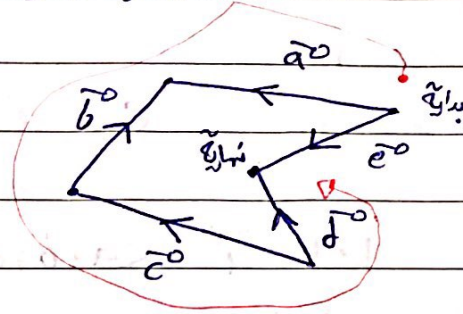
Sol :-



E.X) Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}$ be vectors as in the figure write $\vec{e}$ in terms of $\vec{a}, \vec{b}, \vec{c}, \vec{d}$

SOL:-

$$\begin{aligned}\vec{e} &= \vec{a} + (-\vec{b}) + (-\vec{c}) + \vec{d} \\ &= \vec{a} - \vec{b} - \vec{c} + \vec{d}\end{aligned}$$



E.X) Let A, B, C be 3 pts. Let $\vec{V} = \vec{AB} + \vec{BC} + \vec{CA}$
Find $|\vec{V}|$??

SOL:-

$$\vec{V} = \vec{AB} + \vec{BC} + \vec{CA} = \vec{AA} = \vec{0} \quad \therefore |\vec{V}| = |\vec{0}| = 0$$

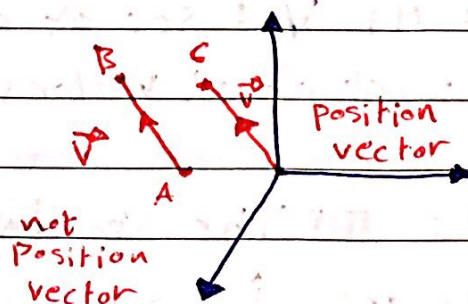
* Def:- Let $A(a, b, c), B(d, e, f)$ and $\vec{V} = \vec{AB}$

(1) The component form of $\vec{V}$ is $\vec{V} = \langle B - A \rangle = \langle d - a, e - b, f - c \rangle$

(2) Let $C = (d - a, e - b, f - c)$ and $f - c >$

$O = (0, 0, 0)$ then $\vec{V} = \vec{OC} = \langle d - a, e - b, f - c \rangle$

this vector is called a position vector of $\vec{V}$



E.X) sketch $\vec{v} = \vec{AB}$ where $A = (2, -1, 3)$, $B = (4, 1, 2)$

SOL:- $\vec{v} = \langle B - A \rangle = \langle 4 - 2, 1 - (-1), 2 - 3 \rangle$

$= \langle 2, 2, -1 \rangle$

$C = (2, 2, -1)$ $O = (0, 0, 0)$ $\vec{v} = \vec{OC}$

* Rule:- Let $\vec{u} = \langle a, b, c \rangle$

$\vec{v} = \langle d, e, f \rangle$

(1) $\lambda \vec{u} = \langle \lambda a, \lambda b, \lambda c \rangle$

(2) $\vec{u} + \vec{v} = \langle a + d, b + e, c + f \rangle$

(3) $\vec{0} = \langle 0, 0, 0 \rangle$

(4) $|\vec{u}| = \sqrt{a^2 + b^2 + c^2}$

E.X) Let $\vec{a} = \langle 1, 2 \rangle$, $\vec{b} = \langle 3, -1 \rangle$, $\vec{c} = \langle 4, 5 \rangle$

find $|2\vec{a} - 3\vec{b} + \vec{c}|$??

SOL:- $2\vec{a} - 3\vec{b} + \vec{c} = \langle 2, 4 \rangle - \langle 9, -3 \rangle + \langle 4, 5 \rangle$

$= \langle 2 - 9 + 4, 4 - (-3) + 5 \rangle = \langle -3, 12 \rangle$

$|2\vec{a} - 3\vec{b} + \vec{c}| = |\langle -3, 12 \rangle| = \sqrt{9 + 144} = \sqrt{153}$

* Def: (1) V_2 : set of all vectors in $\mathbb{R}^2$

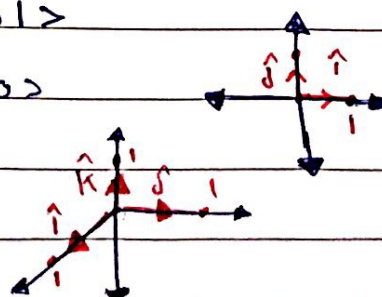
(2) V_3 : set of all vectors in $\mathbb{R}^3$.

* Def:- The standard basis vectors in

(1) V_2 are $\hat{i} = \langle 1, 0 \rangle$, $\hat{j} = \langle 0, 1 \rangle$

(2) V_3 are $\hat{i} = \langle 1, 0, 0 \rangle$, $\hat{j} = \langle 0, 1, 0 \rangle$

$\hat{k} = \langle 0, 0, 1 \rangle$



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* Remark :- $\vec{v}^0 = \langle a, b, c \rangle$ [component form]
 $= \langle a, 0, 0 \rangle + \langle 0, b, 0 \rangle + \langle 0, 0, c \rangle$
 $= a \langle 1, 0, 0 \rangle + b \langle 0, 1, 0 \rangle + c \langle 0, 0, 1 \rangle$
 $= a \hat{i} + b \hat{j} + c \hat{k}$ [basis vectors form]

E.x) (1) $3\hat{i} - 5\hat{k} = \langle 3, 0, -5 \rangle$

(2) $-\hat{i} + 2\hat{j} - 3\hat{k} = \langle -1, 2, -3 \rangle$

(3) $2\hat{i} + 5\hat{j} = \langle 2, 5, 0 \rangle$ in V_3

$2\hat{i} + 5\hat{j} = \langle 2, 5 \rangle$ in V_2

E.x) (1) $\langle 2, 0 \rangle = 2\hat{i}$

(2) $\langle -3, 4, 1 \rangle = -3\hat{i} + 4\hat{j} + \hat{k}$

E.x) $\vec{u}^0 = 2\hat{i} - 3\hat{k}$, $\vec{v}^0 = \langle 2, -1, 4 \rangle$ find $|4\vec{u}^0 - 5\vec{v}^0|$??

sol:- $4\vec{u}^0 - 5\vec{v}^0 = 4\langle 2, 0, -3 \rangle - 5\langle 2, -1, 4 \rangle$
 $= \langle -2, 5, -32 \rangle$

$|4\vec{u}^0 - 5\vec{v}^0| = \sqrt{4 + 25 + (-32)^2}$

* Def:- A vector $\vec{v}^0$ is called a unit vector if $|\vec{v}^0| = 1$.
 If $|\vec{v}^0| = 1 \Rightarrow \vec{v}^0$ written $\vec{v}^0 = \hat{v}$

E.x) (1) $\hat{i}, \hat{j}, \hat{k}$ unit vectors

(2) $|\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}| = \sqrt{(\frac{2}{3})^2 + (-\frac{1}{3})^2 + (-\frac{2}{3})^2} = \sqrt{\frac{9}{9}} = 1$

$\Rightarrow \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$ unit vector

E.x) $|2\hat{u}| = |2||\hat{u}| = 2$

* Def:- Let $\vec{v} \neq \vec{0}$ A unit vector (1) in the same direction of $\vec{v}$ is $\frac{\vec{v}}{|\vec{v}|}$

(2) in the opposite direction of $\vec{v}$ is $-\frac{\vec{v}}{|\vec{v}|}$

E.x) Let $\vec{v} = \langle 2, 3 \rangle$ Find a unit vector: (1) in the same direction of $\vec{v}$ (2) in the opposite direction of $\vec{v}$

Sol:- $|\vec{v}| = \sqrt{4+9} = \sqrt{13}$

(1) The unit vector in the same direction of $\vec{v}$ is

$$\frac{\vec{v}}{|\vec{v}|} = \frac{\vec{v}}{\sqrt{13}} = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$$

(2) The unit vector in the opposite direction of $\vec{v}$ is

$$-\frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{-2}{\sqrt{13}}, \frac{-3}{\sqrt{13}} \right\rangle$$

E.x) Let $\vec{v} = \hat{i} + 2\hat{j} - 3\hat{k}$ Find a vector of length $\frac{\sqrt{14}}{2}\pi$ and (1) in the same direction of $\vec{v}$. (2) in the opposite direction of $\vec{v}$.

Sol:- $\vec{v} = \langle 1, 2, -3 \rangle$

$$|\vec{v}| = \sqrt{1+4+9} = \sqrt{14}$$

$$\frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}} \right\rangle$$

(1) The required (unit) vector is $\frac{\sqrt{14}}{2}\pi \frac{\vec{v}}{|\vec{v}|}$

$$= \left\langle \frac{\sqrt{2}\pi}{2\sqrt{14}}, \frac{\sqrt{2}\pi 2}{2\sqrt{14}}, \frac{\sqrt{2}\pi - 3}{2\sqrt{14}} \right\rangle$$

(2) The required vector is

$$-\frac{\sqrt{2}\pi}{2} \frac{\vec{v}^0}{|\vec{v}^0|} = \left\langle \frac{-\sqrt{2}\pi}{2\sqrt{14}}, \frac{-\sqrt{2}\pi 2}{2\sqrt{14}}, \frac{-\sqrt{2}\pi(-3)}{2\sqrt{14}} \right\rangle$$

* sec 12.3 :- The Dot product.

* Def :- The dot product of $\vec{u}^0 = \langle a, b, c \rangle$ and $\vec{v}^0 = \langle d, e, f \rangle$ is
 $\vec{u}^0 \cdot \vec{v}^0 = ad + be + cf$ (scalar).

e.x) (1) $\langle 3, 2 \rangle \cdot \langle -2, 5 \rangle = 3(-2) + 2(5) = 4$

(2) $(3\hat{i} + 2\hat{k}) \cdot (2\hat{i} - 3\hat{j} + \hat{k}) = 3(2) + 0(-3) + 2(1) = 8$

* properties :-

(1) $\vec{u}^0 \cdot \vec{v}^0 = \vec{v}^0 \cdot \vec{u}^0$ (2) $\vec{u}^0 \cdot \vec{0} = 0$

(3) $\vec{v}^0 \cdot \vec{v}^0 = |\vec{v}^0|^2$ (4) $|\alpha \vec{v}^0 \pm \beta \vec{u}^0|^2 = \alpha^2 |\vec{v}^0|^2 \pm 2\alpha\beta \vec{u}^0 \cdot \vec{v}^0 + \beta^2 |\vec{u}^0|^2$

(5) $(\vec{u}^0 \pm \vec{v}^0) \cdot \vec{w}^0 = \vec{u}^0 \cdot \vec{w}^0 \pm \vec{v}^0 \cdot \vec{w}^0$

(6) $\alpha(\vec{u}^0 \cdot \vec{v}^0) = (\alpha \vec{u}^0) \cdot \vec{v}^0 = \vec{u}^0 \cdot (\alpha \vec{v}^0)$

e.x) If $|2\vec{u}^0 - \vec{v}^0| = 5$, $|\vec{u}^0| = 1$, $|\vec{v}^0| = 3$

Find (1) $\vec{u}^0 \cdot \vec{v}^0$ (2) $|\vec{u}^0 + 3\vec{v}^0|$

SOL:- (1) $|2\vec{u} - \vec{v}|^2 = 5^2$

$$4|\vec{u}|^2 - 4(\vec{u} \cdot \vec{v}) + |\vec{v}|^2 = 25$$

$$4(1)^2 - 4(\vec{u} \cdot \vec{v}) + 9 = 25 \Rightarrow \vec{u} \cdot \vec{v} = -3$$

$$(2) |\vec{u} + 3\vec{v}|^2 = |\vec{u}|^2 + 6\vec{u} \cdot \vec{v} + 9|\vec{v}|^2$$

$$= 1 + 6(-3) + 9(9) = 64 \Rightarrow |\vec{u} + 3\vec{v}| = 8$$

* Rule :-

$$|\vec{u} + \vec{v}|^2 + |\vec{u} - \vec{v}|^2 = 2(|\vec{u}|^2 + |\vec{v}|^2)$$

$$|\vec{u} + \vec{v}|^2 - |\vec{u} - \vec{v}|^2 = 4\vec{u} \cdot \vec{v}$$

proof,

$$|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + 2\vec{u} \cdot \vec{v} + |\vec{v}|^2$$

$$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 - 2\vec{u} \cdot \vec{v} + |\vec{v}|^2$$

ex) If $|\vec{u} + 2\vec{v}| = 6$, $|\vec{u}| = 3$, $|\vec{v}| = 7$

1) find $|2\vec{v} - \vec{u}|$

SOL:-

$$|2\vec{v} - \vec{u}| = |-(2\vec{u} - \vec{u})| = |2\vec{u} - \vec{u}|$$

$$\sim |2\vec{u} + 2\vec{v}|^2 + |2\vec{u} - 2\vec{v}|^2 = 2(|2\vec{u}|^2 + |2\vec{v}|^2)$$

$$= 2(|\vec{u}|^2 + 4|\vec{v}|^2) = 36 + |2\vec{u} - 2\vec{v}|^2 = 2(9 + 4(49))$$

$$= 410 \sim |2\vec{u} - 2\vec{v}|^2 = 410 - 36 = 374$$

$$|2\vec{u} - 2\vec{v}| = \sqrt{374}$$

$$|2\vec{v} - \vec{u}| = \sqrt{374}$$

$$\sqrt{374}$$

E.x) If $|\vec{u} + 2\vec{v}| = 6$, $|\vec{u} - \vec{v}| = 4$
 $\vec{u} \cdot \vec{v} = 3$, find $|\vec{u}|$, $|\vec{v}|$, $|2\vec{u} - 4\vec{v}|$

Sol :-

$$|\vec{u} + 2\vec{v}|^2 = 6^2 \Rightarrow |\vec{u}|^2 + 4\vec{u} \cdot \vec{v} + 4|\vec{v}|^2 = 36$$

$$|\vec{u}|^2 + 4|\vec{v}|^2 = 24 \quad \text{--- (1)}$$

$$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 - 2\vec{u} \cdot \vec{v} + |\vec{v}|^2 = 16$$

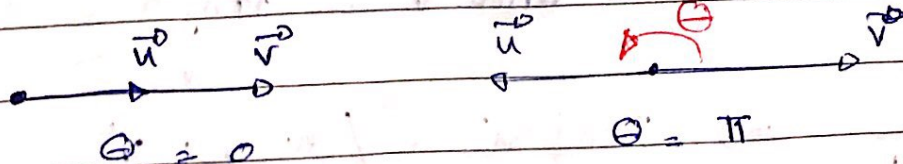
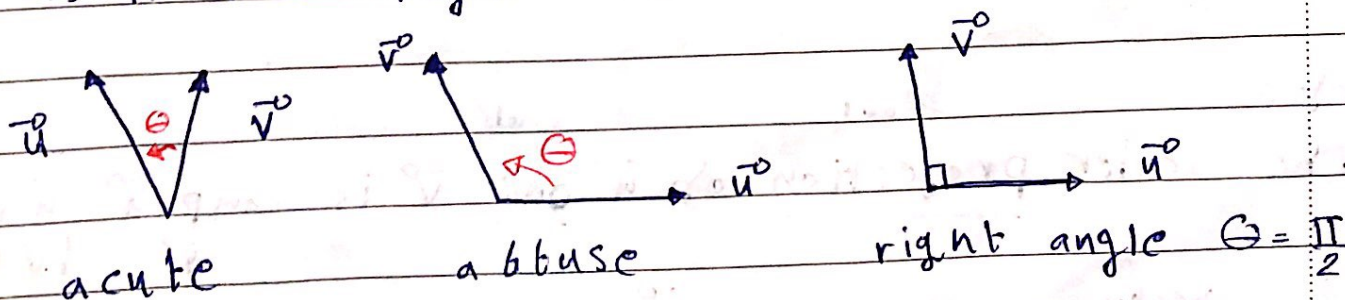
$$\Rightarrow |\vec{u}|^2 + |\vec{v}|^2 = 22 \quad \text{--- (2)}$$

$$\textcircled{1} - \textcircled{2} \Rightarrow 3|\vec{v}|^2 = 2 \Rightarrow |\vec{v}|^2 = \frac{2}{3} \Rightarrow |\vec{v}| = \sqrt{\frac{2}{3}}$$

$$\textcircled{2} \Rightarrow |\vec{u}|^2 = 22 - \frac{2}{3} = \frac{64}{3} \Rightarrow |\vec{u}| = \frac{8}{\sqrt{3}}$$

$$\begin{aligned} (5) \quad |2\vec{u} - 4\vec{v}|^2 &= 4|\vec{u}|^2 - 16\vec{u} \cdot \vec{v} + 16|\vec{v}|^2 \\ &= 4\left(\frac{64}{3}\right) - 16(3) + 16\left(\frac{2}{3}\right) = 48 \end{aligned}$$

* Def :- Let $\vec{u}$, $\vec{v}$ be vectors with the same initial
 the angle Θ between $\vec{u}$, $\vec{v}$ is the smallest angle
 as in the figure

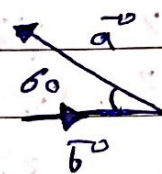
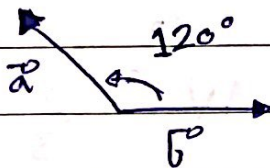
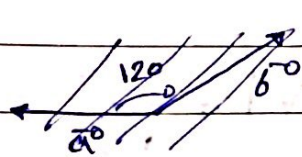


$$0 \leq \Theta \leq \pi$$

* Rule :- $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta$
 $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}$

E.x) $|\vec{a}| = 2$, $|\vec{b}| = 3$ find $\vec{a} \cdot \vec{b}$

Sol:-



$\therefore \vec{a} \cdot \vec{b} = 2 \times 3 \cos 120 = -3$

* Rule :- (1) $\vec{u} \cdot \vec{v} > 0 \Leftrightarrow \theta$ acute angle.

(2) $\vec{u} \cdot \vec{v} < 0 \Leftrightarrow \theta$ obtuse angle.

(3) $\vec{u} \perp \vec{v}$ ($\vec{u}$ perpendicular ~~to~~ orthogonal) to $\vec{v}$,
 $\Leftrightarrow \vec{u} \cdot \vec{v} = 0$

(4) $\vec{u} \parallel \vec{v}$ ($\vec{u}$ parallel to $\vec{v}$) $\Leftrightarrow \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}|$ or
 $- |\vec{u}| |\vec{v}| \Leftrightarrow \theta = 0$ or π

* Def :-

مقدار

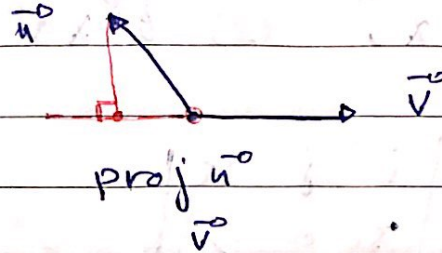
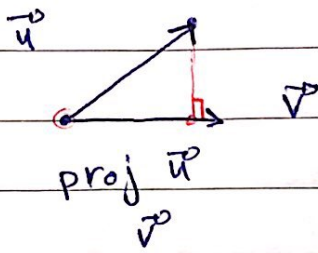
مقدار

(1) The scalar projection of $\vec{u}$ onto $\vec{v}$ is $\text{comp}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$

(2) // vector // " $\vec{u}$ onto $\vec{v}$ is $\text{proj}_{\vec{v}} \vec{u} =$

$\left(\text{comp}_{\vec{v}} \vec{u} \right) \frac{\vec{v}}{|\vec{v}|} \Rightarrow \text{proj}_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v}$

* Remark:



E.x) Find the scalar projection and vector projection of $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$ onto $\vec{b} = \hat{i} - 2\hat{j}$

Sol:-

$$\text{comp}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{3(1) + (-2)(-2) + 1(0)}{\sqrt{1+4}} = \frac{7}{\sqrt{5}}$$

$$\text{proj}_{\vec{b}} \vec{a} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \right) \vec{b} = \frac{7}{5} (\hat{i} - 2\hat{j})$$

$$= \frac{7}{5} \hat{i} - \frac{14}{5} \hat{j}$$

* sec (12.4) cross product.

The cross product of $\vec{u} = \langle a, b, c \rangle$ and $\vec{v} = \langle d, e, f \rangle$ is

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ d & e & f \end{vmatrix}$$

$$= (bf - ce)\hat{i} - (af - cd)\hat{j} + (ae - db)\hat{k} \quad (\text{vector})$$

E.X) Let $\vec{a} = 3\hat{i} - 2\hat{k}$, $\vec{b} = \langle 1, 2, -1 \rangle$

Find (1) $\vec{a} \times \vec{b}$ (2) $\vec{b} \times \vec{a}$

Sol:-

$$(1) \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -2 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= 4\hat{i} + 1\hat{j} + 6\hat{k}$$

$$(2) \vec{b} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 3 & 0 & -2 \end{vmatrix}$$

$$= -4\hat{i} - \hat{j} - 6\hat{k}$$

* properties: Let $\vec{u}, \vec{v}, \vec{w}$ be in V_3 .

$$(1) \vec{u} \times \vec{v} = -\vec{v} \times \vec{u} \quad (2) \vec{u} \times \vec{0} = \vec{0} = \vec{0} \times \vec{u} = \langle 0, 0, 0 \rangle$$

$$(3) (a\vec{u}) \times \vec{v} = \vec{u} \times (a\vec{v}) = a(\vec{u} \times \vec{v})$$

$$(4) \vec{u} \times \vec{u} = \vec{0}$$

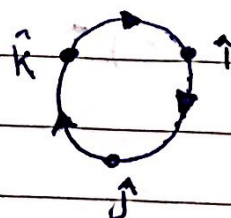
$$(5) (\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$$

$$(6) \vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$$

E.X) (1) $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{i} = -\hat{k}$

(2) $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{j} = -\hat{i}$

(3) $\hat{k} \times \hat{i} = \hat{j}$, $\hat{i} \times \hat{k} = -\hat{j}$



* Remark :-

$$(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})$$

for

$$(\hat{i} \times \hat{j}) \times \hat{j} = \hat{k} \times \hat{j} = -\hat{i}$$

$$\hat{i} \times (\hat{j} \times \hat{j}) = \hat{i} \times \vec{0} = \vec{0}$$

$$\therefore (\hat{i} \times \hat{j}) \times \hat{j} \neq \hat{i} \times (\hat{j} \times \hat{j})$$

* Rule :-

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$$

* Rule :-

$$(1) \quad \vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta$$

$$(2) \quad \vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$

* Rule :-

$$|\vec{a} \times \vec{b}| = \sqrt{|\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2}$$

proof. $\Rightarrow$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\left(\frac{|\vec{a} \times \vec{b}|}{|\vec{a}| |\vec{b}|} \right)^2 + \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right)^2 = 1$$

$$\frac{|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2}{|\vec{a}|^2 |\vec{b}|^2} = 1$$

$$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$$

$$|\vec{a} \times \vec{b}| = \sqrt{|\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2} \quad \#$$

E.X) $\vec{u} = \langle 2, 1, -1 \rangle$, $\vec{v} = 3\hat{i} + 2\hat{k}$

$$|\vec{u} \times \vec{v}| = \sqrt{6 * 13 - (4)^2} = \sqrt{62}$$

* Rule :- (1) $\vec{u} \parallel \vec{v} \Leftrightarrow |\vec{u} \times \vec{v}| = 0 \Leftrightarrow \vec{u} \times \vec{v} = \vec{0}$
 (2) $\vec{u} \parallel \vec{v} \Leftrightarrow$ there is a scalar λ such that $\vec{u} = \lambda \vec{v}$
 or $\vec{v} = \lambda \vec{u}$.

E.X) Determine whether the vectors $\vec{a} = \langle 2, 4, -8 \rangle$
 $\vec{b} = \langle -\frac{1}{2}, -1, 2 \rangle$ parallel or not.

Sol:-

First sol:- $\vec{a} = -4 \vec{b} \Rightarrow \vec{a} \parallel \vec{b}$

2nd sol:-

$$|\vec{a} \times \vec{b}| = \sqrt{(84)(\frac{21}{4}) - (-1 - 4 - 16)^2} = 0 \Rightarrow \vec{a} \parallel \vec{b}$$

E.X) Find the angle between the vectors $\vec{u} = 2\hat{i} - 3\hat{j} + \hat{k}$
 and $\vec{v} = \hat{i} + 3\hat{k}$

Sol:- $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} = \frac{-1}{\sqrt{140}} \Rightarrow \theta = \cos^{-1} \left(\frac{-1}{\sqrt{140}} \right)$

* Def:- (1) 3 pts. A, B, C are called collinear
 if $\vec{AB} \parallel \vec{AC}$ تكونوا على
خط واحد

E.x) Determine whether the pts. are collinear or not:-

(1) $A(1, -1, 2)$, $B(1, 3, -1)$, $C(0, 1, 4)$

(2) $P(2, 4, -3)$, $Q(3, -1, 1)$, $R(4, -6, 5)$

Sol :-

(1) $\vec{AB} = \langle 0, 4, -3 \rangle$ $\vec{AC} = \langle -1, 2, +2 \rangle$ $\Rightarrow \vec{AB} \neq \vec{AC}$ Since there is no scalar λ such $\vec{AB} = \lambda \vec{AC}$
or $|\vec{AB} \times \vec{AC}| \neq 0 \therefore A, B, C$ not collinear.

(2) $\vec{PQ} = \langle 1, -5, 4 \rangle$ $\vec{PR} = \langle 2, -10, 8 \rangle$ $\Rightarrow \vec{PR} = 2\vec{PQ}$ $\therefore P, Q, R$ collinear.

E.x) Find a that makes the pts. $P(2, 4, -3)$

$Q(a, -1, 1)$, $R(4, -2a, 5)$

(1) collinear (2) not collinear

Sol :- $\vec{PQ} = \langle a-2, -5, 4 \rangle$

$\vec{PR} = \langle 2, -2a-4, 8 \rangle$

$\vec{PQ} \parallel \vec{PR} \Rightarrow \vec{PR} = 2\vec{PQ}$

$\Rightarrow \langle 2, -2a-4, 8 \rangle = \langle 2a-4, -10, 8 \rangle$

$2 = 2a-4$

$-2a-4 = -10$

$8 = 8$

$\Rightarrow a = 3$ (1) $a = 3$ (2) $R = \{3\}$

* Rule :- The area of the :-

(1) triangle determined by the pts. A, B, C is

$\frac{1}{2} |\vec{AB} \times \vec{AC}|$

(2) parallelogram (مستطیل) Determined by the pts. A, B, C is $|\vec{AB} \times \vec{AC}|$

E.X) Find the area of the triangle and parallelogram determined by the pts. $A(1, 2, 0), B(-1, 3, 4), C(2, 2, 3)$

SOL:- $\vec{AB} = \langle -2, 1, 4 \rangle \Rightarrow |\vec{AB}|^2 = 21$ Area
 $\vec{AC} = \langle 1, 0, 3 \rangle \Rightarrow |\vec{AC}|^2 = 10$

$$|\vec{AB} \times \vec{AC}| = \sqrt{21(10) - (-2+0+12)^2} = \sqrt{110}$$

(1) Area = $\frac{\sqrt{110}}{2}$ (2) Area = $\sqrt{110}$

*** Rule :-** The scalar triple of $\vec{u} = \langle a, b, c \rangle, \vec{v} = \langle d, e, f \rangle, \vec{w} = \langle g, h, L \rangle$ is $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & L \end{vmatrix}$
 $= a(eL - fh) - b(dL - gf) + c(dh - ge)$

*** Def :-** 4 pts. A, B, C, D are called coplanar if they lie in the same plane.

*** Rule :-** Let A, B, C, D be 4 pts. The volume of the parallelepiped (موشور) determined by A, B, C, D is $|\vec{AB} \cdot (\vec{AC} \times \vec{AD})|$

(2) A, B, C, D coplanar if $\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = 0$

E.x) Determine whether the pts. $P(1,1,1)$, $Q(2,5,-6)$, $R(3,0,5)$, $S(1,0,3)$ coplanar

Sol:-

$$\vec{PQ} = \langle 1, 4, -7 \rangle \quad \vec{PR} = \langle 2, -1, 4 \rangle \quad \vec{PS} = \langle 0, -1, 2 \rangle$$

$$\vec{PQ} \cdot (\vec{PR} \times \vec{PS}) = \begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -1 & 2 \end{vmatrix}$$

$$= 1(-2-4) - 4(4-0) + -7(-2-0) = 2 - 16 + 14 = 0$$

$\therefore \vec{PQ}, \vec{PR}, \vec{PS}$ coplanar. $\therefore P, Q, R, S$ coplanar

E.x) Find the volume of the parallelepiped with vertices $A(-2,1,0)$, $B(4,4,-1)$, $C(-2,2,2)$, $D(2,-1,5)$

Sol:-

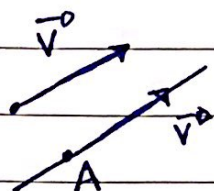
$$\vec{AB} = \langle 6, 3, -1 \rangle \quad \vec{AC} = \langle 0, 1, 2 \rangle \quad \vec{AD} = \langle 4, -2, 5 \rangle$$

$$\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = \begin{vmatrix} 6 & 3 & -1 \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix}$$

$$= 6(5 - -4) - 3(0 - 8) + -1(0 - 4) = 54 + 24 + 4 = 82$$

$$\text{Volume} = |82| = 82$$

* sec (12.5) The eqs. of lines and planes * let L be the line that passes through the pt. $A(x_0, y_0, z_0)$ and parallel to the vector $\vec{v} = \langle a, b, c \rangle$



* L has 3 forms of eqs:

(1) parametric (param.) eqs of L

are: $x = x_0 + at$ $z = z_0 + ct, t \in \mathbb{R}$

$y = y_0 + bt$

(t, eqs)

B is a pt. on $L \iff t = x_0 + at \quad e = y_0 + bt \quad f = z_0 + ct.$

(2) symmetric (symm.) eqs. of L are:-

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}, \quad a \neq 0, b \neq 0, c \neq 0$$

If $a = 0$: $\frac{y - y_0}{b} = \frac{z - z_0}{c}, \quad x = x_0$

If $b = 0$: $\frac{x - x_0}{a} = \frac{z - z_0}{c}, \quad y = y_0$

If $c = 0$: $\frac{x - x_0}{a} = \frac{y - y_0}{b}, \quad z = z_0$

(3) Vector eq. of L is $\vec{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$
 $= \langle x_0, y_0, z_0 \rangle + \langle a, b, c \rangle \cdot t$

E.x) Find param. symm. vector eqs. of the Line that pass through $A(1, 2, 3)$ and parallel to $6\mathbf{i} - 7\mathbf{k}$. Also find (1) 2 pts. on L other than A (2) the pt of intersection (if any) of L with the (a) xy -plane (b) xz -plane (c) yz -plane

Sol:- $\boxed{z=0 \text{ p.s.v}}$ $\boxed{y=0 \text{ p.s.v}}$ $\boxed{x=0 \text{ p.s.v}}$

Param. eqs. :- $x = 1 + 6t = 1 + 6t$

$y = 2 + 0t = 2$

$z = 3 - 7t = 3 - 7t$

Symm. eqs. :- $\frac{x-1}{6} = \frac{y-2}{0} = \frac{z-3}{-7}, y=2$

vector eq. :- $\vec{r}(t) = \langle 1 + 6t, 2, 3 - 7t \rangle$

$= \langle 1, 2, 3 \rangle + \langle 6, 0, -7 \rangle t$

(1) Take $t=0 \Rightarrow x=1, y=2, z=3 \Rightarrow \text{pt. } (1, 2, 3) = A$

$t=1 \Rightarrow x=7, y=2, z=-4 \Rightarrow \text{pt. } B(7, 2, -4)$

$t=2 \Rightarrow x=13, y=2, z=-11 \Rightarrow \text{pt. } C(13, 2, -11)$

(2) (a) $z=0 \Rightarrow 3 - 7t = 0 \Rightarrow t = \frac{3}{7}$

$x = 1 + 6\left(\frac{3}{7}\right) = \frac{25}{7}$

$y = 2$

pt. of intersection $\left(\frac{25}{7}, 2, 0\right)$ (impossible)

(b) $y=0 \Rightarrow 0=2$ $\therefore L$ does not intersect xz -plane

(c) $x=0 \Rightarrow 0 = 1 + 6t \Rightarrow t = -\frac{1}{6}$

$y = 2$

$z = 3 - 7\left(-\frac{1}{6}\right) \Rightarrow z = \frac{25}{6}$ pt. $\left(0, 2, \frac{25}{6}\right)$

E.X) Find param. symm. vector eqs. of L that pass through $A(1, 2, 3)$, $B(-2, 5, 7)$

SOL:-

$$\vec{AB} = \langle -3, 3, 4 \rangle$$

param. eqs. $\therefore x = 1 - 3t$

$$y = 2 + 3t$$

$$z = 3 + 4t$$

symm. $\therefore \frac{x-1}{-3} = \frac{y-2}{3} = \frac{z-3}{4}$

vector eqs: $\vec{r}(t) = \langle 1 - 3t, 2 + 3t, 3 + 4t \rangle$
 $= \langle 1, 2, 3 \rangle + \langle -3, 3, 4 \rangle t$

*** Remark:-** Let L_1, L_2 be two lines Let $\vec{u} \parallel L_1$, $\vec{v} \parallel L_2$

(1) $\vec{u} \parallel \vec{v} \Leftrightarrow L_1 \parallel L_2$

(2) $\vec{u} \nparallel \vec{v} \Leftrightarrow L_1 \nparallel L_2 \Leftrightarrow L_1, L_2$ are intersected or skew (جدا)

E.X) Determine whether the two lines L_1, L_2 parallel intersected or skew If they are intersected find the pt. of intersection

(1) $L_1: x = 2 - 3t, y = 2t, z = 7$

$L_2: x = 5 + 9s, y = 3 - 6s, z = 1$

$$(2) L_1: x=t, y=8-t, z=2+3t$$

$$L_2: x=1+2s, y=2+s, z=5$$

$$(3) L_1: x=1+t, y=-2+3t, z=4-t$$

$$L_2: x=2s, y=3+s, z=-3+4s$$

Sol:- (1) $\vec{u}^0 = \langle -3, 2, 0 \rangle // L_1 \rightarrow -3\vec{u}^0 = \vec{v}^0$

$$\vec{v}^0 = \langle 9, -6, 0 \rangle // L_2$$

$$\therefore \vec{u}^0 // \vec{v}^0 \quad \therefore L_1 // L_2$$

$$(2) \vec{u}^0 = \langle 1, -1, 3 \rangle // L_1 \rightarrow \times \vec{u}^0 \neq \vec{v}^0 \rightarrow L_1 \nparallel L_2$$

$$\vec{v}^0 = \langle 2, 1, 0 \rangle // L_2 \quad \therefore \vec{u}^0 \nparallel \vec{v}^0$$

$\therefore L_1, L_2$ either intersected or skew.

Assume L_1, L_2 intersected:-

$$\Rightarrow t = 1+2s \quad t - 2s = 1 \quad \text{--- (1)}$$

$$3-t = 2+s \quad \Rightarrow -t - s = -1 \quad \text{--- (2)}$$

$$2+3t = 5 \quad 3t = 3 \quad \text{--- (3)}$$

Take eq. (2), (3) to solve, (1) to check

$$(2): -t - s = -1$$

$$(3): 3t = 3 \Rightarrow t = 1$$

$$(2) \Rightarrow -1 - s = -1 \Rightarrow s = 0$$

check:-

$$1 - 0 = 1 \Rightarrow 1 = 1 \quad \text{yes} \Rightarrow L_1, L_2 \text{ intersected}$$

- To find the pt. of intersection:

using value $s = 0 \Rightarrow L_2: x = 1, y = 2, z = 5$

$$\text{pt. } (1, 2, 5)$$

$$(3) \vec{u} = \langle 1, 3, -1 \rangle // L_1 \quad \vec{v} = \langle 2, 1, 4 \rangle // L_2 \quad \Rightarrow \quad \vec{u} \neq \vec{v} \Rightarrow L_1 \neq L_2$$

Assume L_1, L_2 intersected.

$$\begin{aligned} 1+t &= 2s & t-2s &= -1 \quad \text{--- (1)} \\ -2+3t &= 3+s & 3t-s &= 5 \quad \text{--- (2)} \\ 4-t &= -3+4s & -t-4s &= -7 \quad \text{--- (3)} \end{aligned}$$

(1), (3) solve, (2) check:

$$\begin{aligned} (1) \Rightarrow t-2s &= -1 & -6s &= -8 \Rightarrow s = \frac{8}{6} = \frac{4}{3} \\ (3) \Rightarrow -t-4s &= -7 \end{aligned}$$

$$(1) \quad t - 2\left(\frac{4}{3}\right) = -1 \Rightarrow t = -1 + \frac{8}{3} = \frac{5}{3}$$

$$\text{check: } (2) \quad 3t - s = 5 \Rightarrow 3\left(\frac{5}{3}\right) - \frac{4}{3} = 5$$

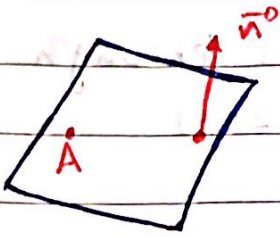
$$\Rightarrow \frac{11}{3} \stackrel{?}{=} 5 \quad \text{No}$$

L_1, L_2 not intersected

L_1, L_2 skew.

* Def :- The eq. of the plane that pass through the pt. $A(x_0, y_0, z_0)$ and with normal (vector) $\vec{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$



E.x) Find the eq. of the plane that passes through the pt. $A(1, -1, 0)$ and with normal vector $\vec{n}^0 = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$. Also, find the intercepts (x, y, z) of the plane with the coordinate axes and finally sketch the plane ??

Sol :-

the eq. $2(x-1) - 3(y+1) + (z-0) = 0$

$$\Rightarrow 2x - 3y + z = 2(1) - 3(-1) + 1(0)$$

$$\Rightarrow 2x - 3y + z = 5 \quad \text{--- (1)}$$

Intercepts :-

x - intercepts : $y = 0, z = 0$

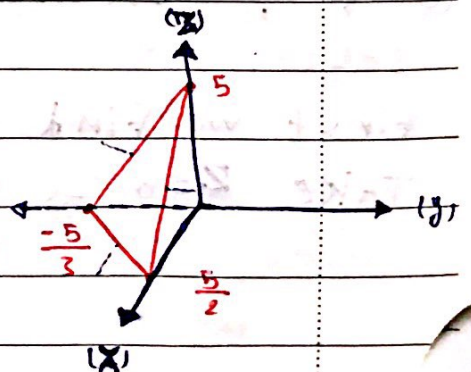
$$\textcircled{1} \Rightarrow 2x - 0 + 0 = 5 \Rightarrow \boxed{x = \frac{5}{2}}$$

y - intercepts : $x = 0, z = 0$

$$\textcircled{1} \Rightarrow 0 - 3y + 0 = 5 \Rightarrow \boxed{y = -\frac{5}{3}}$$

z - intercepts : $x = 0, y = 0$

$$\textcircled{1} \Rightarrow 0 - 0 + z = 5 \Rightarrow \boxed{z = 5}$$



E.x) Find the eq. of the plane that pass through the pts. $A(1, 3, 2)$, $B(3, -1, 6)$, $C(5, 2, 0)$

Sol:-

$$\vec{AB} = \langle 2, -4, 4 \rangle$$

$$\vec{AC} = \langle 4, -1, -2 \rangle$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix}$$

$$= 12i + 20j + 14k \quad \perp \text{ plane}$$

$$\div 2 \Rightarrow 6i + 10j + 7k \quad \perp \text{ plane}$$

$$\text{eq. } 6x + 10y + 7z = 6(1) + 10(3) + 7(2) = 50$$

Ex) Let L be the Line of intersection of the two planes $P_1: x + y - 2z = 3$, $P_2: x - y + z = 1$

(1) Find the param. eqs. of L .

(2) Find the eq. of the plane that pass through the pt. $A(1, 2, 3)$ and (a) pass through the line L
(b) perpendicular to the line L .

Sol:-

First we find two pts. on L :

$$\text{Take } z=0 \Rightarrow P_1: x + y = 3$$

$$P_2: x - y = 1$$

$$2x = 4 \Rightarrow x = 2$$

$$2 + y = 3 \Rightarrow y = 1$$

$$\text{pt. } B(2, 1, 0)$$

Take $z=2 \Rightarrow P_1 : x+y-4=3 \Rightarrow x+y=7$

$P_2 : x-y+2=1 \Rightarrow x-y=-1$

$\Rightarrow 2x=6 \Rightarrow x=3 : 2(3)+y=7 \Rightarrow y=1$

Pt. C (3, 4, 2)

(1) $\vec{BC} = \langle 1, 3, 2 \rangle \parallel L$

param. eqs. of L are:

$x=2+t, y=1+3t, z=0+2t$

(2) (a) A, B, C are in plane

$\vec{AB} = \langle 1, -1, -3 \rangle$

$\vec{AC} = \langle 2, 2, -1 \rangle$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & -1 & -3 \\ 2 & 2 & -1 \end{vmatrix}$$

$= 7i - 5j + 4k$

eq. of plane: $7x - 5y + 4z = 7(2) - 5(1) = 9$

(b) $\vec{BC} \perp$ plane : eq. of plane is: $x+3y+2z=13$

* Rule: The distance from the pt. A (x_0, y_0, z_0)

to the plane: $ax+by+cz+d=0$

is: $\text{dist} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$

E.X) Find the distance from the pt. A (1, 2, 3)

to the plane: (1) $x+y-3z=1$ (2) $x+\frac{y}{2}=2$

Sol: (1) eq $\Rightarrow x + y - 3z - 1 = 0$

$$\text{dist.} = \frac{|1 + 2 - 3(3) - 1|}{\sqrt{(1)^2 + (1)^2 + (-3)^2}} = \frac{7}{\sqrt{11}}$$

(2) eq. $\Rightarrow 2x + y - 4 = 0$

$$\text{dist.} = \frac{|2(1) + 2 - 4|}{\sqrt{(2)^2 + (1)^2 + (0)^2}} = \frac{0}{\sqrt{5}} = 0 \quad (\text{the pt. on plane})$$

* sec(12.6) :- cylinders and Quadric surface.

* Def. :- A cylindrical surface is a surface obtained by moving a curve along a fixed axis.
(منحنى) (محور ثابت)

E.x) $y = x^3$, $z = \cos y$, $y = e^z$

$ax + by + cz = d$ are all cylindrical surface

E.x) $x^2 + y^2 + z^2 = 9$, $x + y + z^2 = 3$, $x + z = \sin y$ are all not cylindrical surface.

* Def. :- A quadric surface is a surface whose eq. is :-

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

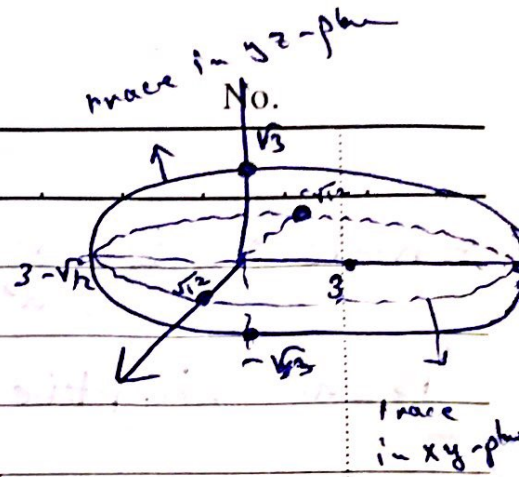
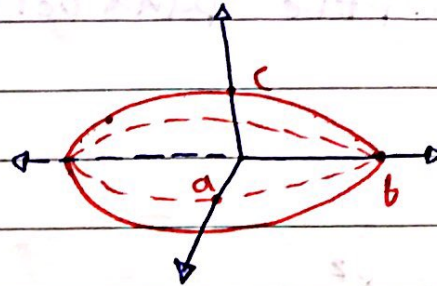
① Ellipsoide (شكل بيضوي)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

(0,0,0) مركزه

$$b > a$$

$$b > c$$



E.x) Identify and Sketch the surface $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$ on your graph show the intercepts and sketch the trace (بف) in the xy-plane.

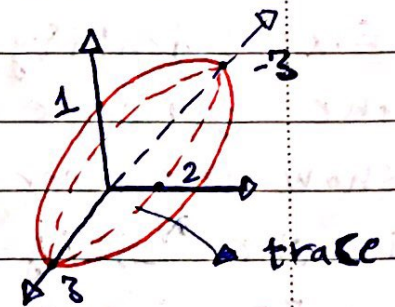
Ellipsoide, center (0,0,0) (شكل بيضوي، مركزه (0,0,0))

Intercepts :-

$$x\text{-intercept} : x = \pm 3$$

$$y\text{-intercept} : y = \pm 2$$

$$z\text{-intercept} : z = \pm 1$$



E.x) Sketch the surface : $\frac{x^2}{4} + y^2 - 6y + z^2 = 5$

$$\frac{x^2}{4} + y^2 - 6y + 9 + z^2 = 12 \Rightarrow \frac{x^2}{4} + (y-3)^2 + z^2 = 12$$

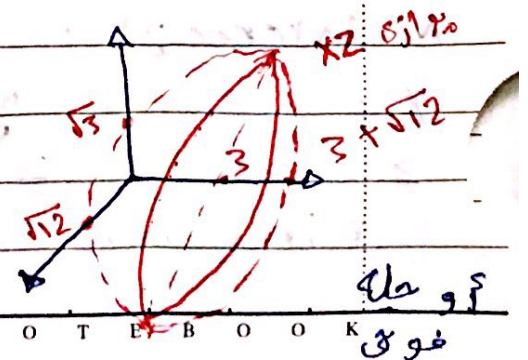
$$\frac{x^2}{48} + \frac{(y-3)^2}{12} + \frac{z^2}{12} = 1 \quad \text{Ellipsoide, center (0,3,0)}$$

Intercepts :-

$$x\text{-intercept} : x = \pm \sqrt{48} = \pm 4\sqrt{3}$$

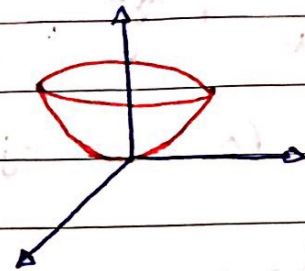
$$y\text{-intercept} : y = 3 \pm \sqrt{12}$$

$$z\text{-intercept} : z = \pm \sqrt{12}$$



* Rule :- $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$, $a > 0$, $b > 0$

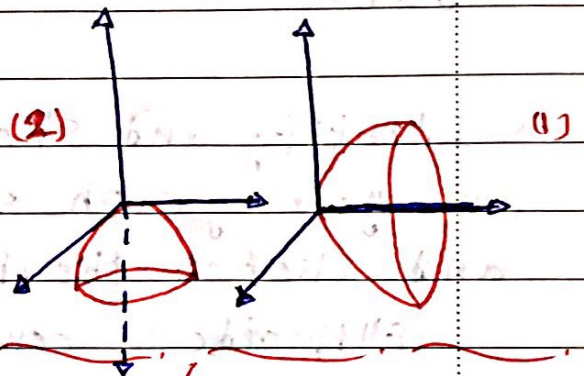
is an Elliptic paraboloid.



E.X) (1) $y = x^2 + \frac{z^2}{6}$

(2) $z = -(x^2 + y^2)$

(3) $x = 2y^2 + 6z^2$



E.X) I identify and sketch the surface. on your graph show the intercepts and trace in the (1) xy -plane

✓ (2) xz -plane (a) $y = 1 - (x^2 + \frac{z^2}{4})$

(b) $y = -1 + (x^2 + \frac{z^2}{4})$

SOL :-

(a) paraboloid $\Rightarrow$ vertex $(0, 1, 0)$

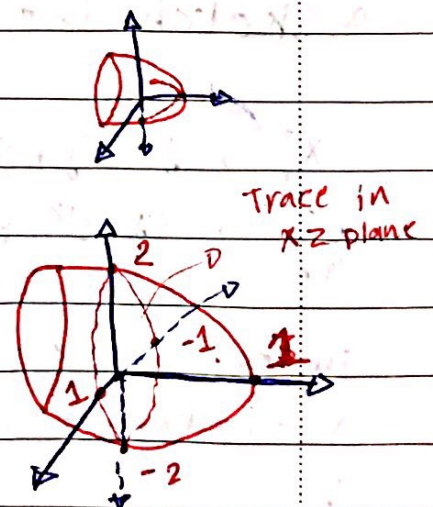
Intercepts :-

x -intercept :- $x = \pm 1$

y - " :- $y = 1$

z - " :- $z = \pm 2$

Direction (-ve) y -axis



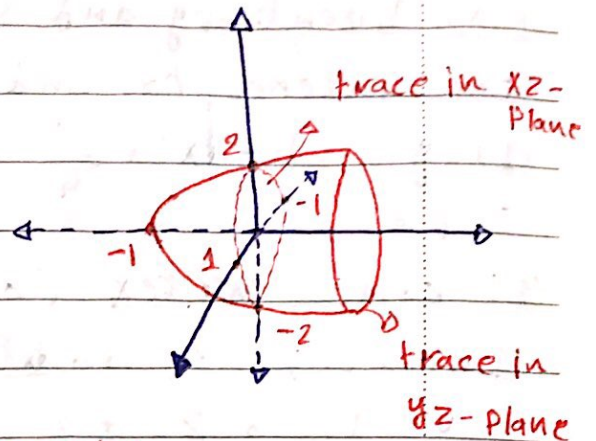
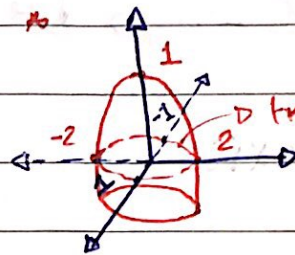
(b) paraboloid, vertex $(0, -1, 0)$

Intercepts: $x = \pm 1$

$y = -1$, $z = \pm 2$

Direction the y -axis

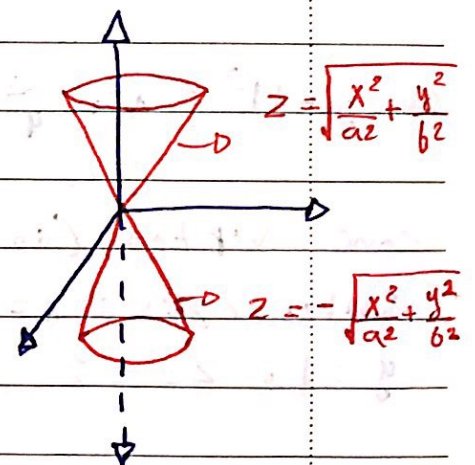
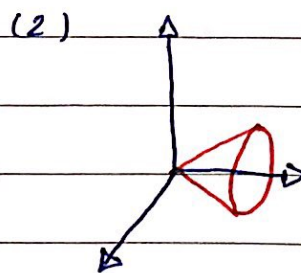
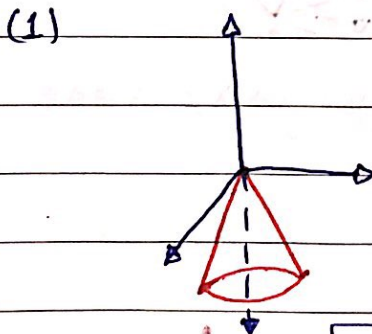
$$* z = 1 - \left(x^2 + \frac{y^2}{4}\right)$$



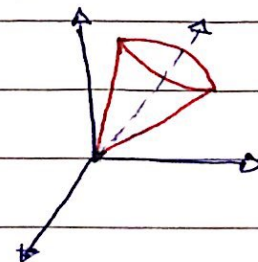
* Remark: An Elliptic Cone has an eq. in the form: $z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$, $a > 0$, $b > 0$

E.X) (1) $z = \sqrt{\frac{x^2}{4} + y^2}$ cone

(2) $y = \sqrt{x^2 + z^2}$ cone



(3) $x = \sqrt{y^2 + 2z^2}$ cone (3)



E.x) Identify and sketch the surface. Also show the intercepts and the trace in the xz -plane

(1) $y = 1 - \sqrt{x^2 + \frac{z^2}{4}}$ (2) $y = -1 + \sqrt{x^2 + \frac{z^2}{4}}$

SOL:-

(1) cone vertex $(0, 1, 0)$

Intercepts: $1 - \sqrt{x^2} = 0 \Rightarrow \sqrt{x^2} = 1 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$y = 1, z = \pm 2$

Direction -ve y -axis

(2) cone vertex $(0, -1, 0)$

Direction +ve y -axis

Intercepts: $x = \pm 1, y = -1$

$z = \pm 2$

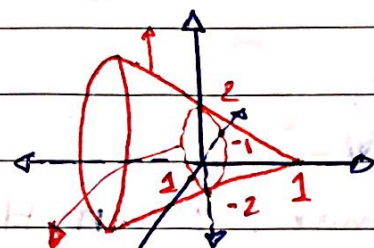
(3) $(y - 1)^2 = x^2 + \frac{z^2}{4}$

cone vertex $(0, 1, 0)$

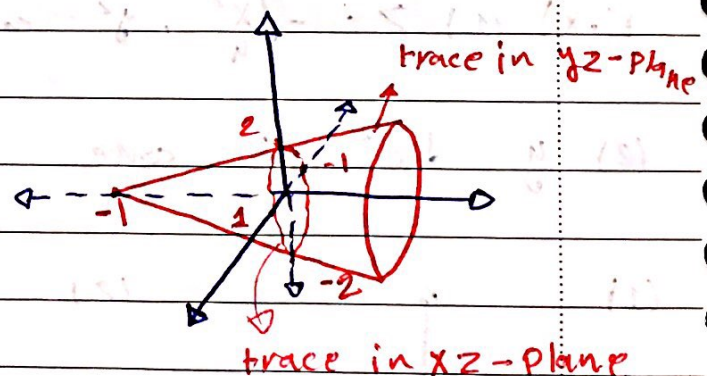
Intercepts: $x = \pm 1$

$y = 1, z = \pm 2$

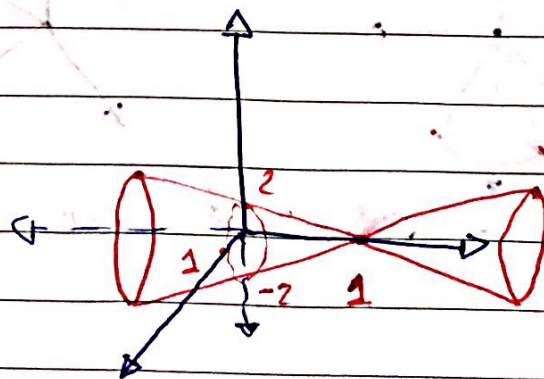
trace in yz -plane



trace in xz -plane



trace in xz -plane



* ch. 13 :- Vector Function

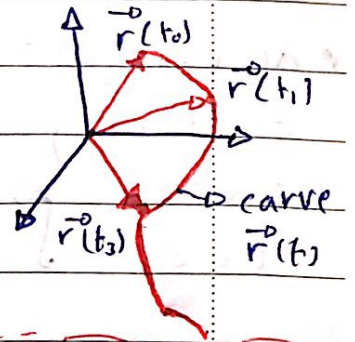
* A vector function

$$\vec{r}^0(x, y, z) = \langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle$$

In particular:-

$$\vec{r}^0(t) = \langle f(t), g(t), h(t) \rangle$$

$$\vec{r}^0'(t) = \langle f'(t), g'(t), h'(t) \rangle$$



* Rule:- The arc length of the curve $\vec{r}^0(t)$, $t_0 \leq t \leq t_1$ is $L = \int_{t_0}^{t_1} |\vec{r}^0'(t)| dt$

E.X1 Find the arc length of the curve $\vec{r}^0(t) = \langle \sin(2t), \cos(2t), t \rangle$
 $0 \leq t \leq \frac{\pi}{2}$

Sol:-

$$\vec{r}^0' = \langle 2\cos 2t, -2\sin 2t, 1 \rangle$$

$$|\vec{r}^0'| = \sqrt{4\cos^2(2t) + 4\sin^2(2t) + 1}$$

$$= \sqrt{4 + 1} = \sqrt{5}$$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{5} dt = \sqrt{5} \frac{\pi}{2}$$

E.x) Find the length of the arc of the circular helix curve (زنجیر دایره‌ای)

$r(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$ from the pt A(1,0,0) to the pt B(1,0,2 π)

Sol:-

$$\begin{aligned} \text{At } (1,0,0) &: \begin{cases} \cos t = 1 \\ \sin t = 0 \\ -t = 0 \end{cases} \Rightarrow t=0 \\ \text{At } (1,0,2\pi) &: \begin{cases} \cos t = 1 \\ \sin t = 0 \\ -t = 2\pi \end{cases} \Rightarrow t=2\pi \end{aligned}$$

$$\vec{r}' = \langle -\sin t, \cos t, 1 \rangle$$

$$|\vec{r}'| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$L = \int_0^{2\pi} \sqrt{2} dt = 2\sqrt{2}\pi$$

* Rule: The curvature (شیوگی) of a curve $\vec{r}(t)$ at a pt. $t=t_0$ is $K(t_0) = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$

E.x) Find the curvature of $\vec{r}(t) = \langle t, t^2, t^3 \rangle$, then find $\vec{r}(t)$ the curvature at the pt. $(-2, 4, -8)$, $(0,0,0)$

Sol:-

$$\vec{r}' = \langle 1, 2t, 3t^2 \rangle, \quad \vec{r}'' = \langle 0, 2, 6t \rangle$$

$$|\vec{r}'| = \sqrt{1 + 4t^2 + 9t^4}, \quad |\vec{r}''| = \sqrt{4 + 36t^2}$$

$$|\vec{r}' \times \vec{r}''| = \sqrt{(1 + 4t^2 + 9t^4)(4 + 36t^2) - (4t + 18t^3)^2}$$

$$\therefore (1) K(t) = \frac{\sqrt{(1 + 4t^2 + 9t^4)(4 + 36t^2) - (4t + 18t^3)^2}}{(1 + 4t^2 + 9t^4)^{\frac{3}{2}}}$$

(2) At pt. $(-2, 4, -8) \Rightarrow t = ?$
 $(t, t^2, t^3) \Rightarrow t = -2, t^2 = 4, t^3 = -8$

$$K(-2) = \frac{\sqrt{(16(9) + 16 + 1)(36(4) + 4) - (4(-2) + 18(-8))^2}}{(9(16) + 16 + 1)^{3/2}} = \dots$$

(3) $K(0) = \frac{\sqrt{4-0}}{1^{3/2}} = 2$

* Remark :- (1) $\vec{r}(t)$ circle of radius $a \Rightarrow$ the curvature is $\frac{1}{a}$.

(2) $\vec{r}(t)$ is a line $\Rightarrow$ the curvature is 0.

E.X) Find the curvature of $y = x^2$ at the pt. $(0,0)$

Sol:-

$$x = t, y = t^2$$

$$\vec{r}(t) = \langle x, y, 0 \rangle = \langle t, t^2, 0 \rangle$$

$$\vec{r}'(t) = \langle 1, 2t, 0 \rangle, \quad \vec{r}''(t) = \langle 0, 2, 0 \rangle$$

$$\text{At } A(0,0) \Rightarrow t=0$$

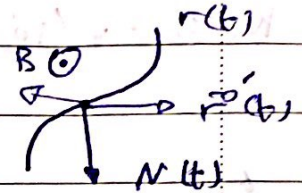
$$K(t) = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{\sqrt{(4t^2+1)4 - (4t)^2}}{(4t^2+1)^{3/2}}$$

$$K(0) = \frac{\sqrt{4}}{1^{3/2}} = 2$$

* Rule :- Let $\vec{r}(t)$ be a curve (1) The unit tangent vector to $\vec{r}(t)$ is $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$

(2) The normal unit vector to $\vec{r}(t)$ is $N(t) = \frac{\vec{r}''(t)}{|\vec{r}''(t)|}$

(3) The binormal vector is $B(t) = T \times N$.



Ex) Find the binormal vector to $\vec{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$

Sol:-

$$\vec{r} = \langle \cos t, \sin t, t \rangle$$

$$\vec{r}' = \langle -\sin t, \cos t, 1 \rangle$$

$$\vec{r}'' = \langle -\cos t, -\sin t, 0 \rangle$$

$$T = \frac{\vec{r}'}{|\vec{r}'|} = \frac{\langle -\sin t, \cos t, 1 \rangle}{\sqrt{\cos^2 t + \sin^2 t + 1}} = \langle \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$$

$$N = \frac{\vec{r}''}{|\vec{r}''|} = \frac{\langle -\cos t, -\sin t, 0 \rangle}{\sqrt{\cos^2 t + \sin^2 t}} = \langle -\cos t, -\sin t, 0 \rangle$$

$$B = T \times N = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{-\sin t}{\sqrt{2}} & \frac{\cos t}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \frac{\sin t}{\sqrt{2}} \mathbf{i} - \frac{\cos t}{\sqrt{2}} \mathbf{j} + \frac{1}{\sqrt{2}} \mathbf{k}$$



اللجنة الأكاديمية لقسم الهندسة المدنية

دفتر

تفاضل وتكامل 3 (سكند)

قصي أبو ديه

اعداد الطالب



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* CH. 14 :- partial Derivatives.

* sec. 14.1 Function and several variables.

* $f(x)$: func in 1 variables. } is several
 $f(x, y)$: " " 2 variables. } variable
 $f(x, y, z)$: " " 3 " }

E.x) Find the domain of the func.

$$(1) f(x, y) = \sqrt{4 - x^2 - y^2}$$

$$(2) f(x, y, z) = \sqrt{4 - x^2 - y^2}$$

$$(3) f(x, y, z) = 3e^{x+y} + \sin z$$

$$(4) f(x, y) = 5$$

$$(5) f(x, y, z) = \frac{1}{2 - (x^2 + y^2 + z^2)}$$

Sol :-

$$(1) \text{Dom}(f) = \{(x, y) \in \mathbb{R}^2 : 4 - x^2 - y^2 \geq 0\}$$

$$= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$$

$$(2) \text{Dom}(f) = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 4\}$$

$$(3) \text{Dom}(f) = \{(x, y, z) \in \mathbb{R}^3\} = \mathbb{R}^3$$

$$(4) \text{Dom}(f) = \{(x, y) \in \mathbb{R}^2\} = \mathbb{R}^2$$

$$(5) \text{Dom}(f) = \{(x, y, z) \in \mathbb{R}^3 : 2 - (x^2 + y^2 + z^2) \neq 0\}$$

* sec 14.2 Limits and continuity

* Let $C: \vec{r}(t)$ be a curve that passes through a pt. $A(x_0, y_0, z_0)$ when $t=t_0$

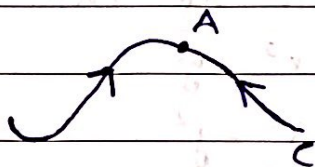
$$\text{Then } \lim_{(x,y,z) \rightarrow (x_0,y_0,z_0)} F(x,y,z) = \lim_{t \rightarrow t_0} F(\vec{r}(t))$$

along C

$$x \rightarrow x_0$$

$$y \rightarrow y_0$$

$$z \rightarrow z_0$$



Ex) Find $\lim_{(x,y) \rightarrow (1,-1)} \frac{x^2 - 3y - 4}{x + y}$ along the following curves:

(1) $C_1: x = 3t + 2, y = t^2 - 5$

(2) $C_2: y = 2x - 3$

Sol:-

(1) $(x,y) \Rightarrow (1,-1) \Rightarrow x \rightarrow 1$

$$y \rightarrow -1$$

$$x = 3t + 2 \Rightarrow t = \frac{x-2}{3}$$

$$x \rightarrow 1 \Rightarrow t \rightarrow -\frac{1}{3}$$

$$\lim_{(x,y) \rightarrow (1,-1)} \frac{x^2 - 3y - 4}{x + y} = \lim_{t \rightarrow -\frac{1}{3}} \frac{(3t+2)^2 - 3(t^2-5) - 4}{3t+2 + t^2-5}$$

$$\lim_{t \rightarrow -\frac{1}{3}} \frac{2(3t+2)(3) - 3(2t)}{3+2t} = \frac{6+12}{-1} = -18$$

$$(2) \quad x=t, \quad y=2t-3 : \quad x \rightarrow 1, \quad y \rightarrow -1$$

$$t=x$$

$$t \rightarrow 1$$

تجربتي x في y و t

$$\lim_{\substack{(x,y) \rightarrow (1,-1) \\ \text{along } c_2}} \frac{x^2 - 3y - 4}{x+y} = \lim_{x \rightarrow 1} \frac{x^2 - 3(2x-3) - 4}{x + 2x - 3} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \rightarrow 1} \frac{2x - 6}{3} = \frac{-4}{3}$$

E.x) Find $\lim_{\substack{(x,y) \rightarrow (1,0) \\ \text{along } y=\ln x}} x \sin(e^y - 1)$

$$(x,y) \rightarrow (1,0)$$

$$x^2 + y^2 - x$$

$$\text{along } y = \ln x$$

Sol:-

$$\lim_{x \rightarrow 1} \frac{x \sin(e^{\ln x} - 1)}{x^2 + (\ln x)^2 - x} \quad \left(\frac{0}{0}\right) = \frac{x \sin(x-1)}{x^2 + (\ln x)^2 - x}$$

$$= \lim_{x \rightarrow 1} \frac{x \cos(x-1) + \sin(x-1)}{2x + 2(\ln x) - 1} = 1$$

* Rule c_1, c_2 two curves pass through pt. A

$$\textcircled{1} \quad \lim_{\substack{(x,y,z) \rightarrow A \\ \text{along } c_1}} F \neq \lim_{\substack{(x,y,z) \rightarrow A \\ \text{along } c_2}} F \Rightarrow \lim_{(x,y,z) \rightarrow A} F \text{ does not exist (DNE)}$$

$$\textcircled{2} \quad \lim_{\text{along } c} F = L \text{ exist} \Leftrightarrow \lim_{(x,y,z) \rightarrow A} F \text{ exist}$$

c (any) curve through A

يجب الحساب بدون مسار

E.X) (1) $\lim_{(x,y,z) \rightarrow (1,2,0)} \frac{x e^{(x+y)z}}{x-2y+z^2} = \frac{-1}{3}$

(2) $\lim_{(x,y) \rightarrow (2,2)} \frac{x^4 - 5x^2y + 6y^2}{x^2 - 2y} \stackrel{\frac{0}{0}}{=} \lim_{(x,y) \rightarrow (2,2)} \frac{(x^2-2y)(x^2-3y)}{(x^2-2y)} = -2$

(3) $\lim_{(x,y) \rightarrow (0,0-3)} \frac{\sin(2x^2y)}{x^2} \stackrel{\frac{0}{0}}{=} \lim_{(x,y) \rightarrow (0,0-3)} \frac{2y \sin(2x^2y)}{2x^2y}$

$= \lim_{(x,y) \rightarrow (0,0-3)} (2y) \lim_{(x,y) \rightarrow (0,0-3)} \frac{\sin(2x^2y)}{2x^2y} \quad \because \theta = 2x^2y$

$= 2(-3) \cdot \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \quad \because \theta \rightarrow 0$

$= -6$

(4) $\lim_{(x,y) \rightarrow (2,-2)} \frac{\tan(x^2-y^2)}{x+y} \stackrel{\frac{0}{0}}{=} \lim_{(x,y) \rightarrow (2,-2)} \frac{(x-y) \tan(x^2-y^2)}{(x+y)(x-y)}$

$= \lim_{(x,y) \rightarrow (2,-2)} \frac{\tan(x^2-y^2)}{x^2-y^2} \cdot \lim_{(x,y) \rightarrow (2,-2)} (x-y) \quad \because \theta = x^2-y^2$

$= \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} \cdot \lim_{(x,y) \rightarrow (2,-2)} (x-y) = 1 \cdot 4 = 4$

$= \lim_{\theta \rightarrow 0} \tan \theta \cdot \lim_{(x,y) \rightarrow (2,-2)} (x-y) = 4$

$$(5) \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{\sqrt{x^2 + y^2} + 1} = \lim_{r \rightarrow 0} \frac{r^2 \cos^2 \theta r \sin \theta}{\sqrt{r^2 + 1} + 1} = \frac{0}{2} = 0$$

Q. 10

(*) Polar

$$x = r \cos \theta, y = r \sin \theta, x^2 + y^2 = r^2, r = \sqrt{x^2 + y^2}$$

$$(x, y) \rightarrow (0, 0) \Rightarrow r \rightarrow 0, 0 \leq \theta < 2\pi$$

$$\lim_{r \rightarrow 0} \frac{r^3 (\sqrt{r^2 + 1} + 1) \cos^2 \theta \sin \theta}{r^2} = 0 (2) = 0$$

E.X) Determine whether the Limit exists or DNE.
if it exists find its limit

$$(1) \lim_{(x,y) \rightarrow (1,1)} \frac{(x-1)^3 (y+1)^2}{(x-1)^6 + (y+1)^4} \quad \left(\frac{0}{0}\right)$$

$$(2) \lim_{(x,y,z) \rightarrow (0,0,0)} \frac{x^2 y^2 z}{x^2 + y^3 + z^4} \quad \left(\frac{0}{0}\right)$$

$$(3) \lim_{(x,y) \rightarrow (0,0)} \frac{1 - \cos \sqrt{x^2 + y^2}}{x^2 + y^2} \quad \left(\frac{0}{0}\right)$$

$$(4) \lim_{(x,y) \rightarrow (0,2)} \frac{x^{2/3} (y-2)^2}{x^2 + (y-2)^3} \quad \left(\frac{0}{0}\right)$$

$$(5) \lim_{(x,y) \rightarrow (0,0)} \frac{x^4 - 4y^4}{x^2 + 2y^2} \quad \left(\frac{0}{0}\right)$$

$$(6) \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + \sin^2(2y)}{2x^2 + y^2} \quad \left(\frac{0}{0}\right)$$

$$(7) \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x^2 + y^2} \quad \left(\frac{0}{0}\right)$$

Sol: (1) $C_1: x = t + 1, y = -1$ pass $(1, -1)$ when $t = 0$

$$\lim_{\substack{t \rightarrow 0 \\ c_1}} F = \lim_{t \rightarrow 0} \frac{(t+1-1)^3 (-1+1)^2}{(t+1-1)^6 + (-1+1)^4} = \lim_{t \rightarrow 0} \frac{t^3 (0)}{t^6 + 0} = 0$$

C2! $x = 0 + 1$, $y = t - 1$ Pass $(1, -1)$ when $t = 0$

عند $\lim$ يصبح الجواب zero لذا نأخذ c آخر

$C_2: x = t^2 + 1, y = t^3 - 1$ Pass $(1, -1)$ when $t = 0$

(*) ٢٠١٥ (مؤلف مشهور في مصر) (أحمد إسماعيل) (٢٠١٥) (تأليف أحمد إسماعيل)

$$\begin{aligned} \lim_{t \rightarrow \infty} F &= \lim_{t \rightarrow \infty} \frac{(t^2+1-1)^3 (t^3-1+1)^2}{(t^2+1-1)^6 + (t^3-1+1)^4} = \lim_{t \rightarrow \infty} \frac{(t^2)^3 (t^3)^2}{(t^2)^6 + (t^3)^4} \\ &= \lim_{t \rightarrow \infty} \frac{t^6 t^6}{t^{12} + t^{12}} = \lim_{t \rightarrow \infty} \frac{t^{12}}{2 t^{12}} = \frac{1}{2} \end{aligned}$$

$$\text{if } \lim_{C_1} F \neq \lim_{C_2} F \Rightarrow \text{Limit D.N.E.}$$

(2) $c_1: x \geq 0, y = 0, z = t$ pass $(0, 0, 0)$ when $t = 0$

$$\lim_{\substack{\text{along} \\ C_1}} F = \lim_{t \rightarrow 0} \frac{0 \cdot (0)^2 \cdot t}{0^2 + 0^3 + t^4} = \lim_{t \rightarrow 0} \frac{0}{t^4} = 0$$

$C_2: x = t^{\frac{12}{12}} + 0, y = t^{\frac{12}{3}} + 0, z = t^{\frac{12}{9}} + 0$ pass $(0,0,0)$ when $t=0$
 $x = t, y = t^4, z = t^3$

$$\lim_{\substack{\text{along} \\ C_2}} F = \lim_{t \rightarrow \infty} \frac{t(t^4)^2(t^3)}{(t)^{12} + (t^4)^3 + (t^3)^4} = \lim_{t \rightarrow \infty} \frac{t^{12}}{3t^{12}} = \frac{1}{3}$$

$$\lim_{\text{along } C_1} F \neq \lim_{\text{along } C_2} F \Rightarrow \text{Limit DNE.}$$

$$(3) \quad x = r \cos \theta, \quad y = r \sin \theta, \quad r = \sqrt{x^2 + y^2} \quad (x, y) \rightarrow (0, 0)$$

$\downarrow$
 $r \rightarrow 0$

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{1 - \cos \sqrt{x^2 + y^2}}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{1 - \cos r}{r^2} = \lim_{r \rightarrow 0} \frac{\sin r}{2r} = \frac{1}{2}$$

$$(4) \quad C_1: \quad x = t, \quad y = 2 \quad \text{pass } (0, 2) \text{ when } t = 0$$

$$\lim_{\text{along } C_1} F = \lim_{t \rightarrow 0} \frac{t^{3/2} (2-2)^2}{t^2 + (2-2)^2} = \lim_{t \rightarrow 0} \frac{0}{t^2} = 0$$

$$C_2: \quad x = t^3, \quad y = t^2 + 2 \quad \text{pass } (0, 2) \text{ when } t = 0$$

$$\lim_{\text{along } C_2} F = \lim_{t \rightarrow 0} \frac{(t^3)^{3/2} (t^2 + 2 - 2)^2}{(t^3)^2 + (t^2 + 2 - 2)^3} = \lim_{t \rightarrow 0} \frac{t^2 t^4}{t^6 + t^6} = \frac{1}{2}$$

$$\therefore \lim_{\text{along } C_1} F \neq \lim_{\text{along } C_2} F \Rightarrow \text{Limit DNE}$$

$$(5) \quad \lim_{(x, y) \rightarrow (0, 0)} F = \lim_{(x, y) \rightarrow (0, 0)} \frac{(x^2 - 2y^2)(x^2 + 2y^2)}{(x^2 + 2y^2)} = 0$$

(6) $C_1: x = t, y = 0$ pass $(0,0)$ when $t=0$

$$\lim_{\text{along } C_1} F = \lim_{t \rightarrow 0} \frac{t^2 + \sin^2 2(0)}{2t^2 + 0^2} = \lim_{t \rightarrow 0} \frac{t^2}{2t^2} = \frac{1}{2}$$

$C_2: x = 0, y = t$ pass $(0,0)$ when $t=0$

$$\lim_{\text{along } C_2} F = \lim_{t \rightarrow 0} \frac{0^2 + \sin^2(2t)}{2(0)^2 + t^2} = \lim_{t \rightarrow 0} \frac{\sin^2(2t)}{t^2}$$

$$= \lim_{t \rightarrow 0} \left(\frac{\sin(2t)}{t} \right)^2 = 4$$

$\therefore \lim_{\text{along } C_1} F \neq \lim_{\text{along } C_2} F \Rightarrow \text{Limit D.N.E}$

(7) $x = r \cos \theta, y = r \sin \theta, x^2 + y^2 = r^2, (x,y) \rightarrow (0,0) \Rightarrow r \rightarrow 0$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{r^3 \cos^3 \theta - r^3 \sin^3 \theta}{r^2}$$

$$= \lim_{r \rightarrow 0} r (\cos^3 \theta - \sin^3 \theta) = 0$$

* Def: A func. $f(x,y)$ is said to be continuous (conts.) at a pt. (a,b) if $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b)$

① $(a,b) \in \text{Dom}(f)$

② $\lim_{(x,y) \rightarrow (a,b)} f = \text{exist}$

E.X) (1) $f(x, y, z) = \frac{3x^2 - z}{x^2 + y^2 + z^2 + 1}$ is conts. on $\text{Dom}(f) = \mathbb{R}^3$

(2) $g(x, y) = \frac{\ln(x - 2y)}{x}$ is conts. on

$\text{Dom}(g) = \{(x, y) \in \mathbb{R}^2 : x - 2y > 0 \text{ and } x \neq 0\}$
 $(0, 1) \notin \text{Dom}(g)$, $(1, 1) \notin \text{Dom}(g)$

(3) $\text{Dom}(f) = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \neq z\}$
 $f(x, y, z) = \frac{1}{x^2 + y^2 - z}$ is conts.

(4) $f(x, y, z) = 3$ is conts. on $\mathbb{R}^3$

E.X) Determine whether the func.

(1) $f(x, y) = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

is conts. on $\mathbb{R}^2$ or not.

(2) $f(x, y) = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 1, & (x, y) = (0, 0) \end{cases}$

conts. on $\mathbb{R}^2$ or not.

Sol:- we must study the continuity of f at $(0,0)$ since f conts. on $\mathbb{R}^2 - \{(0,0)\}$

$$(1) f(0,0) = 0$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x^2 + y^2} = \dots = 0 \quad (\text{by E.X. 2})$$

$$= f(0,0)$$

$\therefore f$ conts. at $(0,0)$ $\wedge f$ conts. on $\mathbb{R}^2$

$$(2) f(0,0) = 1$$

$$\lim_{(x,y) \rightarrow (0,0)} f = \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x^2 + y^2} = \dots = 0 \neq f(0,0)$$

$\therefore f$ is conts. at $(0,0)$ $\wedge f$ conts. only on $\mathbb{R}^2 - \{(0,0)\}$

E.X) Determine whether the func. conts. at the origin $f(x,y) = \begin{cases} x^2 + \sin^2(zy) & , (x,y) \neq (0,0) \\ 2x^2 + y^2 & , (x,y) = (0,0) \end{cases}$

Sol:-

$$f(0,0) = 2$$

$$\lim_{(x,y) \rightarrow (0,0)} f = \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + \sin^2(zy)}{2x^2 + y^2} = \dots = \text{DNE (by E.X. 6)}$$

$\therefore f$ disconts. at $(0,0)$

E.X) Find the value of K that makes the func.:-

$$(1) f(x, y) = \begin{cases} \frac{\sin(Kx^2y)}{x^2} & ; x \neq 0 \\ 6 & , x = 0 \end{cases}$$

conts. $(0, 10)$

$$(2) f(x, y) = \begin{cases} \frac{1 - \cos \sqrt{x^2 + y^2}}{x^2 + y^2} & , (x, y) \neq (0, 0) \\ K & , (x, y) = (0, 0) \end{cases}$$

conts. $(0, 0)$

SOL:-

$$(1) \lim_{(x, y) \rightarrow (0, 10)} f = f(0, 10)$$

$$\lim_{(x, y) \rightarrow (0, 10)} \frac{\sin(Kx^2y)}{x^2} = 6 \Rightarrow \lim_{x \rightarrow 0} \frac{\sin(Kx^2 \cdot 10)}{x^2} = 6$$

$$\lim_{x \rightarrow 0} x \cdot \lim_{x \rightarrow 0} \frac{\sin(Kx^2y)}{x^2y} = 6$$

$$\theta = x^2y$$

$$10 \cdot \lim_{\theta \rightarrow 0} \frac{\sin(K\theta)}{\theta} = 6$$

$$(x, y) \rightarrow (0, 10)$$

$$\theta \rightarrow 0$$

$$10 \cdot K = 6 \Rightarrow K = \frac{6}{10}$$

$$(2) \lim_{(x, y) \rightarrow (0, 0)} f = f(0, 0)$$

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{1 - \cos \sqrt{x^2 + y^2}}{x^2 + y^2} = K \rightarrow \frac{1}{2} = K \text{ (by Ex 3)}$$

* Remark :-

If $\lim_{(x,y) \rightarrow (a,b)} F(x,y) = L$ exist and C is a curve

that pass through (a,b) , then $\lim_{\text{along } C} F = L$

E.x) Find the value of "a" that makes the func. conts. at the origin when $F(x,y) = \begin{cases} \frac{ax^2 - 2y^2}{x^2 + y^2} & (x,y) \neq (0,0) \\ -2 & (x,y) = (0,0) \end{cases}$

Sol:-

F conts. at $(0,0) \Rightarrow \lim_{(x,y) \rightarrow (0,0)} F = F(0,0) = -2$

$$\therefore \lim_{(x,y) \rightarrow (0,0)} \frac{ax^2 - 2y^2}{x^2 + y^2} = -2$$

Take $C: x=t, y=0$ pass $(0,0)$ when $t=0$

$$\therefore \lim_{\text{along } C} F = -2$$

$$\lim_{t \rightarrow 0} \frac{at^2 - 2(0)^2}{t^2 + 0^2} = -2 \Rightarrow \lim_{t \rightarrow 0} \frac{at^2}{t^2} = -2 \Rightarrow a = -2$$

* sec (14.3) partial Derivatives

* Def : The partial derivative of a func. $f(x, y)$ with respect to (w.r.t) (x, y) :

(1) x at a pt. (a, b) is :-

$$\left. \frac{df}{dx} \right|_{(a,b)} = f_x(a, b) = \lim_{x \rightarrow a} \frac{f(x, b) - f(a, b)}{x - a} = \left. \frac{d}{dx} f(x, b) \right|_{x=a}$$

(2) y at a pt. (a, b) is

$$\left. \frac{df}{dy} \right|_{(a,b)} = f_y(a, b) = \lim_{y \rightarrow b} \frac{f(a, y) - f(a, b)}{y - b} = \left. \frac{d}{dy} f(a, y) \right|_{y=b}$$

E.X) If $f(x, y, z) = \cos(2xy) e^{xyz} \ln z$

Sol:-

$$f_x = \cos(2xy) y z e^{xyz} \ln z + e^{xyz} \ln z (-\sin(2xy) \cdot 2y)$$

$$f_y = \cos(2xy) x z e^{xyz} \ln z + e^{xyz} \ln z (-\sin(2xy) \cdot 2x)$$

$$f_z = \cos(2xy) e^{xyz} \frac{1}{z} + \ln z (\cos(2xy) x y e^{xyz})$$

$$* f_z(0, 1, 1) = \cos 0 \cdot \frac{e^0}{1} + \ln 1 (---) = 1$$

E.X) Find $\frac{df}{dx}$, $\frac{df}{dy}$ at $(1, -1)$ where $f(x, y) = \frac{x^2}{xy + zy^2}$

SOL:-

$$\frac{df}{dx} = \frac{(xy + 2y^2)2x - x^2(y)}{(xy + 2y^2)^2}$$

$$\frac{df}{dx} \bigg|_{(1,-1)} = \frac{1(2) - 1(-1)}{1^2} = 3$$

$$\frac{df}{dy} = \frac{-x^2(x + 4y)}{x(xy + 2y^2)^2}$$

$$\frac{df}{dy} \bigg|_{(1,-1)} = \frac{-1(1-4)}{1^2} = 3$$

Ex) If the relation $\frac{x^3 + y^2 + z^3}{1 - 6xyz} = \frac{-1}{5}$ defines a surface z in terms of x, y (that is $z = f(x, y)$) ; Find $\frac{dz}{dx}$, $\frac{dz}{dy}$ at $(1, -1)$

SOL:-

$$\text{Relation} \Rightarrow 5x^3 + 5y^2 + 5z^3 = 6xyz - 1$$

$$5x^3 + 5y^2 + 5z^3 - 6xyz + 1 = 0 \quad \text{--- (1)}$$

At the pt. $(1, -1) \Rightarrow x=1, y=-1, z=?$

$$\Rightarrow 5 + 5 + 5z^3 + 6z + 1 = 0 \Rightarrow 5z^3 + 6z + 11 = 0 \quad \text{--- (2)}$$

$$\therefore z = -1 \quad \therefore \text{pt. } (1, -1, -1)$$

$$\textcircled{1} \Rightarrow 15x^2 + 0 + 15z^2 z_x - (6xy z_x + 6yz) = 0$$

$$\text{pt.} \Rightarrow 15 + 15z_x - (-6z_x + 6) = 0$$

$$9 + 21z_x = 0 \Rightarrow z_x = \frac{-9}{21}$$

$$\therefore \frac{dz}{dx} \bigg|_{(1,-1)} = \frac{-9}{21}$$

$$(1) \Rightarrow 0 + 10y + 15z^2 z_y - (6xy z_y + 6xz) = 0$$

$$p.f. \Rightarrow -10 + 15z_y - (-6z_y - 6) = 0$$

$$-4 + 21z_y = 0 \Rightarrow z_y = \frac{4}{21} \quad \& \quad \frac{dz}{dy} \bigg|_{(1,0,1)} = \frac{4}{21}$$

E.X) (1) If $f(x,y) = \sqrt{x^2 + y^2}$, find $f_x(0,0)$

(2) If $f(x,y) = \sqrt[3]{x^3 - y^3}$ find $f_y(0,0)$

SOL:-

$$(1) f_x(0,0) = \frac{d}{dx} f(x,0) \bigg|_{x=0} = \frac{d}{dx} \sqrt{x^2} \bigg|_{x=0} = \frac{d}{dx} |x| \bigg|_{x=0} = \text{D.N.C}$$

$$(2) f_y(0,0) = \frac{d}{dy} f(0,y) \bigg|_{y=0} = \frac{d}{dy} \sqrt[3]{-y^3} \bigg|_{y=0} = \frac{d}{dy} (-y) \bigg|_{y=0} = -1$$

* Higher partial Derivatives :-

$z = f(x,y)$. The 2nd partial derivatives of f are:-

$f_{xx}, f_{xy}, f_{yx}, f_{yy}$

$$f_{xx} = (f_x)_x = \frac{d^2 f}{dx^2}$$

$$f_{xy} = (f_x)_y = \frac{d^2 f}{dy dx}$$

$$f_{yx} = (f_y)_x = \frac{d^2 f}{dx dy}$$

$$f_{yy} = (f_y)_y = \frac{d^2 f}{dy^2}$$

$$\therefore f_{xxxx} = (((f_x)_x)_y)_x = \frac{d^4 f}{dx dy dx dy dx^2}$$

E.x) Find $f_{xy}(0,0)$ & $f_{yx}(0,0)$ where $f(x,y) = x \sqrt[3]{8x^3 - 2xy^3}$

SOL:-

$$(1) f_x = x \cdot \frac{1}{3} (8x^3 - 2xy^3)^{-2/3} (24x^2) + \sqrt[3]{8x^3 - 2xy^3}$$

$$f_{xy}(0,0) = (f_x)_y \Big|_{(0,0)} = \frac{d}{dy} f_x(0,y) \Big|_{y=0}$$

$$= \frac{d}{dy} \left(0 \cdot \frac{1}{3} (0 - 2xy^3)^{-2/3} (0) + \sqrt[3]{0 - 2xy^3} \right) \Big|_{y=0}$$

$xy \neq 0$ & $y \rightarrow 0$. $\therefore y$

$$= \frac{d}{dy} \sqrt[3]{-2xy^3} \Big|_{y=0} = \frac{d}{dy} (-3y) \Big|_{y=0} = -3$$

$$(2) f_y = x \cdot \frac{1}{3} (8x^3 - 2xy^3)^{-2/3} (-2x(3)y^2)$$

$$f_{yx}(0,0) = \frac{d}{dx} f_y(x,0) \Big|_{x=0}$$

$$= \frac{d}{dx} \left(x \cdot \frac{1}{3} (8x^3)^{-2/3} (-2x(3)0) \right) \Big|_{x=0} = \frac{d}{dx} 0 \Big|_{x=0} = 0$$

* Remark :- f_{xy} need not equal f_{yx} ليس بالضرورة أن يساوي

* claudent's Thm: If $f(x,y)$ has conts. partial derivatives $f_{xy} = f_{yx}$ on a disk D containing a pt. (a,b) , then $f_{xy}(a,b) = f_{yx}(a,b)$

(a,b)

*** Remark :-**

(1) If $f(x, y, z)$ has conts. partial derivatives of all orders (متساوية الرتبة) then $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$, $f_{yz} = f_{zy}$ on $\mathbb{R}^3$

$$\therefore f_{xxxyyyzyx} = f_{zyyyxxxx}$$

E.x) If $f(x, y, z) = x^4 z \sin(xyz)$. Find $f_{xxzyx}(1, 0, -1)$

SOL:-

observe that f has conts. partial derivatives of all orders on $\mathbb{R}^3$.

$$\Rightarrow f_{xxzyx} = f_{yzxxx}$$

$$f_y = x^4 z \cos(xyz) \cdot xz$$

$$f_y(x, 0, z) = x^5 z^2 \cos 0 = x^5 z^2$$

$$f_{yz} = 2x^5 z$$

$$f_{yz}(x, 0, -1) = -2x^5$$

$$f_{yzxxx} = -2(5)(4)(3)x^2$$

$$f_{yzxxx}(1, 0, -1) = -120 = f_{xxzyx}(1, 0, -1) = -120$$

Exercise:-

Find $\frac{\partial^2}{\partial x \partial y} (x e^{3xy})$ at $(1, 0)$

① $3x^2 e^{3xy}$

② $3^2 x^3 e^{3xy}$

③ $3^3 x^4 e^{3xy}$

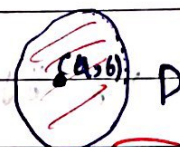
$$= \frac{\partial^2}{\partial x \partial y} (x e^{3xy})$$

① 20

$3^1 x^0 \cdot x^1 e^{3xy}$

* sec (14.4) Tangent planes and linear Approximation

* A func. $f(x,y)$ is differentiable at a pt. (a,b) if f_x, f_y are conts. on a Disk D containing (a,b)



* sec (14.5) The chain Rule.

* Let $w = f(x,y,z)$ and let

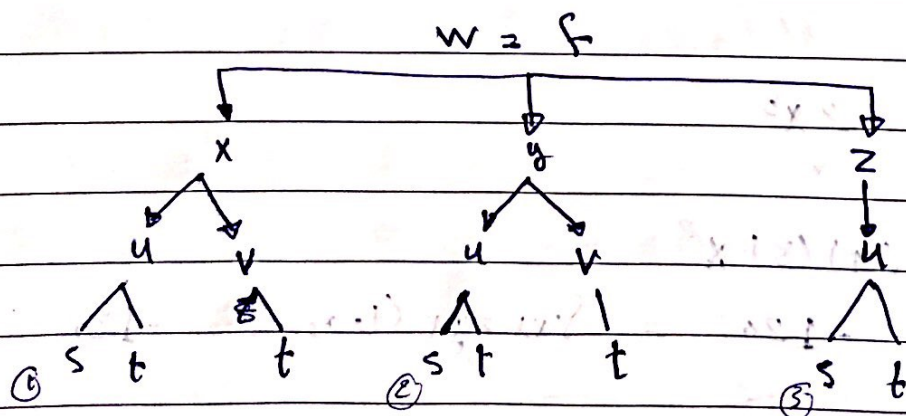
$x = x(u,v) : x$ func. in u,v

$y = y(u,v) : y$ " " u,v

$z = z(u) : z$ " " u only.

$u = u(s,t) : u$ " " s,t

$v = v(t) : v$ " " t only



$$\frac{\partial w}{\partial s} = w_x \cdot x_u \cdot u_s + w_y \cdot y_u \cdot u_s + w_z \cdot \underbrace{\frac{\partial z}{\partial u}}_{z'} \cdot u_s$$

$$\frac{dw}{dt} = f_x \cdot x_u \cdot u_t + f_x \cdot x_v \cdot \frac{dv}{dt} + f_y \cdot y_u \cdot u_t + f_y \cdot y_v \cdot \frac{dv}{dt} + f_z \cdot z' \cdot u_t$$

$$\frac{dx}{dt} = x_u \cdot u_t + x_v \cdot v'$$

$$\frac{dx}{ds} = x_u \cdot u_s$$

E.X1) Let $u = x^4 y + y^2 z^3$, $x = r \sec t$, $y = \cos t$, $z = r^2 \sin t$; Find $\frac{du}{ds}$, $\frac{du}{dt}$, when $r=2$, $s=1$, $t=0$

SOL:-

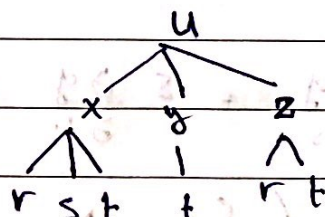
$$\frac{du}{ds} = u_x \cdot x_s = 4x^3 y r \sec t \quad (1)$$

when $r=2$, $s=1$, $t=0$ we need x, y, z

$$x = 2(1) \sec 0 = 2, \quad z = 2^2 \sin 0 = 0$$

$$y = \cos 0 = 1$$

$$(1) \Rightarrow \frac{du}{ds} = 4(2)^3 (1) (2) \sec 0 = 64$$



$$\frac{du}{dt} = u_x \cdot x_t + u_y \cdot y' + u_z \cdot z_t$$

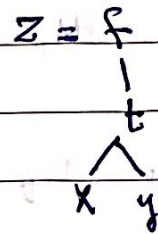
$$= 4x^3 y r \sec t + (x^4 + 2y z^3) (-\sin t) + 3y^2 z^2 r^2 \cos t$$

$$\frac{du}{dt} \Big|_{\substack{r=2 \\ s=1 \\ t=0}} = 4(2)^3 (1) (2) (1) \sec 0 = 64$$

E.x) If $z = f(x-y)$ show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

Sol:-

$$z = f(t), \quad t = x - y$$



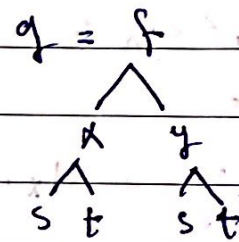
$$\begin{aligned} \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} &= f' t_x + f' t_y \\ &= f'(1) + f'(-1) = 0 \quad \# \end{aligned}$$

E.x) $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ show that $t \frac{\partial g}{\partial s} + s \frac{\partial g}{\partial t} = 0$

Sol:-

$$g(s, t) = f(x, y)$$

$$x = s^2 - t^2, \quad y = t^2 - s^2$$



$$\Rightarrow t \frac{\partial g}{\partial s} + s \frac{\partial g}{\partial t} = t (f_x \cdot x_s + f_y \cdot y_s)$$

$$+ s (f_x \cdot x_t + f_y \cdot y_t)$$

$$= t (f_x \cdot 2s + f_y \cdot (-2s)) + s (f_x \cdot (-2t) + f_y \cdot (2t))$$

$$= 2st f_x - 2st f_y - 2st f_x + 2st f_y = 0 \quad \#$$

* The Implicit Func. Theorem :- الطريقة الإعتراضية

If the relation $F(x, y, z) = 0$ defines implicitly a func. z in terms of x, y then $\frac{\partial z}{\partial x} = - \frac{F_x}{F_z}$

$$\frac{\partial z}{\partial y} = - \frac{F_y}{F_z}$$

E.x) If $\frac{x^3 + y^2 + z^3}{1 - 6xyz} = \frac{-1}{5}$ find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ when

$$x=1, y=-1$$

SOL:-

$$\text{eq.} \Rightarrow \frac{5x^3 + 5y^2 + 5z^3 - 6xyz + 1}{f(x, y, z)} = 0$$

when $x=1, y=-1$ we need $z=??$

$$5 + 5 + 5z^3 + 6z + 1 = 0 \Rightarrow 5z^3 + 6z + 11 = 0 \Rightarrow z = -1$$

$$\frac{\partial z}{\partial x} = - \frac{F_x}{F_z} = - \frac{15x^2 - 6yz}{15z^2 - 6xy}$$

$$\left. \frac{\partial z}{\partial x} \right|_{\substack{x=1 \\ y=-1 \\ z=-1}} = - \frac{15 - 6}{15 + 6} = \frac{-9}{21}$$

$$\frac{\partial z}{\partial y} = - \frac{F_y}{F_z} = - \frac{10y - 6xz}{15z^2 - 6xy}$$

$$\left. \frac{\partial z}{\partial y} \right|_{\substack{x=1 \\ y=-1 \\ z=-1}} = - \frac{-10 + 6}{21} = \frac{4}{21}$$

E.x) Find y' if $x^3 + y^3 = 6xy$ when $x=0$

SOL:-

$$\text{when } x=0 \Rightarrow 0^3 + y^3 = 6(0)y \Rightarrow y=0$$

$$\text{eq.} \Rightarrow \frac{x^3 + y^3 - 6xy}{f(x, y)} = 0$$

$$y' = \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{3x^2 - 6y}{3y^2 - 6x}$$

* sec (14.6) Directional Derivative and Gradient Vector.

* Def:- The Gradient vector of a func. $f(x, y, z)$ is $\nabla f = \langle f_x, f_y, f_z \rangle$

E.x) IF $f(x, y) = 2xe^{3xy}$, find (1) ∇f (2) $\nabla f(1, 0)$

$$(1) \nabla f = \langle f_x, f_y \rangle = \langle 2x \cdot 3y \cdot e^{3xy} + 2e^{3xy}, 2x \cdot 3x \cdot e^{3xy} \rangle \\ = \langle (6xy + 2)e^{3xy}, 6x^2 e^{3xy} \rangle$$

$$(2) \nabla f(1, 0) = \langle (6(1)(0) + 2)e^0, 6(1)^2 e^0 \rangle = \langle 2, 6 \rangle$$

E.x) $f(x, y, z) = x^2 - 3y + z^3$

$$\nabla f = \langle 2x, -3, 3z^2 \rangle$$

* Def: The directional Derivative of a func. $f(x, y)$ at a pt. (x_0, y_0) in the direction of a unit vector $\vec{u} = \langle a, b \rangle$ is

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

* Rule: $D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \hat{u}$

E.X) Let $f(x, y, z) = x e^{2xyz}$. Find the directional derivative of f at the pt. $A(1, 2, 0)$ in the direction:
 (1) of the vector $\vec{V} = 2\hat{i} - 2\hat{j} + \hat{k}$ (2) of the pt. $B(2, 2, 2)$

Sol:-

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$= \langle 2xyz e^{2xyz} + e^{2xyz}, 2x^2 z e^{2xyz}, 2x^2 y e^{2xyz} \rangle$$

$$\nabla f(1, 2, 0) = \langle 1, 0, 4 \rangle$$

$$|\vec{V}| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$$

$$\hat{V} = \frac{\vec{V}}{|\vec{V}|} = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$$

$$D_{\hat{V}} f(1, 2, 0) = \nabla f(1, 2, 0) \cdot \hat{V}$$

$$= \langle 1, 0, 4 \rangle \cdot \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$$

$$= 1 \left(\frac{2}{3} \right) + 0 \left(-\frac{2}{3} \right) + 4 \left(\frac{1}{3} \right) = \frac{2+4}{3} = \frac{6}{3} = 2$$

$$(2) \vec{V} = \vec{AB} = \langle 1, 0, 2 \rangle$$

$$|\vec{V}| = \sqrt{5}$$

$$\hat{V} = \frac{\vec{V}}{|\vec{V}|} = \left\langle \frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}} \right\rangle$$

$$D_{\hat{V}} f(1, 2, 0) = \nabla f(1, 2, 0) \cdot \hat{V}$$

$$= 1 \left(\frac{1}{\sqrt{5}} \right) + 0(0) + 4 \left(\frac{2}{\sqrt{5}} \right) = \frac{9}{\sqrt{5}}$$

*** Remark:-** (1) $D_{\hat{u}} f(x_0, y_0) \equiv$ rate of change of
 f at the pt. (x_0, y_0) in the direction of $\hat{u}$.

(2) The Max. value of the directional derivative of $f(x, y)$ at a pt. (x_0, y_0) is $|\nabla f(x_0, y_0)|$ which holds in the direction of $\nabla f(x_0, y_0)$

(3) The min. value of the directional derivative of $f(x, y)$ at a pt. (x_0, y_0) is $-|\nabla f(x_0, y_0)|$ which holds in the ~~opposite~~ opposite direction $-\nabla f(x_0, y_0)$

$$\vec{u} \cdot \hat{v} = \frac{|\vec{u}| |\vec{v}| \cos \theta}{|\nabla f| \cos \theta}$$

E.x) Find the rate of change of $f(x, y) = x \sin(xy)$ at the pt. $A(1, \frac{\pi}{3})$ in the direction of $B(2, \frac{\pi}{6})$.

Sol:-

$$\begin{aligned} \nabla f &= \langle f_x, f_y \rangle \\ &= \langle xy \cos xy + \sin xy, x^2 \cos xy \rangle \end{aligned}$$

$$\nabla f(1, \frac{\pi}{3}) = \langle \frac{\pi}{3} \cos \frac{\pi}{3} + \sin \frac{\pi}{3}, \cos \frac{\pi}{3} \rangle$$

$$= \langle \left(\frac{\pi}{6} + \frac{\sqrt{3}}{2}\right), \frac{1}{2} \rangle$$

$$\vec{v} = \vec{AB} = \langle 1, \frac{\pi}{6} - \frac{\pi}{3} \rangle = \langle 1, -\frac{\pi}{6} \rangle$$

$$|\vec{v}| = \sqrt{1 + \frac{\pi^2}{36}} = \sqrt{\frac{36 + \pi^2}{36}} = \frac{\sqrt{36 + \pi^2}}{6}$$

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{6}{\sqrt{36 + \pi^2}} \langle 1, -\frac{\pi}{6} \rangle$$

$$\text{rate of change} = D_{\hat{v}} f(1, \frac{\pi}{3}) = \nabla f(1, \frac{\pi}{3}) \cdot \hat{v}$$

$$= \left(\frac{\pi}{6} + \frac{\sqrt{3}}{2} \right) \frac{6}{\sqrt{36+\pi^2}} + \frac{1}{2} \frac{-6\pi}{6\sqrt{36+\pi^2}}$$

E.x) Let $f(x, y) = xe^y$; (1) Find the max. directional derivative of f at the pt. $A(2, 0)$.

(2) Find the direction at which f have max. directional derivative at $(2, 0)$

SOL:-

$$\nabla f = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle \Rightarrow \nabla f(2, 0) = \langle 1, 2 \rangle$$

(1) Max. directional derivative $= |\nabla f(2, 0)| = \sqrt{5}$

(2) direction $= \nabla f(2, 0) = \langle 1, 2 \rangle$

*** Remark :-**

$$D_i f = f_x, \quad D_j f = f_y, \quad D_k f = f_z$$

E.x) Let $\hat{u} = \frac{3i - 4j}{5}$, $\hat{v} = \frac{3i + 4j}{5}$

$$D_{\hat{u}} f(1, 2) = 3, \quad D_{\hat{v}} f(1, 2) = -7$$

(1) Find the directional derivative of f at $A(1, 2)$ in the direction of $B(\frac{1}{2}, 1)$

SOL:-

$$a = f_x(1, 2), \quad b = f_y(1, 2)$$

$$\nabla f(1, 2) = \langle a, b \rangle$$

$$D_{\hat{u}} f = 3 \Rightarrow \nabla f \cdot \hat{u} = 3 \Rightarrow \langle a, b \rangle \cdot \left\langle \frac{3}{5}, \frac{-4}{5} \right\rangle = 3$$

$$\frac{3a - 4b}{5} = 3 \Rightarrow 3a - 4b = 15 \quad (1)$$

$$D_v f = -7 \Rightarrow \frac{3a + 4b}{5} = -7 \Rightarrow 3a + 4b = -35 \quad (2)$$

(1) + (2) :

$$6a = -20 \Rightarrow a = -\frac{10}{3} = f_x(1, 2)$$

$$(1) - (2) \Rightarrow -8b = 50 \Rightarrow b = -\frac{50}{8} = -\frac{25}{4}$$

$$f_y(1, 2) = -\frac{25}{4}$$

$$(2) \quad \nabla f(1, 2) = \left\langle -\frac{10}{3}, -\frac{25}{4} \right\rangle$$

$$\vec{w} = \vec{AB} = \left\langle -\frac{1}{2}, -1 \right\rangle$$

$$|\vec{w}| = \sqrt{\frac{1}{4} + 1} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

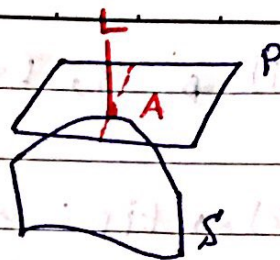
$$\hat{w} = \frac{\vec{w}}{|\vec{w}|} = \frac{2}{\sqrt{5}} \left\langle -\frac{1}{2}, -1 \right\rangle = \left\langle -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle$$

$$D_{\hat{w}} f(1, 2) = \left\langle -\frac{10}{3}, -\frac{25}{4} \right\rangle \cdot \left\langle -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle$$

* Remark : Let $S: f(x, y, z) = 0$ be a surface that pass through a pt. $A(x_0, y_0, z_0)$. Let p : tangent plane to S at A . Let L : be a normal line to S at A . Then.

$$\nabla f(x_0, y_0, z_0) \perp P$$

$$\nabla f(x_0, y_0, z_0) \parallel L.$$



E.x) Find the eqs. of the tangent plane and normal line at the pt. $A(-1, 0, 1)$ for the surfaces:

(1) $Z = 2 - x^2 - 3y^2$ (2) $2x^2y - 3xz^3 = 2y^2z + 3$

Sol :-

(1) $\underbrace{x^2 + 3y^2 + Z - 2 = 0}_F$

$$\nabla F = \langle 2x, 6y, 1 \rangle \quad ; \quad \Delta F(-1, 0, 1) = \langle -2, 0, 1 \rangle$$

eq. of tangent plane : $-2x + 0y + Z = -2(-1) + 1$
 $-2x + Z = 3$

param. eq. of normal line :

$$X = -1 + -2t \quad , \quad y = 0 + 0t \quad , \quad Z = 1 + t$$

(2) $\underbrace{2x^2y - 3xz^3 - 2y^2z - 3 = 0}_F$

$$\nabla F = \langle (4xy - 3z^3), (2x^2 - 4yz), (-9xz^2 - 2y^2) \rangle$$

$$\nabla F(-1, 0, 1) = \langle -3, 2, 9 \rangle$$

eq. of tangent plane : $-3x + 2y + 9z = 3 + 0 + 9 = 12$

Param. eq. of normal line :

$$X = -1 - 3t \quad , \quad y = 2t \quad , \quad Z = 1 + 9t$$

E.x) Find the pt(s) of intersection of the normal line through $(1, 2, 1)$ on $4x^2 + y^2 + 4z^2 = 12$ with $x^2 + y^2 + z^2 = 102$

Sol:-

$$\underbrace{4x^2 + y^2 + 4z^2 - 12}_F = 0$$

$$\nabla F = \langle 8x, 2y, 8z \rangle$$

$$= \langle 8, 4, 8 \rangle \text{ // normal Line}$$

$$\Rightarrow \langle 2, 1, 2 \rangle \text{ // normal Line}$$

$$\text{normal Line : } x = 1 + 2t, y = 2 + t, z = 1 + 2t$$

Now,

$$(1 + 2t)^2 + (2 + t)^2 + (1 + 2t)^2 = 102$$

$$1 + 4t + 4t^2 + 4 + 4t + t^2 + 1 + 4t + 4t^2 = 102$$

$$9t^2 + 12t - 96 = 0 \quad \div 3$$

$$3t^2 + 4t - 32 = 0$$

$$(3t - 8)(t + 4) = 0$$

$$t = -4 \text{ or } \frac{+8}{3}$$

$$t = -4 \Rightarrow x = 1 + 2(-4) = -7$$

$$y = 2 + (-4) = -2$$

$$z = 1 + 2(-4) = -7$$

$$t = \frac{+8}{3} \Rightarrow x = 1 + 2\left(\frac{+8}{3}\right) = \frac{19}{3}$$

$$y = 2 + \frac{+8}{3} = \frac{14}{3}$$

$$z = 1 + 2\left(\frac{+8}{3}\right) = \frac{19}{3}$$

pts. ~~(1, 2, 1)~~

$$(-7, -2, -7) \quad \left(\frac{19}{3}, \frac{14}{3}, \frac{19}{3}\right)$$

sec (14.7) Maximum and Minimum values.

* Def: A func. $f(x, y)$ is said to have a
 (1) Local max. (min.) at a pt. $(a, b) \in \text{Dom}(f)$
 if there is a disk D containing (a, b) and
 $f(a, b) \geq f(x, y)$ for all $(x, y) \in D$
 $(f(a, b) \leq f(x, y))$ The Local max (min.) value
 is $f(a, b)$

(2) ^{extremum} ~~extremum~~ value at (a, b) is $f(x, y)$ have a Local
 max. or min. at (a, b)

* Def: A func. $f(x, y)$ is said to have a critical
 pt. $(a, b) \in \text{Dom}(f)$ if $f_x(a, b) = 0$ and
 $f_y(a, b) = 0$ or $f_x(a, b)$ D.N.E or
 $f_y(a, b)$ D.N.E

E.X/ Find a, b that makes $(1, 2)$ critical pt. of
 $f(x, y) = 3x^3 + ax^2 + 6xy - y^2$

SOL:-

$$f_x = 9x^2 + 2ax + 6y$$

$$f_y = 6x - 2y$$

$$f_x(1, 2) = 0 \Rightarrow 9 + 2a + 12 = 0$$

$$f_y(1, 2) = 0 \Rightarrow 6 - 4 = 0 \Rightarrow b = 4$$

$$9 + 2a + 8 = 0 \Rightarrow a = -\frac{17}{2}$$

* The 2nd Derivative Test :-

⇒ Suppose f_x, f_y both zero at (a, b)

$$\text{Let } Df(a, b) = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2$$

(1) $Df(a, b) < 0$, then f has a saddle pt. at (a, b)

[f has neither a local max. nor a local min. at (a, b)]

(2) $Df(a, b) > 0$: $f_{xx}(a, b) > 0 \Rightarrow f$ has a local min. at (a, b)

* $f_{xx}(a, b) > 0 \Rightarrow f$ has a local min. at (a, b)

* $f_{xx}(a, b) < 0 \Rightarrow f$ has a local max. at (a, b)

* E.X) Find the pts. for which the function

$f(x, y) = 2x^3 + 6xy^2 - 3y^3 - 150x$ have local max., min. saddle pts. What are the local max., and min. values of f ?

SOL :-

$$f_x = 6x^2 + 6y^2 - 150 = 0 \div 6 \Rightarrow x^2 + y^2 = 25 \quad \text{--- (1)}$$

$$f_y = 12xy - 9y^2 = 0 \div 3 \Rightarrow y(4x - 3y) = 0 \quad \text{--- (2)}$$

$$\text{(2)} \Rightarrow y = 0 \text{ or } y = \frac{4x}{3}$$

$$\text{case 1 : } y = 0, x^2 + y^2 = 25 \Rightarrow x^2 = 25 \Rightarrow x = \pm 5$$

$$\Rightarrow (5, 0), (-5, 0)$$

case 2 :- $y = \frac{4x}{3}$, $x^2 + y^2 = 25 \Rightarrow x^2 + \frac{16x^2}{9} = 25$

$$\frac{25}{9} x^2 = 25 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$$

$$x = 3 \Rightarrow y = \frac{4}{3}(3) = 4 \quad \rightarrow \text{pts. } (3, 4), (-3, -4)$$

$$x = -3 \Rightarrow y = \frac{4}{3}(-3) = -4$$

f has 4 critical pts. $(-5, 0), (5, 0), (3, 4), (-3, -4)$

$$f_{xx} = 12x$$

$$f_{yy} = 12x - 18y$$

$$f_{xy} = 12y$$

$$Df = f_{xx} f_{yy} - (f_{xy})^2$$

$$= 12x(12x - 18y) - (12y)^2$$

$$* Df(-5, 0) = 12(-5)(12(-5)) > 0$$

$$\Rightarrow f_{xx}(-5, 0) = 12(-5) < 0$$

$\therefore f$ has Local max. at $(-5, 0)$

$$* Df(5, 0) = 12(5)(12(5)) > 0, f_{xx}(5, 0) = 12(5) > 0 \quad \text{min. Local}$$

$$* Df(3, 4) = 12(3)(12(3) - 18(4)) - (12(4))^2 < 0$$

$\Rightarrow$ saddle pt.

$$* Df(-3, -4) = 12(-3)(12(-3) - 18(-4)) - (12(-4))^2 < 0$$

$\Rightarrow$ saddle pt.

f has 2 saddle pts. at $(3, 4)$, $(-3, -4)$

f has Local min. at $(5, 0)$ The Local min. value is $f(5, 0)$

f has Local max at $(-5, 0)$. The Local max value is $f(-5, 0)$.

E.X) classify the critical pts. of $f(x, y)$

$f(x, y) = x^4 + y^4 - 4xy + 1$ as Local max, min. saddle pts.

Sol:-

$$f_x = 4x^3 - 4y = 0 \Rightarrow y = x^3 \quad \text{--- (1)}$$

$$f_y = 4y^3 - 4x = 0 \Rightarrow x = y^3 \quad \text{--- (2)}$$

$$\text{(1) : (2)} \Rightarrow x = (x^3)^3 \Rightarrow x = x^9 \Rightarrow x = 0, 1, -1$$

$$x = 0 \Rightarrow y = 0^3 \Rightarrow y = 0$$

$$x = 1 \Rightarrow y = 1^3 \Rightarrow y = 1$$

$$x = -1 \Rightarrow y = -1^3 \Rightarrow y = -1$$

critical pts.

$(0, 0), (1, 1), (-1, -1)$

$$f_{xx} = 12x^2$$

$$f_{yy} = 12y^2$$

$$f_{xy} = -4$$

$$\Delta f = f_{xx} f_{yy} - (f_{xy})^2$$

$$= 12x^2 \cdot 12y^2 - 16$$

$$\Delta f(0, 0) = -16 < 0 \Rightarrow \text{saddle pt.}$$

$$\Delta f(1, 1) = 12 \cdot 12 - 16 > 0, \quad f_{xx}(1, 1) = 12 > 0$$

$\therefore$ L. min

$$Df(-1, -1) = 12(12) - 16 > 0 \Rightarrow f_{xx}(-1, -1) = 12 > 0$$

$\therefore$ L. min

* Def: A set D in $\mathbb{R}^2$ is called:

(1) bounded (bdd) if it lies inside a circle

(2) closed if its boundary pts. (نقطه های مرزی) belongs (تعلق دارد) to D

* The Extrem value Theorem:-

Let $f(x, y)$ be conts. on a bdd Closed domain D , then there is pts. $(a, b), (c, d) \in D$ such that

$f(a, b)$ is the absolute max. of f on D

$f(c, d)$ " " " min. of f on D

that is $f(a, b) \geq f(x, y) \rightarrow$ for all $(x, y) \in D$
 $f(c, d) \leq f(x, y)$

* Remark:-

To find the absolute max., min. values of $f(x, y)$ we do the following:

(1) Find the critical pts. of f :

* inside D

* * on the boundary of D

(2) Find the values of f at the pts. in (1)

(3) max. = Largest value in (2)

min. = smallest value in (2)

E.x) Find the absolute max. and min. values of ~~f(x,y)~~

$f(x,y) = 2x^2 + y^2 + 6x$ on the domain:

(1) $D = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 8 \}$

(2) $D = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 8, y \geq 0 \}$

(3) $D = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 8, x \geq 0, y \geq 0 \}$

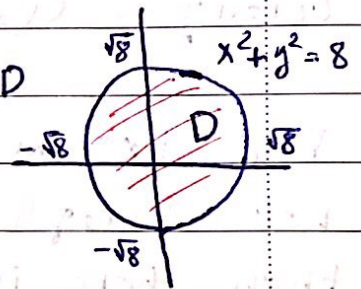
SOL:-

(a) critical pts. $\rightarrow$ inside

$f_x = 4x + 6 = 0 \Rightarrow x = -\frac{3}{2}$ pt. $(-\frac{3}{2}, 0) \in D$

$f_y = 2y = 0 \Rightarrow y = 0$

yes $(-\frac{3}{2})^2 + 0^2 \leq 8$



critical pts. on boundary $\Rightarrow$

$x^2 + y^2 = 8$

عوض معادلة الحدود في شرط اختفاء التدرج

الفرضية x و y

$g(x) = f|_{y^2=8-x^2} = 2x^2 + (8-x^2) + 6x, -\sqrt{8} \leq x \leq \sqrt{8}$ صيغة الرسالة

$g(x) = x^2 + 6x + 8$

$g'(x) = 2x + 6 = 0 \Rightarrow x = -3 \notin [-\sqrt{8}, \sqrt{8}]$

$x = -\sqrt{8} \rightarrow y^2 = 8 - (-\sqrt{8})^2 \rightarrow y = 0$

$x = \sqrt{8} \rightarrow y^2 = 8 - (\sqrt{8})^2 \rightarrow y = 0$

pts. $(-\sqrt{8}, 0)$ و $(\sqrt{8}, 0)$

Pt. | f

$(-\frac{3}{2}, 0) \quad 2(-\frac{3}{2})^2 + 0^2 + 6(-\frac{3}{2}) = -\frac{9}{2}$

$(-\sqrt{8}, 0) \quad 2(-\sqrt{8})^2 + 0^2 + 6(-\sqrt{8}) = 16 - 6\sqrt{8}$

$(\sqrt{8}, 0) \quad 2(\sqrt{8})^2 + 0^2 + 6\sqrt{8} = 16 + 6\sqrt{8}$

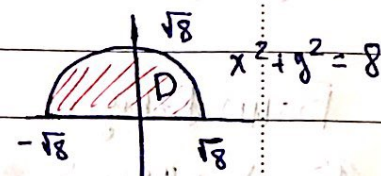
$$\text{absolute max.} = 16 + 6\sqrt{8}$$

$$\text{absolute min.} = -\frac{9}{2}$$

(2) critical pts. inside :-

$$f_x = 4x + 6 = 0 \Rightarrow x = -\frac{3}{2} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{Pt. } \left(-\frac{3}{2}, 0\right) \in D$$

$$f_y = 2y = 0 \Rightarrow y = 0$$



$$\left(-\frac{3}{2}\right)^2 + 0^2 \leq 8 \quad \text{yes}, \quad 0 \geq 0 \quad \text{yes}$$

critical pts. on boundary :-

* (1) f on the boundary of D

$$\text{Boundary : } y = 0$$

$$g_2(x) = f_1 \Big|_{y=0} = 2x^2 + 6x$$

$$g_2(x) = 2x^2 + 6x, \quad -\sqrt{8} \leq x \leq \sqrt{8}$$

$$g_2'(x) = 0 \Rightarrow 4x + 6 = 0 \Rightarrow x = -\frac{3}{2}$$

$$x = -\frac{3}{2} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} y = 0$$

$$x = -\sqrt{8} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} y = 0$$

$$x = \sqrt{8} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} y = 0$$

$$\left(-\frac{3}{2}, 0\right), (-\sqrt{8}, 0), (\sqrt{8}, 0)$$

Pt. f

$$\left(-\frac{3}{2}, 0\right) \quad -\frac{9}{2} = \text{min.}$$

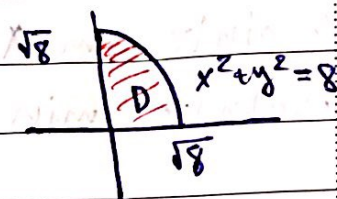
$$(-\sqrt{8}, 0) \quad 16 - 6\sqrt{8}$$

$$(\sqrt{8}, 0) \quad 16 + 6\sqrt{8} = \text{max.}$$

(3) critical pts. inside

$$f_x = 4x + 6 = 0 \Rightarrow x = -\frac{3}{2} \Rightarrow \text{Pt. } (-\frac{3}{2}, 0) \notin D$$

$$f_y = 2y = 0 \Rightarrow y = 0$$



Boundary :-

① $x^2 + y^2 = 8 \Rightarrow g_1(x) = f|_{y^2 = 8 - x^2} = 2x^2 + (8 - x^2) + 6x$

$$g_1(x) = x^2 + 6x + 8, \quad 0 \leq x \leq \sqrt{8}$$

$$g'_1(x) = 0 \Rightarrow 2x + 6 = 0 \Rightarrow x = -3 \notin [0, \sqrt{8}]$$

$$x = 0 \Rightarrow y^2 = 8 - 0^2 = 8 \Rightarrow y = \pm \sqrt{8} \text{ but } y \geq 0 \Rightarrow y = \sqrt{8}$$

$$x = \sqrt{8} \Rightarrow y^2 = 8 - (\sqrt{8})^2 = 0 \Rightarrow y = 0$$

Pts. $(0, \sqrt{8}), (\sqrt{8}, 0)$

② $y = 0$

$$g_2(x) = f|_{y=0} = 2x^2 + 6x, \quad 0 \leq x \leq \sqrt{8}$$

$$g'_2(x) = 4x + 6 = 0 \Rightarrow x = -\frac{3}{2} \notin [0, \sqrt{8}]$$

$$x = 0 \Rightarrow y = 0 \quad \text{Pts. } (0, 0), (\sqrt{8}, 0)$$

$$x = \sqrt{8} \Rightarrow y = 0$$

$$\textcircled{3} \quad x=0$$

$$g_3(y) = f|_{x=0} = 2(0)^2 + y^2 + 6(0)$$

$$g_3(y) = y^2, \quad 0 \leq y \leq \sqrt{8}$$

$$g_3'(y) = 2y = 0 \Rightarrow y = 0$$

$$\left. \begin{array}{l} y=0 \\ y=\sqrt{8} \end{array} \right\} \rightarrow \left. \begin{array}{l} x=0 \\ x=0 \end{array} \right\} \text{pts. } (0,0), (0,\sqrt{8})$$

Pt.	f
(0,0)	0 = min.
(0,\sqrt{8})	8
(\sqrt{8},0)	16 + 6\sqrt{8} = max.



اللجنة الأكاديمية لقسم الهندسة المدنية

دفتر

تفاضل وتكامل 3 (فاينل)

قصي أبو ديه

اعداد الطالب



/groups/Civilttee



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* CH: (15) multiple Integral :-

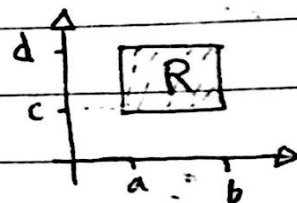
* Sec (15.1) :- Double Integral rectangle.

* Sec (15.2) :- Iterated Integrals :- (do Ex 15.1)

* A rectangle

$$R = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, c \leq y \leq d\}$$

$$= [a, b] \times [c, d]$$



* Fubini's Theorem :- Let $R = [a, b] \times [c, d]$

then $\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx$

$= \int_c^d \int_a^b f(x, y) dx dy$

Double Integral iterated double integral

E.X) compute $I = \iint_R \sqrt{1-x^2} dA$

$P_1 = \{(x, y) : -1 \leq x \leq 1, -2 \leq y \leq 2\}$

$$\int_{-1}^1 \int_{-2}^2 \sqrt{1-x^2} dy dx = 4 \int_{-1}^1 \sqrt{1-x^2} dx = 4 \cdot \frac{1}{2} \pi (1)^2$$

$$= 2\pi$$

* properties :-

$$(1) \iint_R (f+g) dA = \iint_R f dA + \iint_R g dA$$

$$(2) \iint_R c f dA = c \iint_R f dA, \quad c: \text{constant.}$$

$$(3) \int_c^d \int_a^b f(x) g(y) dx dy = \left(\int_a^b f(x) dx \right) \cdot \left(\int_c^d g(y) dy \right)$$

$x \rightarrow \tilde{x} \text{ to } x_c$
 $y \rightarrow \tilde{y} \text{ to } y_c$

$$\text{E.x) } \int_0^{\pi/2} \int_1^{\pi} \sin x \cos y dy dx = \left(\int_0^{\pi/2} \sin x dx \right) \left(\int_1^{\pi} \cos y dy \right)$$

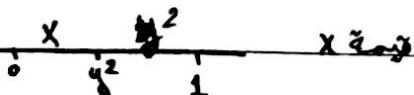
$$= \left(-\cos x \right)_0^{\pi/2} \left(\sin y \right)_1^{\pi}$$

$$= -[\cos \frac{\pi}{2} - \cos 0] [\sin \pi - \sin 1] = 1(-\sin 1) = -\sin 1$$

$$\text{E.x) let } f(x,y) = \begin{cases} y^2, & x \geq y^2 \\ x, & x < y^2 \end{cases}$$

$$\text{find } I = \int_0^1 \int_0^1 f(x,y) dx dy$$

Sol :-



$x < y^2$ $x > y^2$

$$I = \int_0^1 \left[\int_0^{y^2} f dx + \int_{y^2}^1 f dx \right] dy$$

$$= \int_0^1 \left[\int_0^{y^2} x dx + \int_{y^2}^1 y^2 dx \right] dy$$

$$= \int_0^1 \left[\frac{x^2}{2} y^2 + y^2 (1-y^2) \right] dy$$

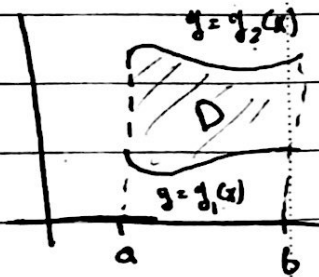
$$= \int_0^1 \left[\frac{y^4}{2} + y^2 - y^4 \right] dy$$

$$= \int_0^1 \left(-\frac{1}{2} y^4 + y^2 \right) dy = \left[-\frac{y^5}{10} + \frac{y^3}{3} \right]_0^1 = -\frac{1}{10} + \frac{1}{3} = \frac{2}{30}$$

* sec (15.3) Double Integrals over nonrectangular regions.

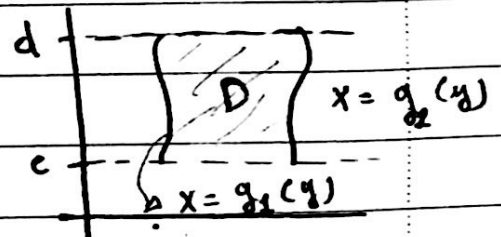
* Type 1 region :-

$$\iint_D f(x,y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) dy dx$$



* Type 2 region :-

$$\iint_D f(x,y) dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y) dx dy$$



E.X) compute $I = \iint_D (x+2y) dA$; D is the region bdd.

by $y = 2x^2$, $y = 1+x^2$

Sol :-

$$I = \int_{-1}^1 \int_{2x^2}^{1+x^2} (x+2y) dy dx$$

↳ Pts. of Intersections

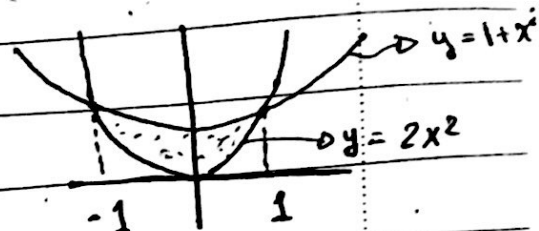
$$2x^2 = 1+x^2$$

$$x^2 = 1$$

$$x = \pm 1$$

* لمعرفة المنحنى الأكبر والأصغر نأخذ نقطة اختبار x أكبر

أصغر $y = 1 + 0^2 = 1$ و $y = 2(0)^2 = 0$ أصغر x take $x=0 \Rightarrow y = 2(0)^2 = 0$



$$I = \int_{-1}^1 \left[x(1+x^2-2x^2) + y^2 \right]_{2x^2}^{1+x^2} dx$$

$$= \int_{-1}^1 [x - x^3 + (1+x^2)^2 - (2x^2)^2] dx$$

$$= \int_{-1}^1 [x - x^3 + 1 + 2x^2 + x^4 - 4x^4] dx$$

$$= \int_{-1}^1 [-3x^4 - x^3 + 2x^2 + x + 1] dx = \dots$$

E.x) Evaluate $I = \iint_D xy \, dA$; where D is the region

enclosed by $y = x-1$, $y^2 = 2x+6$

Sol:-

Type 2 : $x = y+1$, $x = \frac{y^2-6}{2}$

حدد التكامل y الخارجيا

$$\frac{y^2-6}{2} = y+1 \Rightarrow y^2-6 = 2y+2 \Rightarrow y^2-2y-8=0$$

$$\Rightarrow (y-4)(y+2)=0 \therefore y = -2, 4$$

تعيده المنحنى الأكبر والأصغر take $y=0$

$$x = 0+1 = 1 \text{ أكبر}$$

$$x = \frac{0^2-6}{2} = -3 \text{ أصغر}$$

$$I = \int_{-2}^4 \int_{\frac{y^2-6}{2}}^{y+1} xy \, dx \, dy$$

$$= \int_{-2}^4 \left[\frac{x^2 y}{2} \right]_{\frac{y^2-6}{2}}^{y+1} dy$$

$$= \int_{-2}^4 \frac{y}{2} \left[(y+1)^2 - \left(\frac{y^2-6}{2} \right)^2 \right] dy$$

$$= \int_{-2}^4 \frac{y}{2} \left[y^2 + 2y + 1 - \frac{y^4 - 12y^2 + 36}{4} \right] dy$$

$$= \frac{1}{8} \int_{-2}^4 \left[4y^3 + 8y^2 + 4y - y^5 + 12y^3 - 36y \right] dy = \dots$$

$$\star \int f' e^f = e^f, \quad \int e^{ax+b} dx = \frac{e^{ax+b}}{a}$$

Ex) Evaluate

$$(1) \int_0^{1/2} \int_{2x}^1 \sin y^2 dy dx$$

Type II question D & E part ①

Type II question ②

$$(2) \int_0^2 \int_{y/2}^1 e^{x^2} dx dy$$

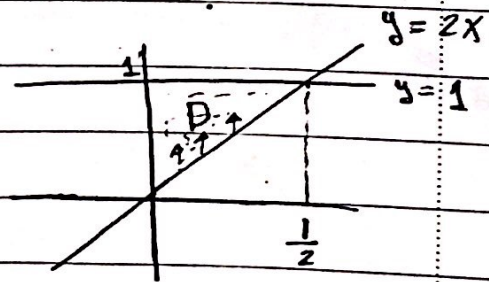
Sol:-

$$(1) 0 \leq x \leq \frac{1}{2}$$

$$2x \leq y \leq 1$$

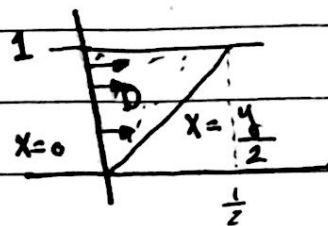
$$y = 2x$$

$$y = 1$$



Type 1 $\rightarrow$ Type 2

$$I = \int_0^1 \int_0^{y/2} \sin y^2 dx dy$$



$$= \int_0^1 \frac{y}{2} \sin y^2 dy = \frac{1}{4} \int_0^1 2y \sin y^2 dy$$

$$= \frac{1}{4} (-\cos y^2) \Big|_0^1 = -\frac{1}{4} (\cos 1 - \cos 0) = \frac{1 - \cos 1}{4}$$

(2) Type 2 $\rightarrow$ $dA = dx dy$

y varies x

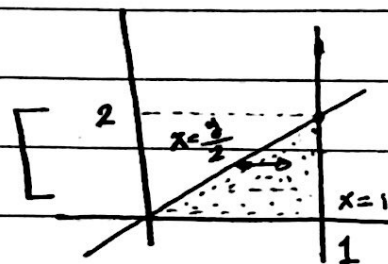
$$x = \frac{y}{2} \rightarrow x = 1$$

$$0 \leq y \leq 2$$

نوع 2
نوع 1
نوع 3

$$D: x = \frac{y}{2}, x = 1, y = 0$$

$$0 \leq y \leq 2$$

Type 2 $\rightarrow$ Type 1

$$D: y = 2x, y = 0$$

$$\frac{y}{2} = 1 \Rightarrow y = 2$$

$$x = 1, x = 0$$

نوع 2
نوع 1

$$I = \iint_D e^{x^2} dA$$

$$= \int_0^1 \int_0^{2x} e^{x^2} dy dx = \int_0^1 2x e^{x^2} dx = e^{x^2} \Big|_0^1$$

$$= e^1 - e^0 = e - 1$$

* Theorem:- Let S be a solid (جسم) bounded from above by the surface $z = f_1(x, y)$

and from below by the surface $z = f_2(x, y)$

Let D be the projection of S on the xy -plane

The volume (حجم) of S is $V = \iint_D [f_1 - f_2] dA$

E.x) set up (but do not evaluate) the volume as a double integral of the volume of the tetrahedron (مخروط) bounded by the planes $x+2y+z=2$, $x=2y$, the xy -plane and the yz -plane

Sol:-

surfaces :-

$$z = 2 - x - 2y, \quad z = 0$$

region D :- $x=2y, x=0, 2-x-2y=0$ $\rightarrow$ $z=0$ $\rightarrow$ xy -plane

$$y = \frac{x}{2}, \quad y = \frac{2-x}{2}$$

* $x=0, x=1$ $\rightarrow$ $z=0$ $\rightarrow$ xy -plane

$$0 \leq x \leq 1$$

اعتبار $(0,0) \rightarrow$ $z=0$ $\rightarrow$ xy -plane

في $z=2, z=0$ $\rightarrow$ $z=0$ $\rightarrow$ xy -plane

$$V = \iint_D [2 - x - 2y - 0] dA$$

$$= \int_0^1 \int_{x/2}^{(2-x)/2} (2-x-2y) dy dx$$

E.X) Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy -planes bounded by $y = 2x$, $y = x^2$

Sol :-

surface :- $z = x^2 + y^2$ (upper) , $z = 0$ (Lower)

D : $y = 2x$, $y = x^2$, $(x^2 + y^2 = 0 \Rightarrow \text{Pt.})$ fair is curve

Type 1: $dA = dy dx$

$$y = 2x, y = x^2$$

$$x \text{ var} \Rightarrow 2x = x^2 \Rightarrow x^2 - 2x = 0 \Rightarrow x = 0, x = 2$$

$$0 \leq x \leq 2$$

$$V = \iint_D [x^2 + y^2 - 0] dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2) dy dx$$

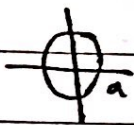
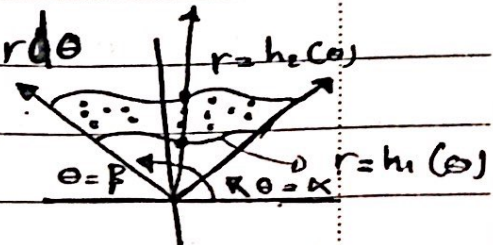
$$= \int_0^2 \left[x^2 y + \frac{y^3}{3} \right]_{x^2}^{2x} dx = \int_0^2 \left[(x^2 \cdot 2x + \frac{(2x)^3}{3}) - (x^2 x^2 + \frac{(x^2)^3}{3}) \right] dx$$

$$= \int_0^2 \left[2x^3 + \frac{8x^3}{3} - x^4 - \frac{x^6}{3} \right] dx = \dots$$

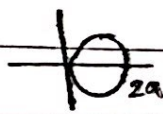
Sec (15.4) Double Integrals in polar coordinates.

* Let $D = \{(r, \theta) : h_1(\theta) \leq r \leq h_2(\theta), \alpha \leq \theta \leq \beta, \beta - \alpha \leq 2\pi\}$

$$\iint_D f(x, y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta$$



$$r = a$$



$$r = 2a \cos \theta$$



$$r = 2b \sin \theta$$

D: two sides of angle

$$dA = r dr d\theta$$

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned} \quad \left| \quad \begin{aligned} \frac{y}{x} &= \tan \theta \\ x^2 + y^2 &= r^2 \end{aligned} \right.$$

Ex) compute $I = \iint_D (3x + 4y^2) dA$ when D is the

region in the upper half plane bdd. by $x^2 + y^2 = 1$

$$x^2 + y^2 = 4$$

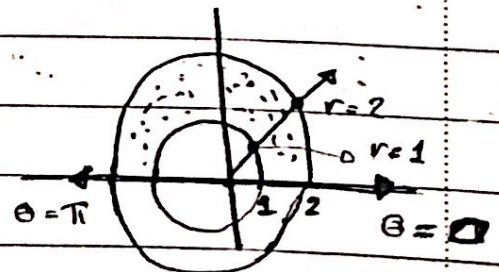
SOL:-

we use Type 3: polar, $dA = r dr d\theta$

$$I = \int_0^{\pi/2} \int_1^2 (3r \cos \theta + 4r^2 \sin^2 \theta) r dr d\theta$$

$$= \int_0^{\pi/2} \int_1^2 (3r^2 \cos \theta + 4r^3 \sin^2 \theta) dr d\theta$$

$$= \int_0^{\pi/2} \left[r^3 \cos \theta + r^4 \sin^2 \theta \right]_1^2 d\theta$$



$$I = \int_0^{\pi} [(8\cos\theta + 16\sin^2\theta) - (\cos\theta + \sin^2\theta)] d\theta$$

$$= \int_0^{\pi} (7\cos\theta + 15\sin^2\theta) d\theta$$

$$= \int_0^{\pi} (7\cos\theta + \frac{15}{2} - \frac{15}{2}\cos 2\theta) d\theta$$

$$= \left[7(\sin\theta) + \frac{15\theta}{2} - \frac{15}{2} \frac{\sin(2\theta)}{2} \right]_0^{\pi} = \dots$$

Ex) compute $I = \iint_R \arctan\left(\frac{y}{x}\right) dA$

$$R = \{ (x,y) : 1 \leq x^2 + y^2 \leq 4, 0 \leq y \leq \sqrt{3}x \}$$

SOL:-

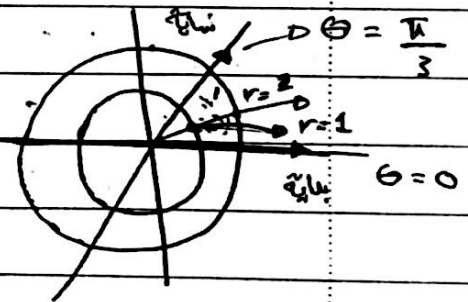
$$x^2 + y^2 = 1, x^2 + y^2 = 4, y = 0, y = \sqrt{3}x$$

Polar

$$I = \int_0^{\frac{\pi}{3}} \int_1^2 \tan^{-1}(\tan\theta) r dr d\theta$$

$$= \int_0^{\pi/3} \int_1^2 \theta r dr d\theta$$

$$= \left(\int_0^{\pi/3} \theta d\theta \right) \left(\int_1^2 r dr \right) = \dots$$



$$\theta \Rightarrow \sqrt{3}x = y$$

$$\frac{y}{x} = \sqrt{3}$$

$$\tan\theta = \sqrt{3}$$

$$\therefore \theta = \frac{\pi}{3}$$

E.x) Find the Volume of the solid bdd. by

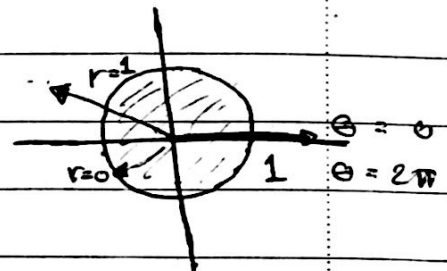
$$Z = 1, Z = 2 - x^2 - y^2$$

Sol:-

Surfaces: $Z = 1, Z = 2 - x^2 - y^2$

region D: $x^2 + y^2 = 1$

Type 3: $dA = r dr d\theta$



$$V = \iint_D [2 - x^2 - y^2 - 1] dA$$

$$= \iint_D (1 - (x^2 + y^2)) dA$$

$$= \int_0^{2\pi} \int_0^1 (1 - r^2) r dr d\theta$$

$$= \left(\int_0^{2\pi} d\theta \right) \left(\int_0^1 (r - r^3) dr \right) = \dots$$

$$= 2\pi \left(\frac{r^2}{2} - \frac{r^4}{4} \right) \Big|_0^1 = \frac{\pi}{2}$$

E.x) Evaluate

(1) $I = \int_0^3 \int_0^{\sqrt{9-y^2}} \sin(x^2 + y^2) dx dy$

(2) $I = \int_0^1 \int_x^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2} dy dx$

Sol: ① $D: X=0 \rightarrow X=\sqrt{9-y^2}$

also $0 \leq y \leq 3$

$x^2 + y^2 = 9$

$$I = \iint_D \sin(x^2 + y^2) \, dA$$

$$I = \int_0^{\pi/2} \int_0^3 \sin(r^2) r \, dr \, d\theta$$

$$= \frac{\pi}{2} \int_0^3 r \sin(r^2) \, dr = \frac{\pi}{4} \int_0^3 2r \sin r^2 \, dr$$

$$= \frac{\pi}{4} (-\cos r^2) \Big|_0^3 = \frac{\pi}{4} (1 - \cos 9)$$



② $y=x \rightarrow y=\sqrt{2x-x^2}$

$0 \leq x \leq 1$

$y^2 = 2x - x^2 \Rightarrow x^2 - 2x - y^2 = 0$

$x^2 - 2x + 1 + y^2 = 1$

$(x-1)^2 + y^2 = 1$

Intersection:-

$x = \sqrt{2x-x^2}$

$x^2 = 2x - x^2$

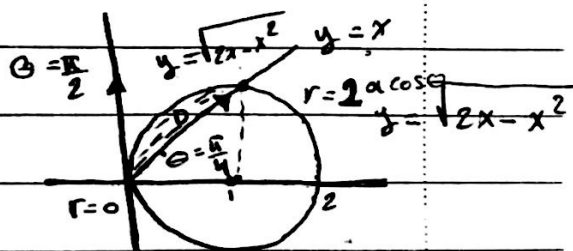
$2x^2 = 2x$

$x^2 = x$

$x = 0, 1$

$I = \iint_D \sqrt{x^2 + y^2} \, dA \quad dA = r \, dr \, d\theta$

$$I = \int_{\pi/4}^{\pi/2} \int_0^{2\cos\theta} r^2 \, dr \, d\theta = \dots$$



$y = x$

$\frac{y}{x} = 1$

$\tan \theta = 1$

$\theta = \frac{\pi}{4}$

Ex) compute :-

$$(1) I_1 = \int_0^{\infty} e^{-x^2} dx$$

$$(2) I_2 = \int_{-\infty}^{\infty} e^{-x^2} dx$$

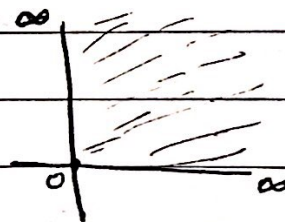
Sol :-

$$(1) I_1^2 = I_1 \cdot I_1 = \left(\int_0^{\infty} e^{-x^2} dx \right) \left(\int_0^{\infty} e^{-x^2} dx \right)$$

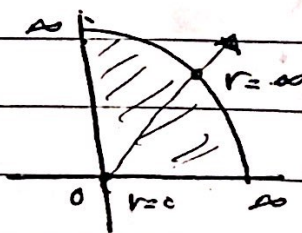
$$= \left(\int_0^{\infty} e^{-x^2} dx \right) \left(\int_0^{\infty} e^{-y^2} dy \right)$$

$$= \int_0^{\infty} \int_0^{\infty} e^{-x^2} e^{-y^2} dx dy = \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy$$

$$= \iint_D e^{-(x^2+y^2)} dA$$



$$= \int_0^{\pi/2} \int_0^{\infty} e^{-r^2} r dr d\theta$$



$$= \left(\int_0^{\pi/2} d\theta \right) \left(\int_0^{\infty} r e^{-r^2} dr \right)$$

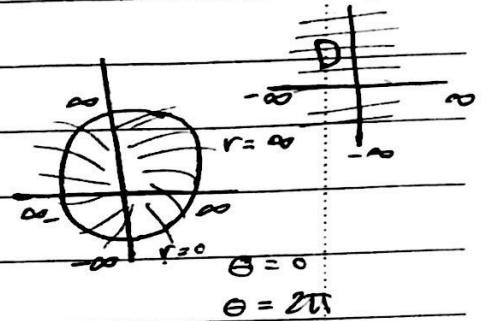
$$= \frac{\pi}{2} \cdot \left[-\frac{1}{2} e^{-r^2} \right]_0^{\infty} = \frac{\pi}{4} [e^{-\infty} - e^{-0}] = \frac{\pi}{4} [0 - 1]$$

$$= \frac{\pi}{4} \quad \therefore I_1 = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$$

$$(2) I_2^2 = I_2 \cdot I_2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy \right) \\ = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right) \cdot \left(\int_{-\infty}^{\infty} e^{-y^2} dy \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy$$

$$= \iint_D e^{-(x^2+y^2)} dA$$

$$= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta$$



$$= \left(\int_0^{2\pi} d\theta \right) \cdot \left(\int_0^{\infty} r e^{-r^2} dr \right) = +2\pi \cdot \left[-\frac{1}{2} e^{-r^2} \right]_0^{\infty}$$

$$= -\pi [0 - 1] = \pi \quad \therefore I_2 = \sqrt{\pi}$$

* Rule :- The area of the region D is $A = \iint_D 1 dA$

E.x) Find the area of the region D :

(1) bdd. by $x^2 + y^2 = 4$, $y = x$ and in the 3rd quadrant.

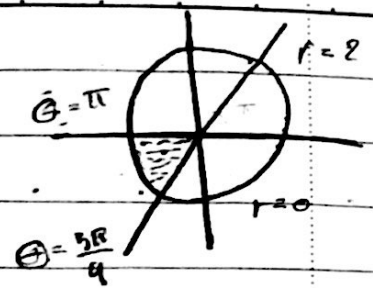
(2) enclosed by $y = x$, $x = 1$, the x-axis.

sol:-

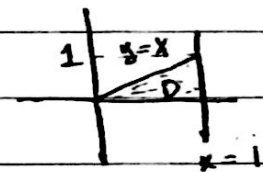
$$(1) A = \iint_D 1 \, dA$$

$$= \int_{\pi}^{\frac{5\pi}{4}} \int_0^2 r \, dr \, d\theta = \left(\frac{5\pi}{4} - \pi \right) \left[\frac{r^2}{2} \right]_0^2$$

$$= \frac{\pi}{4} \cdot \frac{4}{2} = \frac{\pi}{2}$$



$$(2) A = \frac{1}{2} \cdot (1) (1) = \frac{1}{2}$$



* sec (15.7) Triple Integrals.

$$dv = dz \, dy \, dx, \quad dv = dz \, dx \, dy, \quad dv = dy \, dx \, dz$$

$$dv = dy \, dz \, dx, \quad dv = dx \, dy \, dz, \quad dv = dx \, dz \, dy$$

* Fubini's thm: Let $f(x, y, z)$ be conts. on a rectangular box

$$B = \{ (x, y, z) : a \leq x \leq b, c \leq y \leq d, e \leq z \leq f \}$$

$$= [a, b] \times [c, d] \times [e, f]$$

$$\iiint_B f \, dv = \int_a^b \int_c^d \int_e^f f \, dz \, dy \, dx$$

E.X) compute $I = \iiint_S x y z^2 dv$

when $S = [-2, 3] \times [1, 2] \times [-1, 1]$

Sol:-

$$I = \int_{-1}^1 \int_{-2}^3 \int_1^2 x y z^2 dy dx dz = \left(\int_1^2 y dy \right) \left(\int_{-2}^3 x dx \right) \left(\int_{-1}^1 z^2 dz \right)$$

* Theorem: Let S be a solid such that:-

(1) $g_1(x, y) \leq z \leq g_2(x, y)$ and D is the projection of S on the xy -plane $\Rightarrow \iiint_S f dv = \iint_D \left[\int_{g_1}^{g_2} f dz \right] dA$

(2) $g_1(x, z) \leq y \leq g_2(x, z)$ and D is the projection of S on the xz -plane $\Rightarrow \iiint_S f dv = \iint_D \left[\int_{g_1}^{g_2} f dy \right] dA$

(3) $g_1(y, z) \leq x \leq g_2(y, z)$ and D is the projection of S on the yz -plane $\Rightarrow \iiint_S f dv = \iint_D \left[\int_{g_1}^{g_2} f dx \right] dA$

E.X) Evaluate $I = \iiint_E z dv$, where E is the solid

tetrahedron bdd by $x=0, y=0, z=0, x+y+z=1$

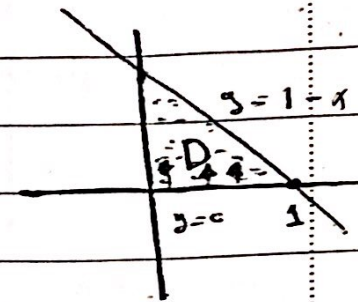
Sol:-

surfaces:- $z=0, z=1-x-y$

region D :- $x=0, y=0, x+y=1$ $\rightarrow$ triangle

$$= \iint_D \left[\int z \, dz \right] dA$$

D is



* $\int z \, dz$ is the volume element dz at height z
 $z=0$ to $z=1$ (0,0) is the origin

$$\Rightarrow I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2} \right]_0^{1-x-y} dy \, dx = \frac{1}{2} \int_0^1 \int_0^{1-x} (1-x-y)^2 dy \, dx$$

$$= \frac{1}{2} \int_0^1 \left[\frac{(1-x-y)^3}{3(-1)} \right]_0^{1-x} dx = -\frac{1}{6} \int_0^1 (1-x-1+x)^3 - (1-x-0)^3 dx$$

$$= -\frac{1}{6} \int_0^1 (-x)^3 - (1-x)^3 dx = -\frac{1}{6} \left[\frac{-x^4}{4} - \frac{(1-x)^4}{4} \right]_0^1 = -\frac{1}{24} [(-1)^4 - (1)^4]$$

$$= \frac{1}{24}$$

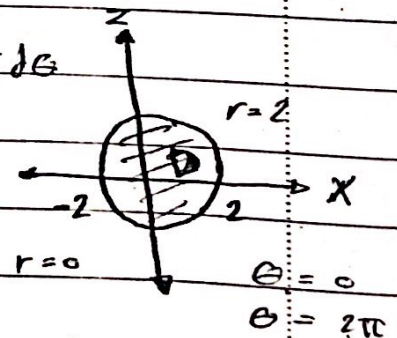
E.x) compute $I = \iiint_T \sqrt{x^2 + z^2} \, dv$ where T is the

region bdd by the paraboloid $y = x^2 + z^2$ and $y = 4$
 sol:-

surfaces :- $y = x^2 + z^2$, $y = 4 \Rightarrow dv = dy \, dA$

region :- $x^2 + z^2 = 4$

$$dA = r \, dr \, d\theta$$



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$$I = \iint_D \left[\int_{x^2+z^2}^4 [x^2+z^2] dy \right] dA$$

في اعتبار اسطوانة نصف (0,0)

$$y = 0^2 + 0^2 = 0$$

$$y = 4$$

$$= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r \, dy \, dr \, d\theta$$

الزوايا x, z

$$= \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r^2 \, dy \, dr \, d\theta = \dots$$

* Rule :- The volume of the solid S is

$$V = \iiint_S 1 \, dv$$

E.x) set up (but do not evaluate) the volume as a triple integral of the solid enclosed by

$$x + 2y + z = 2, \quad x = 2y, \quad x = 0, \quad z = 0$$

Sol :-

$$\text{surfaces :- } z = 2 - 2y - x, \quad z = 0 \Rightarrow dv = dz \, dA$$

$$\text{region D :- } x = 2y, \quad x = 0, \quad 2 - x - 2y = 0$$

$$V = \iiint_S 1 \, dv$$

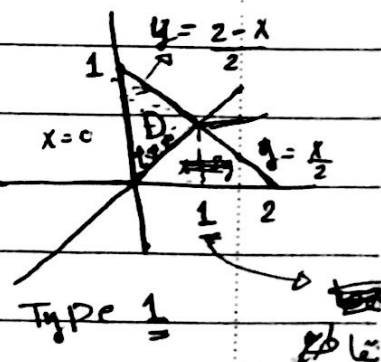
$$= \iint_D \left[\int_0^{2-x-2y} 1 \, dz \right] dA$$

$$\frac{2-x}{2} = \frac{x}{2}$$

$$2-x = x$$

$$2x = 2$$

$$x = 1$$



$$= \int_0^1 \int_{\frac{x}{2}}^{\frac{2-x}{2}} \int_0^{2-x-2y} 1 \, dz \, dy \, dx$$

* sec 15.8 : Triple Integrals in cylindrical coordinates

$A(x, y, z)$

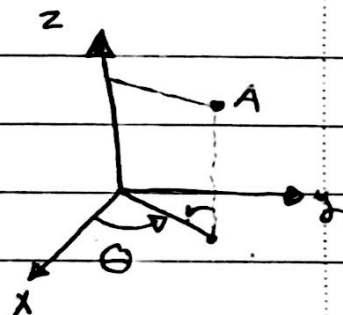
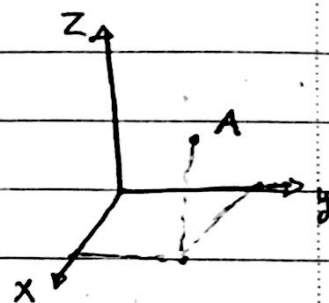
* Let ~~$A(x, y, z)$~~ be a pt. in rectangular coordinates

* The cylindrical coordinates of

A are $A(r, \theta, z)$

$$r = \sqrt{x^2 + y^2} \Leftrightarrow r^2 = x^2 + y^2$$

$$\tan \theta = \frac{y}{x}, \quad x = r \cos \theta, \quad y = r \sin \theta$$



E.X) Let $A(3, -3, 7)$ be in rectangular coord.

(1) Find the cylindrical coord. of A

(2) Sketch the pt. A showing its cylindrical coordinates on the graph

sol:-

$$x = 3, \quad y = -3, \quad z = 7$$

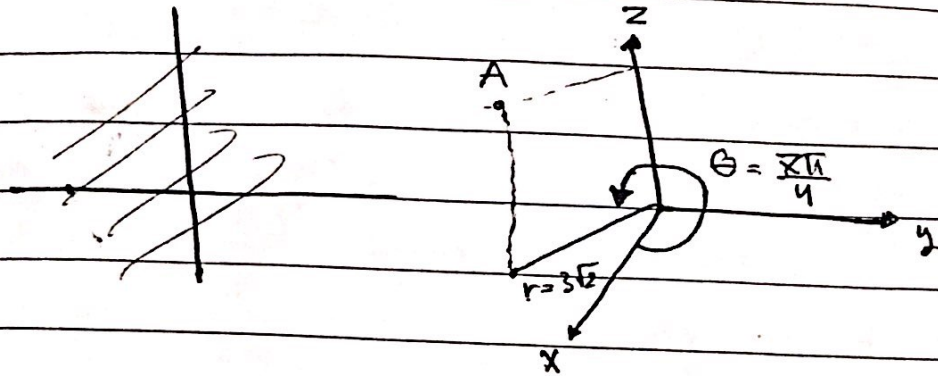
$$(1) \quad r = \sqrt{3^2 + (-3)^2} = \sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}$$

$$\tan \theta = \frac{y}{x} = \frac{-3}{3} = -1 \Rightarrow \theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

$$(2) \quad \text{Sketch of } A \text{ showing cylindrical coordinates}$$

$$\sim A(3\sqrt{2}, \frac{7\pi}{4}, 7)$$

(2)



Ex) If $A(2, \frac{\pi}{6}, -2)$ is in cylindrical coordinates find the rectangular coord. of A

sol:-

$$r = 2, \theta = \frac{\pi}{6}, z = -2$$

$$x = r \cos \theta = 2 \cos \frac{\pi}{6} = 2 \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$y = r \sin \theta = 2 \sin \frac{\pi}{6} = 2 \cdot \frac{1}{2} = 1$$

$$\sim A(\sqrt{3}, 1, -2)$$

rect.

cyl.

$$(-2, 0, 1) \rightarrow (2, \pi, 1)$$

$$(2, 0, 1) \rightarrow (2, 0, 1)$$

$$(0, 2, 1) \rightarrow (2, \frac{\pi}{2}, 1)$$

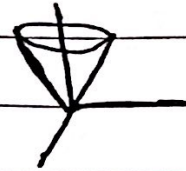
$$(0, -2, 1) \rightarrow (2, \frac{3\pi}{2}, 1)$$

Ex) Describe the surface whose eq. is given in cylindrical coord. by (1) $z = r$ (2) $r = 5$ (3) $\theta = \frac{\pi}{4}$

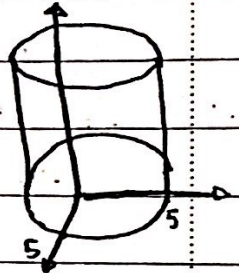
(4) $\theta = 0$ (5) $\theta = \frac{\pi}{2}$

Sol:-

(1) $z = r \Rightarrow z = \sqrt{x^2 + y^2} \Rightarrow \text{cone}$



(2) $r = 5 \Rightarrow \sqrt{x^2 + y^2} = 5 \Rightarrow x^2 + y^2 = 25$ cylinder



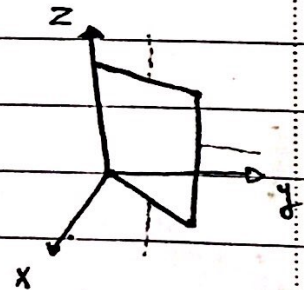
(3) $\theta = \frac{\pi}{4} \Rightarrow \tan \theta = \tan \frac{\pi}{4}$

$\frac{y}{x} = 1 \Rightarrow y = x$

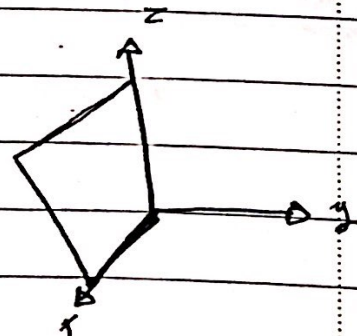
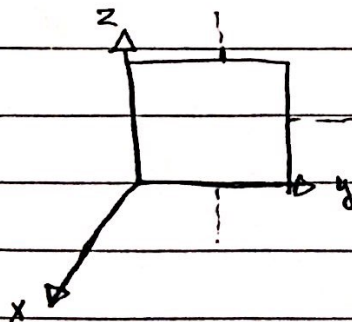
$\theta = \frac{\pi}{4}$ is a half-plane in the 1st octant

(4) $\theta = 0 \Rightarrow \tan \theta = \tan 0$

$\frac{y}{x} = 0 \Rightarrow y = 0$ is xz -plane when $x \geq 0$



(5) $\theta = \frac{\pi}{2}$ is the yz -plane $y \geq 0$



* In cylindrical coordinates $dv = dz dA$
 $dv = dz r dr d\theta$

E.x) Find the volume of the solid bdd by $z = x^2 + y^2$
 and $z = 9$

sol:-

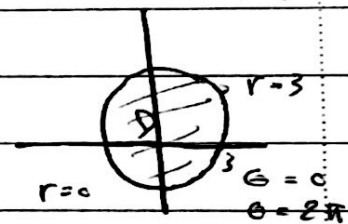
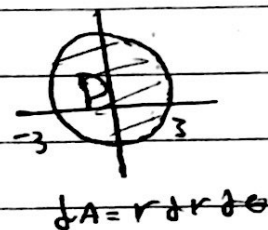
surfaces:- $z = x^2 + y^2$, $z = 9 \Rightarrow dv = dz dA$

region D:- $x^2 + y^2 = 9$

$$V = \iiint_S 1 dv$$

$$= \iint_D \left[\int_{x^2+y^2}^9 1 dz \right] dA$$

$$= \int_0^{2\pi} \int_0^3 \int_{r^2}^9 1 dz r dr d\theta$$



E.x) Evaluate

$$(1) I_1 = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 (x^2+y^2) dz dy dx$$

$$(2) I_1 = \int_{-2}^0 \int_{-\sqrt{4-x^2}}^0 \int_{x^2+y^2}^2 \sqrt{x^2+y^2} dz dy dx$$

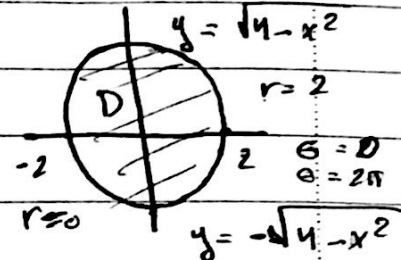
$$(3) I_3 = \int_0^2 \int_0^{\sqrt{4-y^2}} \int_{x^2+y^2}^4 \sqrt{x^2+y^2} dz dx dy$$

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sol:-

(1) region D: $y = -\sqrt{4-x^2}$, $y = \sqrt{4-x^2}$
 $-2 \leq x \leq 2$ by $\theta \rightarrow$ Type 1

$$I_1 = \iint_D \left[\int_{\sqrt{x^2+y^2}}^2 (x^2+y^2) dz \right] dA$$

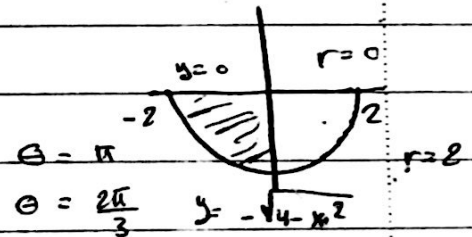


$$= \int_0^{2\pi} \int_0^2 \int_{\sqrt{x^2+y^2}}^2 r^2 dz r dr d\theta = \dots$$

$$dr = dz dA$$

(2) region D:- $y = -\sqrt{4-x^2}$, $y = 0$
 $-2 \leq x \leq 0$

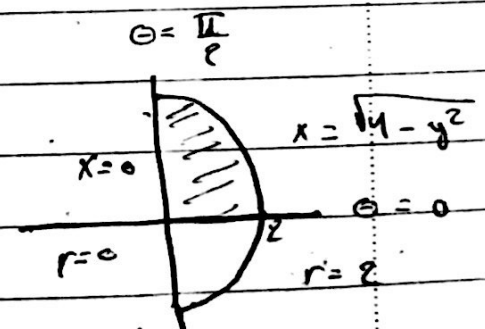
$$I_2 = \iint_D \left[\int_{\sqrt{x^2+y^2}}^2 \sqrt{x^2+y^2} dz \right] dA$$



$$= \int_{\pi}^{3\pi/2} \int_0^2 \int_{\sqrt{x^2+y^2}}^2 r dz r dr d\theta = \dots$$

(3) region D:- $x=0$, $x = \sqrt{4-y^2}$
 $0 \leq y \leq 2$

$$I_3 = \iint_D \left[\int_{\sqrt{x^2+y^2}}^4 \sqrt{x^2+y^2} dz \right] dA$$



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$$= \int_0^{\pi/2} \int_0^2 \int_{r^2}^4 r^2 dz dr d\theta = \dots$$

* sec (15.9) Triple Integrals in spherical coordinates.

* Let $A(x, y, z)$ be a pt. in rectangular coordinates. The spherical coordinates of A are: $A(\rho, \theta, \phi)$

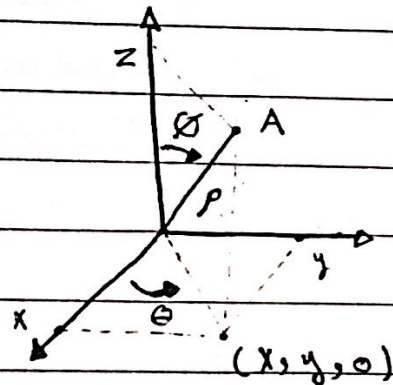
where $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$

$$0 \leq \theta \leq 2\pi : \tan \theta = \frac{y}{x}$$

$$0 \leq \phi \leq \pi \quad / \quad r = \rho \sin \phi$$

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi \Rightarrow \cos \phi = \frac{z}{\rho}$$



Ex) If $A(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 1)$, $B(0, 2\sqrt{3}, -2)$

$C(0, 0, -2)$, $D(0, 0, 3)$ are in rectangular coord.

(1) Find their spherical coord. (2) plot the pt.

A and on your graph show its spherical coord.

sol:-

(1) A : $x = -\frac{\sqrt{3}}{2}$, $y = \frac{\sqrt{3}}{2}$, $z = 1$

$$\rho = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + 1^2} = \sqrt{\frac{3}{4} + \frac{3}{4} + 1} = 2$$

$$\tan \theta = \frac{y}{x} = \frac{\sqrt{3}/2}{-\sqrt{3}/2} = -1$$

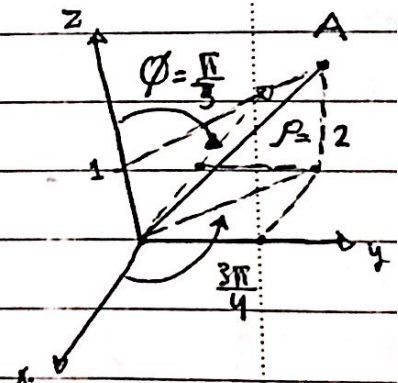
$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\begin{matrix} \text{④} & \text{②} \\ x & y \end{matrix}$

$x < 0$
 $y > 0$

$$\cos \phi = \frac{z}{\rho} = \frac{1}{2} \Rightarrow \phi = \frac{\pi}{3}$$

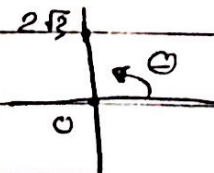
∴ A $(2, \frac{3\pi}{4}, \frac{\pi}{3})$



B : $x = 0$, $y = 2\sqrt{3}$, $z = -2$

$$\rho = \sqrt{0 + 12 + 4} = 4$$

$$\tan \theta = \frac{y}{x} = \frac{2\sqrt{3}}{0} \Rightarrow \text{undefined}$$



$$\cos \phi = \frac{z}{\rho} = \frac{-2}{4} = -\frac{1}{2}$$

$\phi = \frac{2\pi}{3}$

$\theta = \frac{\pi}{2}$

$$\phi = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \quad \text{or} \quad \phi = \cos^{-1}\left(-\frac{1}{2}\right)$$

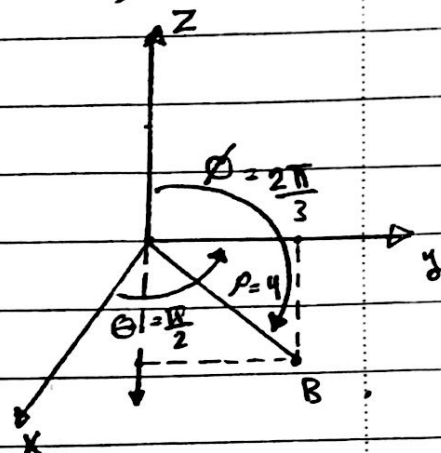
$$\therefore B \left(4, \frac{\pi}{2}, \frac{2\pi}{3}\right) \quad \text{or} \quad \left(4, \frac{\pi}{2}, \cos^{-1}\left(-\frac{1}{2}\right)\right)$$

C:- $x=0, y=0, z=-2$

$$\rho = \sqrt{4} = 2$$

$$\tan \theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1} \frac{0}{0} = ??$$

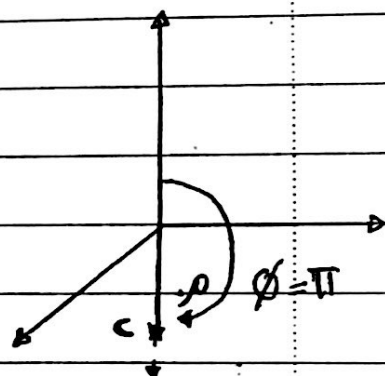
$(0,0)$ $\hat{i} \hat{j} \hat{k}$ θ



$$\therefore \theta \in [0, 2\pi]$$

$$\cos \phi = -\frac{2}{2} \Rightarrow \phi = \cos^{-1}(-1) \Rightarrow \phi = \pi$$

$$\therefore C (2, \theta, \pi) \quad \text{and} \quad \theta \in [0, 2\pi]$$



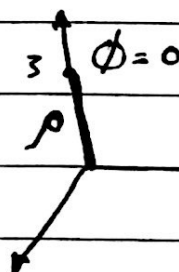
D:- $x=0, y=0, z=3$

$$\rho = \sqrt{9} = 3$$

$$\tan \theta = \frac{y}{x} = \frac{0}{0} \Rightarrow ?? \quad \text{and} \quad \theta \in [0, 2\pi]$$

$$\cos \phi = \frac{3}{3} = 1 \Rightarrow \phi = \cos^{-1}(1) = \phi = 0$$

$$\therefore D (3, \theta, 0) \quad \text{and} \quad \theta \in [0, 2\pi]$$



E.X) convert the eq. into rectangular coord.

(1) $\rho = 5$ (2) $\rho = \sin \phi \sin \theta$

(3) $\phi = 0$ (4) $\phi = \pi$

(5) $\phi = \frac{\pi}{2}$ (6) $\frac{2\pi}{3}$

Sol :-

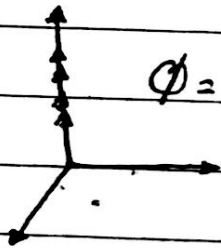
(1) $\rho = 5 \Rightarrow \sqrt{x^2 + y^2 + z^2} = 5 \Rightarrow x^2 + y^2 + z^2 = 25$

(2) $\rho = \sin \phi \sin \theta \Rightarrow \rho^2 = \rho \sin \phi \sin \theta$

$x^2 + y^2 + z^2 = y \Rightarrow x^2 + y^2 - y + z^2 = 0$

$x^2 + y^2 - y + \frac{1}{4} + z^2 = \frac{1}{4} \Rightarrow x^2 + (y - \frac{1}{2})^2 + z^2 = \frac{1}{4}$
 sphere centered at $(0, \frac{1}{2}, 0)$ of radius $\frac{1}{2}$

(3)



$\phi = 0 \Rightarrow$ positive Z-axis

$\cos \phi = \cos 0 \Rightarrow \frac{z}{\rho} = 1 \Rightarrow z = \rho$

$z^2 = x^2 + y^2 + z^2 \Rightarrow x^2 + y^2 = 0$

(4) $\phi = \pi$ negative Z-axis

(5) $\phi = \frac{\pi}{2}$ xy-plane

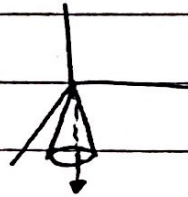
$\cos \phi = \cos \frac{\pi}{2} \Rightarrow \frac{z}{\rho} = 0 \Rightarrow z = 0$

$$6) \cos \phi = \cos \frac{2\pi}{3} \Rightarrow \frac{z}{\rho} = -\cos \frac{\pi}{3} \Rightarrow \frac{z}{\rho} = -\frac{1}{2}$$

$$z = -\frac{1}{2} \rho \Rightarrow 2z = -\sqrt{x^2 + y^2 + z^2}$$

$$4z^2 = x^2 + y^2 + z^2 \Rightarrow 3z^2 = x^2 + y^2 \Rightarrow z^2 = \frac{x^2 + y^2}{3}$$

$$z = -\sqrt{\frac{x^2 + y^2}{3}}$$



Ex) convert the eq. into spherical coord.

$$(1) x^2 + y^2 + z^2 = a, \quad a > 0$$

$$(2) \sqrt{x^2 + y^2} = 2$$

$$(3) z^2 = x^2 + y^2$$

$$(4) z = -\sqrt{x^2 + y^2}$$

Sol:-

$$(1) \rho^2 = a \Rightarrow \rho = a$$

$$(2) r = 2$$

$$\rho \sin \phi = 2 \Rightarrow \rho = \frac{2}{\sin \phi} \Rightarrow \rho = 2 \csc \phi$$

$$(3) z^2 = r^2 \Rightarrow \rho^2 \cos^2 \phi = \rho^2 \sin^2 \phi \Rightarrow \rho^2$$

$$\cos^2 \phi = \sin^2 \phi \Rightarrow \tan^2 \phi = 1 \Rightarrow \tan \phi = 1 \text{ or } \tan \phi = -1$$

$$\phi = \frac{\pi}{4}$$

$$\phi = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$(4) z = -r \Rightarrow \rho \cos \phi = -\rho \sin \phi$$

$$\Rightarrow \tan \phi = -1 \Rightarrow \phi = \frac{3\pi}{4}$$

$$\sqrt{x^2+y^2} \Rightarrow \phi = \frac{\pi}{4} \quad / \quad -\sqrt{x^2+y^2} \Rightarrow \frac{3\pi}{4}$$

⇐ Debat

$$\sqrt{\frac{x^2}{3} + \frac{y^2}{3}} \Rightarrow \phi = \frac{\pi}{3} \quad / \quad -\sqrt{\frac{x^2}{3} + \frac{y^2}{3}} \Rightarrow \frac{2\pi}{3}$$

$$\sqrt{3x^2 + 3y^2} \Rightarrow \frac{\pi}{6} \quad / \quad -\sqrt{3x^2 + 3y^2} \Rightarrow \frac{5\pi}{6}$$

* Rule :- $dv = \rho \sin\phi \, d\rho \, d\theta \, d\phi$

origin

أرأس الجسم .

* ρ :- أطول شعاع داخل الجسم من الأصل O .

* ϕ :- الزاوية بين شعاع ρ والسطح عند دخول الشعاع الجسم .

* θ :- الانحناء بين شعاع ρ والسطح عند خروج الشعاع من الجسم .

* θ :- أنسب الجسم داخل xy -plane المنطبق التي ملئت .

* إذا كانت الجسم z فإن $\theta \in [0, 2\pi]$

origin

* ϕ :- زوايا شعاع منطبق من O إلى X و Y (الوجه z) حركة شعاع ليصل

رأس الجسم .

ϕ :- الزاوية بين شعاع ρ والسطح عند دخول الشعاع الجسم

θ :- الزاوية بين شعاع ρ والسطح عند خروج الشعاع من الجسم



E.X) Find the Volume of the solid bdd. by

$$z = -\sqrt{3x^2 + 3y^2} \text{ and } x^2 + y^2 + z^2 = 16$$

Sol :-

inside

from above

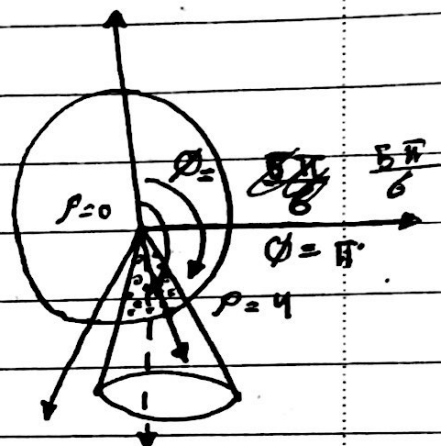
spherical

$$z = -\sqrt{3x^2 + 3y^2} \Rightarrow \phi = \frac{5\pi}{6}$$

$$x^2 + y^2 + z^2 = 16 \Rightarrow \rho = 4$$

$$V = \iiint 1 \, dv$$

$$= \int_{\frac{5\pi}{6}}^{\pi} \int_0^{2\pi} \int_0^4 \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \dots$$



$$\rho : 0 \rightarrow 4$$

$$\theta : 0 \rightarrow 2\pi$$

$$\phi : \frac{5\pi}{6} \rightarrow \pi$$

E.X) compute $I = \iiint_G e^{-(x^2+y^2+z^2)^{3/2}} \, dv$

where G is the region bdd between $x^2 + y^2 + z^2 = 4$

$x^2 + y^2 + z^2 = 9$ and above the cone $z = \sqrt{x^2 + y^2}$

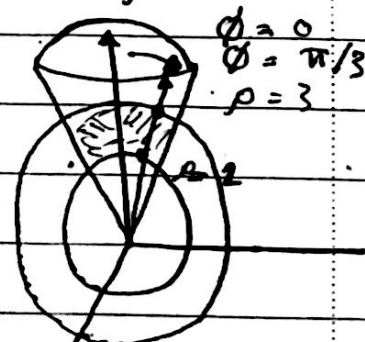
Sol :-

spherical

$$x^2 + y^2 + z^2 = 4 \Rightarrow \rho = 2$$

$$x^2 + y^2 + z^2 = 9 \Rightarrow \rho = 3$$

$$z = \sqrt{x^2 + y^2} \Rightarrow \phi = \frac{\pi}{3}$$



N O T E B O O K

$$\rho: 2 \rightarrow 3, \quad \theta: 0 \rightarrow 2\pi, \quad \phi: 0 \rightarrow \frac{\pi}{3}$$

$$I = \int_0^{\pi/3} \int_0^{2\pi} \int_2^3 e^{-\rho^2} \rho^{3/2} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi/3} \int_0^{2\pi} \int_2^3 \rho^2 e^{-\rho^3} \sin \phi \, d\rho \, d\theta \, d\phi$$

$$= \left(-\frac{1}{3} \int_2^3 -3\rho^2 e^{-\rho^3} \right) \left(\int_0^{2\pi} 1 \, d\theta \right) \left(\int_0^{\pi/3} \sin \phi \, d\phi \right)$$

$$= -\frac{1}{3} \left[e^{-\rho^3} \right]_2^3 \cdot (2\pi) \cdot (-\cos \phi) \Big|_0^{\pi/3}$$

Ex) convert to spherical coordinates

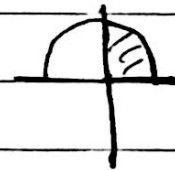
$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy \, dz \, dy \, dx$$

Sol :-

surfaces :- $z = \sqrt{x^2 + y^2}, \quad z = \sqrt{2 - x^2 - y^2}$

region D :- $y = 0 \rightarrow y = \sqrt{1 - x^2}$
 $0 \leq x \leq 1$

D : $x = 0, y = 0, y = \sqrt{1 - x^2}$



ملاحظة (ملاحظة) :-

$$\sqrt{x^2 + y^2} = \sqrt{2 - x^2 - y^2}$$

$$x^2 + y^2 = 2 - x^2 - y^2 \Rightarrow 2x^2 + 2y^2 = 2$$

$$x^2 + y^2 = 1$$

$$\therefore G: Z = \sqrt{x^2 + y^2}, \quad Z = \sqrt{2 - x^2 - y^2}, \quad x=0, y=0$$

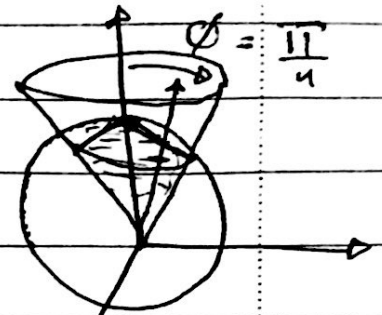
1st octant

$$\phi = \frac{\pi}{4}$$

$$\rho: 0 \rightarrow \sqrt{2}$$

$$\theta: 0 \rightarrow \pi/2$$

$$\phi: 0 \rightarrow \pi/4$$



$$I = \int_0^{\pi/4} \int_0^{\pi/2} \int_0^{\sqrt{2}} (\rho \sin \phi \cos \theta) (\rho \sin \phi \sin \theta) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

المساحة المطلوبة *

ch. 15 sec 15.2 (25) المساحة

المساحة (15) $\Rightarrow$ المساحة المطلوبة ch. 12 sec 12.6
ch. 13 Arc Length + Curvature
ch. 14 $\rightarrow$ sec. 14.6 + sec 14.7