



اللجنة الأكاديمية للهندسة المدنية

دفتر

كالكولاس 1

عمر ابو الرب

Contact us :

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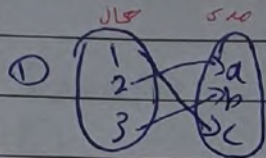
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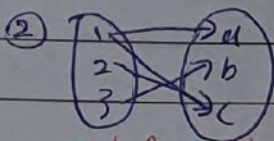


Lecture 1

example:



Function



not function

الافتراض (function) : هو علاقة تربط كل عنصر بالجال بعنصر واحد فقط بالترتيب

Domain = مجال = input = x

Range = المدى = output = y

$y = f(x)$ و $f(x)$ هو

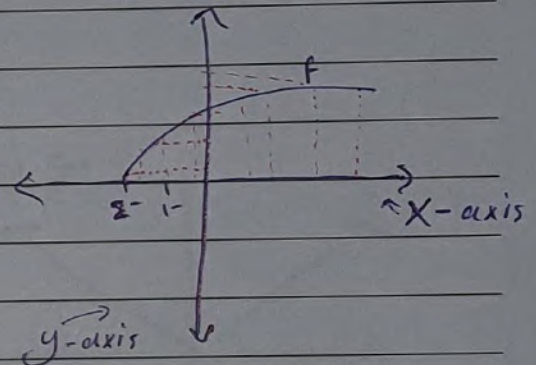
Example : $f(x) = \sqrt{x+2}$ $f(x) \in \mathbb{R}$

Find:

① $f(2) = \sqrt{2+2} = \sqrt{4} = 2$

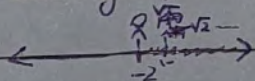
② $f(-4) = \sqrt{-4+2} = \sqrt{-2}$ undefined

③ $f(m+3) = \sqrt{m+3+2} = \sqrt{m+5}$



④ Domain $f = [-2, \infty)$

⑤ Range $f = [0, \infty)$



من 0 إلى

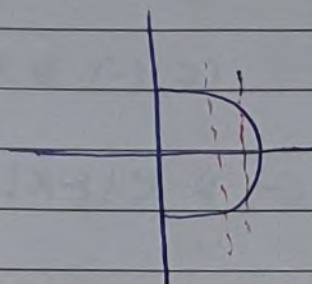
لا يحدد المجال من الرسم نقطة
رسم الافتراض ان كل عنصر / البيان
لا يحدد المجال من الرسم
نقطة الافتراض ان كل عنصر / البيان

Lecture 2

اختبار الخط العمودي : the vertical line test :

Example :

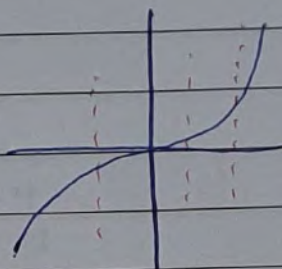
①



not function

نقطة واحدة
بأكثر من نقطة
من خط عمودي واحد
→ (نقطة)

②



function

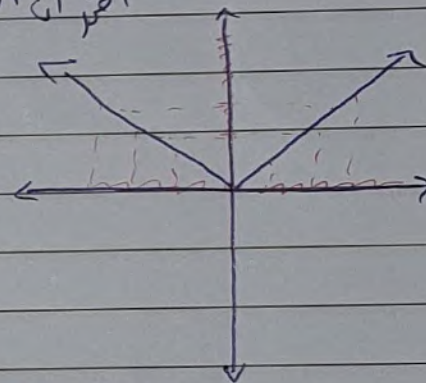
اختبار القيمة المطلقة : the absolute value function :

$$f(x) = |x| = \begin{cases} -x & x \leq 0 \\ x & x > 0 \end{cases}$$

$$f(0) = |0| = 0$$

$$f(2) = |2| = 2$$

$$f(-2) = |-2| = 2$$



Domain : $(-\infty, \infty)$: Real numbers : $\mathbb{R}$ الأعداد الحقيقية

Range : $[0, \infty)$

1) $|ab| = |a||b|$

2) $|a/b| = |a|/|b|$

3) $\sqrt{x^2} = |x|$

4) $(\sqrt{x})^2 = x$

5) $|x| = a \rightarrow x = -a \text{ or } x = a$

6) $|x| \leq a \rightarrow -a \leq x \leq a$

7) $|x| \geq a \rightarrow x \geq a \text{ or } x \leq -a$

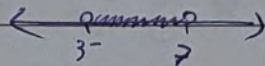
Example: Find the values of x

① $|x| \leq 3 \rightarrow x \leq 3, x \geq -3$

② $|x-2| < 5 \rightarrow \begin{matrix} 5- \\ +2 \end{matrix} < x-2 < \begin{matrix} 5 \\ +2 \end{matrix}$

$-3 < x < 7$

$x \in (-3, 7)$



③ $|x-3| \geq 4 \rightarrow \begin{matrix} x-3 \geq 4 \\ +3 \end{matrix}$

$\begin{matrix} x-3 \leq -4 \\ +3 \end{matrix}$

$x \geq 7$

$x \leq -1$

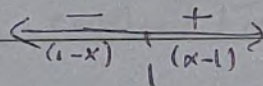
$x \in (-\infty, -1] \cup [7, \infty)$



Example: Rewrite (إعادة كتابة)

① $f(x) = |x-1|$

$x-1 \geq 0 \rightarrow x \geq 1$
 $\begin{matrix} + \\ +1 \end{matrix}$

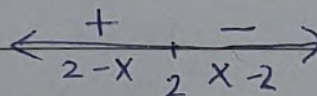


~~$f(x) = \begin{cases} 1-x, & x < 1 \\ x-1, & x \geq 1 \end{cases}$~~

$f(x) = \begin{cases} 1-x, & x < 1 \\ x-1, & x \geq 1 \end{cases}$

② $f(x) = |2-x|$

$2-x \geq 0 \rightarrow x \leq 2$



$f(x) = \begin{cases} 2-x, & x \leq 2 \\ x-2, & x > 2 \end{cases}$

Lecture 3

- Polynomials: متعددات الحدود

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 + a_1 + a_0 \quad \boxed{a_n \neq 0}$$

$$a_n x^n = 5x^2$$

$$a_n = 5 = \text{العدد الثابت}$$

$$a_n, a_{n-1}, \dots, a_2, a_1, a_0 \in \mathbb{R} \rightarrow (-\infty, \infty)$$

the number $a_n, a_{n-1}, a_{n-2}, \dots, a_2, a_1, a_0$ are constant called the coefficients of the polynomial المتغيرات

the degree of the polynomial is n

n is nonnegative integer عدد صحيح غير سالب

the domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$

nonnegative

غير سالب

real

integer

natural

nonpositive

سالب

Example:

$$① f(x) = 3x^5 + 2x - 4 = 3x^5 + 0x^4 + 0x^3 + 0x^2 + 2x - 4$$

polynomial with degree 5, constant term = -4 الحد الثابت

$$② f(x) = \sqrt{x} + ax^4 + 1 = x^{\frac{1}{2}} + ax^4 + 1$$

not polynomial

$$③ f(x) = ax^{-5} - 4$$

not polynomial

$$④ f(x) = a = ax^0$$

polynomial

$$⑤ f(x) = |x| + 5x + 7 = \begin{cases} -x + 5x + 7, & x \leq 0 \\ x + 5x + 7, & x > 0 \end{cases}$$

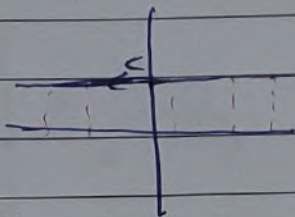
$$f(x) = \begin{cases} 4x + 7, & x \leq 0 \\ 6x + 7, & x > 0 \end{cases} \quad \text{not polynomial}$$

→
② Constant function;

$$f(x) = c$$

$$\text{Domain } f \subseteq \mathbb{R} = (-\infty, \infty)$$

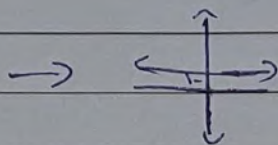
$$\text{Range } f \subseteq \{c\}$$



$$f(x) = 5$$

$$f(x) = 0$$

$$f(x) = -1$$



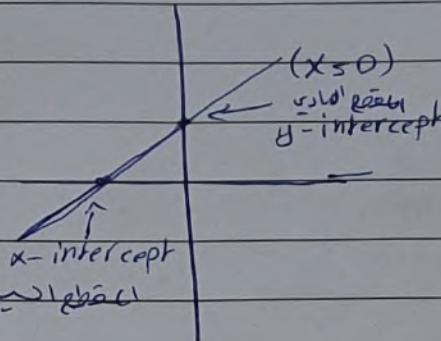
Lecture 4

- Linear Function:

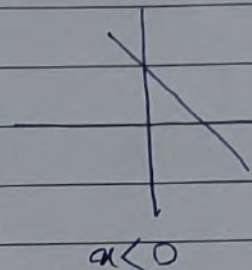
$$f(x) = ax + b$$

$$\text{Domain } f = \mathbb{R}$$

$$\text{Range } f = \mathbb{R} \quad (y = f(x) = 0)$$



$a > 0$



$a < 0$

Example: $f(x) = 2x + 4$

x-intercept ($y=0$)

$$0 = 2x + 4$$

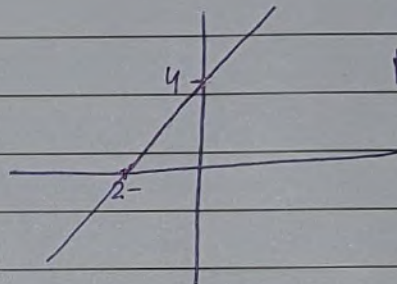
$$4 = 2x$$

$$2 = x$$

y-intercept ($x=0$)

$$y = 2(0) + 4$$

$$y = 4 = f(0)$$



$$\text{Domain} = \mathbb{R}$$

$$\text{Range} = \mathbb{R}$$

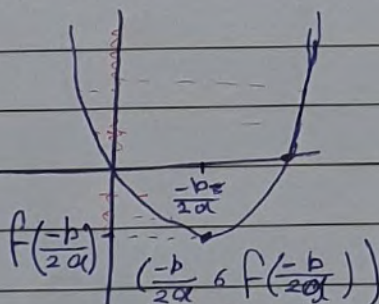
- Quadratic Function: $f(x) = ax^2 + bx + c$

$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

$$\text{Domain} = \mathbb{R}$$

If $a > 0$:

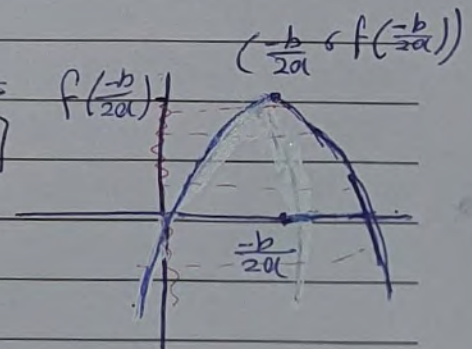
$$\text{Range } f = [f(\frac{-b}{2a}), \infty)$$



نقطة الصدارة

If $a < 0$

$$\text{Range } f = (-\infty, f(\frac{-b}{2a})]$$



→
Example:

① $f(x) \leq x^2 - 6x + 7$

$a \leq 1, b \leq -6, c \leq 7$

$a > 0 \rightarrow$ parabola opening upwards

Domain $f \leq \mathbb{R}$

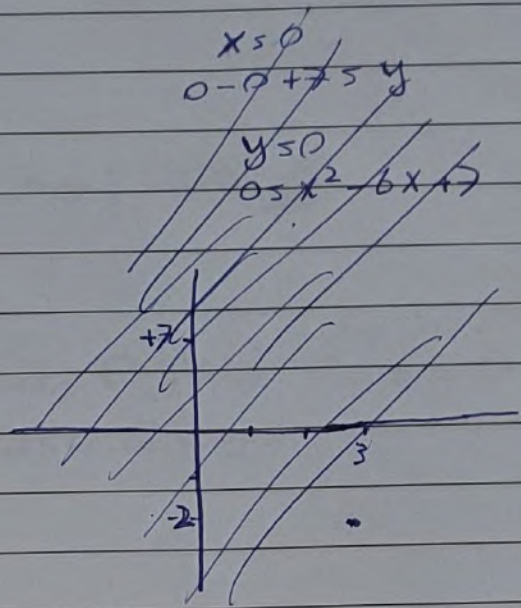
Range $f \leq [f(\frac{-b}{2a}), \infty)$

$\leq [f(\frac{-(-6)}{2(1)}), \infty)$

$\leq [f(3), \infty)$

$\leq [3^2 - 6(3) + 7, \infty)$

$\leq [-2, \infty)$



② $f(x) \leq 3 + 2x - x^2$

$a \leq -1 < 0, b \leq 2, c \leq 3$

Domain $f \leq \mathbb{R}$

Range $f \leq (-\infty, f(\frac{-b}{2a})]$

$\leq (-\infty, f(\frac{-2}{2(-1)})]$

$\leq (-\infty, f(1)]$

$\leq (-\infty, 3 + 2 - 1]$

$\leq (-\infty, 4]$

Lecture 5

- The quadratic expression: $\frac{1}{2}x^2 + 3x + \frac{1}{2}$

$$ax^2 + bx + c \leq 0 \quad a \neq 0$$

$ax^2 + bx + c = 0$ $a \neq 0$
 Discriminant $\Delta = b^2 - 4ac$

القانون العام

$$X_5 = \frac{-b \pm \sqrt{\Delta}}{2a}$$

Example: Find the value(s) of X :

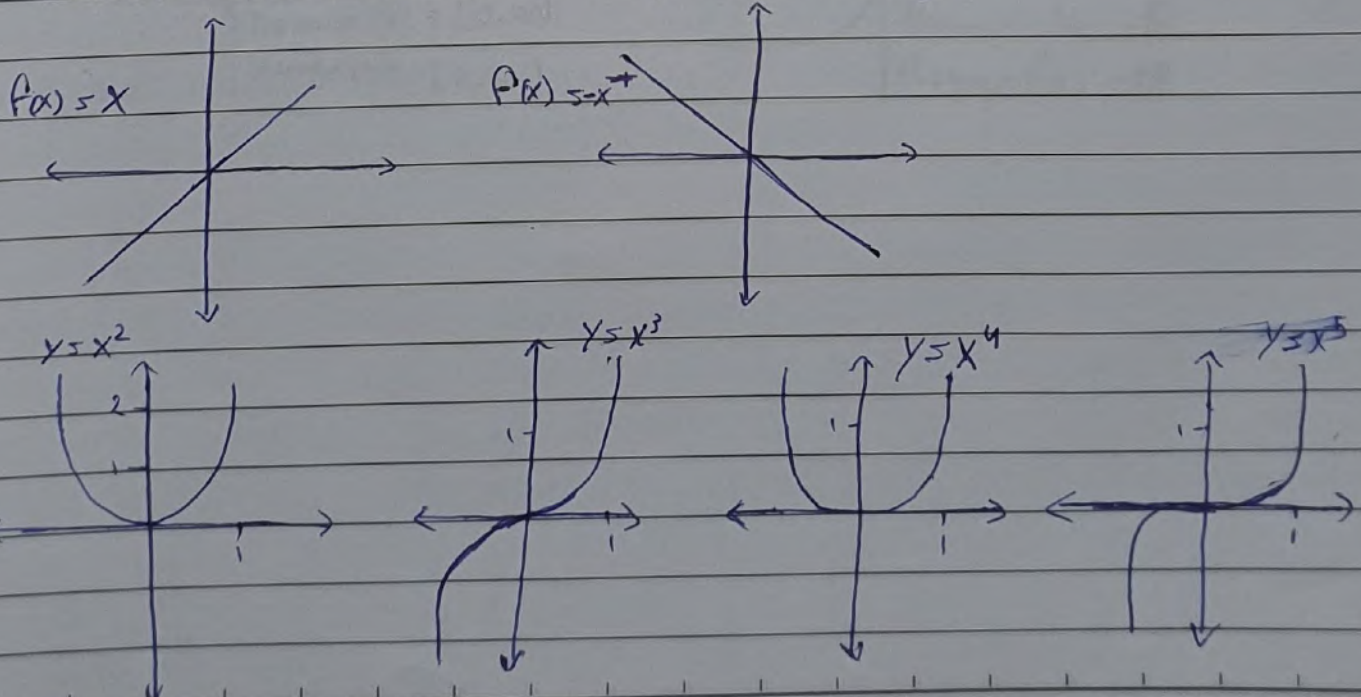
① $x^2 + x + 12 \leq 0 \rightarrow \Delta = 1^2 - (4)(1)(12) = 1 - 48 < 0$ $\Delta < 0$ has no solutions
as $a > 0$, $b > 0$, $c > 0$

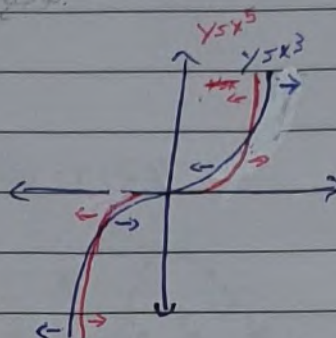
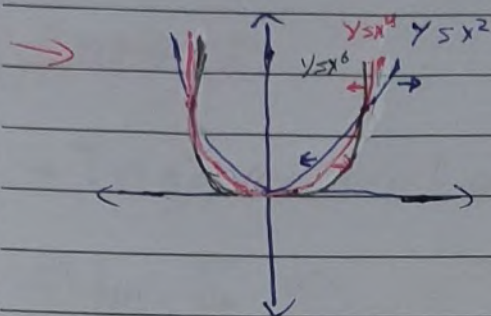
② $x^2 - 3x - 4 \leq 0$ $a=1, b=-3, c=-4$

$$\Delta = (-3)^2 - 4(1)(-4) = 9 + 16 = 25 > 0 \text{ ymplw}$$

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{3 \pm 5}{2} \rightarrow \begin{matrix} \frac{8}{2} s^4 \\ -\frac{2}{2} s^{-1} \end{matrix} \rightarrow x \in \{-1, 4\} : (x+1)(x-4)$$

— Power function: $f(x) = x^n$, $n = \text{positive integer}$



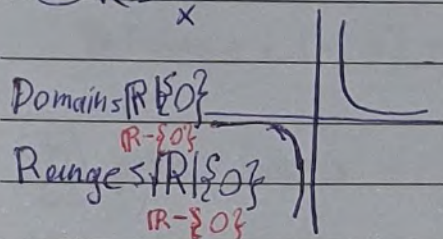


- Rational Function : $\frac{y}{x}$ ان النسبة

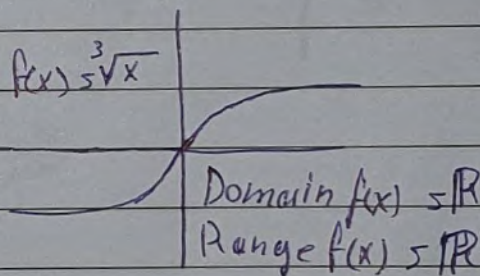
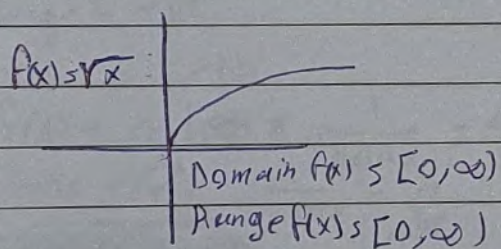
$$f(x) = \frac{Q(x)}{P(x)} = \frac{\text{كثير حدود}}{\text{كثير حدود}}$$

Example : ① $f(x) = \frac{x^2 + 5x}{x^2 - 6x + 1}$ Rational Function

② $f(x) = \frac{1}{x}$



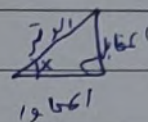
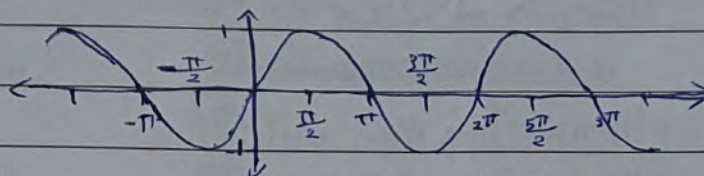
- Root function : $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$, n is positive integer



Lecture ②

- Trigonometric functions: الدوال المثلثية

① $f(x) = \sin x$ ^{جيب} _{الوتر}

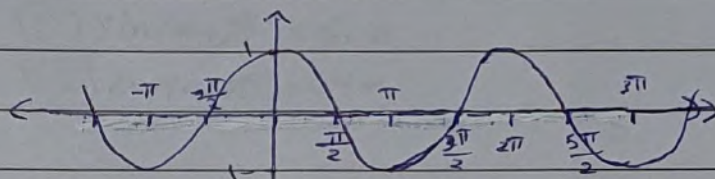


Domain $\mathbb{R}$

Range $f = [-1, 1] \rightarrow -1 \leq \sin x \leq 1 \rightarrow |\sin x| \leq 1$

② $f(x) = \cos x$ ^{جيب} _{الوتر}

Domain $\mathbb{R}$

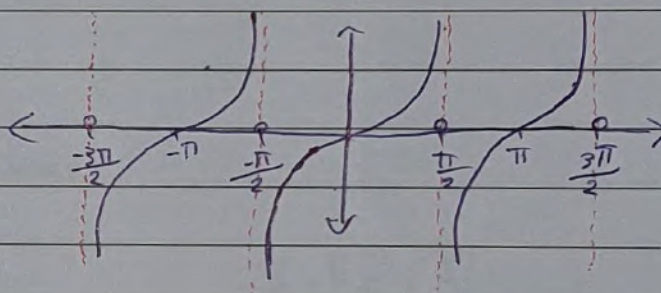


Range $f = [-1, 1] \rightarrow -1 \leq \cos x \leq 1 \rightarrow |\cos x| \leq 1$

③ $f(x) = \tan x = \frac{\sin x}{\cos x}$ ^{جيب} _{الوتر}

Domain $f \in \mathbb{R} \setminus \{\pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots\}$

Range $f = \mathbb{R}$



④ $f(x) = \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$ ^{جيب} _{الوتر}

⑤ $f(x) = \sec x = \frac{1}{\cos x}$ ^{جيب} _{الوتر}

~~⑥ $f(x) = \csc x = \frac{1}{\sin x}$ ^{جيب} _{الوتر}~~

⑦ $f(x) = \csc x = \frac{1}{\sin x}$ ^{جيب} _{الوتر}

المطابقان

$$1) \sin^2 \alpha + \cos^2 \alpha = 1$$

$$2) \sin(2\alpha) = 2 \sin \alpha \cos \alpha$$

$$3) \cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha$$

$$= \cos^2 \alpha - 1$$

$$= 1 - 2 \sin^2 \alpha$$

$$4) \sec^2 \alpha = 1 + \tan^2 \alpha$$

$$5) \csc^2 \alpha = 1 + \cot^2 \alpha$$

~~$$6) \sin^2 \alpha$$~~

$$6) \sin^2 \alpha = \frac{1}{2} (1 - \cos(2\alpha))$$

$$7) \cos^2 \alpha = \frac{1}{2} (1 + \cos(2\alpha))$$

~~$$8) \sin \alpha = \frac{1}{2} (1 + \cos(2\alpha))$$~~

$$8) \sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)]$$

$$9) \sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$10) \cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

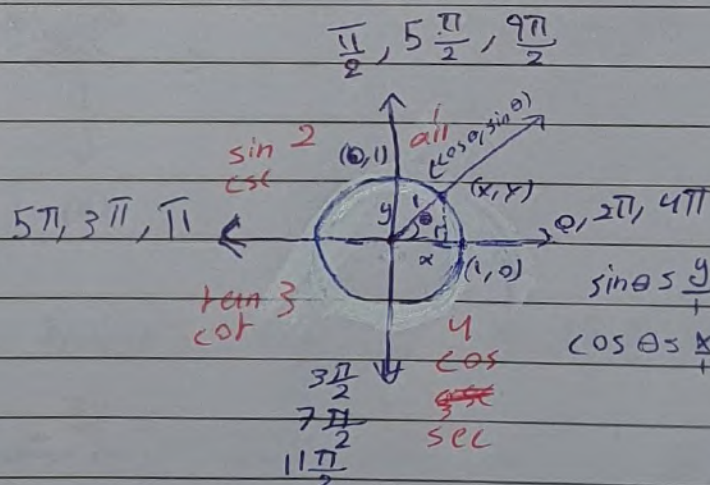
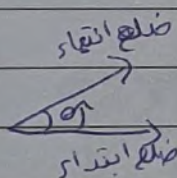
$$11) \sin(\alpha + 2\pi) = \sin \alpha$$

$$12) \cos(\alpha + \pi) = -\cos \alpha$$

Note: ^{ملاحظة}

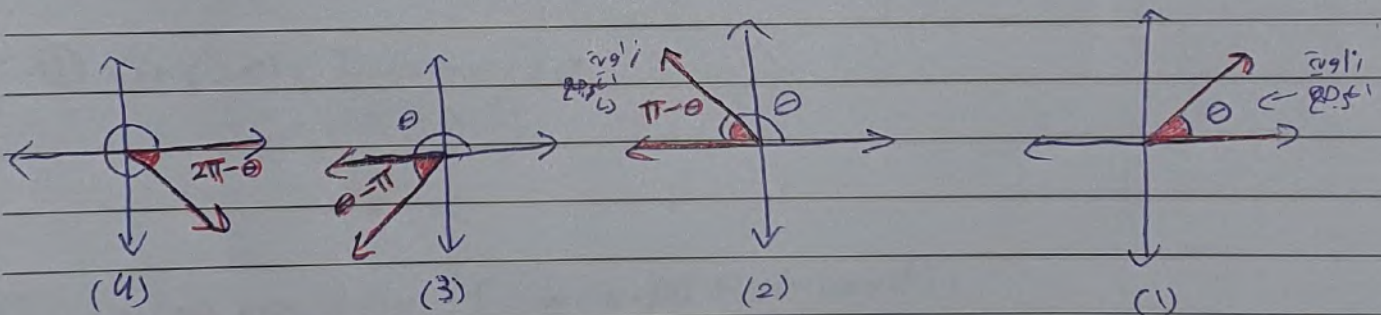
1) $\cos x$, $\sec x$ even functions ^{زوج}

2) $\sin x$, $\tan x$, $\cot x$, $\csc x$ odd functions ^{فرد}



Lecture 7

x	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\csc x$
$0, 2\pi$	0	1				
$90^\circ, \frac{\pi}{2}$	1	0				
$180^\circ, \pi$	0	-1				
$270^\circ, \frac{3\pi}{2}$	-1	0				
$30^\circ, \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$				
$45^\circ, \frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$				
$60^\circ, \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$				



⊗ زاوية 120° هي زاوية مارة مرسومة بين محور السينات وخط انقطاع الزاوية

للتحويل جديني $\frac{\pi}{180} \times$ / ار $\frac{180}{\pi} \times$ (جديني)

$$120^\circ \rightarrow 120 \cdot \frac{\pi}{180} = \frac{2\pi}{3}$$

$$3 \cdot \frac{\pi}{4} \rightarrow 3 \cdot \frac{\pi}{4} \cdot \frac{180}{\pi} = 135^\circ$$

$3 \times 45 = 135^\circ$

→ Example:

$$\textcircled{1} \cos\left(\frac{3\pi}{4}\right) = -\cos\left(\frac{4\pi}{4} - \frac{3\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right)$$

30 deg
45 deg

$$= -\frac{1}{\sqrt{2}}$$

$$\textcircled{2} \sin\left(11\frac{\pi}{6}\right) = -\sin\left(2\pi - 11\frac{\pi}{6}\right) = -\sin\left(\frac{12\pi}{6} - \frac{11\pi}{6}\right)$$

30 deg

$$= -\sin\frac{\pi}{6} = -\frac{1}{2}$$

$$\textcircled{3} \tan\left(4\frac{\pi}{3}\right) = \tan\left(\frac{4\pi}{3} - \frac{3\pi}{3}\right) = \tan\frac{\pi}{3} = \sqrt{3}$$

60 deg

H.W.:

$$\textcircled{1} \sin\theta = \frac{1}{2}, \cos\theta = \frac{\sqrt{3}}{2} \text{ Find } \sin(2\theta)$$

$$\textcircled{4} \text{ Find: } \textcircled{1} \sin\left(5\frac{\pi}{8}\right)$$

$$\textcircled{2} \text{ find } \sin(3x) \cos(5x)$$

$$\textcircled{2} \sin\left(3\frac{\pi}{4}\right) \textcircled{3} \cot\left(4\frac{\pi}{3}\right)$$

$$\textcircled{3} \text{ If } \sin\theta = \frac{1}{2}, \frac{\pi}{2} < \theta < \pi$$

$$\textcircled{4} \csc\left(5\frac{\pi}{4}\right) \textcircled{5} \cos\left(\frac{\pi}{4}\right)$$

$$\text{find } \cos\theta, \csc\theta, \sec\theta, \tan\theta, \cot\theta$$

$$\textcircled{5} \cos\left(11\frac{\pi}{6}\right) \textcircled{6} \sin^2\left(\frac{\pi}{2}\right)$$

$$\textcircled{1} \sin(2\theta) = 2\sin\theta \cos\theta$$

$$= 2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}}{2}$$

$$\textcircled{2} \sin(3x) \cos(5x) = \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)]$$

$$= \frac{1}{2} [\sin(-2x) + \sin(7x)]$$

$$\textcircled{3} \cos\theta = -\frac{\sqrt{3}}{2} / \csc\theta = 2 / \sec\theta = \frac{2}{\sqrt{3}} / \tan\theta = \frac{1}{\sqrt{3}} / \cot\theta = -\sqrt{3}$$

$$\textcircled{4} \textcircled{1} 5\frac{\pi}{6} = 5 \times 30 = 150$$

$$\sin 5\frac{\pi}{6} = \frac{1}{2}$$

$$\textcircled{2} \sin\left(5\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$\textcircled{3} 4\frac{\pi}{3} = 4 \times 60 = 240$$

$$\cot\left(4\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}$$

$$\textcircled{4} 5\frac{\pi}{4} = 5 \times 45 = 225$$

$$\csc\left(5\frac{\pi}{4}\right) = -\sqrt{2}$$

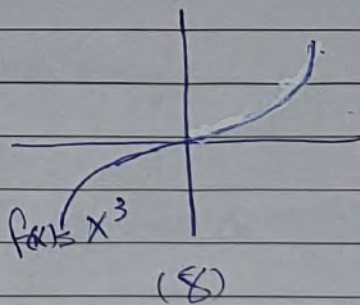
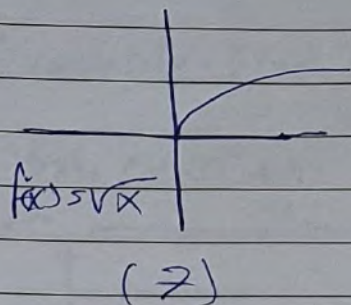
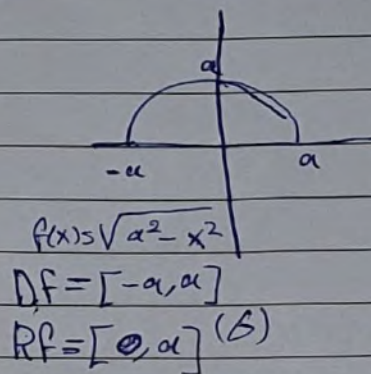
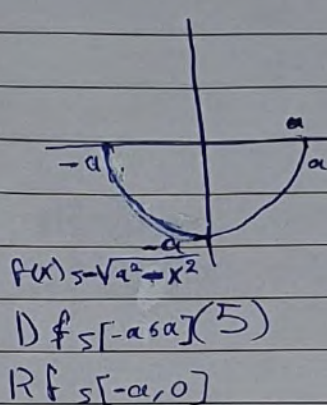
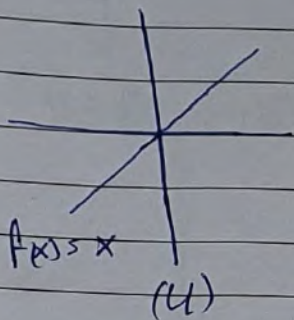
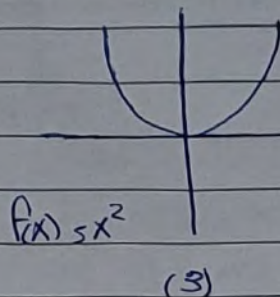
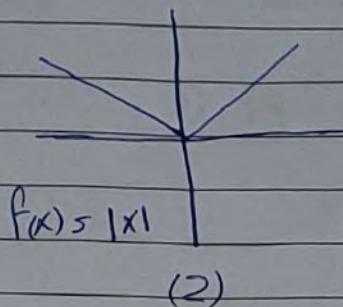
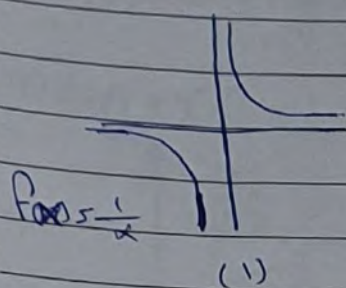
$$\textcircled{5} \cos\left(11\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$\textcircled{6} \sin^2\left(\frac{\pi}{2}\right) = \frac{1}{2} (1 - \cos(2\frac{\pi}{2}))$$

$$= \frac{1}{2} (1 - \cos(\pi))$$

$$= \frac{1}{2} (1 - (-1)) = \frac{1}{2} (2) = 1$$

Lecture 8



Definition: Suppose $c > 0$, $y \leq f(x)$

1) $y = f(x) + c$, shift the graph $y = f(x)$ a distance c units upward.

2) $y = f(x) - c$, " " " " " " " " downward

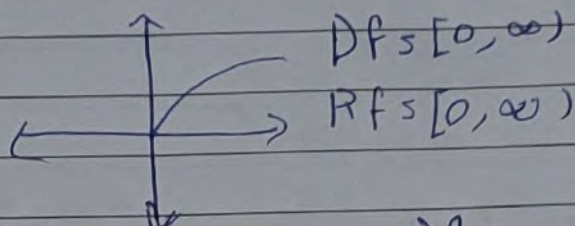
3) $y = f(x - c)$, " " " " " " " " to the right

4) $y = f(x + c)$, " " " " " " " " to the left

5) $y = -f(x)$, reflect " " about the x -axis

6) $y = f(-x)$, " " " " " " y -axis

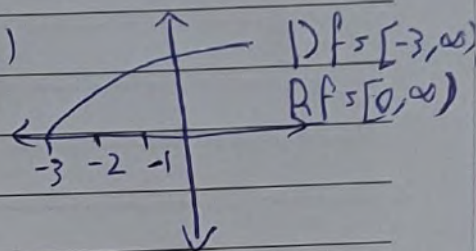
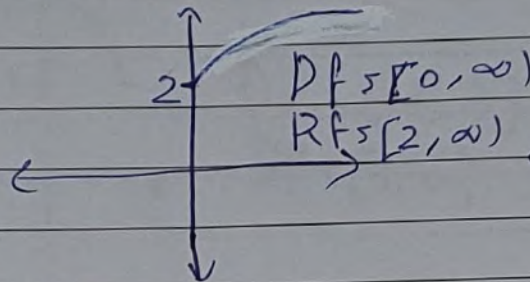
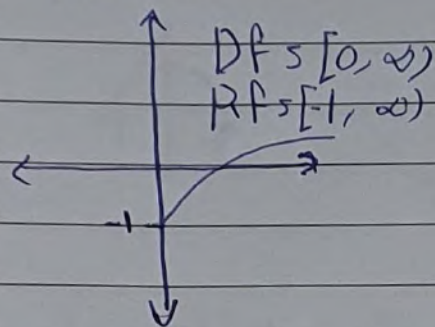
Example: $y = \sqrt{x}$



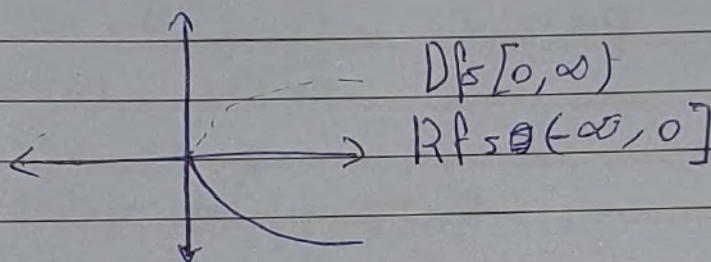
1) $f(x) = \sqrt{x} - 1$

2) $f(x) = \sqrt{x+2}$

3) $f(x) = \sqrt{x+3}$

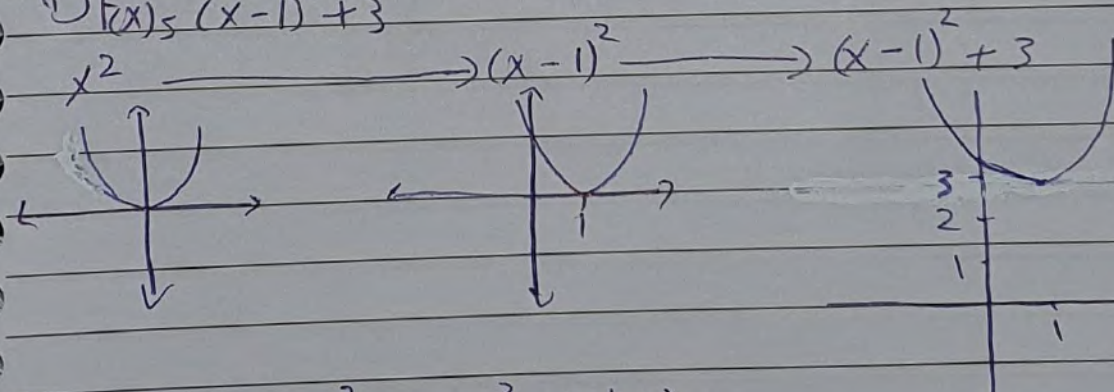


4) $f(x) = -\sqrt{x}$



Example: find domain and range

1) $f(x) = (x-1)^2 + 3$



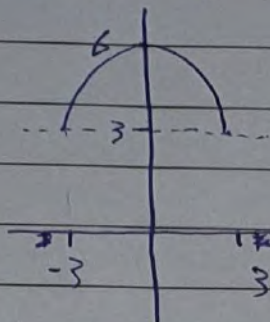
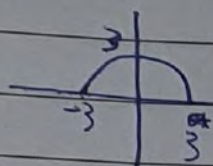
* $f(x) = (x-1)^2 + 3 = x^2 - x + 1 + 3$

$= x^2 - x + 4$
نقطه التقاطع المنخفض هو 3.5

$Df \subseteq \mathbb{R}$
 $Rf \subseteq [3, \infty)$

② $f(x) \leq \sqrt{9-x^2} + 3$

$\sqrt{9-x^2} \longrightarrow \sqrt{9+x^2} + 3$

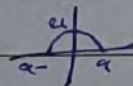


$DF \leq [-3, 3]$
 $RF \leq [3, 6]$

$f(x) \leq \sqrt{a^2+x^2}$

$DF \leq [-a, a]$

$RF \leq [0, a]$



~~W~~ H.W. Find domain and range

1) $f(x) \leq 4 - |x-2| \leq -|x-2| + 4$

2) $f(x) \leq -\sqrt{4-x^2}$

3) $f(x) \leq x^2 + 7$

4) $f(x) \leq -|x+8| + 9$

5) $f(x) \leq |x-1| - 4$

Lecture 9

Definition: if f and g functions with domains D_f, D_g

- (1) $(f+g)(x) = f(x) + g(x)$ with $D_{f+g} = D_f \cap D_g$
- (2) $(f \cdot g)(x) = f(x) \cdot g(x)$ with $D_{f \cdot g} = D_f \cap D_g$
- (3) $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ with $D_{\frac{f}{g}} = D_f \cap D_g - \{x: g(x) = 0\}$

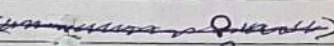
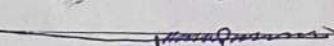
Example: $f(x) = \sqrt{x+2}$, $g(x) = \frac{1}{x}$

$$1) (f-g)(x) = f(x) - g(x) = \sqrt{x+2} - \frac{1}{x} = \frac{x\sqrt{x+2} - 1}{x}$$

$$2) (f-g)(4) = f(4) - g(4) = \sqrt{4+2} - \frac{1}{4} = \sqrt{6} - \frac{1}{4} = \frac{4\sqrt{6} - 1}{4}$$

or $(f-g)(4) = \frac{4\sqrt{4+2} - 1}{4} = \frac{4\sqrt{6} - 1}{4}$

$$3) D_{fg} = D_f \cap D_g$$

$D_f = [-2, \infty) \cap \mathbb{R} \setminus \{0\}$ D_f  D_g  D_{fg} 	$f(x) = \sqrt{x+2}$ $x+2 \geq 0$ $x \geq -2$ $D_f = [-2, \infty)$	$g(x) = \frac{1}{x}$ $D_g = \mathbb{R} - \{0\}$
--	--	--

Example: Find domain

$$1) f(x) = \frac{x}{x^2-9} \rightarrow 1) \quad x^2-9 \neq 0$$

$$D_1 = \mathbb{R}$$

$$D_2 = \mathbb{R}$$

$$D_f = D_1 \cap D_2 \setminus \{x: x^2-9=0\}$$

$$x^2-9=0 \quad \text{factorial}$$

$$= \mathbb{R} \cap \mathbb{R} \setminus \{x: x^2-9=0\} = \mathbb{R} - \{x: x^2-9=0\}$$

$$(x-3)(x+3)=0$$

$$x \neq 3, -3$$

$$2) f(x) = \frac{1}{\sqrt{1-x}} \rightarrow 2) \quad D_f = D_1 \cap D_2 \setminus \{x: x=1\}$$

$$D_1 = \mathbb{R}$$

$$= \mathbb{R} \cap (-\infty, 1) \cup (1, \infty)$$

$$D_2 = (-\infty, 1) \cup (1, \infty)$$

$$D_1 = \mathbb{R}$$

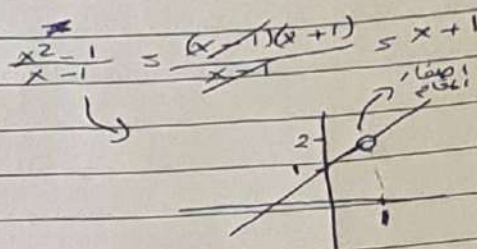
$$D_2 = (-\infty, 1) \cup (1, \infty)$$

$$\{x: x=1\}$$

Lecture 10

Example 1 Find domain f:

① $f(x) = \frac{x^2-1}{x-1} \rightarrow \mathbb{R}$
 $\frac{x^2-1}{x-1} \rightarrow \mathbb{R}$
 $x-1=0$
 $x=1$



$\mathbb{R} \setminus \{1\} = \mathbb{R} \cap \mathbb{R} \setminus \{1\}$
 $= \mathbb{R} - \{1\} = (-\infty, 1) \cup (1, \infty)$

② $f(x) = \sqrt{|x-5|-2}$

$|x-5|-2 \geq 0 \rightarrow |x-5| \geq 2$

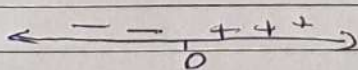
$x-5 \geq 2$ or $x-5 \leq -2$
 $x \geq 7$ $x \leq 3$

$\text{Df} = (-\infty, 3] \cup [7, \infty)$

③ $f(x) = \sqrt{\frac{x}{x+1}}$

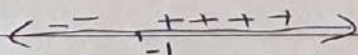
$\frac{x}{x+1} \geq 0$

$x \geq 0 \rightarrow$

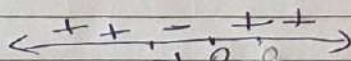


$\text{Df} = (-\infty, -1) \cup [0, \infty)$

$x+1=0$
 $x=-1$



$\frac{x}{x+1}$



$$f(x) = \frac{\sqrt{x}}{\sqrt{x+1}} \rightarrow \textcircled{1}$$

$D, x \geq 0$ 

$$D_2 \text{ s } x+1 > 0 \quad \leftarrow \begin{array}{c} \text{++++++} \\ -1 \end{array}$$

$D_1 \leftarrow \text{fuzz}$

$$D_2 \leftarrow \text{Hess} \rightarrow$$

$D_f \leftarrow \begin{matrix} -1 & 0 \end{matrix}$

$$Df = \mathbb{R} - \{1\} = \text{مجال الدالة}$$

Lecture 11

Example: Find domain

$$1) f(x) = 2x+1 \rightarrow 2x+1 \in \mathbb{R} \rightarrow D_{2x+1} = \mathbb{R}$$

$$3-|x| \rightarrow 3-|x| \rightarrow D_{3-|x|} = \mathbb{R}$$

test, test

$$\begin{matrix} 3-|x| \geq 0 \\ +|x| \quad +|x| \end{matrix}$$

$$3 \geq |x| \rightarrow x \leq 3, -3$$

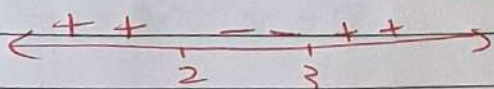
$$D_f \subseteq D_{2x+1} \cap D_{3-|x|} = \mathbb{R} \cap \mathbb{R} - \{\pm 3\} = \mathbb{R} - \{\pm 3\}$$

$$2) f(x) = \sqrt{x^2-5x+6}$$

$$x^2-5x+6 \geq 0$$

$$(x-3)(x-2) = 0$$

$$x \leq 2, x \geq 3$$



$$D_f \subseteq (-\infty, 2] \cup [3, \infty)$$

$$3) f(x) = \sqrt{x^2-5x+6} + \frac{2x+1}{3-|x|}$$

$$D_f \subseteq D_2 \cap D_1$$

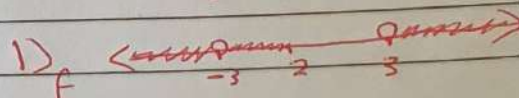
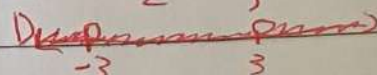
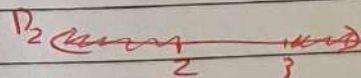
$$\subseteq ((-\infty, 2] \cup [3, \infty)) \cap (\mathbb{R} - \{\pm 3\})$$

$$\subseteq (-\infty, -3) \cup (-3, 2] \cup (3, \infty)$$

$$\subseteq (-\infty, 2] \cup (3, \infty) - \{-3\}$$

$$\subseteq \mathbb{R} - \{-3, (2, 3)\}$$

$$\subseteq \mathbb{R} - \{-3, [2, 3]\}$$



Lecture 12

Example: Find the domain and the Range

① $f(x) = \sqrt{x-1} + 5$
 $x-1 \geq 0 \Rightarrow x \geq 1$
 $\mathbb{R}$

$D_f = [1, \infty) \cap \mathbb{R}$; $R_f = [5, \infty)$

$D_f = [1, \infty)$

$\sqrt{x} \rightarrow \sqrt{x-1} \rightarrow \sqrt{x-1} + 5$

② $f(x) = \frac{3}{5 + \cos x} \rightarrow \mathbb{R}$

$5 + \cos x \neq 0$
 $\cos x \neq -5$
 $-1 \leq \cos x \leq 1$

$D_f = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$

ملاحظة

$-1 \leq \cos x \leq 1$
 $+5 \quad +5 \quad +5$
 $4 \leq 5 + \cos x \leq 6$
 $\frac{1}{4} \geq \frac{1}{5 + \cos x} \geq \frac{1}{6}$
 $\frac{3}{4} \geq \frac{3}{5 + \cos x} \geq \frac{3}{6}$

المتباينة
 كل قلبها الإفتوان
 أو عند الصفر

$\frac{3}{4} \geq f(x) \geq \frac{1}{2}$

$R_f = [\frac{1}{2}, \frac{3}{4}]$

③ $f(x) = -3 \sin^2 x$

$D_f = \mathbb{R}$

للعدد

$-1 \leq \sin x \leq 1$

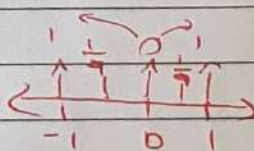
$0 \leq \sin^2 x \leq 1$

$-3 \leq -3 \sin^2 x \leq 0$

$-3 \leq f(x) \leq 0$

$-3 \leq f(x) \leq 0$

$R_f = [-3, 0]$



H.W (4) + (5)

(4) $f(x) = -x^2 + 5$

$D_f = \mathbb{R}$

$R_f = (-\infty, 5]$

$x^2 \rightarrow -x^2 \rightarrow -x^2 + 5$

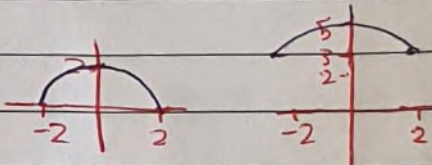


(5) $f(x) = \sqrt{4-x^2} + 3$

$D_f = [-2, 2]$

$R_f = [3, 5]$

$\sqrt{4-x^2} \rightarrow \sqrt{4-x^2} + 3$



Example: If Range of f is $[1, 2]$ find the range of $g(x) = \frac{1}{1+(f(x))^2}$?

$$1 \leq f(x) \leq 2$$

$$1 \leq f(x)^2 \leq 4$$

$$2 \leq f(x)^2 + 1 \leq 5$$

$$\frac{1}{2} \geq \frac{1}{1+(f(x))^2} \geq \frac{1}{5}$$

$$\frac{1}{5} \leq \frac{1}{1+(f(x))^2} \leq \frac{1}{2}$$

$$\frac{1}{5} \leq g(x) \leq \frac{1}{2}$$

$R_g = \left[\frac{1}{5}, \frac{1}{2}\right]$

Lecture 13

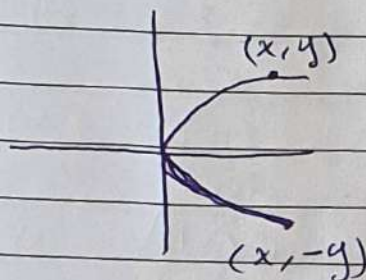
- Symmetry

التماثل

لفرقة اذا كان في عنوان

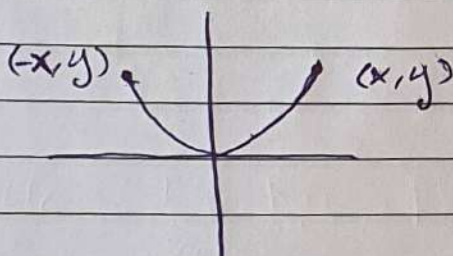
متساوي في العنوان
y - مكان في عنوان
ويجب ان يكون العلاقة

1) About the x-axis ($y \rightarrow -y$)



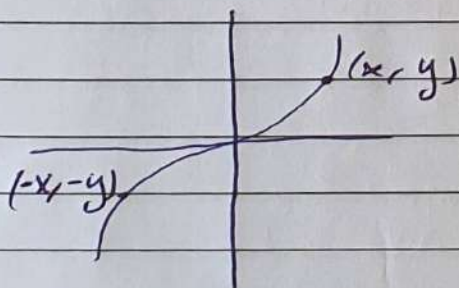
$$(x, y), (x, -y)$$

2) About the y-axis ($x \rightarrow -x$)



A graph is said to be symmetric about the y-axis if whenever (x, y) is on the graph so $(-x, y)$

3) About the origin ($x \rightarrow -x, y \rightarrow -y$)



$$(x, y), (-x, -y)$$

Example:

1) Is $x^3 = y^4 + 3$ symmetric about x-axis? ($y \rightarrow -y$)

$$x^3 = (-y)^4 + 3 \rightarrow$$

$x^3 = y^4 + 3$ symmetric about x-axis

→ 2) Is $x^4 + 5y^2 = 9$ symmetric about origin?

$$(-x)^4 + 5(-y)^2 = 9$$

$$x^4 + 5y^2 = 9 \quad \text{symmetric about origin}$$

3) Is $y^3 = x^2 - 7$ symmetric about origin? $\left(\begin{matrix} x \rightarrow -x \\ y \rightarrow -y \end{matrix} \right)$

$$(-y)^3 = (-x)^2 - 7$$

$$-y^3 = -x^2 - 7$$

$$-y^3 = -(x^2 + 7)$$

$$y^3 = x^2 + 7$$

$$y^3 = x^2 + 7$$

not symmetric about origin

Lecture (14)

Even and odd Function

$$① f(-x) = f(x) \quad \text{even function} \quad \text{متناظرة للمحور الأفقي}$$

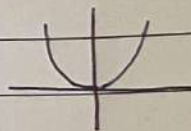
$$② f(-x) = -f(x) \quad \text{odd function} \quad \text{متناظرة للمحور العمودي}$$

Example: classify the functions as even or odd or neither:

$$① f(x) = x^2$$

~~for~~

$$f(-x) = -x^2 = x^2 = f(x) \quad \text{even function}$$

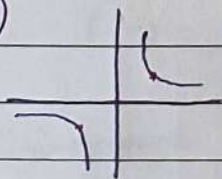


$$② f(x) = \frac{x^5 - x}{1 + x^2}$$

$$f(-x) = \frac{(-x)^5 - (-x)}{1 + (-x)^2} = \frac{-x^5 + x}{1 + x^2} = \frac{-(x^5 - x)}{1 + x^2} = -f(x)$$

$$f(-x) = -f(x) \quad \text{odd function}$$

③



odd function (متناظرة للمحور العمودي)

$$④ f(x) = x^5 + x$$

$$f(-x) = (-x)^5 + (-x) = -x^5 - x = -(x^5 + x)$$

$$f(-x) = -f(x) \quad \text{odd function}$$

~~for~~

Notes:

① $\cos x$, $\sec x$ even function

$$\cos(-x) = \cos x$$

$$\sec(-x) = \sec x$$

② $\sin x$, $\csc x$, $\tan x$, $\cot x$ odd function

$$\sin(-x) = -\sin x$$

$$\csc(-x) = -\csc x$$

$$\tan(-x) = -\tan x$$

⑤ $f(x) = \frac{\tan x}{x^3 - x}$

$$f(-x) = \frac{\tan(-x)}{(-x)^3 - (-x)}$$

$$= \frac{-\tan x}{-x^3 + x} = \frac{-\tan x}{-(x^3 - x)} = \frac{\tan x}{x^3 - x} = f(x) \text{ even function}$$

H.W ① + ②

⑥ $f(x) = \frac{x^4 + 1}{\csc x}$

$$f(-x) = \frac{(-x)^4 + 1}{\csc(-x)} = \frac{x^4 + 1}{-\csc x} = -f(x) \text{ odd function}$$

~~⑦ $2x - x^2$~~

⑦ $f(x) = 2x - x^2$

$$f(-x) = 2(-x) - (-x)^2$$

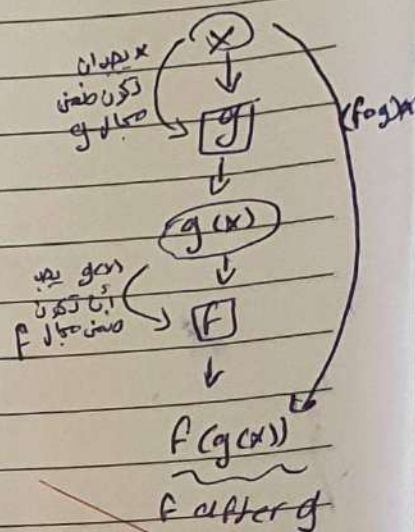
$$= -2x - x^2$$

$$= -(2x + x^2) \text{ neither}$$

Lecture 15

- composition of functions ترتيب الاغترانك
 - The composition of f with g $(f \circ g)(x) = f(g(x))$ f after g
 - The composition of g with f $(g \circ f)(x) = g(f(x))$ g after f

Domain $f \circ g = \{x \in D_{g(x)} \text{ and } g(x) \in D_f\}$
 $= \{x \in D_{\text{الاولى}} \text{ and } g(x) \in D_{\text{الثانية}}\}$



Domain $f \circ g = D_{\text{الاولى}} \cap D_{\text{الثانية}}$
التابعين من
الترتيب بعد
التبعية

Example :- $f(x) = \sqrt{x}$, $g(x) = \sqrt{2-x}$ Find each function and its domain:

① $(f \circ g)(x) = f(g(x))$
 $= f(\sqrt{2-x})$
 $= \sqrt{\sqrt{2-x}} = ((2-x)^{\frac{1}{2}})^{\frac{1}{2}} = (2-x)^{\frac{1}{4}} = \sqrt[4]{2-x}$

Domain $f \circ g = \{x \in D_{g(x)} \text{ and } g(x) \in D_{f(x)}\}$
 $= \{x \in (-\infty, 2] \cap \sqrt{2-x} \in [0, \infty)\}$
 $= \{x \in (-\infty, 2] \text{ and } (\sqrt{2-x})^2 \geq 0\}$
 $= \{x \in (-\infty, 2] \text{ and } 2-x \geq 0\}$
 $= \{x \in (-\infty, 2] \text{ and } (-\infty, 2]\}$
 $= (-\infty, 2] \cap (-\infty, 2] = (-\infty, 2]$

$f(x) = \sqrt{x} \quad x \geq 0$
 $D_f = [0, \infty)$
 $g(x) = \sqrt{2-x}$
 $2 \geq x$
 $D_g = (-\infty, 2]$

Domain $f \circ g \subseteq D_f \cap D_g$

المنطق $\rightarrow$ المنطق
التركيب بعد التبسيط

$$= (-\infty, 2] \cap (-\infty, 2] = (-\infty, 2]$$

$D_f \rightarrow \sqrt[4]{2-x}$

المنطق $\rightarrow$ المنطق

$2-x \geq 0$

$2 \geq x$

$-\infty, 2]$

② $(g \circ f)(x) \subseteq g(f(x))$

$$= g(\sqrt{x})$$

$$\subseteq \sqrt{2-\sqrt{x}}$$

$D_{g \circ f} \subseteq D_f \cap D_g$

المنطق $\rightarrow$ المنطق

$$= [0, \infty) \cap (-\infty, 4]$$

$$= [0, 4]$$

$\sqrt{2-\sqrt{x}}$

$2-\sqrt{x} \geq 0$

$2 \geq \sqrt{x}$

$4 \geq x$

$D_{g \circ f} = (-\infty, 4]$

or $D_{g \circ f} \subseteq \{x \in D_f \text{ and } f(x) \in D_g\}$

$\subseteq$

$= [0, 4]$

Find $(g \circ f)(4) = g(f(4)) \subseteq g(3) \subseteq \sqrt{2-3} \subseteq \text{undefined}$

لأنه لا يمكن أن تكون
من المجال

$$4 \in [0, 4]$$

أيضا يمكن
طرحه عن تابع
فإنه انقطع
المجال

H.W.

③ + 9

③ $(f \circ f)(x) \subseteq f(f(x)) \subseteq f(\sqrt{x}) \subseteq \sqrt{\sqrt{x}} \subseteq \sqrt[4]{x}$

$D_{f \circ f} \subseteq \{??\} \rightarrow [0, \infty)$

④ $(g \circ g)(x) \subseteq g(g(x)) \subseteq g(\sqrt{2-x}) \subseteq \sqrt{2-\sqrt{2-x}}$

$D_{g \circ g} = ?? \rightarrow [-2, 2]$

Lecture 16

Example: $f(x) = \frac{3}{x}$, $g(x) = \frac{5x}{x-2}$ find $(g \circ f)(x)$ and Domain $(g \circ f)(x)$

$$(g \circ f)(x) = g(f(x))$$

$$\begin{aligned} &= g\left(\frac{3}{x}\right) \\ &= \frac{5\left(\frac{3}{x}\right)}{\frac{3}{x} - 2} = \frac{\frac{15}{x}}{\frac{3 - 2x}{x}} = \frac{15}{x} \cdot \frac{x}{3 - 2x} = \frac{15}{3 - 2x} \end{aligned}$$

$$\textcircled{*} f(x) = \frac{3}{x} \rightarrow D_f = \mathbb{R} \setminus \{0\}$$

$$g(x) = \frac{5x}{x-2} \rightarrow \mathbb{R}$$

$$x - 2 = 0 \rightarrow x = 2 \rightarrow D_g = \mathbb{R} \setminus \{2\}$$

$$\text{Domain}(g \circ f) = D_f \cap D_{g \circ f}$$

$$= D_{\frac{3}{x}} \cap D_{\frac{15}{3-x}}$$

$$\begin{aligned} &= \mathbb{R} \setminus \{0\} \cap \mathbb{R} \setminus \left\{\frac{3}{2}\right\} \\ &= \mathbb{R} \setminus \left\{0, \frac{3}{2}\right\} \end{aligned}$$

$$\textcircled{*} \frac{15}{3-2x} \rightarrow \mathbb{R}$$

$$3 - 2x = 0$$

$$x = \frac{3}{2}$$

$$D_{\frac{15}{3-2x}} = \mathbb{R} \setminus \left\{\frac{3}{2}\right\}$$

Example: Find domain $f(x) = \sqrt{3 - \sqrt{x}}$?

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$$\begin{aligned} g(x) = \sqrt{x}, \quad h(x) = \sqrt{3-x} \rightarrow f(x) &\rightarrow (h \circ g)(x) = h(g(x)) \\ &= h(\sqrt{x}) \\ &= \sqrt{3 - \sqrt{x}} \end{aligned}$$

$$D_f = D_{\sqrt{x}} \cap D_{\sqrt{3-\sqrt{x}}}$$

$$= [0, \infty) \cap (-\infty, 9]$$

$$= [0, 9]$$

$$\begin{aligned} \textcircled{*} \sqrt{x} &\rightarrow x \geq 0 \rightarrow [0, \infty) \\ \textcircled{*} \sqrt{3-\sqrt{x}} &\rightarrow 3 - \sqrt{x} \geq 0 \rightarrow 3 \geq \sqrt{x} \rightarrow 9 \geq (\sqrt{x})^2 \\ &\rightarrow 9 \geq x \rightarrow (-\infty, 9] \end{aligned}$$

← 0
0 9

Lecture 17

Example: $(f \circ g)(x) = 6x^2 - 10x + 5$, $f(x) = 2x + 1$ Find $g(x)$?

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) = 6x^2 - 10x + 5 \\ 2g(x) + 1 &= 6x^2 - 10x + 5 \end{aligned}$$

$$\frac{2g(x)}{2} = \frac{6x^2 - 10x + 4}{2}$$

$$g(x) = 3x^2 - 5x + 2$$

$$g(0) = 3(0)^2 - 5(0) + 2 = 2$$

Ex: $(f \circ g)(x) = x^2 + 5$, $g(x) = 2x + 3$ find $f(x)$?

$$(f \circ g)(x) = f(g(x)) = x^2 + 5$$

let

$$y = 2x + 3$$

$$\frac{y-3}{2} = x$$

$$\frac{y-3}{2} = x$$

$$f(2x+3) = x^2 + 5$$

$$f(y) = \left(\frac{y-3}{2}\right)^2 + 5$$

$$f(2) = \left(\frac{2-3}{2}\right)^2 + 5$$

$$f(m) = \left(\frac{m-3}{2}\right)^2 + 5$$

$$f(x) = \left(\frac{x-3}{2}\right)^2 + 5$$

$$f(x) = x^2$$

$$\downarrow$$

$$f(z) = z^2$$

$$\downarrow$$

$$f(y) = y^2$$

Ex: If $D_f = [1, 3]$ and $g(x) = -2x + 1$ Find domain $f \circ g$?

$$\text{Domain } f \circ g = \{x \in D \text{ and } g(x) \in D_f\}$$

$$= \{x \in \mathbb{R} \text{ and } -2x + 1 \in [1, 3]\}$$

$$= \{x \in \mathbb{R} \text{ and } 1 \leq -2x + 1 \leq 3\}$$

$$= \{x \in \mathbb{R} \text{ and } 0 \leq \frac{-2x}{-2} \leq \frac{2}{-2}\}$$

$$= \{x \in \mathbb{R} \text{ and } 0 \geq x \geq -1\}$$

$$= \mathbb{R} \cap [-1, 0]$$

$$= [-1, 0]$$

H.w

* ① $f(x) = x^2 + 3$, $g(x) = \sqrt{3-x}$ Find $(f \circ g)(x)$, $D_{f \circ g}$
 $f \circ g(x)$, $(f \circ g)(4)$, $(g \circ g)(x)$, $D_{g \circ g}$, $g \circ f$, $D_{g \circ f}$?

② $(f \circ g)(x) = 3x^2 + 3x + 2$, $f(x) = 3x + 5$ Find $g(x)$?

③ $(f \circ g)(x) = x^2 + 6x + 6$, $g(x) = x + 1$ Find $f(x)$?

④ $f(x) = \frac{x}{x+1}$, $g(x) = x^3$, $h(x) = x + 2$ Find

$$= (f \circ g \circ h)(x) = f(g(h(x)))$$

$$= (h \circ g)(x)$$

$$= (f \circ g)(x)$$

$$= (f \circ f)(x)$$

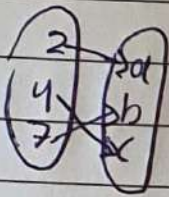
Lecture 18

one to one function: f is called one to one function if $f(x_1) = f(x_2)$ whenever $x_1 \neq x_2$

f is called one to one function if $f(x_1) = f(x_2)$ whenever $x_1 \neq x_2$

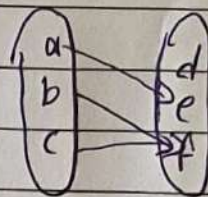
Ex:

(1)



one to one
function

(2)



not one to one

$b \neq c$
but $f(b) = f(c) = e$

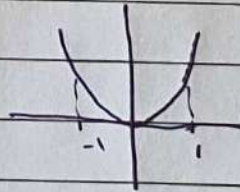
Remark: Test for one to one function: If $f(a) = f(b) \rightarrow (a=b)$ then f is one to one function

Ex: Is $f(x) = x^2$ one to one function?

not one to one function

$-1 \neq 1$ but $f(-1) = f(1) = 1$

$-2 \neq 2$ but $f(-2) = f(2) = 4$



Ex: Is $f(x) = 3x - 2$ one to one function?

Let $f(a) = f(b)$

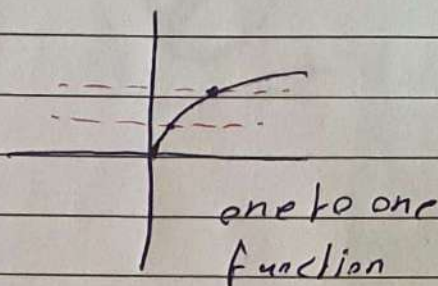
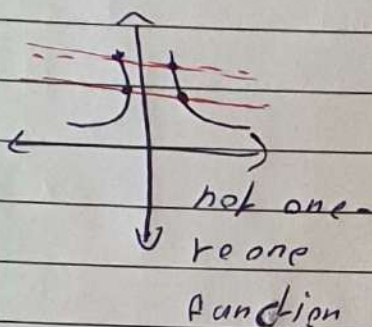
$$3a - 2 = 3b - 2$$

$$\frac{3a}{3} = \frac{3b}{3}$$

$a = b$ $\therefore f$ is one to one function

→ Horizontal line test : اختبار الخط الأفقي

العمودي ↓ حل العلاقة
الأفقى ↓ حل الدقتان
one to one
أتم لا

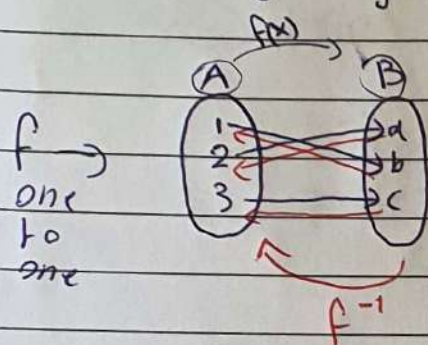


Lecture 19

- Inverse Function الوظيفة العكسية

Theorem: f has an inverse $\iff f$ one to one function
if and only if

Definition: let f is one to one, then its inverse is defined by $f^{-1}(y) = x \iff y = f(x)$



$f(1) = a$	$f^{-1}(a) = 1$
$f(2) = b$	$f^{-1}(b) = 2$
$f(3) = c$	$f^{-1}(c) = 3$

$$f(1) = a \iff f^{-1}(a) = 1$$

$$f(2) = b \iff f^{-1}(b) = 2$$

$$f(3) = c \iff f^{-1}(c) = 3$$

$D_{f^{-1}} = R_f$
 $R_{f^{-1}} = D_f$

$$f^{-1}(f(2)) = f^{-1}(b) = 2$$

$$f^{-1}(f(a)) = f^{-1}(a) = 1$$

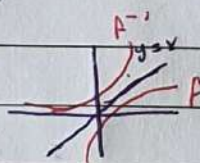
$$f^{-1}(c) = 3$$

$$f(f^{-1}(c)) = f(3) = c$$

Remark:

- ① Domain f^{-1} is Range $f \subseteq B$
- ② Range f^{-1} is Domain $f \subseteq A$
- ③ $(f \circ f^{-1})(x) = x \iff f(f^{-1}(x)) = x \quad \forall x \in B$
- ④ $(f^{-1} \circ f)(x) = x, \quad \forall x \in A$
- ⑤ f and f^{-1} are symmetric about $y=x$ متماثلين

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المرآتية



لكل $\forall \rightarrow$ for all
 يوجد $\exists \rightarrow$ there exist

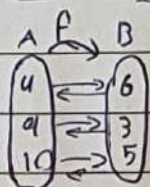
Ex: given that f has an inverse $f(4) = 6$, $f(9) = 3$, $f(5) = 10$ find:

1) $f^{-1}(6) = 4$

* $f(4) = 6$

~~$f^{-1}(f(4)) = f^{-1}(6)$~~

$4 = f^{-1}(6)$



2) $f(10) = 5$

$f^{-1}(f(x)) = x \quad \forall x \in D_f$

① $f^{-1}(5) = 10$

~~$f^{-1}(f(5)) = f^{-1}(10)$~~

$5 = f(10)$

③ $f^{-1}(3) = 9$

Lecture 20

Ex: find $f^{-1}(x)$ for:

① $f(x) = \sqrt{3x-2}$

$(y)^2 = (\sqrt{3x-2})^2$

$y^2 = 3x-2$

$y^2 + 2 = 3x$

$\frac{y^2 + 2}{3} = x$

$x = \frac{y^2 + 2}{3}$

$\downarrow \quad \downarrow$

$f^{-1}(x) = \frac{x^2 + 2}{3}$

② $f(x) = \frac{x+1}{x-1}$

$y = \frac{x+1}{x-1}$

$y(x-1) = x+1$

$yx - y = x+1$

$yx - x = y+1$

$x(y-1) = y+1$

$x = \frac{y+1}{y-1}$

$\downarrow \quad \downarrow$

$\downarrow \quad \downarrow$

$f^{-1}(x) = \frac{1+x}{x-1}$

Find $D_{f^{-1}}$

$f^{-1}(x) = \frac{1+x}{x-1} \rightarrow \mathbb{R}$

$x-1 \neq 0$

$x \neq 1$

$D_{f^{-1}} = \mathbb{R} \cap \mathbb{R} \setminus \{1\} = \mathbb{R} \setminus \{1\} = \text{Range } f$

Find $R_{f^{-1}} = \text{Domain } f = \mathbb{R} \setminus \{1\}$

Ex: find Domain and range for $f(x) = \frac{x+2}{2x-1}$

$\text{Domain } f = \mathbb{R} \cap \mathbb{R} \setminus \{\frac{1}{2}\} = \mathbb{R} \setminus \{\frac{1}{2}\}$

$\text{Range } f(x) = \text{Domain } f^{-1}(x)$

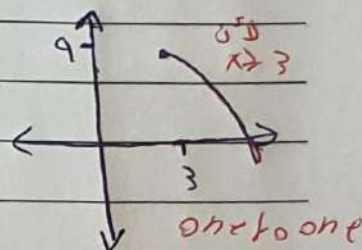
$R_f = D_{f^{-1}} = \mathbb{R} \cap \mathbb{R} \setminus \{\frac{1}{2}\} = \mathbb{R} \setminus \{\frac{1}{2}\}$

$y = \frac{x+2}{2x-1}$

$x = \frac{y+2}{2y-1} \rightarrow f^{-1}(x) = \frac{x+2}{2x-1}$

Lecture 21

Ex: $f(x) = 6x - x^2, x \geq 3$ Find $f^{-1}(x)$?



$$y = 6x - x^2$$

$$-y = -6x + x^2$$

$$-y = x^2 - 6x + 9 - 9$$

$$-y = (x-3)(x-3) - 9$$

$$-y = (x-3)^2 - 9$$

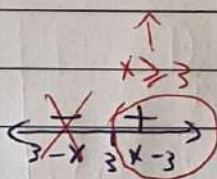
$$\sqrt{-y+9} = \sqrt{(x-3)^2}$$

$$\sqrt{9-y} = |x-3|$$

$$\sqrt{9-y} = x-3$$

$$x = \sqrt{9-y} + 3$$

$$f^{-1}(x) = \sqrt{9-x} + 3$$



$$f \circ f^{-1}(x) = x$$

$$f^{-1} \circ f(x) = x$$

Ex: Determine whether f and g are inverse functions

$$f(x) = x^3 + 3x^2 + 3x + 1 = (x+1)^3$$

$$g(x) = x^{\frac{1}{3}} - 1$$

$$(f \circ g)(x) \stackrel{?}{=} x$$

$$(g \circ f)(x) \stackrel{?}{=} x$$

$$\Rightarrow g = f^{-1}$$

$$f = g^{-1}$$

$$(f \circ g)(x) = f(g(x)) = f(x^{\frac{1}{3}} - 1)$$

$$= (x^{\frac{1}{3}} - 1 + 1)^3 = (x^{\frac{1}{3}})^3 = x \quad \checkmark$$

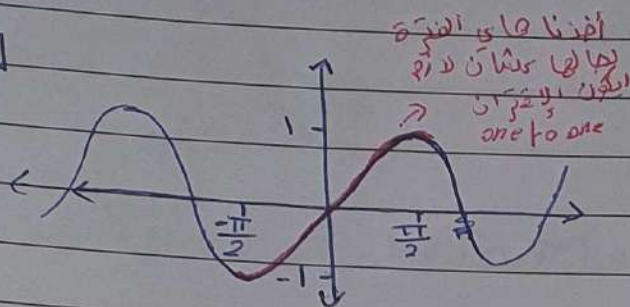
$$(g \circ f)(x) = g(f(x)) = g((x+1)^3) = ((x+1)^3)^{\frac{1}{3}} - 1 = x+1-1 = x \quad \checkmark$$

$\therefore f, g$ are inverse

Lecture 22

Inverse Trigonometric Function

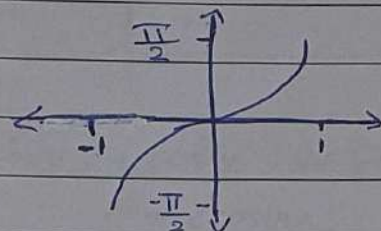
III



$$f(x) = \sin x, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$D_{\sin x} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$R_{\sin x} = [-1, 1]$$



$$f(x) = \sin^{-1} x, x \in [-1, 1]$$

$$= \arcsin$$

$$D_{\sin^{-1} x} = [-1, 1]$$

$$R_{\sin^{-1} x} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Remark:

- ① $D_{\sin^{-1}} = [-1, 1]$
- ② $R_{\sin^{-1}} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- ③ $\sin(\sin^{-1} x) = x, \forall x \in [-1, 1]$
- ④ $\sin^{-1}(\sin x) = x, \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- ⑤ $\sin^{-1}(-x) = -\sin^{-1} x$ ($\sin^{-1} x$ odd function)

$$\sin^{-1} x \neq \frac{1}{\sin x}$$

$$f^{-1} \neq \frac{1}{f}$$

Ex: find

$$① \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \rightarrow \text{العدد الجانبي}$$

$$② \sin^{-1}\left(-\frac{1}{2}\right) = -\sin^{-1}\left(\frac{1}{2}\right) = -\frac{\pi}{6}$$

$$③ \sin^{-1}\left(\sin \frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$④ \sin^{-1}\left(\sin 11\frac{\pi}{6}\right) = \sin^{-1}\left(-\sin\left(2\pi - 11\frac{\pi}{6}\right)\right)$$

العدد الجانبي

$$= \sin^{-1}\left(-\sin \frac{\pi}{6}\right)$$

$$= -\sin^{-1}\left(\sin \frac{\pi}{6}\right)$$

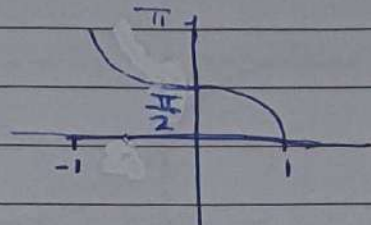
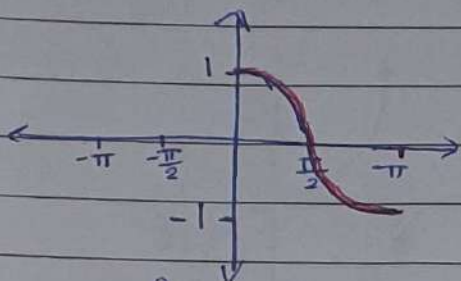
$$= -\frac{\pi}{6}$$

$$\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\sin^{-1}(\sin x) = x, \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Lecture 23



$$f(x) = \cos x, x \in [0, \pi], x \in [0, \pi] \quad \Bigg| \quad f(x) = \cos^{-1} x, x \in [-1, 1] \\ = \arccosine$$

Remark:

- ① $D_{\cos^{-1}x} = [-1, 1]$
- ② $R_{\cos^{-1}x} = [0, \pi]$
- ③ $\cos(\cos^{-1}x) = x, \forall x \in [-1, 1]$
- ④ $\cos^{-1}(\cos x) = x, \forall x \in [0, \pi]$
- ⑤ $\cos^{-1}(-x) = \pi - \cos^{-1}x$ ($\cos^{-1}x$ not even and not odd)

Ex: Find

$$① \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

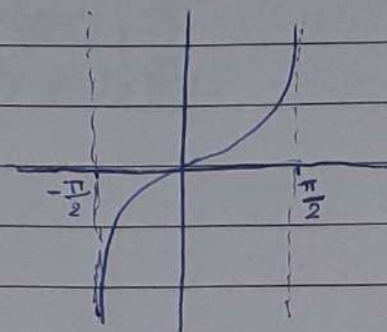
$$② \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$③ \cos^{-1}\left(\cos \frac{11\pi}{6}\right) = \cos^{-1}\left(\cos\left(2\pi - \frac{\pi}{6}\right)\right) \\ = \cos^{-1}\left(\cos \frac{\pi}{6}\right) \\ = \frac{\pi}{6}$$

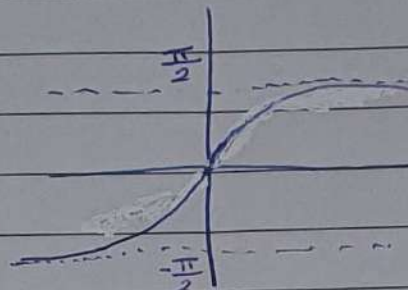
 $\cos^{-1}(\cos x) = x, \forall x \in [0, \pi]$ $\frac{11\pi}{6} \notin [0, \pi]$

$$④ \cos^{-1}\left(\cos 7\frac{\pi}{6}\right) = \cos^{-1}\left(-\cos\left(7\frac{\pi}{6} - \pi\right)\right) \\ = \cos^{-1}\left(-\cos\left(\frac{\pi}{6}\right)\right) \\ = \pi - \cos^{-1}\left(\cos\left(\frac{\pi}{6}\right)\right) \\ = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Lecture 29



$$f(x) = \tan x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$



$$f(x) = \tan^{-1}(x), x \in \mathbb{R} \\ = \arctan x$$

Remark:

$$1) D_{\tan^{-1}} = \mathbb{R}$$

$$2) R_{\tan^{-1}} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$3) \tan(\tan^{-1}x) = x \quad \forall x \in \mathbb{R}$$

$$4) \tan^{-1}(\tan x) = x \quad \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$5) \tan^{-1}(-x) = -\tan^{-1}(x) \quad (\tan^{-1}x \text{ odd function})$$

Ex: Find

$$1) \tan^{-1}(1) = \frac{\pi}{4}$$

$$2) \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$3) \tan^{-1}(\tan \frac{\pi}{6}) = \frac{\pi}{6}$$

★

$$\textcircled{*} f(x) = \csc^{-1}, D_{\csc^{-1}} = (-\infty, -1] \cup [1, \infty)$$

$$R_{\csc^{-1}} = \left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$$

★

$$\textcircled{*} f(x) = \sec^{-1}, D_{\sec^{-1}} = (-\infty, -1] \cup [1, \infty)$$

$$R_{\sec^{-1}} = \left[0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right]$$

★

$$\textcircled{*} f(x) = \cot^{-1}x$$

$$D_{\cot^{-1}} = \mathbb{R}$$

$$R_{\cot^{-1}} = (0, \pi)$$



Ex: find

$$① \sec^{-1}(2) = \frac{\pi}{3}$$

$$② \cot^{-1}(\sqrt{3}) = \frac{\pi}{6}$$

$$③ \cot^{-1}(-\sqrt{3}) = \frac{5\pi}{6}$$

Lecture 25

Ex: Find domain $f(x) = \sin^{-1}(3x+1)$?

$$-1 \leq 3x+1 \leq 1$$

$$D_f = \left[-\frac{2}{3}, 0\right]$$

$$-\frac{2}{3} \leq 3x \leq 0$$

$$-\frac{2}{3} \leq x \leq 0$$

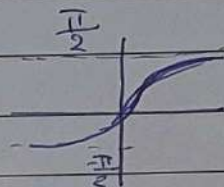
Ex: Find D+R $f(x) = \frac{\pi}{2} + \tan^{-1}x$

$$D_f = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$$

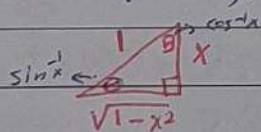
$$-\frac{\pi}{2} < \tan^{-1}x < \frac{\pi}{2}$$

$$0 < \tan^{-1}x + \frac{\pi}{2} < \pi$$

$$R \setminus (0, \pi)$$

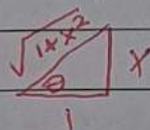


$$\sin(\sin^{-1}x) = x = \frac{\text{adj}}{\text{hyp}}$$

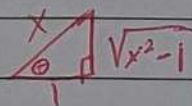


$$\cos(\cos^{-1}x) = x = \frac{\text{adj}}{\text{hyp}}$$

$$\tan(\tan^{-1}x) = x = \frac{\text{opp}}{\text{adj}}$$



$$\sec(\sec^{-1}x) = x = \frac{\text{hyp}}{\text{adj}}$$

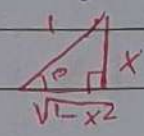


Ex: Find

① $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

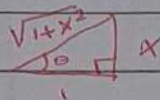
② $\tan^{-1}(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}$

المقابل
 $\sin(\sin^{-1} x) = x = \frac{\text{المقابل}}{\text{المثلث}}$



③ $\sec(\tan^{-1} x) = \sqrt{1+x^2}$

$\tan(\tan^{-1} x) = x$



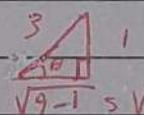
④

④ $\cos^2(\tan^{-1} x) = (\cos(\tan^{-1} x))^2$
 $= \left(\frac{1}{\sqrt{1+x^2}} \right)^2 = \frac{1}{1+x^2}$

~~$\cos^2 x = (\cos x)^2$~~
 $\cos^2 x = (\cos x)^2$
 $\cos x^2 \neq \cos^2 x$

⑤ $\cot(\sin^{-1}(\frac{1}{3})) = \frac{\sqrt{8}}{1}$

$\sin(\sin^{-1}(\frac{1}{3})) = \frac{1}{3} = \frac{\text{المقابل}}{\text{المثلث}}$



⑥ $\sin(2\sin^{-1}(\frac{3}{5})) = 2\sin(\sin^{-1}(\frac{3}{5}))\cos(\sin^{-1}(\frac{3}{5}))$
 $= 2(\frac{3}{5})(\frac{4}{5})$

$\frac{5}{5} = 1$
 $\sqrt{25-9} = \sqrt{16} = 4$
 $\sin(\sin^{-1}(\frac{3}{5})) = \frac{3}{5}$
 $\frac{4}{5} = \frac{\text{المقابل}}{\text{المثلث}}$

⑦ $\sin(\sin^{-1} x + \cos^{-1} y) =$

⑧ $\tan(\sin^{-1}(\frac{1}{5})) =$

⑨ $\sin(\sec^{-1}(2)) =$

★ ★ ★

① $\sin(2a) = 2\sin a \cos a$

② $\sin(a+b) = \sin a \cos b + \sin b \cos a$

③ $\cos(a+b) = \cos a \cos b - \sin a \sin b$

Lecture 26

Exponential and Logarithmic function:

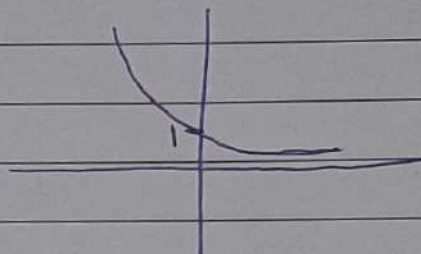
⊗ Definition: Exponential function

$$f(x) = b^x, b > 0, b \neq 1$$

If $b > 1$



If $0 < b < 1$



Ex: $f(x) = 3^x$

Domain: $\mathbb{R}$

Range: $(0, \infty)$

Ex: $f(x) = \left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$

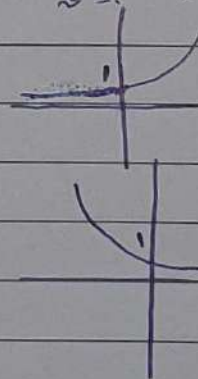
Domain: $\mathbb{R}$

Range: $(0, \infty)$

★ The natural exponential function:

$f(x) = e^x$

$e \approx 2.718...$



$D_f = \mathbb{R}$

$R_f = (0, \infty)$

$f(x) = e^{-x} = (e^{-1})^x = \left(\frac{1}{e}\right)^x$

Ex: find domain:

① $f(x) = e^{\sqrt{x}}$

$D_f = D_{\sqrt{x}} = [0, \infty)$

$\text{Domain}(e^{g(x)}, b^{g(x)}) = \text{Domain}(g(x))$

$$② f(x) = \frac{x}{1+e^x} \rightarrow \mathbb{R}$$

$$\mathbb{R} \leftarrow \frac{1+e^x}{\mathbb{R}} \Rightarrow \mathbb{R}$$

المقام، البسط

$e^x \neq -1$
لا يوجد مقام، البسط

$$e^x \rightarrow \neq \text{zero}$$

$$\rightarrow \neq \text{denominator}$$

$$D_f = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$$

Lecture 2

★ Logarithmic function

$$2^5 = 32 \Leftrightarrow \log_2 32 = 5$$

الأساس ← الأس ← البعد ← البعد ← البعد

$$\log_2 16 = 4 \Leftrightarrow 2^4 = 16$$

$$x^y = z \Leftrightarrow \log_x z = y$$

Ex: find the value: $\log_3 3$

$$\log_3 3 = x$$

$$3^x = 3 \rightarrow (3^2)^x = 3$$

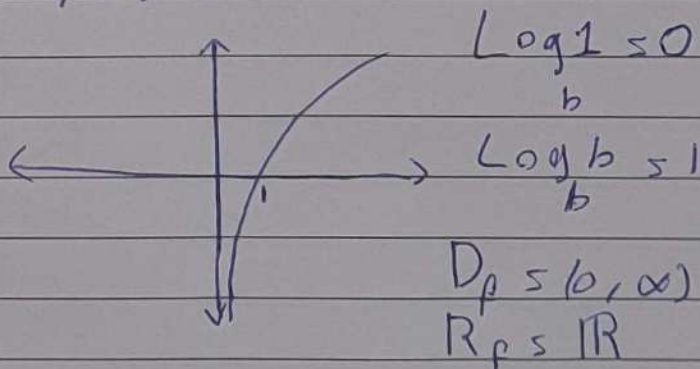
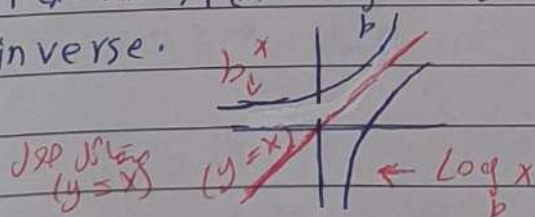
$$3^{2x} = 3^1$$

$$2x = 1$$

$$\therefore \log_3 3 = \frac{1}{2}$$

★ $f(x) = \log_b x$, $b > 0$, $b \neq 1$, $x > 0$

Theorem: If $b > 0$, $b \neq 1$ then $f(x) = b^x$, $f(x) = \log_b x$ are inverse.



$$\log x = \log_{10} x \quad \text{common log}$$

★ $\text{Log}_e x = \ln x$ natural Log

Ex: $\log 10 \leq \log 10 \leq 1$

$$② \ln e = \log_e e = 1$$
$$\textcircled{3} \ln 1 = \log_e 1 = 0$$

Ex: Find domain:

① $f(x) = \log(x^2 - 9) \rightarrow > 0$

$$x^2 - 4 > 0$$

$$(x-3)(x+3) > 0$$

$$D_f = (-\infty, 3) \cup (3, \infty)$$

② $f(x) = \frac{1}{\ln(4-x)}$ $\xrightarrow{\text{PR}}$ $\frac{1}{\log(4-x)} \rightarrow > 0$
 $4-x > 0$
 $x < 4$
 $(-\infty, 4)$

$$\ln(4-x) \leq 0 \quad \text{p} \hat{q}_6 / \hat{q}_0$$

$$\log_e(u-x) \leq 0$$

$$e^0 = 4 - X$$

154-X

$$X \subseteq Y \rightarrow \begin{matrix} \text{top} \\ \text{bottom} \end{matrix}$$

$$\begin{aligned} & \ln(1-x) \\ & \ln(1-x) \\ & \ln 1 = 0 \end{aligned}$$

$$D_f = \mathbb{R} \cap (-\infty, 4) \cup [5, \infty)$$

$$S = (-\infty, 4) \cup \{3\} \quad S = (-\infty, 3) \cup (3, 4) \cup \{4\}$$

Lecture 28

Logarithm properties:

$$1) b^{\log_b x} = x$$

$$\ln x = x$$

$$\log_b b^x = x$$

$$(f \circ f)(x) = x$$

$$(f \circ f)(x) = x$$

$$2) \log_b b^x = x$$

$$\ln e^x = x$$

$$3) \log_b a = \frac{\ln a}{\ln b}$$

$$4) \log_b ac = \log_b a + \log_b c$$

$$5) \log_b \frac{a}{c} = \log_b a - \log_b c$$

$$\log_b \frac{a}{c} \neq \log_b a - \log_b c$$

$$6) \log_b a^r = r \log_b a$$

$$\Rightarrow \log_b x = \log_b y \Leftrightarrow x = y$$

Ex: Find the values of x

$$1) \log \sqrt{x} = 2 \rightarrow 10^2 = \sqrt{x} \rightarrow (10^2)^2 = (\sqrt{x})^2 \rightarrow x = 10^4$$

$$10^{\log \sqrt{x}} = 10^2 \rightarrow \sqrt{x} = 10^2 \rightarrow x = 10^4$$

$$2) \log(x-2) + \log(2x-3) = 2 \log x$$

$$\log(x-2)(2x-3) = \log x^2$$

$$(x-2)(2x-3) = x^2$$

$$2x^2 - 3x - 4x + 6 = x^2$$

$$x^2 - 7x + 6 = 0$$

$$\Rightarrow (x-1)(x-6) = 0 \rightarrow x = 1, 6$$

$$x = 6$$

$$\log x \neq 0$$

$$x > 0$$

$$x \neq 1$$

$$(1)^2 \rightarrow 1 + 6 = 7$$

Lecture 29

Ex: Find the value of x :

$$\textcircled{1} \log_x 4 = 2 \Rightarrow x^2 = 4 \rightarrow \sqrt{x^2} = 2 \quad x = 2 \checkmark$$

$$|x| = 2 \rightarrow x = \pm 2 \rightarrow x = -2 \times$$

$$\textcircled{2} 5^x = 7$$

$\log_5 5^x = \log_5 7$ $x = \log_5 7$	$\textcircled{1} \quad \ln 5^x = \ln 7$ $x \ln 5 = \ln 7$ $x = \frac{\ln 7}{\ln 5} \rightarrow \log_5 7$	$\textcircled{3} \quad x = \log_5 7$
--	--	--------------------------------------

$$\textcircled{3} (x^2 - 1)(x - 5) \cdot x^3 \cdot \log_x 2x = 0$$

$$x^2 - 1 = 0 \rightarrow x = \pm 1 \quad \text{Note: } x=1 \text{ is not allowed for log base}$$

$$x - 5 = 0 \rightarrow x = 5 \checkmark$$

$$x^3 = 0 \rightarrow x = 0 \times$$

$$\log_x 2x = 0 \rightarrow x^0 = 2x \rightarrow 2x = 1 \rightarrow x = \frac{1}{2} \checkmark$$

$$x = 5, \frac{1}{2}$$

$$\textcircled{4} e^{2x} - 6e^x - 7 = 0$$

$$(e^x)^2 - 6e^x - 7 = 0$$

$$e^x(e^x + 1)(e^x - 7) = 0$$

$$e^x \neq 0 \quad e^x = 7 \rightarrow \ln e^x = \ln 7$$

$$x = \ln 7$$

$$e^x \neq 0$$

$$⑤ \quad -2 \ln x < 2$$

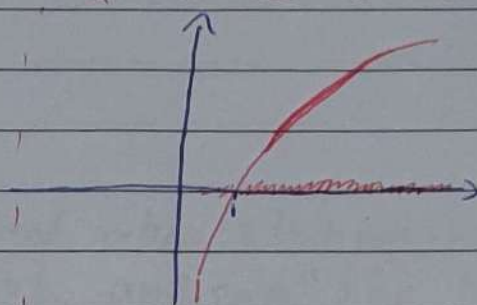
$$\frac{-2 \ln x}{-2} < \frac{2}{-2}$$

$$\ln x > -1$$

بما أن $\ln x > -1$ (بما أن $\ln x > -1$)

$$e^{\ln x} > e^{-1}$$

$$x > \frac{1}{e}$$



$$\ln x > 0 \rightarrow x > 1$$

$$\ln x < 0 \rightarrow 0 < x < 1$$

$$\frac{1}{\ln 2} x > \frac{5}{\ln 2}$$

$$x > \frac{5}{\ln 2}$$

Ex: $f(x) = \ln(x-1) + 5$ find $f^{-1}(x)$?

$$y = \ln(x-1) + 5$$

$$y - 5 = \ln(x-1)$$

$$e^{y-5} = x-1$$

$$e^{y-5} + 1 = x$$

$$x = e^{y-5} + 1$$

$$\downarrow$$

$$f^{-1}(x)$$

$$f^{-1}(x) = e^{x-5} + 1$$

Ex: find the value of:

$$① \log_2 8 + \log_4 2$$

$$② \ln(\ln e^b) = \ln(b)$$

$$P = \frac{\ln 8}{\ln 2} + \frac{\ln 2}{\ln 4}$$

Lecture 30

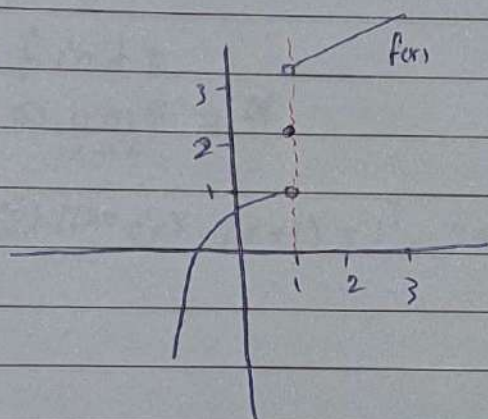
The limit of a function:

Definition: Suppose $f(x)$ is defined when x is near the number a then we write $\lim_{x \rightarrow a} f(x) = L$ and say "the limit of $f(x)$ as x goes to a equals L "

If $\lim_{x \rightarrow a^+} f(x) = L$ and $\lim_{x \rightarrow a^-} f(x) = L$ then $\lim_{x \rightarrow a} f(x) = L$ exist

If $\lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x)$ then $\lim_{x \rightarrow a} f(x)$ does not exist.

Ex:



Find:

① $f(1) = 2$

② $\lim_{x \rightarrow 1^+} f(x) = 3$

③ $\lim_{x \rightarrow 1^-} f(x) = 1$

④ $\lim_{x \rightarrow 1} f(x) = \text{does not exist}$

$\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$

does not exist

* Limit laws: suppose that c is a constant and the limits $\lim_{x \rightarrow a} f(x)$, $\lim_{x \rightarrow a} g(x)$, then:

① $\lim_{x \rightarrow a} (f(x) \pm g(x)) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$

② $\lim_{x \rightarrow a} (c f(x)) = c \lim_{x \rightarrow a} f(x)$

③ $\lim_{x \rightarrow a} (f(x) \cdot g(x)) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$

④ $\lim_{x \rightarrow a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ if $\lim_{x \rightarrow a} g(x) \neq 0$

$$5) \lim_{x \rightarrow a} c = c$$

$$6) \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}, \text{ } n \text{ positive integer. (If } n \text{ even we assume that } \lim_{x \rightarrow a} f(x) > 0 \text{)}$$

~~7) limit~~

$$\Rightarrow \lim_{x \rightarrow a} f(x) = f(a), \text{ If } f \text{ is a polynomial or a rational function and } a \in D_f$$

$$8) \text{ If } f(x) \leq g(x) \text{ then } \lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$$

Ex: find:

$$\textcircled{1} \lim_{x \rightarrow 5} 9 = 9$$

$$\textcircled{2} \lim_{x \rightarrow 1} (x^3 - 2x + 1) = 1^3 - 2(1) + 1 = 0$$

Lecture 31

$$\lim_{x \rightarrow a} \frac{f}{g}$$

كسر
مقام
مقام

$$\lim_{x \rightarrow a} \frac{f}{g} = \text{كسر}$$

$$\text{Ex: } \lim_{x \rightarrow -2} \frac{2x+4}{3x-2} = \frac{2(-2)+4}{3(-2)-2} = \frac{0}{-8} = 0$$

$$\lim_{x \rightarrow a} \frac{f}{g} = \text{مقام}$$

تقريب
القوت
بمواضع
القيمة
المادة
التعريف

Ex: Find

$$1) \lim_{x \rightarrow -4} \frac{2x+8}{x^2+x-12} = \lim_{x \rightarrow -4} \frac{2(x+4)}{(x+4)(x-3)} = \frac{2}{-7} = \boxed{\frac{-2}{7}}$$

$$2) \lim_{x \rightarrow 0} \frac{x}{1-\sqrt{x+1}} \left(\frac{1+\sqrt{x+1}}{1+\sqrt{x+1}} \right)$$

$$= \lim_{x \rightarrow 0} \frac{x(1+\sqrt{x+1})}{1-(x+1)} = \lim_{x \rightarrow 0} \frac{x(1+\sqrt{x+1})}{-x} = -1(1+\sqrt{0+1}) = -1(2)$$

$$= -2$$

$$3) \lim_{x \rightarrow -4} \frac{\frac{1}{4} + \frac{1}{x}}{4+x} = \lim_{x \rightarrow -4} \frac{\frac{1 \cdot x + 4(1)}{4x}}{4+x} = \lim_{x \rightarrow -4} \frac{x+4}{4x} \div (4+x)$$

$$= \lim_{x \rightarrow -4} \frac{x+4}{4x} \times \frac{1}{4+x} = \lim_{x \rightarrow -4} \frac{1}{4x} = \boxed{\frac{-1}{16}}$$

Lecture 32

Ex: Find

$$\textcircled{1} \lim_{x \rightarrow 1} \frac{\sqrt[4]{x+15} - 2}{x-1} \quad \left(\frac{0}{0}\right)$$

$$\text{let } y = \sqrt[4]{x+15} \Rightarrow y^4 - 15 = x$$

$$\lim_{y \rightarrow 2} \frac{y-2}{y^4-15-1} = \lim_{y \rightarrow 2} \frac{y-2}{y^4-16}$$

$$\begin{matrix} x \rightarrow 1 \\ y \rightarrow 2 \end{matrix}$$

$$= \lim_{y \rightarrow 2} \frac{y-2}{(y^2-4)(y^2+4)} = \lim_{y \rightarrow 2} \frac{\cancel{y-2}}{(\cancel{y-2})(y+2)(y^2+4)}$$

$$= \frac{1}{(2+2)(4+4)} = \frac{1}{32}$$

$$\textcircled{2} \lim_{x \rightarrow 0^+} \frac{|x|}{x} \quad \left(\frac{0}{0}\right)$$

$$\begin{matrix} |x| \\ x < 0 \end{matrix} \leftarrow \frac{-x}{-} \quad \begin{matrix} x \\ x > 0 \end{matrix} \leftarrow \frac{+x}{+}$$

$$= \lim_{x \rightarrow 0^+} \frac{x}{x} = \lim_{x \rightarrow 0^+} 1 = 1$$

$$\lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1 \rightarrow \lim_{x \rightarrow 0} \frac{|x|}{x} \text{ d.n.e.}$$

$$\textcircled{3} \lim_{x \rightarrow 2} \frac{|x-3|}{2x-6} = \frac{|2-3|}{2(2)-6} = \frac{1}{-2}$$

$$\textcircled{4} \lim_{x \rightarrow 3^+} \frac{|x-3|}{2x-6} = \lim_{x \rightarrow 3^+} \frac{x-3}{2x-6}$$

$$\begin{matrix} |x-3| \\ x = 3 \end{matrix}$$

$$\leftarrow \frac{3-x}{-} \quad \frac{x-3}{+} \rightarrow$$

$$= \lim_{x \rightarrow 3^+} \frac{x-3}{2(x-3)} = \frac{1}{2}$$

lecture 33

$$\lim_{x \rightarrow a} \frac{f}{g} = \frac{0}{0} \begin{cases} \rightarrow +\infty \\ \rightarrow -\infty \\ \rightarrow \text{d.n.e} \end{cases}$$

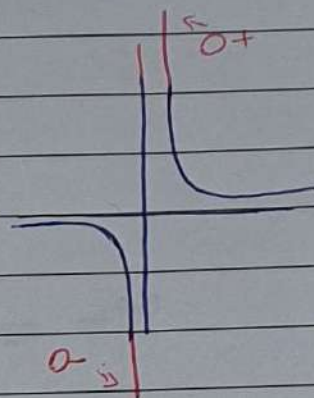
Ex: Find

$$\textcircled{1} \lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$$

1 $\leftarrow \begin{array}{c} + + + + \end{array} \rightarrow$

$x=0$ $\leftarrow \begin{array}{c} - - + + \end{array} \rightarrow$
0

$\frac{1}{x}$ $\leftarrow \begin{array}{c} - - + + \end{array} \rightarrow$
0



$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x \rightarrow 0} \frac{1}{x} = \text{d.n.e}$$

$$\textcircled{2} \lim_{x \rightarrow 2} \frac{x}{x^2-4} = \text{d.n.e}$$

$x < 0$ $\leftarrow \begin{array}{c} - + \end{array} \rightarrow$
0

$x > \pm 2$ $\leftarrow \begin{array}{c} + - + \end{array} \rightarrow$
-2 2

$\frac{x}{x^2-4}$ $\leftarrow \begin{array}{c} + - + \end{array} \rightarrow$
-2 0 2

$$\lim_{x \rightarrow 2^+} \frac{x}{x^2-4} = +\infty$$

$$\lim_{x \rightarrow 2^-} \frac{x}{x^2-4} = -\infty$$

النقطة

$$\textcircled{3} \lim_{x \rightarrow 0} \frac{x}{x^2-4} = \frac{0}{0-4} = \frac{0}{-4} = 0$$

$$\textcircled{3} \lim_{x \rightarrow 5} \frac{1}{(x-5)^2} = +\infty$$

$$\begin{array}{c} 1 \quad \leftarrow + + + \\ (x-5)^2 \rightarrow 0 \quad \leftarrow + + \\ x < 5 \quad \quad \quad 5 \end{array}$$

$$\frac{1}{(x-5)^2} \leftarrow + + \rightarrow$$

$$\lim_{x \rightarrow 5+} \frac{1}{(x-5)^2} = +\infty$$

$$\lim_{x \rightarrow 5-} \frac{1}{(x-5)^2} = +\infty$$

$$\textcircled{4} \lim_{x \rightarrow 2\pi-} x \cdot \csc x = \lim_{x \rightarrow 2\pi-} \frac{x}{\sin x} \quad \left(\frac{\infty}{0} \right) = -\infty$$

$$x > 0 \quad \leftarrow - - + + + \rightarrow$$

$$\sin x > 0 \quad \leftarrow + + - - - \rightarrow$$

$$\begin{array}{l} x > 0 \\ x < \pi \\ x < 2\pi \\ x > 3\pi \end{array} \quad \left| \quad \frac{x}{\sin x} \leftarrow + + - - - \rightarrow \right.$$

$$\textcircled{5} \lim_{x \rightarrow \pi-} \cot x$$

Lecture 34

Definition: The line $x = a$ is called a **Vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \rightarrow a} f(x) = \infty$$

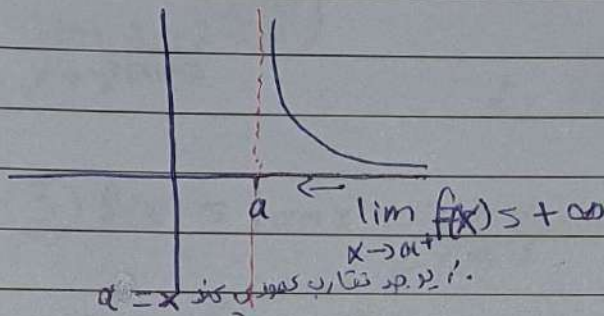
$$\lim_{x \rightarrow a^-} f(x) = \infty$$

$$\lim_{x \rightarrow a^+} f(x) = \infty$$

$$\lim_{x \rightarrow a} f(x) = -\infty$$

$$\lim_{x \rightarrow a^-} f(x) = -\infty$$

$$\lim_{x \rightarrow a^+} f(x) = -\infty$$



Ex: Find the Vertical asymptote of:

$$f(x) = \frac{x-1}{x^2-1}$$

الخطوط المقع

$$x^2-1 = 0 \Rightarrow x = \pm 1$$

$$x \neq \pm 1$$

$$\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} = \lim_{x \rightarrow 1} \frac{x-1}{(x-1)(x+1)} = \frac{1}{2}$$

$$x \neq -1$$

$$\lim_{x \rightarrow -1} \frac{x-1}{x^2-1} = \lim_{x \rightarrow -1} \frac{1}{x+1} = \text{d.n.e}$$

$\therefore x = -1$ vertical asymptote

إذا كان المقام
صفرًا والعدد
ليس صفرًا

$$\begin{matrix} \leftarrow + + + \rightarrow \\ \leftarrow - + \rightarrow \\ -\infty \quad -1 \quad +\infty \end{matrix}$$

$$\textcircled{2} f(x) = \frac{x-2}{|x|-2}$$

$$|x|-2 = 0 \rightarrow x = -2, +2$$

$$\text{at } x = 2$$

$$\lim_{x \rightarrow 2} \frac{x-2}{|x|-2} = \frac{x-2}{x-2} = 1$$

$$|x| \begin{matrix} \leftarrow -x & x \\ x < 0 & 0 & + \end{matrix}$$

$$\text{at } x = -2$$

$$\lim_{x \rightarrow -2} \frac{x-2}{|x|-2} \left(\frac{-0}{0} \right)$$

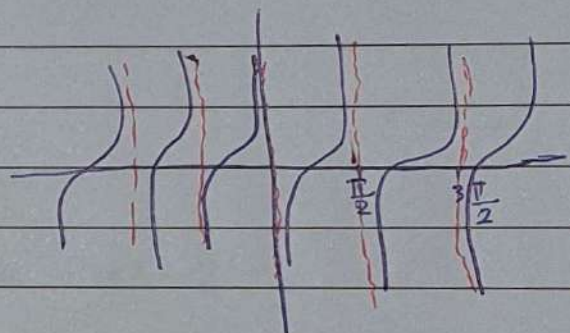
$\therefore$ at $x = -2$ vertical asymptote

$$\textcircled{3} f(x) = \tan x = \frac{\sin x}{\cos x}$$

$$x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots$$

$$\text{at } x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots \text{ vertical asymptote}$$

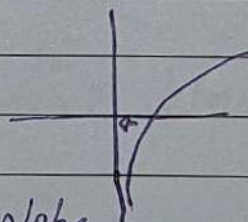
$$x = \pm (2n+1) \frac{\pi}{2}$$



$$\textcircled{4} f(x) = \ln x$$

$$\lim_{x \rightarrow 0^+} \ln x = -\infty$$

$\therefore$ at $x = 0$ vertical asymptote



$$\textcircled{5} f(x) = \ln(x^2 - a)$$

$$x^2 - a = 0 \rightarrow x = \pm \sqrt{a}$$

at $x = \pm \sqrt{a}$ vertical asymptote

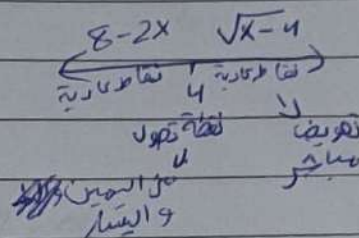
$$\textcircled{6} f(x) = \cot x$$

lecture 35

Exp

$$f(x) = \begin{cases} \sqrt{x-4}, & x > 4 \\ 8-2x, & x \leq 4 \end{cases}$$

~~find~~



find:

$$\textcircled{1} \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (8 - 2x) = 8 - 2(0) = 8$$

$$\textcircled{2} \lim_{x \rightarrow 4} f(x) = 0$$

$$\lim_{x \rightarrow 4+} f(x) = \lim_{x \rightarrow 4+} \sqrt{x-4} = \sqrt{4-4} = 0$$

$$\lim_{x \rightarrow 4-} f(x) = \lim_{x \rightarrow 4-} (8 - 2x) = 8 - 2(4) = 0$$

$$\text{Ex: } g(x) = \begin{cases} x+1, & x \neq 1 \\ 1, & x = 1 \end{cases}$$

$$\lim_{x \rightarrow 1+} x+1 = \lim_{x \rightarrow 1-} x+1 = \lim_{x \rightarrow 1} g(x) = 1+1 = 2$$

$$\text{Ex: } f(x) = \begin{cases} 5, & x \leq 2 \\ x+a, & x > 2 \end{cases} \text{ find } a \text{ if } \lim_{x \rightarrow 2} f(x) \text{ exist?}$$

$$\lim_{x \rightarrow 2-} f(x) = \lim_{x \rightarrow 2+} f(x)$$

$$5 = 2+a$$

$$a = 3$$

Ex: find

$$① \lim_{x \rightarrow 0^-} \sqrt{x} \text{ d.n.e}$$

$$[0, \infty) \leftarrow \text{|||||}$$

$$② \lim_{x \rightarrow 1} \sqrt{x} = \sqrt{1} = 1$$

$$③ \lim_{x \rightarrow 2^+} \sqrt{x-2} = \sqrt{0} = 0$$

$$[2, \infty)$$

$$④ \lim_{x \rightarrow 2} \sqrt{x-2} \text{ d.n.e}$$

$$\text{Theorem: } \lim_{x \rightarrow 0} \frac{\sin(ax)}{bx} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0} \frac{ax}{\sin(bx)} = \frac{a}{b}$$

$$\lim_{x \rightarrow 0} \frac{\tan(ax)}{bx} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0} \frac{ax}{\tan(bx)} = \frac{a}{b}$$

Ex: Find

$$1) \lim_{x \rightarrow 0} \frac{x}{\cos x} = \frac{0}{\cos 0} = \frac{0}{1} = 0$$

$$2) \lim_{x \rightarrow 0} \frac{\sin x}{x} \stackrel{\frac{0}{0}}{=} 1$$

$$3) \lim_{x \rightarrow 0} \frac{\tan 4x}{5x} \stackrel{\frac{0}{0}}{=} \frac{4}{5}$$

$$4) \lim_{x \rightarrow 0} \frac{x \sin x - \tan(2x)}{\sin(3x) - 5x} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0} \frac{\cancel{x} \sin x - \tan(2x)}{\frac{\sin(3x)}{x} - \frac{5x}{x}}$$

$$= \frac{0 - 2}{3 - 5} = \frac{-2}{-2} = 1$$

$$5) \lim_{x \rightarrow 5} \frac{\sin(x-5)}{x-5}$$

$$\left\{ \begin{array}{l} \text{let } y = x-5, \quad x \rightarrow 5 \\ y \rightarrow 0 \end{array} \right.$$

$$\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$$

$$6) \lim_{x \rightarrow 5} \frac{\sin(x^2-25)}{x+5} \left(\frac{x+5}{x+5} \right)$$

$$= \lim_{x \rightarrow 5} x+5 \cdot \lim_{x \rightarrow 5} \frac{\sin(x^2-25)}{x^2-25}$$

$$\left\{ \begin{array}{l} \text{let } y = x^2-25 \\ x \rightarrow 5 \\ y \rightarrow 0 \end{array} \right.$$

$$= 10 \cdot \lim_{y \rightarrow 0} \frac{\sin y}{y}$$

$$= 10$$

lecture 3.6

The Squeeze Theorem: If $f(x) \leq g(x) \leq h(x)$ when x is near a and $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$

then $\lim_{x \rightarrow a} g(x) = L$

Ex B IF $3x \leq f(x) \leq x^3 + 2$ $\forall x \in (-2, 2)$
find $\lim_{x \rightarrow 1} f(x)$?

$$\lim_{x \rightarrow 1} 3x = 3, \quad \lim_{x \rightarrow 1} x^3 + 2 = 3$$

by the squeeze theorem

$$\therefore \lim_{x \rightarrow 1} f(x) = 3$$

show that

$$\text{Ex: Find } \lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) = 0?$$

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

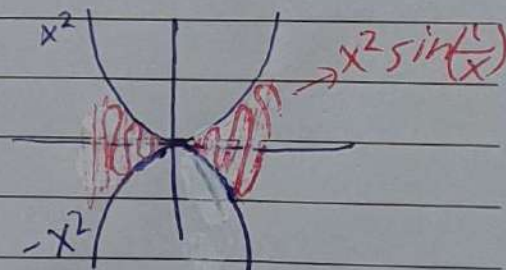
$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

$f(x)$ $g(x)$ $h(x)$

$$\lim_{x \rightarrow 0} -x^2 = 0, \quad \lim_{x \rightarrow 0} x^2 = 0$$

$\therefore$ by squeeze theorem

$$\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$



Lecture 37

Continuity

استمرارية

Definition: continuity at number في نقطة f is continuous at $x=a$ if1) $f(a)$ defined2) $\lim_{x \rightarrow a} f(x)$ exist3) $\lim_{x \rightarrow a} f(x) = f(a)$ Ex: Is $f(x)$ continuous at $x=3$

$$\textcircled{1} f(x) = \begin{cases} \frac{x^2-9}{x-3} & , x \neq 3 \\ 6 & , x = 3 \end{cases}$$

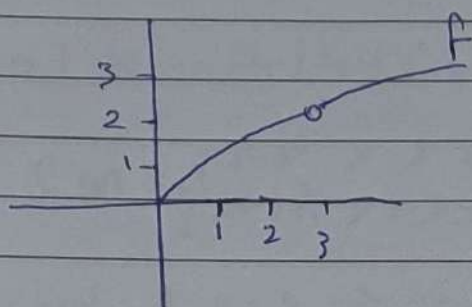
$$\textcircled{1} f(3) = 6$$

$$\textcircled{2} \lim_{x \rightarrow 3} \frac{x^2-9}{x-3} = \lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{x-3} = 6$$

$$\textcircled{3} f(3) = \lim_{x \rightarrow 3} f = 6$$

 $\therefore f$ cont at $x=3$

②



1) $f(3)$ undefined

$\therefore f$ not cont (discont) at $x=3$

Theorems

The following types of functions are ~~continuous~~ continuous at every number in their domains.

~~Polynomials, rational functions~~

Polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, exponential functions, logarithmic functions.

Ex: when f is continuous:

$$f(x) = \frac{\ln x + \tan^{-1} x}{x^2 - 9} \xrightarrow{(0, \infty)} \mathbb{R} \xrightarrow{(0, \infty)} \mathbb{R}$$

(القسمة) $\neq \pm 3$

$$D_f = (0, \infty) \cap \mathbb{R} \setminus \{\pm 3\} = (0, \infty) \setminus \{\pm 3\}$$

$\therefore f$ cont on $(0, \infty) \setminus \{\pm 3\}$

or $\mathbb{R}$

Ex: Find k such that f continuous every where

$$\textcircled{1} f(x) = \begin{cases} kx^2, & x \leq 2 \\ 2x+k, & x > 2 \end{cases}$$

$$\because f \text{ cont at } x=2 \rightarrow \lim_{x \rightarrow 2} f \text{ exist} \Rightarrow \lim_{x \rightarrow 2^+} 2x+k = \lim_{x \rightarrow 2^-} kx^2$$

$$2(2)+k \leq k \cdot 4$$

$$4+k \leq 4k$$

$$4 \leq 4k - k$$

$$4 \leq 3k$$

$$k \geq \frac{4}{3}$$

$$\textcircled{2} f(x) = \begin{cases} \frac{\sin(kx)}{5x}, & x < 0 \\ 2, & x \geq 0 \end{cases}$$

$$\lim_{x \rightarrow 0^+} 2 \leq \lim_{x \rightarrow 0^-} \frac{\sin(kx)}{5x} \Rightarrow 2 \leq \frac{k}{5} \Rightarrow k \geq 10$$

Ex: where f is not cont (dis cont)

$$1) f(x) = \frac{1}{\sqrt{x-2}} \begin{matrix} \rightarrow \mathbb{R} \\ \rightarrow [2, \infty) \end{matrix}$$

def, $\forall x \geq 2$

$$D_f \subseteq \mathbb{R} \cap [2, \infty) \setminus \{2\} = (2, \infty)$$

f cont on $(2, \infty)$

f not cont $\subseteq (-\infty, 2] \subseteq \mathbb{R} \setminus (2, \infty)$

~~$$\frac{x+16}{x} \cdot \frac{2x+3}{x}$$~~

$(-\infty, 4) \rightarrow 2x+3$ poly cont
 $(4, \infty) \rightarrow 7 + \frac{16}{x+4}$ cont on
 $\mathbb{R} \setminus \{-4\}$

$2x+3$
 $2 + \frac{16}{x+4}$

$$f(4) = 2(4) + 3 = 11$$

$$\lim_{x \rightarrow u^+} f \neq f(u) \quad \therefore f \text{ not cont at } x = u$$

f cont on $\mathbb{R} \setminus \{4\}$

Lecture 38

continuity at an interval

الاتصال على فترة

~~f is continuous~~f is continuous on $[a, b]$ if

① f cont on (a, b)

① الفترة المفتوحة

② $\lim_{x \rightarrow a^+} f(x) = f(a)$

② الطرف الأيمن من الفترة المغلقة

③ $\lim_{x \rightarrow b^-} f(x) = f(b)$

③ نقاط التماس

Ex: $f(x) = \begin{cases} x^2 - 5x, & 0 \leq x < 1 \\ \frac{1}{x}, & 1 \leq x \leq 4 \end{cases}$

is f continuous on $[0, 4]$

$$\begin{array}{c} x^2 - 5x \quad \frac{1}{x} \\ \hline 0 \quad 1 \quad 4 \end{array}$$

الفترة المفتوحة
 $(0, 1) \rightarrow f(x) = x^2 - 5x$ poly cont on $\mathbb{R}$, cont $(0, 1)$

الفترة المفتوحة
 $(1, 4) \rightarrow f(x) = \frac{1}{x}$ cont on $\mathbb{R} \setminus \{0\}$ $0 \notin (1, 4)$, f cont on $(1, 4)$

نقاط التماس
at $x=1$

1) $f(1) = \frac{1}{1} = 1$

2) $\lim_{x \rightarrow 1^+} f(x) = \frac{1}{1} = 1$

3) $\lim_{x \rightarrow 1^-} f(x) = 1 - 5 = -4 \quad \therefore \lim_{x \rightarrow 1} f(x) \text{ d.n.e}$

f not cont at $x=1$

الطرف الأيسر

at $x=0$

1) $f(0) = 0$

2) $\lim_{x \rightarrow 0^+} (x^2 - 5x) = 0$

3) $\lim_{x \rightarrow 0^+} f(x) = 0 \quad \therefore f \text{ cont at } x=0$

at $x=4$

1) $f(4) = \frac{1}{4}$ 2) $\lim_{x \rightarrow 4^-} \frac{1}{x} = \frac{1}{4}$

3) $f(4) = \lim_{x \rightarrow 4^-} f(x) = \frac{1}{4}$

 $\therefore f \text{ cont at } x=4$ f cont on $[0, 4]$

⇒

Theorem: If f and g are continuous at a and c is a constant, then the following functions are also continuous at $x = a$:

- 1) $f + g$ 2) $f - g$ 3) cf 4) $\frac{f}{g}$ 5) $\frac{f}{g}$ if $g(a) \neq 0$

Ex:

$$f(x) = 5x^2$$

$$g(x) = x - 1$$

is $\frac{f}{g}$ cont at $x = 2$?

$f(x) = 5x^2$ poly cont on $\mathbb{R}$. cont at $x = 2$

$g(x) = x - 1$ " " " " " " "

by theorem $\frac{f}{g}$ cont at $x = 2$, $g(2) \neq 0 \neq 1$

Lecture (39)

 $f(g(x))$

Theorem: If 1) $\lim_{x \rightarrow c} g(x) = L$
 2) f continuous at $x=L$ then

$$\lim_{x \rightarrow c} f(g(x)) = f(\lim_{x \rightarrow c} g(x)) = f(L)$$

Ex: ① $\lim_{x \rightarrow 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right) = \sin^{-1}\left(\lim_{x \rightarrow 1} \frac{1-\sqrt{x}}{1-x}\right)$ $f = \sin^{-1}$
 $g = \frac{1-\sqrt{x}}{1-x}$ cont at $x = \frac{1}{2}$

$$= \sin^{-1}\left(\lim_{x \rightarrow 1} \frac{1-\sqrt{x}}{1-x} \left(\frac{1+\sqrt{x}}{1+\sqrt{x}}\right)\right) = \sin^{-1}\left(\lim_{x \rightarrow 1} \frac{1-x}{1-x} \left(\frac{1}{1+\sqrt{x}}\right)\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

② $\lim_{x \rightarrow 1} e^{\frac{x^2-1}{x-1}} = e^{\lim_{x \rightarrow 1} \frac{x^2-1}{x-1}} = e^{\lim_{x \rightarrow 1} \frac{(x+1)(x-1)}{x-1}} = e^2$ نقطة
و هنا نكون
في الشروط

③ $\lim_{x \rightarrow 1} \cos\left(\frac{x^2-1}{x-1}\right)$ (H.W)

Theorem: If 1) g continuous at $x=c$
 2) f continuous at $x=g(c)$

Then $f \circ g$ cont at $x=c$

Ex: $f(x) = \frac{1}{x} \rightarrow$ cont on $\mathbb{R} \setminus \{0\}$

$g(x) = \frac{1}{2x} \rightarrow$ cont on $\mathbb{R} \setminus \{0\}$

Is $f \circ g$ cont at $x=0$

① g cont at $x=0$? No $\Rightarrow g$ not cont at $x=0$

$\therefore f \circ g$ not cont at $x=0$

→ 2) is $f \circ g$ cont at $x=1$

① g cont at $x=1$? yes g cont at $x=1$

② f cont at $x=g(1)=\frac{1}{2}$? No f not cont at $x=\frac{1}{2}$

$\therefore f \circ g$ not cont at $x=1$

3) Is $f \circ g$ cont at $x=2$

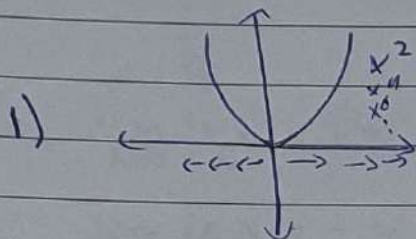
① g cont at $x=2$? yes

② f cont at $g(2)=\frac{1}{4}$? Yes

$f \circ g$ cont at $x=2$

Lecture 41

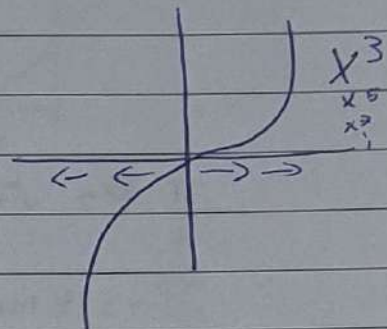
Limits of infinity



$$\rightarrow \lim_{x \rightarrow +\infty} x^2 = +\infty$$

$$\rightarrow \lim_{x \rightarrow -\infty} x^2 = +\infty$$

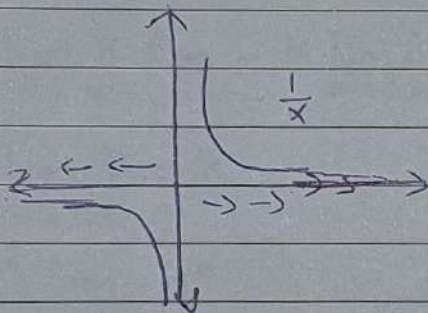
2)



$$\rightarrow \lim_{x \rightarrow +\infty} x^3 = +\infty$$

$$\rightarrow \lim_{x \rightarrow -\infty} x^3 = -\infty$$

3)



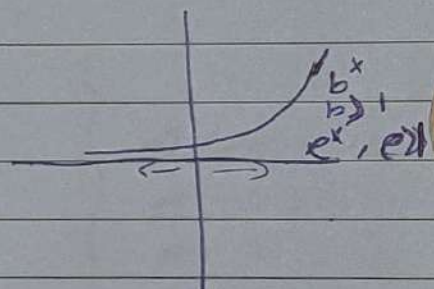
$$\rightarrow \lim_{x \rightarrow +\infty} \frac{1}{x} = 0$$

$$\rightarrow \lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

$$\rightarrow \lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$$

$$\rightarrow \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

4)



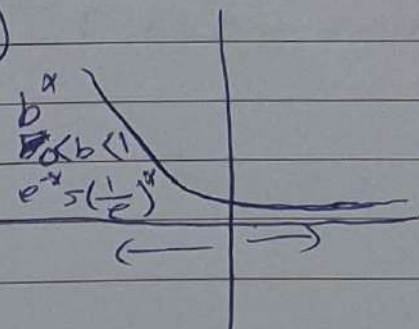
$$\lim_{x \rightarrow +\infty} b^x = +\infty$$

$$\lim_{x \rightarrow -\infty} b^x = 0$$

$$\lim_{x \rightarrow +\infty} e^x = +\infty$$

$$\lim_{x \rightarrow -\infty} e^x = 0$$

5)



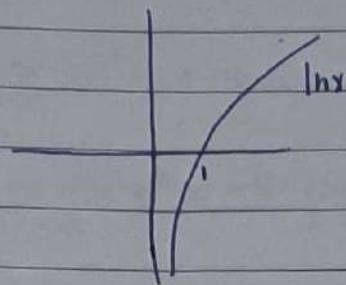
$$\lim_{x \rightarrow +\infty} e^{-x} = 0$$

$$\lim_{x \rightarrow -\infty} e^{-x} = +\infty$$

$$\lim_{x \rightarrow +\infty} b^{-x} = 0$$

$$\lim_{x \rightarrow -\infty} b^{-x} = +\infty$$

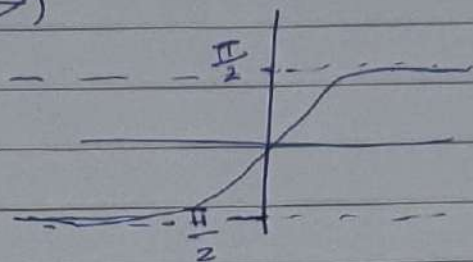
6)



$$\lim_{x \rightarrow +\infty} \ln x = +\infty$$

$$\lim_{x \rightarrow 0^+} \ln x = -\infty$$

7)



$$\lim_{x \rightarrow +\infty} \tan^{-1} x = \frac{\pi}{2}$$

$$\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2}$$

⊗ Theorem:

$$1) \lim_{x \rightarrow \pm \infty} (f(x))^n = (\lim_{x \rightarrow \pm \infty} f(x))^n \quad n \text{ positive integer}$$

$$2) \lim_{x \rightarrow \pm \infty} k f(x) = k \lim_{x \rightarrow \pm \infty} f(x)$$

$$3) \lim_{x \rightarrow \pm \infty} k = k$$

$$4) \lim_{x \rightarrow +\infty} x^n = +\infty, \quad n = 1, 2, 3, \dots$$

$$5) \lim_{x \rightarrow -\infty} x^n = \begin{cases} -\infty, & n = 1, 3, 5, \dots \\ +\infty, & n = 2, 4, 6, \dots \end{cases}$$

$$-2(+\infty) = -\infty$$

$$-3(-\infty) = +\infty$$

$$+4(+\infty) = +\infty$$

$$+1(-\infty) = -\infty$$

$$6) \lim_{x \rightarrow \pm \infty} (a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1} + \underbrace{a_n x^n}_{\text{dominant term}}) = \lim_{x \rightarrow \pm \infty} a_n x^n$$

$$7) \text{ If } r > 0 \text{ is a rational number then } \lim_{x \rightarrow \pm \infty} \frac{1}{x^r} = 0$$

→
Ex:

$$1) \lim_{x \rightarrow \infty} x^8 = +\infty$$

$$2) \lim_{x \rightarrow -\infty} -4x^2 = -4 \lim_{x \rightarrow -\infty} x^2 = -\infty$$

$$3) \lim_{x \rightarrow -\infty} -3x^5 = -3 \lim_{x \rightarrow -\infty} x^5 = +\infty$$

$$4) \lim_{x \rightarrow -\infty} (4x^2 - 5x^6 + 1) = \lim_{x \rightarrow -\infty} -5x^6 = -5 \lim_{x \rightarrow -\infty} x^6 = -\infty$$

$$5) \lim_{x \rightarrow -\infty} 2^x = 0$$

$$6) \lim_{x \rightarrow +\infty} 3^{-x} = \lim_{x \rightarrow +\infty} (3^{-1})^x = \lim_{x \rightarrow +\infty} \left(\frac{1}{3}\right)^x = 0$$

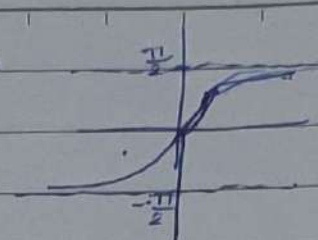
$$7) \lim_{x \rightarrow \infty} \left(\frac{\pi}{e}\right)^x = +\infty$$

$$8) \lim_{x \rightarrow -\infty} 5 \frac{1}{x^4} = 5 \lim_{x \rightarrow -\infty} \frac{1}{x^4} = 0$$

$$9) \lim_{x \rightarrow \infty} e^{-2/x} = e^{-2 \lim_{x \rightarrow \infty} \frac{1}{x}} = e^{-2(0)} = e^0 = 1$$

Lecture (42)

tan



Exo

$$1) \lim_{x \rightarrow \infty} \tan^{-1}(e^x) \stackrel{+}{\sim} \tan^{-1}(\lim_{x \rightarrow \infty} e^x) = \frac{\pi}{2}$$

$$2) \lim_{x \rightarrow 0^+} \tan^{-1}(\ln x) \stackrel{-}{\sim} \tan^{-1}(\lim_{x \rightarrow 0^+} \ln x) = -\frac{\pi}{2}$$

$$3) \lim_{x \rightarrow \infty} \frac{5 + x^2 - 3x}{5 + 10x} \stackrel{+}{\sim} \lim_{x \rightarrow \infty} \frac{x^2}{10x} \stackrel{+}{\sim} \frac{1}{10} \lim_{x \rightarrow \infty} x = \infty$$

$$4) \lim_{x \rightarrow -\infty} \frac{5 - x^2}{3 + 2x^3 + 10x} \stackrel{-}{\sim} \lim_{x \rightarrow -\infty} \frac{-x^2}{2x^3} \stackrel{-}{\sim} -\frac{1}{2} \lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

كثيرا : $\frac{\infty}{\infty}$
كثيرا : $\frac{0}{0}$

$$5) \lim_{x \rightarrow \infty} \frac{3 + 4x^5 - 2x^3}{4 + x^5} \stackrel{+}{\sim} \lim_{x \rightarrow \infty} \frac{4x^5}{x^5} \stackrel{+}{\sim} \lim_{x \rightarrow \infty} 4 = 4$$

(1) $\frac{-\infty}{+\infty} < \frac{0}{0} < \frac{+\infty}{+\infty}$
(2) $\frac{+\infty}{-\infty} < \frac{0}{0} < \frac{-\infty}{-\infty}$

$$6) \lim_{x \rightarrow +\infty} \ln\left(\frac{x-1}{x}\right) \stackrel{+}{\sim} \ln\left(\lim_{x \rightarrow +\infty} \frac{x-1}{x}\right) = \ln 1 = 0$$

$$= \ln\left(\lim_{x \rightarrow +\infty} \frac{x}{x}\right) = \ln 1 = 0$$

$$7) \lim_{x \rightarrow \infty} \frac{1 - e^x}{1 + 2e^x} \stackrel{\frac{-\infty}{+\infty}}{\sim}$$

let $y = e^x, x \rightarrow +\infty$
 $y \rightarrow \infty$

$$\lim_{y \rightarrow \infty} \frac{1 - y}{1 + 2y} \stackrel{-}{\sim} \lim_{y \rightarrow \infty} \frac{-y}{2y} = -\frac{1}{2}$$

H.W

$$q) \lim_{x \rightarrow \infty} \frac{e^{-x} - 3e^x}{e^{-2x} + 5e^x}$$

$$8) \lim_{x \rightarrow -\infty} \frac{e^{2x} - e^{-x}}{e^{3x} - e^{-2x}} \stackrel{\frac{\infty}{\infty}}{\sim}$$

$y = e^x, x \rightarrow -\infty$
 $y \rightarrow 0$

$$\lim_{y \rightarrow 0} \frac{y^2 - y^{-1}}{y^3 - y^{-2}} = \lim_{y \rightarrow 0} \frac{y^2 - \frac{1}{y}}{\frac{y^3 - 1}{y^2}} = \lim_{y \rightarrow 0} \frac{y^3 - 1}{y} \div \left(\frac{y^5 - 1}{y^2}\right)$$

$$= \lim_{y \rightarrow 0} \left(\frac{y^3 - 1}{y} \cdot \frac{y^2}{y^5 - 1}\right) = \lim_{y \rightarrow 0} \frac{y(y^3 - 1)}{y^5 - 1} = \lim_{y \rightarrow 0} \frac{y^4 - y}{y^5 - 1} = \frac{0 - 0}{0 - 1} = 0$$

Lecture 143

Ex:

$$1) \lim_{x \rightarrow -\infty} \frac{\sqrt{5x^2-2}}{x+3}$$

$$= \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2(5-\frac{2}{x^2})}}{x(1+\frac{3}{x})} = \frac{1}{1}$$

$$= \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2} \sqrt{5-\frac{2}{x^2}}}{x(1+\frac{3}{x})} = \lim_{x \rightarrow -\infty} \frac{|x| \sqrt{5-\frac{2}{x^2}}}{x(1+\frac{3}{x})}$$

$$= \lim_{x \rightarrow -\infty} \frac{-x \sqrt{5-\frac{2}{x^2}}}{x(1+\frac{3}{x})} = \frac{\sqrt{5-0}}{1+0} = -\sqrt{5}$$

بما ان $x < 0$
 فـ $|x| = -x$
 و $\lim_{x \rightarrow -\infty} \frac{2}{x^2} = 0$

$$\rightarrow \sqrt{x^2} = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

$$\rightarrow \sqrt{x^4} = (\sqrt{x^2})^2 = (|x|)^2 = x^2$$

$$\rightarrow \sqrt{x^6} = (\sqrt{x^2})^3 = (|x|)^3 = \begin{cases} -x^3, & x < 0 \\ x^3, & x \geq 0 \end{cases}$$

$$|x| \leftarrow \begin{array}{ccc} -x & & x \\ & \leftarrow 0 \rightarrow & \end{array}$$

$$\lim_{x \rightarrow -\infty} |x| = \lim_{x \rightarrow -\infty} -x, \quad \lim_{x \rightarrow \infty} |x| = \lim_{x \rightarrow \infty} x$$

$$\lim_{x \rightarrow -\infty} \frac{2}{x^2} = \frac{2}{\infty} = 0 \quad \left(\frac{\pm \infty}{\pm \infty} = 0 \right)$$

$$2) \lim_{x \rightarrow -\infty} \frac{\sqrt{4x^4+x}}{x^2-8} = \lim_{x \rightarrow -\infty} \frac{\sqrt{x^4(4+\frac{1}{x^3})}}{x^2(1-\frac{8}{x^2})} = \lim_{x \rightarrow -\infty} \frac{\sqrt{x^4} \sqrt{4+\frac{1}{x^3}}}{x^2(1-\frac{8}{x^2})}$$

$$= \lim_{x \rightarrow -\infty} \frac{x^2 \sqrt{4+\frac{1}{x^3}}}{x^2(1-\frac{8}{x^2})} = \frac{\sqrt{4+0}}{1-0} = \sqrt{4} = 2$$

3) ~~lim~~

$$3) \lim_{x \rightarrow +\infty} (\sqrt{x^2-3x} - x) \left(\frac{\sqrt{x^2-3x} + x}{\sqrt{x^2-3x} + x} \right)$$

$$= \lim_{x \rightarrow \infty} \frac{x^2-3x-x^2}{\sqrt{x^2-3x}+x} = \frac{-3x}{\sqrt{x^2} \sqrt{1-\frac{3}{x}} + x} = \lim_{x \rightarrow \infty} \frac{-3x}{x(\sqrt{1-\frac{3}{x}} + 1)}$$

$$= \frac{-3}{\sqrt{1-0} + 1} = \frac{-3}{2}$$

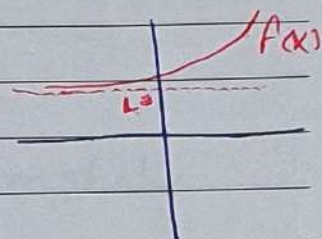
H.W

$$\lim_{y \rightarrow +\infty} \frac{2-y}{\sqrt{y}+6y^2}$$

⊗ Definitions

The line $y=L$ is called horizontal asymptote of the curve $y=f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = L$$



Ex: Find horizontal asymptote for:

1) $f(x) = \frac{\sqrt{x^2+7} - 4x}{x+3}$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2+7} - 4x}{x+3} = \lim_{x \rightarrow \infty} \frac{\sqrt{x^2} \sqrt{1+\frac{7}{x^2}} - 4x}{x(1+\frac{3}{x})} \quad \text{where } |x|=x$$

$$= \lim_{x \rightarrow \infty} \frac{x(\sqrt{1+\frac{7}{x^2}} - 4)}{x(1+\frac{3}{x})} = \frac{\sqrt{1+0} - 4}{1+0} = 1 - 4 = -3$$

∴ at $y = -3$ horizontal asymptote

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+7} - 4x}{x+3} = \lim_{x \rightarrow -\infty} \frac{-x\sqrt{1+\frac{7}{x^2}} - 4x}{x(1+\frac{3}{x})} = \lim_{x \rightarrow -\infty} \frac{-x(\sqrt{1+\frac{7}{x^2}} + 4)}{x(1+\frac{3}{x})}$$

$$= \frac{-\sqrt{1+0} - 4}{1+0} = -5$$

∴ at $y = -5$ horizontal asymptote

2) $f(x) = e^x$

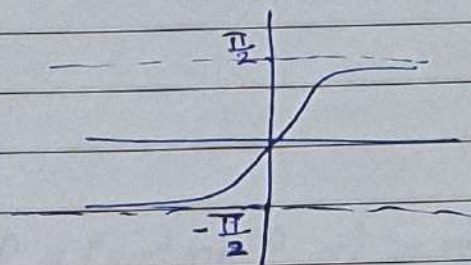
$$\lim_{x \rightarrow \infty} e^x = \infty$$

$$\lim_{x \rightarrow -\infty} e^x = 0 \quad \therefore \text{at } y = 0 \text{ horizontal asymptote}$$

3) $f(x) = \tan^{-1} x$

$$\lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2}$$

$$\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2}$$



at $y = \pm \frac{\pi}{2}$ horizontal asymptote

Lecture (44)

Derivatives: مشتقات

Definition: The derivative of a function f at a number a denoted by $f'(a)$ is:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

المشتقة الأولى	$y'(a)$
المشتقة الثانية	$\frac{dy}{dx} \Big _{x=a}$
المشتقة الثالثة	$\frac{d^2y}{dx^2} \Big _{x=a}$
المشتقة الرابعة	$\frac{d^3y}{dx^3} \Big _{x=a}$

The derivative of f with respect to x :

$$\left. \begin{array}{l} f'(x) \\ y'(x) \\ \frac{dy}{dx} \\ \frac{df}{dx} \end{array} \right\} \begin{array}{l} f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} \end{array}$$

Ex: $f(x) = x^2$ Find $f'(x)$, $f'(2)$?

$$① \quad f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{h(2x + h)}{h} = 2x + 0$$

$$= 2x$$

$$② \quad f'(2) = 2(2) = 4$$

→ Derivation rules:

$$\boxed{1} \quad f(x) = c \quad / \quad c \text{ constant} \\ f'(x) = 0$$

$$\text{Ex: } 1) f(x) = 3 \rightarrow f'(x) = 0 \\ 2) f(x) = e^3 \rightarrow f'(x) = 0$$

$$\boxed{2} \quad f(x) = x \\ f'(x) = 1$$

$$\boxed{3} \quad f(x) = x^r \\ f'(x) = r x^{r-1}$$

$$\text{Ex: } ① \quad f(x) = \sqrt{x} = x^{\frac{1}{2}} \\ f'(x) = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$$

$$② \quad f(x) = x^7 \\ f'(x) = 7x^6 \\ f'(1) = 7(1)^6 = 7$$

$$\boxed{4} \quad \frac{d}{dx}(cf(x)) = c f'(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{c}\right) = \frac{1}{c} f'(x)$$

$$\text{Ex: } ① \quad f(x) = 3x^4 \\ f'(x) = 3(4)x^{4-1} = 12x^3 \\ ② \quad f(x) = \frac{x^6}{5} \rightarrow f'(x) = \frac{6x^5}{5}$$

$$\boxed{5} (f \pm g)' = f' \pm g'$$

Ex: $f(x) = 3x^2 - 2x^4 + x$ find $f'(x)$

$$f'(x) = 21x^6 - 8x^3 + 1$$

$$\boxed{6} (f \cdot g)' = f' \cdot g + f \cdot g'$$

Ex: $y = (x^2 + 1)(x^3 - 2x)$ find $\frac{dy}{dx} \Big|_{x=1}$

$$y' = 2x(x^3 - 2x) + (x^2 + 1)(3x^2 - 2)$$

$$y'(1) = 2(1)(1^3 - 2(1)) + (1^2 + 1)(3(1)^2 - 2)$$

$$= -2 + 2 = 0$$

$$\boxed{7} \left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$$

Ex: ① $y = \frac{3x+4}{x^2+1}$ find $y'(1)$?

$$\frac{dy}{dx} = \frac{3(x^2+1) - 2x(3x+4)}{(x^2+1)^2}$$

$$y'(1) = \frac{3(2) - 2(2)}{(2)^2} = \frac{6 - 4}{4} = \frac{2}{4} = \frac{1}{2}$$

② $f(2) = -3$, $f'(2) = 4$, $g(2) = 1$, $g'(2) = 5$ Find

$$a) \left(3 \frac{f}{g}\right)'(2) = \frac{3 \cdot f'(2) \cdot g(2) - 3g'(2) \cdot f(2)}{(g(2))^2}$$

$$= \frac{3(4)(1) - (5)(-3)}{1^2} = \frac{12 + 15}{1} = 27$$

$$b) \left(3 \frac{f}{g}(2)\right)' = 0$$

constant

Lecture 45

$$\Rightarrow \boxed{8} \left(\frac{c}{f}\right)' = \frac{-c \cdot f'}{f^2} \quad , c \rightarrow \text{constant}$$

Ex:

$$1) f(x) = \frac{2}{3x+1}$$

$$f'(x) = \frac{-2(3)}{(3x+1)^2} = \frac{-6}{(3x+1)^2}$$

$$2) \frac{d}{dx} \left(x^2 \cdot \frac{d}{dx} \left(\frac{1}{x} \right) \right)$$

$$\frac{d}{dx} \left(x^2 \cdot \frac{-1(1)}{x^2} \right)$$

$$\frac{d}{dx} \left(\cancel{x^2} \cdot \frac{-1}{\cancel{x^2}} \right)$$

$$\frac{d}{dx} (-1) = 0$$

~~9~~ 9 f f'

$$1) \sin x \quad \cos x$$

$$2) \cos x \quad -\sin x$$

$$3) \tan x \quad \sec^2 x$$

$$4) \cot x \quad -\csc^2 x$$

$$5) \sec x \quad \sec x \tan x$$

$$6) \csc x \quad -\csc x \cot x$$

→

Ex: $f(x) = \tan x + 3x^2 \cos x - \frac{5x}{\sec x}$ find $f'(x)$

$$f'(x) = \sec^2 x + 6x(\cos x) + 3x^2(-\sin x) - \left(\frac{5 \sec x - (\sec x \tan x)(5x)}{\sec^2 x} \right)$$

$$f'(x) = \sec^2 x + 6x \cos x - 3x^2 \sin x - \frac{5 \sec x}{\sec^2 x} + \frac{5x \sec x \tan x}{\sec^2 x}$$

10 chain rule:

If $y = f(z)$

$z = g(x)$

then

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

Ex:

if $y = 3z^3 + 2$

$z = x^2 + 4$

find $\frac{dy}{dx}$

$$\frac{dy}{dz} = 9z^2$$

$$\frac{dz}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = 9z^2 (2x)$$

$$= 9(x^2 + 4)^2 (2x) = 18x(x^2 + 4)^2$$

② if $x = \cos n$, $\frac{dn}{dm} = 3$ find $\frac{dx}{dm}$ when $n = \frac{3\pi}{2}$

$$\frac{dx}{dm} = \frac{dx}{dn} \cdot \frac{dn}{dm}$$

$$\frac{dx}{dn} = -\sin n \cdot 3 = -3 \sin n$$

$$\left. \frac{dx}{dn} \right|_{n = \frac{3\pi}{2}} = -3 \sin \frac{3\pi}{2} = -3(-1) = 3$$

→ Lecture (48)

$$\boxed{III} \quad (f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Ex:

$$1) f(x) = x^2 \rightarrow f'(x) = 2x$$

$$2) g(x) = 3x + 1 \rightarrow g'(x) = 3$$

$$\begin{aligned} \text{Find } 1) (g \circ f)'(x) &= g'(f(x)) \cdot f'(x) \\ &= g'(x^2) \cdot 2x \\ &= 3 \cdot 2x = 6x \end{aligned}$$

$$2) (f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

$$= f'(3x+1) \cdot 3$$

$$= 2(3x+1) \cdot 3 = 6(3x+1) = 18x + 6$$

$$\text{Ex: } f'(1) = 5, g(2) = 1, g'(2) = -3, \text{ find}$$

$$(f \circ g)'(2) = f'(g(2)) \cdot g'(2)$$

$$= f'(1) \cdot (-3)$$

$$= 5 \cdot (-3) = -15$$

$$\text{Ex: if } f(2x) = 3x^2 + 8x \text{ Find } f'(2)$$

$$\text{(نقطة التفاضل)} \quad \frac{d}{dx} (f(2x) = 3x^2 + 8x)$$

$$f'(2) \text{ المطلوب } f'(2x) \cdot 2 = 6x + 8$$

$$\begin{aligned} 2x &= 2 \\ x &= 1 \end{aligned} \quad f'(2x) \cdot 2 = \frac{6(1) + 8}{2}$$

$$f'(2) = 3 + 4 = 7$$

Ex: If $\frac{d}{dx}(f(3x)) = 6x$ find $\frac{d}{dx}(f(x))$

~~3x~~ $f'(3x) \cdot 3 = 6x$

$f'(3x) = 2x$

$f'(y) = \frac{2}{3}y$

let $y = 3x$

$\frac{y}{3} = x$

[12] $\frac{d}{dx}(f(x)^n) = n(f(x))^{n-1} \cdot f'(x)$

Ex: 1) $f(x) = (x^3 + 5)^{27}$ Find $f'(x)$

$f'(x) = 27(x^3 + 5)^{26} \cdot (3x^2) = \dots$

2) $y = f^5(3x+4)$ find $\frac{dy}{dx}$

$y' = 5f^4(3x+4) \cdot f'(3x+4) \cdot 3$

$= 15(f(3x+4))^4 \cdot f'(3x+4)$

$f^5 = (f(x))^5$
 $f^{(5)} \leftarrow$ derivado da derivada
 $f^{(5)} \leftarrow$ derivado da derivada
 $f^2 = (f(x))^2$
 $f^{(2)} = f''$

3) $y = \csc^3 x$ find y'

$y = (\csc x)^3$

$y' = 3(\csc x)^2 \cdot (-\cot x \csc x)$

$\frac{dy}{dx} = -3(\cos x)^3 \cdot \cot x$

<u>f(x)</u>	<u>f'(x)</u>
$\sin(f(x))$	$f'(x) \cos(f(x))$
$\cos(f(x))$	$-f'(x) \sin(f(x))$
$\tan(f(x))$	$f'(x) \sec^2(f(x))$
$\cot(f(x))$	$-f'(x) \csc^2(f(x))$
$\sec(f(x))$	$f'(x) \sec(f(x)) \tan(f(x))$
$\csc(f(x))$	$-f'(x) \csc(f(x)) \cot(f(x))$

~~f(x)~~

~~f(x)~~

Ex: find $\frac{dy}{dx}$

1) $f(x) = \cot x^2$

$$f'(x) = 2x (-\csc^2(x^2))$$

$$= -2x \csc^2(x^2)$$

2) $y = \cos^2(3x^5) = (\cos(3x^5))^2$

$$\frac{dy}{dx} = 2 (\cos(3x^5))' \cdot (15x^4 \cdot (\sin(3x^5)))$$

$$= -30x^4 \sin(3x^5) \cos(3x^5)$$

$$= -15x^4 \sin(6x^5) \rightarrow \sin(2x) = 2 \sin x \cos x$$

3) $y = \sin(\cos(\tan x))$

$$\frac{dy}{dx} = \sec^2 x \cdot (-\sin(\tan x)) \cdot \cos(\cos(\tan x))$$

$$\boxed{14} \quad \frac{d}{dx} (\sqrt{f(x)}) = \frac{f'(x)}{2\sqrt{f(x)}}$$

Ex:

1) $y = \sqrt{\tan x}$ find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\sec^2 x}{2\sqrt{\tan x}}$$

Lecture (47)

15 Implicit Differentiation

الاشتقاق الضمني

$$y = x^2 + x + 5 \quad \text{explicit} \quad \text{مُصرَّح$$

$$yx + y^2 + 1 = x \quad \text{implicit} \quad \text{مُخْفِي$$

$$\frac{d}{dx}(x) = 1 \cdot \frac{dx}{dx} = 1$$

$$\frac{d}{dx}(y) = 1 \cdot \frac{dy}{dx} = y'$$

$$\frac{d}{dx}(y^2) = 2y \frac{dy}{dx} = 2yy'$$

$$\frac{d}{dx}(x^2 + y^2) = 2x \cdot \frac{dx}{dx} + 2y \cdot \frac{dy}{dx} = 2x + 2y \frac{dy}{dx} = 2x + 2yy'$$

$$\frac{d}{dx}(xy) = 1 \cdot \frac{dx}{dx} \cdot y + x \cdot 1 \cdot \frac{dy}{dx} = y + xy'$$

$$\frac{d}{dx}\left(\frac{x}{y}\right) = \frac{1 \cdot \frac{dx}{dx} \cdot y - x \cdot 1 \cdot y'}{y^2} = \frac{y - xy'}{y^2}$$

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1 \cdot \frac{dy}{dx} \cdot x - 1 \cdot y}{x^2} = \frac{y'x - y}{x^2}$$

Ex:

$$\textcircled{1} x^2 + xy = 1 \quad \text{find } \frac{dy}{dx}$$

$$(2x + 1y) + x \cdot 1 \cdot \frac{dy}{dx} = 0$$

$$\frac{x \frac{dy}{dx}}{x} = \frac{-2x - y}{x} \rightarrow \frac{dy}{dx} = 2 - \frac{y}{x}$$

$$2) x^2 + xy = 1 \text{ find } \frac{d^2 y}{dx^2}$$

$$\frac{dy}{dx} = -2 - \frac{y}{x} = \frac{-2x - y}{x}$$

$$\frac{d^2 y}{dx^2} = 0 - \left(\frac{\frac{dy}{dx} \cdot x - y}{x^2} \right)$$

$$\frac{d^2 y}{dx^2} = - \left(\frac{(-2x - y) \cdot x - y}{x^2} \right)$$

$$\frac{d^2 y}{dx^2} = \frac{2x + y}{x^2}$$

$$\frac{d^2 y}{dx^2} = \frac{2x + 2y}{x^2}$$

$y', f', \frac{dy}{dx}$ ← المشتقة الأولى

$y'', f'', \frac{d^2 y}{dx^2}$ ← المشتقة الثانية

$y''', f''', \frac{d^3 y}{dx^3}$ ← المشتقة الثالثة

$$3) \text{ If } \sin(xy^2) + 3xy = 1 - y \text{ find } \frac{dy}{dx} \text{ when } x=0$$

$$\text{when } x=0 \rightarrow \sin(0 \cdot y^2) + 3 \cdot 0 \cdot y = 1 - y$$

$$0 + 0 = 1 - y \rightarrow y = 1 \Rightarrow \left. \frac{dy}{dx} \right|_{x=0, y=1} = ??$$

$$(1 \cdot y^2 + x \cdot y \cdot y') \cos(xy^2) + 3 \cdot y + 3x y' = 0 - y'$$

$$(1 \cdot (1)^2 + 0 \cdot 1 \cdot y') \cos(0 \cdot (1)^2) + 3 \cdot 1 + 3(0) y' = -y'$$

$$1 \cdot 1 + 3 = -y'$$

$$4 = -y' \rightarrow y' = -4$$

$$\left. \frac{dy}{dx} \right|_{x=0, y=1} = -4$$

4) If $x^2 + y^2 = a$ show that $\frac{d^2 y}{dx^2} = -\frac{a}{y^3}$

$$2x + 2y y' = 0$$

$$2y y' = -2x$$

$$y' = \frac{-2x}{2y} = -\frac{x}{y}$$

$$y'' = \frac{-1 \cdot y - y'(-x)}{y^2}$$

$$y'' = \frac{-y - (-\frac{x}{y})(-x)}{y^2} = \frac{(-y - \frac{x^2}{y})}{y^2} = -\frac{y}{y^3}$$

توی پر
مقدار

$$= -\frac{(y^2 + x^2)}{y^3} \rightarrow a \quad | y^2 + x^2 = a$$

$$\frac{d^2 y}{dx^2} = -\frac{a}{y^3} \quad \#$$

Lecture (48)

$$\boxed{16} \quad f(x) = \log_a(g(x)) \rightarrow f'(x) = \frac{g'(x)}{g(x) \cdot \ln a}$$

$$f(x) = \ln(g(x)) \rightarrow f'(x) = \frac{g'(x)}{g(x)} \quad \text{since } \ln e = 1$$

Ex:

$$1) \quad y = \log_{10}(x^2+1)$$

$$\frac{dy}{dx} = \frac{2x}{(x^2+1) \ln 10}$$

$$2) \quad y = \log_7(\cot x)$$

$$y' = \frac{-\csc^2 x}{\cot x \cdot \ln 7}$$

$$3) \quad y = (\ln x)^3$$

$$\frac{dy}{dx} = 3(\ln x)^2 \cdot \frac{1}{x} = \frac{3}{x} (\ln x)^2$$

$$4) \quad \cos(\ln x) = y$$

$$\frac{dy}{dx} = \frac{1}{x} \cdot -\sin(\ln x)$$

$$\frac{dy}{dx} = \frac{-\sin(\ln x)}{x}$$

$$5) \quad y = \ln\left(\frac{x^2 \sin x}{x+1}\right) = \ln(x^2 \sin x) - \ln(x+1) \\ = \ln(x^2) + \ln(\sin x) - \ln(x+1)$$

$$\frac{dy}{dx} = \frac{2x}{x^2} + \frac{\cos x}{\sin x} \cdot \frac{1}{x+1} = \frac{2}{x} + \cot x - \frac{1}{x+1}$$

$$6) y = \log_x 2 = \frac{\ln 2}{\ln x} \rightarrow \text{constant}$$

$$\frac{dy}{dx} = \frac{-\ln 2 \cdot \frac{1}{x}}{(\ln x)^2} = \frac{-\ln 2}{x(\ln x)^2}$$

$$7) y = (x^2+1)^{\sin x}$$

$$\ln y = \ln(x^2+1)^{\sin x}$$

$$\ln y = \sin x \ln(x^2+1)$$

$$\frac{dy}{dx}$$

$$\frac{dy}{y} = \cos x \cdot \ln(x^2+1) + \sin x \left(\frac{2x}{x^2+1} \right)$$

$$\frac{dy}{dx} = \left(\cos x \cdot \ln(x^2+1) + \frac{2x \sin x}{x^2+1} \right) y$$

$$\frac{dy}{dx} = \left(\cos x \cdot \ln(x^2+1) + \frac{2x \sin x}{x^2+1} \right) (x^2+1)^{\sin x}$$

$$\boxed{1.7} \quad f(x) = a^{g(x)} \rightarrow f'(x) = g'(x) \cdot a^{g(x)} \cdot \ln a$$

$$f(x) = e^{g(x)} \rightarrow f'(x) = g'(x) \cdot e^{g(x)}$$

Ex:

$$1) y = e^{3x}$$

$$y' = 3 \cdot e^{3x}$$

$$2) y = 2^{\sin x}$$

$$y' = \cos x \cdot 2^{\sin x} \cdot \ln 2$$

$$3) y = \ln(\cos(e^x))$$

$$\frac{dy}{dx} = \frac{-1 \cdot e^x \sin(e^x)}{\cos(e^x)} = -e^x \cdot \tan(e^x)$$

$$4) 2^{3^{x^2}} = y$$

$$\begin{aligned} y' &= a^{bx} \cdot b \cdot \ln a \\ &= 2^{3^{x^2}} \cdot \ln 2 \\ &= 2^{3^{x^2}} \cdot \ln 2 \cdot 3^{x^2} \cdot 2x \\ &= 2x \ln 3 \cdot \ln 2 \cdot 3^{x^2} \cdot 2^{3^{x^2}} \end{aligned}$$

$$\begin{aligned} a^{bx} &= 3^{x^2} \\ a^{bx} &= 2x \cdot 3^{x^2} \cdot \ln 3 \end{aligned}$$

Lecture (44)

[10]

<u>f</u>	<u>f'</u>	<u>f</u>	<u>f'</u>
$\sin^{-1}(g(x))$	$\frac{g'(x)}{\sqrt{1-(g(x))^2}}$	$\cos^{-1}(g(x))$	$-\frac{g'(x)}{\sqrt{1-(g(x))^2}}$
$\tan^{-1}(g(x))$	$\frac{g'(x)}{1+g^2(x)}$	$\cot^{-1}(g(x))$	$-\frac{g'(x)}{1+g^2(x)}$
$\sec^{-1}(g(x))$	$\frac{g'}{g\sqrt{g^2-1}}$	$\csc^{-1}(g(x))$	$-\frac{g'}{g\sqrt{g^2-1}}$

Ex:

1) $y = \sin^{-1}(5x)$

$$y' = \frac{5}{\sqrt{1-25x^2}}$$

2) $y = \tan^{-1}(3x^2)$

$$y' = \frac{6x}{1+4x^4}$$

3) $y = \log(\cos^{-1}x)$

$$y' = \frac{\frac{1}{\sqrt{1-x^2}}}{\cos^{-1}x \cdot \ln 10} = \frac{-1}{\ln 10 \cdot \cos^{-1}x \cdot \sqrt{1-x^2}}$$

4) $y = 2^{\cot^{-1}x}$

$$y' = \frac{-1}{1+x^2} \cdot 2^{\cot^{-1}x} \cdot \ln 2 = \frac{-\ln 2 \cdot 2^{\cot^{-1}x}}{1+x^2}$$

★ Higher derivatives:

$$f', f'', f''', f^{(4)}, f^{(5)}, f^{(6)}, \dots$$

Ex:

① $f(x) = 5x^7 + 2x^5 + 3x$ find $f^{(4)}(x)$

$$f'(x) = 5(7)x^6 + 2(5)x^4 + 3$$

$$f''(x) = 5(7)(6)x^5 + 2(5)(4)x^3 + 0$$

$$f'''(x) = 5(7)(6)(5)x^4 + 2(5)(4)(3)x^2$$

$$f^{(4)}(x) = 5(7)(6)(5)(4)x^3 + 2(5)(4)(3)(2)x$$

2) $f(x) = 2^x$ find $f^{(100)}(x)$

$$f'(x) = 2^x \cdot \ln 2$$

$$f''(x) = \ln 2 \cdot 2^x \ln 2 = (\ln 2)^2 \cdot 2^x$$

$$f'''(x) = (\ln 2)^2 \cdot 2^x \cdot \ln 2 = (\ln 2)^3 \cdot 2^x$$

$$f^{(4)}(x) = (\ln 2)^3 \cdot 2^x \cdot \ln 2 = (\ln 2)^4 \cdot 2^x$$

$$f^{(5)}(x) = (\ln 2)^5 \cdot 2^x$$

⋮

$$f^{(100)}(x) = (\ln 2)^{100} \cdot 2^x$$

$$\boxed{f^{(n)}(x) = (\ln 2)^n \cdot 2^x}$$

Lecture (50)

* Differentiability: قابلية التفاضل

Theorem: The function f is differentiable at $x=a$ if $f'(a)$ exist

Theorem:

① If f is differentiable at $x=a$ then f is continuous at $x=a$

② If f is not continuous at $x=a$ then f is not differentiable at $x=a$

Ex:

① $f(x) = \begin{cases} x^2+1, & x < 1 \\ 2x, & x > 1 \end{cases}$ if f is differentiable at $x=1$

① $f(1) = 2$ ② $\lim_{x \rightarrow 1^+} 2x = 2 = \lim_{x \rightarrow 1^-} (x^2+1) = 2$

③ $f(1) = \lim_{x \rightarrow 1} f = 2$

$f'(x) = \begin{cases} 2x, & x < 1 \\ 2, & x > 1 \\ 2, & x = 1 \end{cases} \Rightarrow f'_+(1) = 2, f'_-(1) = 2(1) = 2$
 $f'_+(1) = f'_-(1) = 2$

$\therefore f'(1) = 2$ exist

$\therefore f$ differentiable at $x=1$

② $f(x) = \begin{cases} x^2 + 1, & x \leq 1 \\ x^3 + 1, & x > 1 \end{cases}$ is f differentiable at $x = 1$

check $\rightarrow \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) = f(1) = 2$

$\therefore f$ cont at $x = 1$

$f'(x) = \begin{cases} 2x, & x < 1 \\ 3x^2, & x > 1 \end{cases}$
 $x = 1$

$f'_+(1) = 3(1)^2 = 3$

$f'_-(1) = 2(1) = 2$

$\rightarrow f'_+(1) \neq f'_-(1), \therefore f'(1) \text{ d.n.e}$

$\therefore f$ not differentiable at $x = 1$

③ $f(x) = \begin{cases} x^2 + 1, & x > 1 \\ 2x + 5, & x \leq 1 \end{cases}$ is f differentiable at $x = 1$

$\lim_{x \rightarrow 1^+} f(x) \neq \lim_{x \rightarrow 1^-} f(x) \quad (2 \neq 7) \rightarrow \lim_{x \rightarrow 1} f(x) \text{ d.n.e}$

$\therefore f$ not cont at $x = 1$

$\therefore f$ not differentiable at $x = 1$

Q4) $f(x) = \begin{cases} 3x^2, & x \leq 1 \\ ax+b, & x > 1 \end{cases}$ is differentiable at $x=1$ find a and b ?

f differentiable at $x=1$, $f'(1)$ exist
 f " " " then f cont at $x=1$

$$f'(x) = \begin{cases} 6x, & x \leq 1 \\ a, & x > 1 \end{cases}, f \text{ differentiable } f'_+(1) = f'_-(1)$$

$$6(1) = a$$

$$\boxed{a=6}$$

f differentiable $\therefore f$ cont at $x=1$

$$f(1) = \lim_{x \rightarrow 1+} f(x)$$

$$3(1)^2 = \lim_{x \rightarrow 1+} (ax+b) \rightarrow 3 = 6(1) + b \rightarrow \boxed{b = -3}$$

~~Ex 3~~

Ex: ① $f(4) = 3$, $f'(4) = a$ Find

$$\lim_{h \rightarrow 0} \frac{f(4+h) - 3}{5h} = \frac{1}{5} \lim_{h \rightarrow 0} \frac{f(4+h) - \boxed{f(4)}}{h} = f'(4) \quad \left(\begin{array}{l} \text{تعريف الاشتقاق} \\ \text{derivative} \end{array} \right)$$

$$= \frac{1}{5} f'(4) = \frac{1}{5} \cdot a = \frac{a}{5}$$

② $\lim_{x \rightarrow 4} \frac{f(x) - f(4)}{x - 4} = 2$ then may equal to;

a) $8\sqrt{x}$ $\xrightarrow{f'}$ $\frac{8(1)}{2\sqrt{x}} \rightarrow f'(4) = \frac{8}{2\sqrt{4}} = 2$ (if $f(4) = 2$ then $f(x) = ??$)

b) $-8\sqrt{x}$ $\xrightarrow{f'}$ $-\frac{8(1)}{2\sqrt{x}} \rightarrow f'(4) = -2$

c) $2 \cos(x-4)$ $\xrightarrow{f'}$ $-2 \sin(x-4) \rightarrow f'(4) = 0$

d) $x^2 - a$ $\xrightarrow{f'}$

$2x \rightarrow f'(4) = 8$

lecture (51)

★ Equation of tangent line at $x=a$:

- Point of tangency $(a, f(a))$

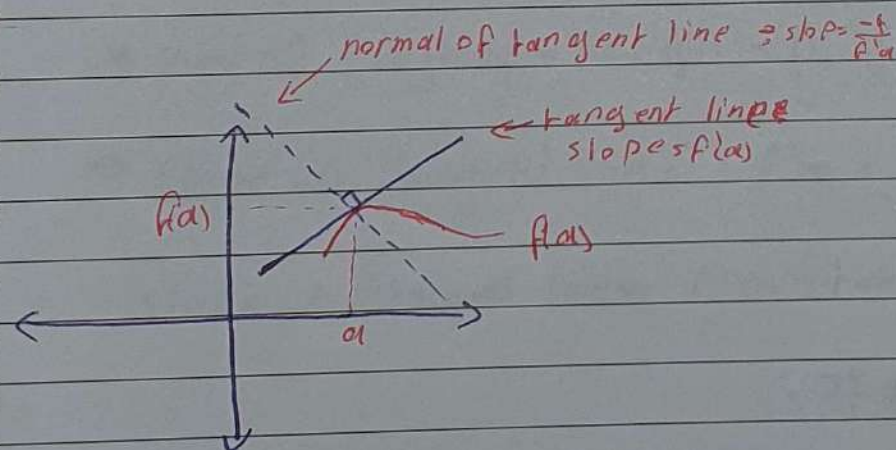
- slope of tangent line $= f'(a) = \frac{dy}{dx} \Big|_{(a, f(a))}$

- equation of tangent line:

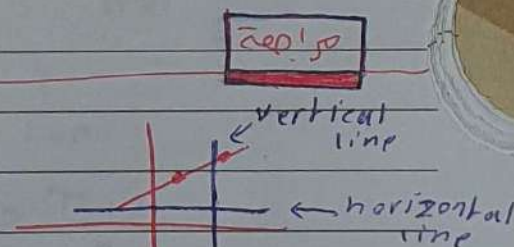
$$y - f(a) = f'(a)(x - a)$$

- slope of normal (perpendicular) $= \frac{-1}{f'(a)} = \frac{-1}{\text{slope of tangent line}}$

- equation of normal: $y - f(a) = \frac{-1}{f'(a)}(x - a)$



$$f'(a) \cdot \frac{-1}{f'(a)} = -1$$



$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y - y_1 = m(x - x_1)$$

slope horizontal line = 0

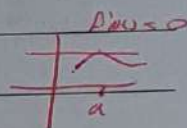
slope vertical line = undefined

★ Remark:

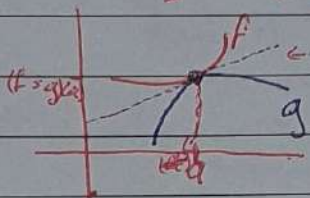
1) If $L_1 \parallel L_2 \rightarrow m_1 = m_2$ (parallel $\parallel$) slope = slope

2) If $L_1 \perp L_2 \rightarrow m_1 \times m_2 = -1$

3) The slope of horizontal lines is zero
The slope of horizontal tangent lines $f'(a)$ is zero



4) $f(a) = y(a)$
 $f'(a) = y'(a)$



Ex:

1) Find the equation of tangent line for ~~the function~~

$$y = x \tan^{-1} x \text{ at } x=1$$

• point of tangency $= (1, y(1)) = (1, \frac{\pi}{4})$

• slope of tangent line: $f'(x) = \tan^{-1} x + x \cdot \frac{1}{1+x^2}$

$$\text{slope of tangent line} = f'(1) = \tan^{-1}(1) + \frac{1}{1+1^2}$$

$$= \frac{\pi}{4} + \frac{1}{2} = \frac{\pi+2}{4}$$

• equation of tangent line:

$$y - f(a) = f'(a)(x - a)$$

$$y - \frac{\pi}{4} = \frac{\pi+2}{4}(x-1) \Rightarrow y = \dots$$

• equation of normal: $y - f(a) = \frac{-1}{f'(a)}(x - a) \rightarrow y - \frac{\pi}{4} = \frac{-4}{\pi+2}(x-1)$

2) find the slope of tangent line and slope of normal

$$2x^2 + 3e^{x+1} + 3xy + y^2 = 9 \text{ at } (-1, 1)$$

diff
grad

$$4x + 3e^{x+1} + 3y + 3xy' + 2yy' = 0$$

$$4(-1) + 3(e^{-1}) + 3(1) + 3(-1)y'_{(-1,1)} + 2(1)y'_{(-1,1)} = 0$$

$$-4 + 3 + 3 - 3y'_{(-1,1)} + 2y'_{(-1,1)} = 0$$

$$2 - y'_{(-1,1)} = 0$$

slope
of tangent

$$\boxed{y'_{(-1,1)} = 2}$$

slope of tangent
 $= y'_{(-1,1)}$

slope of normal
 $= \frac{-1}{y'_{(-1,1)}}$

$$\text{slope of normal} = -\frac{1}{2}$$

$$\text{equation of tangent} \Rightarrow (y - 1) = 2(x + 1)$$

$$// \quad // \quad \text{normal} \Rightarrow (y - 1) = -\frac{1}{2}(x + 1)$$

3) find all values of x at which the tangent line of the curve

1) $y = x^5 e^x$ horizontal $\rightarrow$ slope = 0

$$y' = 5x^4(e^x) + x^5 \cdot e^x = 0$$

$$x^4(e^x)(5+x) = 0$$

$\downarrow$ $\downarrow$
 $x = 0$ $x = -5$
 $e^x \neq 0$

at $x = 0$, $x = -5$ horizontal tangent line

(امثلة على التفاضل) (Examples of differentiation)

Ex: ① find $\frac{dy}{dx}$ for
[a] $f(x) = \sqrt{x^x}$

$$y = (x^x)^{\frac{1}{2}} = x^{\frac{x}{2}}$$

$$\ln y = \ln(x^{\frac{x}{2}})$$

$$\ln y = \frac{x}{2} \ln x \rightarrow \frac{y'}{y} = \frac{1}{2} \cdot \ln x + \frac{x}{2} \cdot \frac{1}{x}$$

$$y' = \left(\frac{1}{2} \ln x + \frac{1}{2} \right) y$$

$$y' = \left(\frac{1}{2} \ln x + \frac{1}{2} \right) \sqrt{x^x}$$

$$[b] y = \sin^2(\cos(\sqrt{\sinh(\pi x)}) = (\sin(\cos(\sqrt{\sinh(\pi x)})))^2$$

$$y' = 2(\sin(\cos(\sqrt{\sinh(\pi x)}))) \cdot \cos(\cos(\sqrt{\sinh(\pi x)})) \cdot -\sin(\sqrt{\sinh(\pi x)})$$

$$\cdot \frac{\pi \cosh(\pi x)}{2 \sqrt{\sinh(\pi x)}}$$

② find a such ~~that~~ $y = a \sin(3t)$ satisfies $y'' + 2y = 4 \sin(3t)$

$$y' = 3a \cdot (\cos(3t))$$

$$y'' + 2y = 4 \sin(3t)$$

$$-9a \sin(3t) + 2a \sin(3t) = 4 \sin(3t)$$

$$y'' = -9a \sin(3t)$$

$$-7a \sin(3t) = 4 \sin(3t)$$

$$\rightarrow a = -\frac{4}{7}$$

③ If $g(x) = xf(x)$, $f(3) = 4$, $f'(3) = 2$ find the equation of tangent line of $g(x)$ at $x = 3$

also $(3, g(3)) = (3, 3f(3)) = (3, 3(4)) = (3, 12)$

step: $g'(x) = 1 \cdot f(x) + x \cdot f'(x)$

$$g'(3) = f(3) + 3 \cdot f'(3)$$

$$= 4 + 3(2) = 4 + 6 = 10$$

equation of tangent

$$y - g(3) = g'(3)(x - 3)$$

$$y - 12 = 10(x - 3)$$

$$y = 10x - 18$$

④ If $f(x) = \frac{1}{2-x}$ find $f^{(n)}(x)$

$$f'(x) = \frac{-1(-1)}{(2-x)^2} = \frac{1}{(2-x)^2}$$

$$f''(x) = \frac{-1 \cdot 2(2-x)^{-3} \cdot (-1)}{(2-x)^4} = \frac{2}{(2-x)^3} = \frac{2!}{(2-x)^3}$$

$$f'''(x) = \frac{-2 \cdot 3(2-x)^{-4} \cdot (-1)}{(2-x)^6} = \frac{6}{(2-x)^4} = \frac{3!}{(2-x)^4}$$

$$f^{(4)}(x) = \frac{(1)(2)(3)(4)}{(2-x)^5} = \frac{4!}{(2-x)^5}$$

$$f^{(n)}(x) \leq \frac{n!}{(2-x)^{n+1}}$$

مضروب

$$n! = n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1$$

$$3! = (3)(2)(1)$$

$$5! = (5)(4)(3)(2)(1)$$

!

(5) show that $\frac{d}{dx} \left(\sqrt[4]{\frac{1+\tanh x}{1-\tanh x}} \right) = \frac{1}{2} e^{\frac{x}{2}}$

$$y = \sqrt[4]{\frac{1+\tanh x}{1-\tanh x}} = \left(\frac{1+\tanh x}{1-\tanh x} \right)^{\frac{1}{4}}$$

$$y = \left(\frac{1 + \frac{\sinh x}{\cosh x}}{1 - \frac{\sinh x}{\cosh x}} \right)^{\frac{1}{4}} \Rightarrow \text{crossed out}$$

$$y = \left(\frac{\frac{\cosh x + \sinh x}{\cosh x}}{\frac{\cosh x - \sinh x}{\cosh x}} \right)^{\frac{1}{4}} = \left(\frac{\cosh x + \sinh x}{\cosh x - \sinh x} \right)^{\frac{1}{4}}$$

$$y = \left(\frac{e^x}{e^{-x}} \right)^{\frac{1}{4}} = y = (e^{2x})^{\frac{1}{4}}$$

$$y = e^{\frac{x}{2}}$$

$$y' = \frac{1}{2} e^{\frac{x}{2}} \quad \text{crossed out}$$

$$\begin{aligned} \cosh x - \sinh x &= e^{-x} \\ \cosh x + \sinh x &= e^x \end{aligned}$$

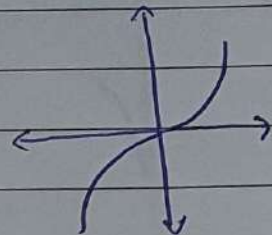
Lecture 52

★ Hyperbolic Functions:

1) hyperbolic sine $\sinh x = \frac{e^x - e^{-x}}{2}$

$D_{\sinh x} = \mathbb{R}$ $\lim_{x \rightarrow +\infty} \sinh x = +\infty$

$R_{\sinh x} = \mathbb{R}$ $\lim_{x \rightarrow -\infty} \sinh x = -\infty$



$\sinh(0) = 0$

$\frac{e^0 - e^{-0}}{2} = \frac{0}{2} = 0$

$\sinh x \geq 0 \quad \forall x \geq 0$

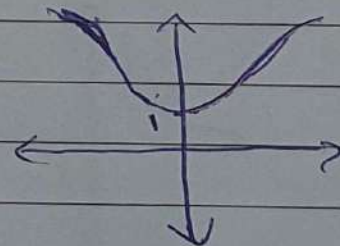
$\sinh x \leq 0 \quad \forall x \leq 0$

odd

2) Hyperbolic cosine $\cosh x = \frac{e^x + e^{-x}}{2}$

$D_{\cosh x} = \mathbb{R}$ $\lim_{x \rightarrow +\infty} \cosh x = +\infty$

$R_{\cosh x} = [1, \infty)$ $\lim_{x \rightarrow -\infty} \cosh x = +\infty$



$\cosh(0) = 1$

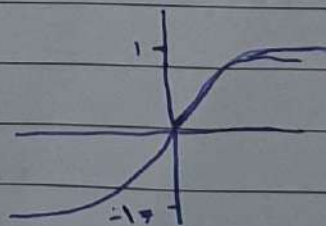
$\cosh x \geq 1 \quad \forall x$

even

3) hyperbolic tangent $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$D_{\tanh x} = \mathbb{R}$ $\lim_{x \rightarrow +\infty} \tanh x = 1$ $\tanh(0) = 0$

$R_{\tanh x} = (-1, 1)$ $\lim_{x \rightarrow -\infty} \tanh x = -1$ $x \geq 0 \rightarrow 0 \leq \tanh x < 1$
 $x \leq 0 \rightarrow -1 < \tanh x \leq 0$

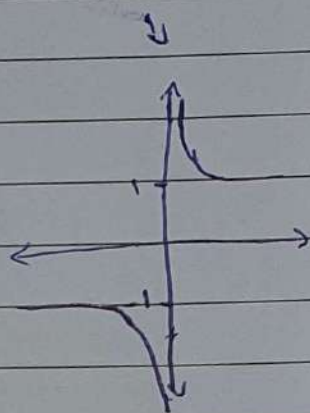


[4] Hyperbolic cotangent ~~→~~ $\coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x}$

$$D_{\coth x} = \mathbb{R} \setminus \{0\}$$

$$R_{\coth x} = (-\infty, -1) \cup (1, \infty)$$

$$\begin{cases} \lim_{x \rightarrow +\infty} \coth x = 1 \\ \lim_{x \rightarrow -\infty} \coth x = -1 \end{cases}$$

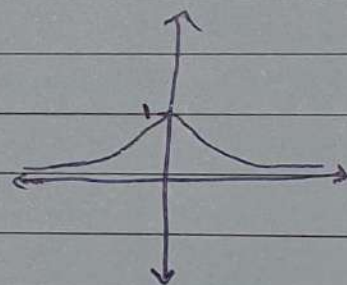


[5] Hyperbolic secant ~~→~~ $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

$$D_{\operatorname{sech} x} = \mathbb{R}$$

$$R_{\operatorname{sech} x} = (0, 1]$$

$$\begin{cases} \lim_{x \rightarrow +\infty} \operatorname{sech} x = 0 \\ \lim_{x \rightarrow -\infty} \operatorname{sech} x = 0 \\ 0 < \operatorname{sech} x \leq 1 \end{cases}$$



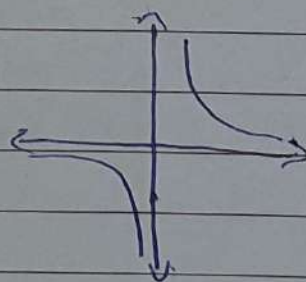
[6] Hyperbolic cosecant

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$D_{\operatorname{csch} x} = \mathbb{R} \setminus \{0\}$$

$$R_{\operatorname{csch} x} = \mathbb{R} \setminus \{0\}$$

$$\begin{cases} \lim_{x \rightarrow +\infty} \operatorname{csch} x = 0 \\ \lim_{x \rightarrow -\infty} \operatorname{csch} x = 0 \end{cases}$$



✗ Remark:

[1] $\cosh x, \operatorname{sech} x$ even function

→

$$\cosh(-x) = \cosh x$$

$$\operatorname{sech}(-x) = \operatorname{sech} x$$

[2] $\sinh x, \tanh x, \coth x, \operatorname{csch} x$

odd function

$$\sinh(-x) = -\sinh x$$

$$\tanh(-x) = -\tanh x$$

$$\coth(-x) = -\coth x$$

$$\operatorname{csch}(-x) = -\operatorname{csch} x$$

~~★~~

f

f'

$$\sinh x$$

$$\cosh x$$

$$\cosh x$$

$$\sinh x$$

$$\tanh x$$

$$\operatorname{sech}^2 x$$

$$\coth x$$

$$-\operatorname{csch}^2 x$$

$$\operatorname{sech} x$$

$$-\operatorname{sech} x \tanh x$$

$$\operatorname{csch} x$$

$$-\operatorname{csch} x \coth x$$

~~★~~ Theorem: $\frac{d}{dx}$

$$1) \cosh x + \sinh x = e^x$$

$$2) \cosh x - \sinh x = e^{-x}$$

$$3) \cosh^2 x - \sinh^2 x = 1$$

$$4) 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$5) \coth^2 x - 1 = \operatorname{csch}^2 x$$

$$6) \sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$7) \cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

$$8) \sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$$

★

$$9) \cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$$

$$10) \cos(2x) = \cosh^2 x + \sinh^2 x$$

$$11) \cos(2x) = 2 \sinh^2 x + 1 = 2 \cosh^2 x - 1$$

$$12) \sinh^2 x = \frac{\cosh(2x) - 1}{2}$$

$$13) \cosh^2 x = \frac{\cosh(2x) + 1}{2}$$

Lecture 53

Ex:

① find $\sinh(2\ln 2) = \sinh(\ln 2^2) = \sinh(\ln 4)$

$$= \frac{e^{\ln 4} - e^{-\ln 4}}{2} = \frac{4 - e^{\ln \frac{1}{4}}}{2} = \frac{4 - \frac{1}{4}}{2} = \frac{16 - 1}{4(2)} = \frac{15}{8}$$

② find $\cosh(\ln x)$

$$\cosh(\ln x) = \frac{e^{\ln x} + e^{-\ln x}}{2} = \frac{e^{\ln x} + e^{\ln x^{-1}}}{2} = \frac{x + \frac{1}{x}}{2}$$

$$= \frac{x + \frac{1}{x}}{2} = \frac{x^2 + 1}{2x}$$

③ If $\sinh x + \cosh x = 5$ find x

$$\downarrow$$

$$e^x = 5$$

$$\ln e^x = \ln 5$$

$$x = \ln 5$$

$$\sinh x + \cosh x = e^x$$

④ Given that $\operatorname{csch} x = -\frac{1}{3}$ find $\sinh x, \cosh x, \coth x, \tanh x, \operatorname{sech} x$

$$\operatorname{csch} x = \frac{1}{\sinh x} = -\frac{1}{3} \rightarrow \sinh x = -3$$

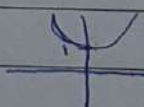
$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh^2 x - \sinh^2 x = 1 \quad \coth x = \pm \sqrt{10}$$

$$\cosh^2 x - (-3)^2 = 1 \quad \therefore \cosh x = \sqrt{10} > 1$$

$$\cosh^2 x - 9 = 1$$

$$\cosh^2 x = 10$$



$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{-3}{\sqrt{10}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{1}{\sqrt{10}}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{-\sqrt{10}}{3}$$

Ex 2 find $\frac{dy}{dx}$ for

$$\textcircled{1} y = \sinh(4x-8)$$

$$y' = 4 \cosh(4x-8)$$

f	f'
$\sinh x$	$\cosh x$
$\sinh(g(x))$	$g'(x) \cosh(g(x))$

$$\textcircled{2} y = \cosh^3(2x) = (\cosh(2x))^3$$

$$\frac{dy}{dx} = 3(\cosh(2x))^2 \cdot 2 \sinh(2x)$$

$$= 6 \sinh(2x) \cosh^2(2x)$$

$$\textcircled{3} y = \ln(\tanh(2x))$$

$$\frac{dy}{dx} = \frac{2 \cdot \operatorname{sech}^2(2x)}{\tanh(2x)}$$

$$\textcircled{4} y = \operatorname{sech}(e^{2x})$$

~~$$y' = 2 \cdot \sinh(e^{2x})$$~~

$$y' = -2 e^{2x} \operatorname{sech}(e^{2x}) \tanh(e^{2x})$$

B

$$\textcircled{5} y = \sinh(\cos(3x))$$

$$\frac{dy}{dx} = 3 \cdot -\sin(3x) \cdot \cosh(\cos(3x))$$

$$\textcircled{6} y = (\cosh x)^{\sinh x}$$

$$\ln y = \ln (\cosh x)^{\sinh x}$$

$$\ln y = \sinh x \cdot \ln(\cosh x)$$

$$y \left(\frac{y'}{y} \right) = \left(\cosh x \cdot \ln(\cosh x) + \sinh x \cdot \frac{\sinh x}{\cosh x} \right) y$$

$\sinh(0) = 0$
 $\cosh(0) = 1$

$$y' = \left(\cosh x \ln(\cosh x) + \frac{\sinh^2 x}{\cosh x} \right) (\cosh x)^{\sinh x}$$

$$\textcircled{7} y = \frac{(\sinh x)(\ln x)(\tan^{-1} x)}{\cos(2x)}$$

$$\ln y = \ln \left(\frac{(\sinh x)(\ln x)(\tan^{-1} x)}{\cos(2x)} \right)$$

$$\ln y = \ln(\sinh x) + \ln(\ln x) + \ln(\tan^{-1} x) - \ln(\cos(2x))$$

$$y \left(\frac{y'}{y} \right) = \left(\frac{\cosh x}{\sinh x} + \frac{1}{\ln x} + \frac{1}{1+x^2} - 2 \frac{\sin(2x)}{\cos(2x)} \right) y$$

$$y' = \left(\coth x + \frac{1}{\ln x} + \frac{1}{(1+x^2)\tan^{-1} x} + 2 \tan(2x) \right) \left(\frac{(\sinh x)(\ln x)(\tan^{-1} x)}{\cos(2x)} \right)$$

Lecture 54

Applications of Differentiation

$f(x)$ (find domain)
↓

① $f'(x)$
↓
regime

a critical number $a \in D_f$

$f'(a) = 0$ or $f'(a) = \text{d.n.e.}$
 $f'(a) = 0$ → $f'(a) = 0$ or $f'(a) = \text{d.n.e.}$
 $f'(a) = \text{d.n.e.}$ → $f'(a) = \text{d.n.e.}$ or $f'(a) = \text{d.n.e.}$

↑ ↓ ↑ ↓ ↑ ↓ ↑ ↓

⊗ f increasing:

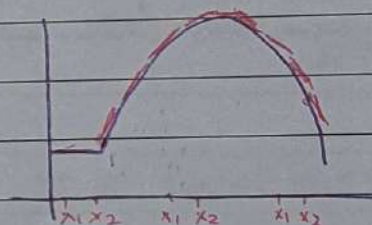
* if $x_2 > x_1 \rightarrow f(x_2) > f(x_1)$
 * $++++$ $f' > 0$

⊗ f decreasing:

* if $x_2 > x_1 \rightarrow f(x_2) < f(x_1)$
 * $-----$ $f' < 0$

⊗ f constant:

$f(x_1) = f(x_2) \forall x \quad f' = 0$



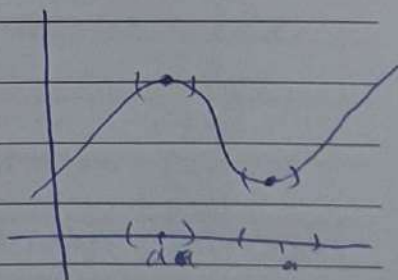
~~absolute max if $f(a) \geq f(x) \forall x \in D_f$~~

⊗ a local max if $f(a) \geq f(x)$ when x is near a

a local min if $f(a) \leq f(x)$ when x is near a

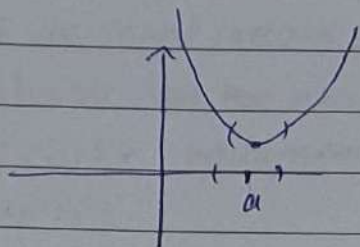
⊗ absolute max if $f(a) \geq f(x) \forall x \in D_f$

absolute min if $f(a) \leq f(x) \forall x \in D_f$



Ex:

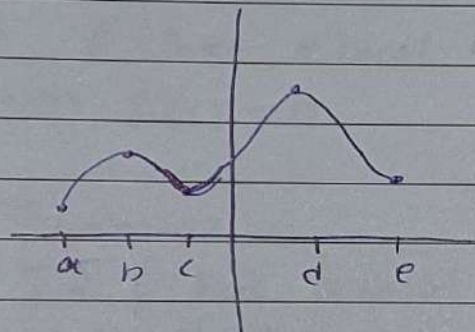
1)



at $x=a$ local min $= f(a)$
 $x=a$ absolute min $= f(a)$
 no absolute max

$D_f = \mathbb{R}$

2)

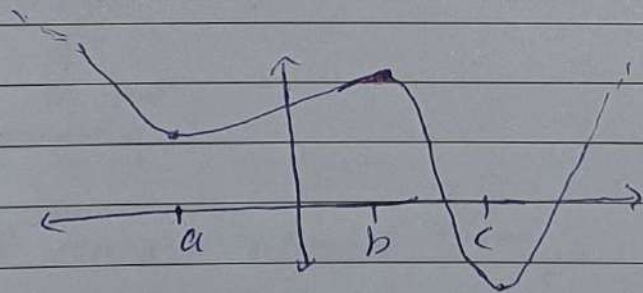


at $x=c$ local min $= f(c)$
 at $x=b$ local max $= f(b)$
 $x=d$ local ~~max~~ min $= f(d)$
 at $x=d$ absolute max $= f(d)$
 at $x=a$ absolute mins $f(a)$

$D_f = [a, e]$

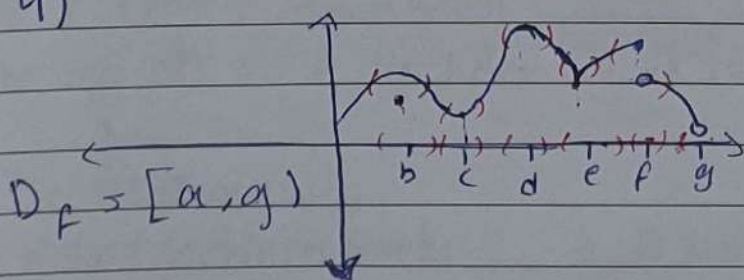
في الفترة ان العلاقة لا يمكن للاكطار ان تكون
 قيمة كظم محلية (فقط كظم او صفر كظم)

3)



at $x=a$ local min $= f(a)$
 $x=c$ " " $= f(c)$
 at $x=b$ local max $= f(b)$
 no absolute max
 at $x=c$ absolute min $= f(c)$

4)



$D_f = [a, g]$

at $x=c$ local min $= f(c)$
 $x=e$ " " $= f(e)$
 $x=b$ " " $= f(b)$
 at $x=c$ local max $= f(c)$
 $x=f$ " " $= f(f)$
 at $x=d$ absolute max $= f(d)$
 No absolute min

* The extreme value theorem:

If f is continuous on a closed interval $[a, b]$. Then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$

* Fermat's theorem:

If f has a local maximum or minimum at c and $f'(c)$ exists then $f'(c) = 0$

* Theorem:

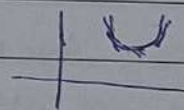
If f has a local maximum or minimum at c then c is a critical number of f .

(2) $f''(x)$:

↓
Concavity

* f concave upward $+++$ f''

(The graph of f lies above all of its tangents on an interval)



* f concave downward $---$ f''

(The graph of f lies below all of its tangents on an interval)



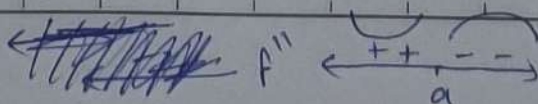
↓
نقطة انعطاف

* a inflection point:

1) $a \in D_f$

2) f cont at $x=a$

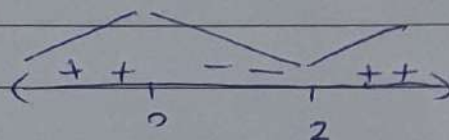
3) the curve changes from concave upward to downward



Lecture (55)

Ex: 1

1) $f(x) = x^3 - 3x^2 + 5$



$D_f = \mathbb{R}$

$f'(x) = 3x^2 - 6x = 0$

$3x(x-2) = 0$

 $x \leq 0, x \leq 2$ critical numberscritical points $(0, 5), (2, 1)$ f ~~the~~ increasing $(-\infty, 0) \cup (2, \infty)$ f decreasing $(0, 2)$ $\rightarrow$ at $x \leq 0$ local max $= f(0) = 5$ $\rightarrow$ at $x \leq 2$ local min $= f(2) = 1$

$f''(x) = 6x - 6 = 0$

$x \leq 1 \rightarrow (1, 3)$

 f concave upward $(1, \infty)$ f // downward $(-\infty, 1)$ ① $1 \in D_f = \mathbb{R}$ ② f cont at $x \leq 1$ ③ f'' changes about $x \leq 1$ $(1, 3) \rightarrow$ inflection point

$$f'' \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array}$$

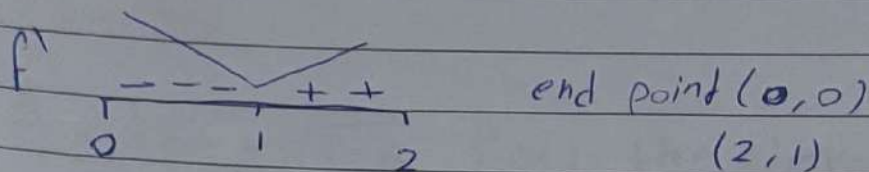
2) $f(x) = x^4 - 4x, x \in [0, 2]$

$D_f = [0, 2]$

$f' = 4x^3 - 4 = 0$

$4(x^3 - 1) = 0 \rightarrow x \leq 1$ critical number

 $(1, -3)$ critical point



f Decreasing $(0, 1)$

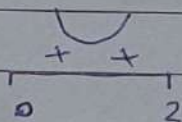
f Increasing $(1, 2)$

at $x=1$ local min $= f(1) = -3$

$x=1$ absolute min $= f(1) = -3$

at $x=2$ absolute max $= f(2) = 1$

$$f'' \geq 2x^2 \geq 0 \rightarrow x \geq 0$$



f concave up $(0, 2)$

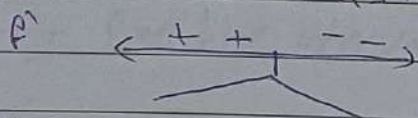
no inflection point

3) $f(x) = x e^{-x}$

1) $f: \mathbb{R} \rightarrow \mathbb{R}$

$$\begin{aligned} f'(x) &= e^{-x} + x(-e^{-x}) \\ &= e^{-x}(1-x) \geq 0 \\ e^{-x} &\neq 0 \end{aligned}$$

$$1-x \geq 0 \rightarrow x \leq 1 \rightarrow (1, e^{-1}) \text{ critical point}$$

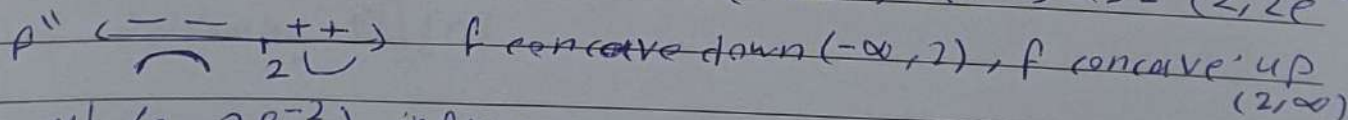


f increasing $(-\infty, 1)$

f decreasing $(1, \infty)$

at $x=1$ local + absolute ~~max~~ max $= e^{-1}$

$$\begin{aligned} f''(x) &= -e^{-x} - (e^{-x} + x(-e^{-x})) \\ &= -2e^{-x} + x e^{-x} = 0 \rightarrow e^{-x}(-2+x) = 0 \rightarrow x=2 \quad (2, 2e^{-2}) \end{aligned}$$



at $(2, 2e^{-2})$ inflection point

Lecture 56

Ex: $f(x) = \sqrt{9-x^2}$ find the interval of concavity
 $D_f = [-3, 3]$

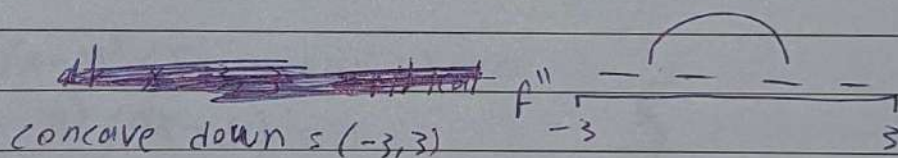
$$f'(x) = \frac{-2x}{2\sqrt{9-x^2}} = \frac{-x}{\sqrt{9-x^2}} \rightarrow x \leq 0 \rightarrow \text{critical numbers}$$

$$f''(x) = \frac{-\sqrt{9-x^2} - \frac{-x}{\sqrt{9-x^2}}(-x)}{(\sqrt{9-x^2})^2} = -\left(\frac{\sqrt{9-x^2}}{1} + \frac{x^2}{\sqrt{9-x^2}}\right)$$

$$= \frac{-(9-x^2) + x^2}{\sqrt{9-x^2}} = \frac{-9}{(\sqrt{9-x^2})^3} = \frac{-9}{(9-x^2)^{3/2}}$$

$$= \frac{-9}{(\sqrt{9-x^2})^3} \neq 0$$

$$\rightarrow (9-x^2)^{3/2} > 0 \quad x \in (-3, 3)$$



Ex: $f(x) = \frac{1}{x^3}$ find the intervals of increasing and decreasing

$$D_f = \mathbb{R} \setminus \{0\}$$

$$f(x) = x^{-3} \rightarrow f'(x) = -3x^{-4} = \frac{-3}{x^4} \rightarrow x^4 > 0 \rightarrow x > 0 \notin D_f$$

f decreasing $\mathbb{R} \setminus \{0\}$

Ex: $f(x) = x^2 - x - \ln x$ find the critical numbers of function

$$D_f = (0, \infty)$$

$$f'(x) = 2x - 1 - \frac{1}{x} = \frac{2x^2 - x - 1}{x} \rightarrow 2x^2 - x - 1 = 0 \rightarrow x = 1 \in D_f$$

$x \rightarrow x = 0 \notin D_f$

$\therefore (1, 0)$ critical point

Ex: $f(x) = (x+1)^5 (x-2)^4$ find the critical numbers of f
 $D_f = \mathbb{R}$

$$f'(x) = 5(x+1)^4 (x-2)^4 + (x+1)^5 (4(x-2)^3) = 0$$

$$(x+1)^4 (x-2)^3 (5(x-2) + 4(x+1)) = 0$$

$$(x+1)^4 (x-2)^3 (5x - 10 + 4x + 4) = 0$$

$$(x+1)^4 (x-2)^3 (9x - 6) = 0$$

$$\downarrow \quad \quad \quad \downarrow$$

$x = -1 \in D_f$ $x = \frac{6}{9} = \frac{2}{3} \in D_f$
 $x = 2 \in D_f$

critical numbers

critical points $(-1, f(-1))$, $(2, f(2))$, $(\frac{2}{3}, f(\frac{2}{3}))$

Lecture 57

Ex: $f(x) = \cos x + \sin x, x \in [0, 2\pi]$

Find the intervals of increasing and decreasing + absolute max and min.

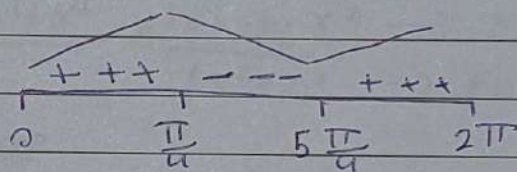
$$f'(x) = -\sin x + \cos x = 0 \rightarrow \sin x = \cos x$$

$$x = \frac{\pi}{4}, 5\frac{\pi}{4}$$

~~$$\left(\frac{\pi}{4}, \sqrt{2}\right), \left(5\frac{\pi}{4}, -\sqrt{2}\right)$$~~

critical points

$(0, 1) \quad (2\pi, 1)$
end points



at $x = \frac{\pi}{4}$ local max $f(\frac{\pi}{4}) = \sqrt{2}$

$x = 5\frac{\pi}{4}$ // min $= -\sqrt{2}$

at $x = \frac{\pi}{4}$ absolute max $= \sqrt{2}$

$x = 5\frac{\pi}{4}$ // min $= -\sqrt{2}$

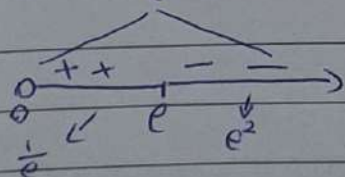
Ex: $f(x) = \frac{\ln x}{x}$ find the intervals of increasing and decreasing

$$D_f = (0, \infty)$$

$$f'(x) = \frac{\frac{1}{x} \cdot x - 1 \cdot \ln x}{x^2} = \frac{1 - \ln x}{x^2} \rightarrow 1 - \ln x \leq 0 \rightarrow \ln x \geq 1 \rightarrow x \geq e \in D_f$$

$$x^2 \rightarrow x^2 = 0 \rightarrow x \leq 0 \notin D_f$$

at $(e, \frac{1}{e}) = (e, \frac{1}{e}) = (e, e^{-1})$ critical points



f increasing $(0, e)$
decreasing (e, ∞)

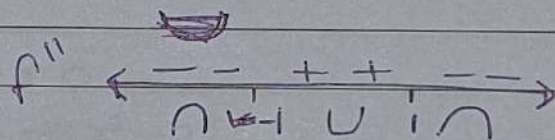
Ex: $f(x) = \frac{x^2}{x^2+3}$ find the intervals of concavity and inflection points.

$$D_f = \mathbb{R}$$

$$f'(x) = \frac{2x(x^2+3) - 2x(x^2)}{(x^2+3)^2} = \frac{\cancel{2x^3} + 6x - \cancel{2x^3}}{(x^2+3)^2} = \frac{6x}{(x^2+3)^2}$$

$$f''(x) = \frac{6(x^2+3)^2 - 2(x^2+3)(2x)(6x)}{(x^2+3)^4} = \frac{\cancel{(x^2+3)}(6(x^2+3) - 24x^2)}{(x^2+3)^3}$$

$$= \frac{6x^2 + 18 - 24x^2}{(x^2+3)^3} = \frac{18 - 18x^2}{(x^2+3)^3} \rightarrow x \in (-1, 1)$$



f concave up $(-1, 1)$
 down $(1, \infty) \cup (-\infty, -1)$ } inflection points at $(-1, f(-1))$ $(1, f(1))$

Ex:

$f(x) = (x-1)^3$ Find the intervals of increasing and decreasing.

$$D_f = \mathbb{R} / f'(x) = 3(x-1)^2 \geq 0 \rightarrow x \geq 1$$



f increasing on $\mathbb{R}$
 $(1, 0)$ critical point

Ex: $f(x) = |2x-8|$ Find the critical numbers

$$D_f = \mathbb{R}$$

$$f(x) = \begin{cases} 2x-8, & x \geq 4 \\ 8-2x, & x < 4 \end{cases}$$

$$f'(x) = \begin{cases} 2, & x > 4 \\ -2, & x < 4 \\ \text{d.n.e.}, & x = 4 \end{cases}$$

at $x=4$ critical number
 $(4,0)$ critical point

$f' \leftarrow \begin{matrix} -2 & 2 \end{matrix} \rightarrow$
 $\quad \quad \quad 4$

$f'(4) = 2$
 $f'(4) = -2$
 $f'(4) \text{ d.n.e.}$

Ex: If at $x=1$ we have extreme values for
 $f(x) = ax^3 - 2x + 3$ find a ?

$$f'(1) = 0$$

$$f'(x) = 3ax^2 - 2$$

$$f'(1) = 3a(1) - 2$$

$$0 = 3a - 2$$

$$3a = 2 \rightarrow a = \frac{2}{3}$$

Ex: If $(1,3)$ extreme point for $f(x) = x^2 + ax + b$ find
 a, b ?

$$f(1) = 3 \quad / \quad f'(1) = 0$$

$$f'(x) = 2x + a$$

$$f'(1) = 2(1) + a$$

$$0 = 2 + a \rightarrow \boxed{a = -2}$$

$$f(x) = x^2 - 2x + b$$

$$f(1) = (1)^2 - 2(1) + b$$

$$3 = 1 - 2 + b \rightarrow b = 4$$

Lecture (58)

~~The second~~ The second derivative test:suppose f'' continuous near c

- ① If $f'(c) = 0$ and $f''(c) > 0$ then f has local ~~max~~^{min} at c
 ② " " " " " $f''(c) < 0$ " " " " max etc

Ex: $f(x) = x^4 - 4x^3$ Find ~~the~~ intervals of concavity and points of inflection and local max and min.

$$D_f = \mathbb{R} \quad / \quad f'(x) = 4x^3 - 12x^2$$

$$4x^3 - 12x^2 = 0$$

$$4x^2(x-3) = 0$$

$$\downarrow \quad \downarrow$$

$$\boxed{x=0} \quad \boxed{x=3}$$

at $x=3$

$$f''(3) = 36 > 0$$

 $\therefore$ local min $= f(3) =$

$$f''(x) = 12x^2 - 24x$$

$$12x^2 - 24x = 0$$

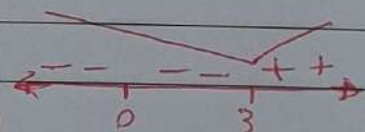
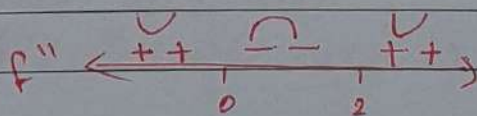
$$12x(x-2) = 0$$

$$\downarrow \quad \downarrow$$

$$\boxed{x=0} \quad \boxed{x=2}$$

at $x=0$

$f''(0) = 0$ (the second derivative test gives no information about $x=0$)



⊗ f concave up $(-\infty, 0) \cup (2, \infty)$ ~~and~~ f does not have local max or min at $x=0$
 f concave down $(0, 2)$

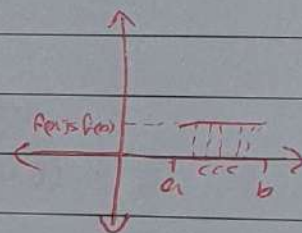
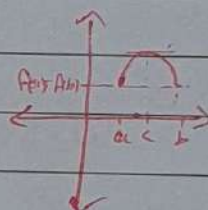
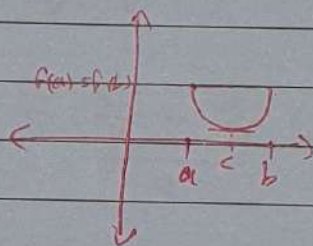
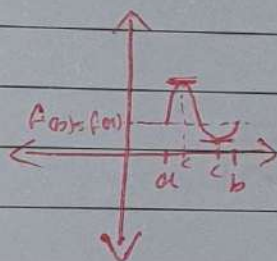
⊗ at $(0, f(0))$, $(2, f(2))$ inflection points

Lecture 15.1

★ Roll's Theorem: let $y = f(x)$ that satisfies

- ① f continuous on $[a, b]$
- ② f differentiable on (a, b)
- ③ $f(a) = f(b)$

Then $\exists \underset{\text{at least}}{c} \in (a, b)$ such that $f'(c) = 0$



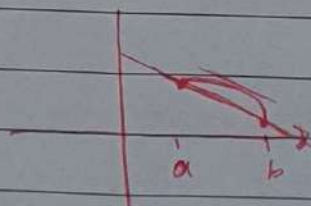
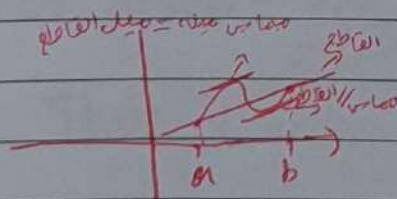
(M.V.T)

★ The mean value theorem: let $y = f(x)$ that satisfies

- ① f cont on $[a, b]$
- ② f differentiable on (a, b)

Then

Then $\exists$ at least $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$



$$\frac{dy}{dx} = \frac{f(b) - f(a)}{b - a}$$

Ex: Find the values of c that satisfy the inclusion of M.V.T

① $f(x) = \frac{1}{x-1}$ on $[2, 5]$

① f cont on $[2, 5]$

② $f'(x) = \frac{-1}{(x-1)^2}$ diff on $(2, 5)$

then by M.V.T $\exists$ at least $c \in (2, 5)$ such that

$$f'(c) = \frac{f(5) - f(2)}{5 - 2}$$

$$\frac{-1}{(c-1)^2} = \frac{\frac{1}{4} - \frac{1}{9}}{3} \Rightarrow \frac{-1}{(c-1)^2} = \frac{-5}{36}$$

$$(c-1)^2 = \frac{36}{5} \Rightarrow |c-1| = \frac{6}{\sqrt{5}} \Rightarrow c-1 = \pm \frac{6}{\sqrt{5}} \Rightarrow c = 1 \pm \frac{6}{\sqrt{5}}$$

$c = 1 + \frac{6}{\sqrt{5}} \approx 3.1$ ✓
 $c = 1 - \frac{6}{\sqrt{5}} \approx -1.2$ ✗
 $-1 \notin (2, 5)$

$$\boxed{c = 3}$$

② $f(x) = \sqrt{25-x^2}$ on $[-3, 5]$

① f cont on $[-3, 5]$

② $f'(x) = \frac{-x}{\sqrt{25-x^2}} = \frac{-x}{\sqrt{25-x^2}}$ diff on $(-3, 5)$

Then $\exists$ at least $c \in (-3, 5)$ such that

$$f'(c) = \frac{f(5) - f(-3)}{5 - (-3)}$$

$$\frac{-c}{\sqrt{25-c^2}} = \frac{0-4}{8} \rightarrow \frac{c}{\sqrt{25-c^2}} \neq \frac{1}{2}$$

$$4c^2 = (\sqrt{25-c^2})^2$$

$$4c^2 \leq 25 - c^2$$

$$4c^2 + c^2 \leq 25 \rightarrow \frac{5c^2}{8} \leq \frac{25}{8} \rightarrow c^2 \leq 5 \rightarrow c \leq \pm \sqrt{5}$$

$$2c \leq \sqrt{25-c^2}$$

$$\rightarrow \sqrt{5} = \sqrt{25-5}$$

$$\ominus \neq \oplus$$

لا نه كند تعريفها في
يُنتج

$$c = -\sqrt{5} \quad c = \sqrt{5}$$

$$\boxed{\therefore c = \sqrt{5}}$$

lecture 60

Indeterminate Forms and L'Hospital's Rule:

$$\boxed{\frac{0}{0}, \frac{+\infty}{+\infty}, \frac{+\infty}{-\infty}, \frac{-\infty}{+\infty}, \frac{-\infty}{-\infty}} \quad \text{نصف البسط} \quad \text{نصف المقام}$$

Ex: ① $\lim_{x \rightarrow +\infty} \frac{e^{3x}}{x^2}$ $\left(\frac{+\infty}{+\infty}\right)$

$$\leq \lim_{x \rightarrow +\infty} \frac{3e^{3x}}{2x} \quad \left(\frac{+\infty}{+\infty}\right)$$

↓ نصف البسط

$$\leq \lim_{x \rightarrow +\infty} \frac{9e^{3x}}{2} = \frac{9}{2} \lim_{x \rightarrow +\infty} e^{3x} = +\infty$$

② $\lim_{x \rightarrow 0^+} \frac{\ln(\sin x)}{\ln(\tan x)}$

$$\lim_{x \rightarrow 0^+} \ln(\sin x) = \ln(\lim_{x \rightarrow 0^+} \sin x) = -\infty$$

$$\lim_{x \rightarrow 0^+} \ln(\tan x) = \ln(\lim_{x \rightarrow 0^+} \tan x) = -\infty$$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{\cos x}{\sin x}}{\frac{1}{\tan x}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{\tan x}}{\sec^2 x \cdot \frac{1}{\tan x}}$$

$$= \lim_{x \rightarrow 0^+} \frac{1}{\sec^2 x} = \lim_{x \rightarrow 0^+} \frac{1}{\frac{1}{\cos^2 x}} = \lim_{x \rightarrow 0^+} \cos^2 x = 1$$

~~known defined~~

2 $0 \cdot \infty, +\infty - \infty, -\infty + \infty$;

$-\infty - \infty = -\infty \checkmark$
 $+\infty + \infty = +\infty \checkmark$

Ex 8

① $\lim_{x \rightarrow 0^+} x \ln x$ $(0 \cdot -\infty)$

$\lim_{x \rightarrow \infty} x = 0$

$= \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}}$ $(\frac{-\infty}{+\infty})$

$\lim_{x \rightarrow 0^+} \ln x = -\infty$

$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}}$

$= \lim_{x \rightarrow 0^+} \frac{1}{x} \cdot \frac{x^2}{-1} = \lim_{x \rightarrow 0^+} \frac{-1}{x} = -\infty$

$\lim_{x \rightarrow 0^+} -x = 0$

هذا الشكل
بالطريقة التي
التي

$\frac{a}{b} = \frac{a \cdot \frac{1}{b}}{1}$

$a \cdot b = \frac{a}{\frac{1}{b}}$

$x = e^{\ln x}$

$x = \ln e^x$

$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

$\sec x = \frac{1}{\cos x}$

~~② $\lim_{x \rightarrow \frac{\pi}{2}^-} (\sec x - \tan x)$~~

② $\lim_{x \rightarrow \frac{\pi}{2}^-} (\sec x - \tan x)$ $(+\infty - +\infty)$

$\lim_{x \rightarrow \frac{\pi}{2}^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) = \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{(1 - \sin x)}{\cos x}$ $(\frac{0}{0})$

~~$\lim_{x \rightarrow \frac{\pi}{2}^-} \frac{1}{\cos x}$~~
 $\lim_{x \rightarrow \frac{\pi}{2}^-} \frac{-\cos x}{-\sin x} = \frac{0}{1} = 0$

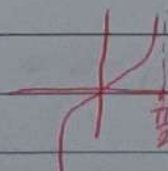
~~$\lim_{x \rightarrow \frac{\pi}{2}^-} \sec x = \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{1}{\cos x} = +\infty$~~

بالطريقة +++

$\cos x = 0$
 $x = \frac{\pi}{2}, \frac{5\pi}{2}$

$\lim_{x \rightarrow \frac{\pi}{2}^-} \tan x = +\infty$

$\lim_{x \rightarrow \frac{\pi}{2}^-} \tan x = +\infty$



$$(3) \lim_{x \rightarrow \frac{\pi}{2}^-} \sec(3x) \cos(5x) \quad (\infty \cdot 0)$$

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \frac{1}{\cos(3x)} \cdot \cos(5x)$$

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \cos(5x) \quad (\frac{0}{0})$$

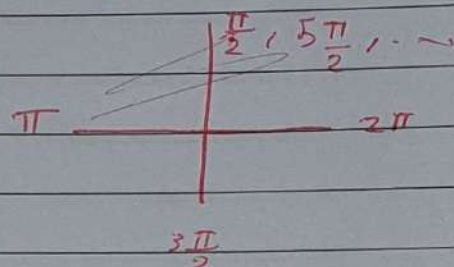
$$\lim_{x \rightarrow \frac{\pi}{2}^-} \frac{-5 \sin(5x)}{-3 \sin(3x)} = \frac{5}{3} \frac{\sin(5 \frac{\pi}{2})}{\sin(3 \frac{\pi}{2})}$$

$$= \frac{5}{3} \frac{(1)}{(-1)} = -\frac{5}{3}$$

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \sec(3x) = \infty$$

$$= \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{1}{\cos(3x)} \quad \begin{matrix} \nearrow +\infty \\ \searrow -\infty \end{matrix}$$

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \cos(5x) = 0$$



$$(4) \lim_{x \rightarrow +\infty} (x - \ln(x^2 + 1)) \quad (+\infty - \infty)$$

$$= \lim_{x \rightarrow +\infty} (\ln e^x - \ln(x^2 + 1))$$

$$= \lim_{x \rightarrow +\infty} \ln \left(\frac{e^x}{x^2 + 1} \right) = \ln \left(\lim_{x \rightarrow +\infty} \frac{e^x}{x^2 + 1} \right) \quad (\frac{+\infty}{+\infty})$$

$$= \ln \left(\lim_{x \rightarrow +\infty} \frac{e^x}{2x} \right) = \ln \left(\lim_{x \rightarrow +\infty} \frac{e^x}{2} \right) = +\infty$$

$$(5) \lim_{x \rightarrow \infty} (5^x - 2^x) \quad (+\infty - +\infty)$$

$$\lim_{x \rightarrow \infty} 5^x \left(1 - \frac{2^x}{5^x} \right) = \lim_{x \rightarrow \infty} 5^x \left(1 - \left(\frac{2}{5} \right)^x \right) \quad +\infty (1-0)$$

$$= +\infty$$

$$\lim_{x \rightarrow +\infty} b^x = +\infty, \quad b > 1$$



$$\lim_{x \rightarrow \infty} 5^x = +\infty$$

$$\lim_{x \rightarrow \infty} \left(\frac{2}{5} \right)^x = 0$$

Lecture (61)

3 $1^{\pm\infty}, \infty^0, 0^0$

Ex:

① $\lim_{x \rightarrow 0^+} (1+3x)^{\frac{-5}{x}}$

$\lim_{x \rightarrow 0^+} (1+3x) = 1, \lim_{x \rightarrow 0^+} \frac{-5}{x} = -\infty$ $\frac{0}{0}$

① $y = (1+3x)^{\frac{-5}{x}}$

② $\ln y = \ln(1+3x)^{\frac{-5}{x}}$

$\lim_{x \rightarrow 0^+} \frac{-5}{x} \cdot \ln(1+3x)$

$\lim_{x \rightarrow 0^+} \frac{-5}{x} = -\infty$

$\lim_{x \rightarrow 0^+} \ln(1+3x) = 0$

$= \lim_{x \rightarrow 0^+} \frac{-5 \ln(1+3x)}{x}$

③ $= \lim_{x \rightarrow 0^+} \frac{-5 \left(\frac{3}{1+3x} \right)}{1} = \frac{-15}{1} = -15$

④ $\therefore \lim_{x \rightarrow 0^+} (1+3x)^{\frac{-5}{x}} = e^{-15}$

② $\lim_{x \rightarrow 0^+} x^x$

$y = x^x$
 $\ln y = \ln x^x \rightarrow \ln y = x \ln x$

$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}}$

$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} \frac{1}{x} \cdot x^2 = \lim_{x \rightarrow 0^+} x = 0$

$\therefore \lim_{x \rightarrow 0^+} x^x = e^0 = 1$

Remark:

$$\textcircled{1} \lim_{x \rightarrow +\infty} \left(1 + \frac{a}{x}\right)^{bx} = e^{ab}$$

$$\textcircled{2} \lim_{x \rightarrow 0} \left(1 + ax\right)^{\frac{b}{x}} = e^{ab}$$

Ex:

$$\textcircled{1} \lim_{x \rightarrow +\infty} \left(1 + \frac{5}{x}\right)^{10x} = e^{50}$$

$$\textcircled{2} \lim_{x \rightarrow 0} (1-x)^{\frac{2}{x}} = e^{-1 \cdot 2} = e^{-2}$$

$$\textcircled{3} \lim_{x \rightarrow \infty} \left(\frac{x+4}{x}\right)^{5x} = \lim_{x \rightarrow \infty} \left(\frac{x}{x} + \frac{4}{x}\right)^{5x} = \lim_{x \rightarrow \infty} \left(1 + \frac{4}{x}\right)^{5x} = e^{20}$$

$$\textcircled{4} \lim_{x \rightarrow \infty} \left(\frac{x+2}{x+3}\right)^x = \lim_{x \rightarrow \infty} \frac{\left(\frac{x+2}{x}\right)^x}{\left(\frac{x+3}{x}\right)^x} = \frac{\lim_{x \rightarrow \infty} \left(1 + \frac{2}{x}\right)^x}{\lim_{x \rightarrow \infty} \left(1 + \frac{3}{x}\right)^x} = \frac{e^2}{e^3} = e^{2-3} = e^{-1} = \frac{1}{e}$$

Lecture (62)

★ Indefinite Integrals: $\int f(x) \cdot dx$

$$\int f(x) \cdot dx = \underbrace{F(x)}_{\text{antiderivative}} + C \quad , \quad C \text{ constant}$$

Ex: Find general antiderivative of

① $H(x) = \sec^2 x$

$$H(x) = \tan x + C \quad \leftarrow \text{general antiderivative } \int \sec^2 x \cdot dx = \tan x + C$$

② $g(x) = \cos x + \frac{1}{1+x^2}$

$$G(x) = \sin x + \tan^{-1} x + C \quad \leftarrow$$

$$\int \cos x + \frac{1}{1+x^2} \cdot dx = \sin x + \tan^{-1} x + C$$

★ Definite Integrals: $\int_a^b f(x) \cdot dx$

$$\int_a^b f(x) \cdot dx = \left[f(x) \right]_a^b = f(b) - f(a)$$

$\xleftarrow{\text{the upper limit}} b$
 $\xleftarrow{\text{the lower limit}} a$

III $\int k \cdot dx = kx + C, k, C \text{ constants}$

Ex: $\int 5 \cdot dx = 5x + C$

$$\boxed{2} \int x^r \cdot dx = \frac{x^{r+1}}{r+1} + C, \quad r \neq -1$$

Ex:

$$\textcircled{1} \int_1^2 \frac{1}{\sqrt{x}} dx = \int_1^2 x^{-\frac{1}{2}} \cdot dx = \left[\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right]_1^2 = 2\sqrt{x} \Big|_1^2$$

$$= 2\sqrt{2} - (2\sqrt{1}) = 2\sqrt{2} - 2$$

$$\textcircled{2} \int \frac{5x^3 - x}{x} \cdot dx = \int \left(\frac{5x^3}{x} - \frac{x}{x} \right) dx$$

$$= 5 \int x^2 \cdot dx - \int 1 \cdot dx$$

$$= 5 \frac{x^{2+1}}{2+1} - x + C$$

$$= \frac{5}{3} x^3 - x + C$$

$$\textcircled{3} \int \sqrt{x} (1+x) \cdot dx = \int (\sqrt{x} + \sqrt{x} \cdot x) \cdot dx$$

$$= \int (x^{\frac{1}{2}} + x^{\frac{3}{2}}) \cdot dx = \int x^{\frac{1}{2}} \cdot dx + \int x^{\frac{3}{2}} \cdot dx$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C = \frac{2x^{\frac{3}{2}}}{3} + \frac{2x^{\frac{5}{2}}}{5} + C$$

$$\boxed{3} \int (ax+b)^r \cdot dx = \frac{(ax+b)^{r+1}}{a(r+1)} + C, \quad r \neq -1$$

Ex:

$$\textcircled{1} \int (5x+3)^3 \cdot dx = \frac{(5x+3)^4}{(5)(4)} + C = \frac{(5x+3)^4}{20} + C$$

$$\textcircled{2} \int_0^1 \frac{1}{\sqrt{7-3x}} \cdot dx = \int_0^1 \frac{1}{(7-3x)^{\frac{1}{2}}} \cdot dx$$

$$= \int_0^1 (7-3x)^{-\frac{1}{2}} \cdot dx = \frac{(7-3x)^{-\frac{1}{2}+1}}{(-3)(-\frac{1}{2}+1)} \Bigg|_0^1$$

$$= \frac{-2(7-3x)^{\frac{1}{2}}}{3} \Bigg|_0^1 = \frac{-2}{3} \sqrt{7-3x} \Bigg|_0^1 = -\frac{4}{3} + \frac{2}{3} \sqrt{7}$$

$$= \frac{2\sqrt{7} - 4}{3}$$

Lecture 6.3

$$\boxed{4} \int \cos(ax+b) \cdot dx = \frac{1}{a} \sin(ax+b) + C$$

$$\int \sin(ax+b) \cdot dx = -\frac{1}{a} \cos(ax+b) + C$$

$$\int \sec^2(ax+b) \cdot dx = \frac{1}{a} \tan(ax+b) + C$$

$$\int \csc^2(ax+b) \cdot dx = -\frac{1}{a} \cot(ax+b) + C$$

$$\int \sec(ax+b) \tan(ax+b) \cdot dx = \frac{1}{a} \sec(ax+b) + C$$

$$\int \csc(ax+b) \cot(ax+b) \cdot dx = -\frac{1}{a} \csc(ax+b) + C$$

Ex: Find

~~① $\int \sin(5x) \cdot dx$~~

$$\textcircled{1} \int \sin(5x+7) \cdot dx = -\frac{1}{5} \cos(5x+7) + C$$

$$\textcircled{2} \int 2 \sin x \cdot \cos x \cdot dx = \int \sin(2x) \cdot dx = -\frac{1}{2} \cos(2x) + C$$

$$\textcircled{3} \int \frac{\sin(2x)}{\sin x} \cdot dx = \int \frac{2 \sin x \cos x}{\sin x} \cdot dx$$

$$= 2 \int \cos x \cdot dx = 2 \sin x + C$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\sin(2x) = 2 \sin x \cos x$$

$$\boxed{5} \int c \cdot f(x) \cdot dx = c \int f(x) \cdot dx$$

$$\int (f \pm g) \cdot dx = \int f(x) \cdot dx \pm \int g(x) \cdot dx$$

$$\text{Ex: } \textcircled{1} \int (\sec^2 x - 5 \cos(4x)) \cdot dx = \int \sec^2 x \cdot dx - 5 \int \cos(4x) \cdot dx$$

$$= \tan x - 5 \cdot \frac{1}{4} \sin(4x) + C$$

$$\begin{aligned}
 \textcircled{2} \int \sin^2(3x) dx &= \int \frac{1 - \cancel{\sin} \cos(2(3x))}{2} dx = \frac{1}{2} \int (1 - \cos(6x)) dx \\
 &= \frac{1}{2} \int 1 \cdot dx - \frac{1}{2} \int \cos(6x) \cdot dx \\
 &= \frac{1}{2} x - \frac{1}{2} \cdot \frac{1}{6} \sin(6x) + C = \frac{x}{2} - \frac{1}{12} \sin(6x) + C
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{3} \int \frac{\sec x + \cos x}{2 \cos x} \cdot dx &= \int \frac{\sec x}{2 \cos x} dx + \int \frac{\cos x}{2 \cos x} \cdot dx \\
 &= \frac{1}{2} \int \sec x \cdot \frac{\sec x}{\cos x} \cdot dx + \int \frac{1}{2} \cdot dx
 \end{aligned}$$

$$\int \sec^2 x \cdot dx + \int \frac{1}{2} dx = \frac{1}{2} \tan x + \frac{1}{2} x + C$$

$$\textcircled{4} \int \frac{2}{1 - \cos x} \cdot dx = 2 \int \frac{1}{1 - \cos x} \left(\frac{1 + \cos x}{1 + \cos x} \right) \cdot dx$$

$$= 2 \int \frac{1 + \cos x}{1 - \cos^2 x} = 2 \int \frac{1 + \cos x}{\sin^2 x} \cdot dx$$

$$= 2 \int \frac{1}{\sin^2 x} dx + 2 \int \frac{\cos x}{\sin^2 x} dx$$

$$= 2 \int \csc^2 x \cdot dx + 2 \int \cot x \cdot \csc x \cdot dx$$

$$= -2 \cot x + -2 \csc x + C$$

$$\textcircled{5} \int \frac{\cos}{\sin^2 x} \cdot dx = \int \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} \cdot dx = \int \cot x \cdot \csc x \cdot dx$$

$$= -\csc x + C$$

$$\begin{aligned}
 \sin^2 x + \cos^2 x &= 1 \\
 1 - \cos^2 x &= \sin^2 x \\
 1 - \sin^2 x &= \cos^2 x
 \end{aligned}$$

Lecture 6.4

$$\boxed{6} \quad \int_a^a f(x) \cdot dx = 0$$

$$\int_a^b c \cdot dx = c(b-a)$$

$$\int_a^b f(x) \cdot dx = - \int_b^a f(x) \cdot dx$$

$$\int_a^b f(x) \cdot dx = \int_a^c f(x) \cdot dx + \int_c^b f(x) \cdot dx$$

$$\int_{-a}^a \text{odd function} = 0$$

$$\int_{-a}^a \text{even function} = 2 \int_0^a \text{even function}$$

Ex:

$$\textcircled{1} \text{ If } \int_1^2 3f(x) \cdot dx = 6, \int_3^2 (5-f(x)) \cdot dx = 4$$

$$\text{find } \int_1^3 f(x) \cdot dx ?$$

$$\int_1^2 3f(x) \cdot dx = 6 \rightarrow \int_1^2 f(x) \cdot dx = 2$$

$$\int_3^2 (5-f(x)) \cdot dx = 4$$

$$\int_3^2 5 \cdot dx - \int_3^2 f(x) \cdot dx = 4$$

$$-5 + 5 - \int_3^2 f(x) \cdot dx = 4$$

$$- \int_3^2 f(x) \cdot dx = 4$$

$$\int_2^3 f(x) \cdot dx = 4$$

$$\int_3^2 (5-f(x)) \cdot dx = 4$$

$$\int_3^2 5 \cdot dx - \int_3^2 f(x) \cdot dx = 4$$

$$5(2-3) - \int_3^2 f(x) \cdot dx = 4$$

$$\int_2^3 f(x) \cdot dx = 4$$

$$\int_1^3 f(x) \cdot dx = \int_1^2 f(x) \cdot dx + \int_2^3 f(x) \cdot dx$$

$$= 2 + 4 = 6$$

odd function

$$\textcircled{2} \int_{-1}^1 \frac{\sin(2x)}{x^4 + x^2 + 1} dx = 0$$

$$f(x) \rightarrow -f(x) \rightarrow \text{odd}$$

$$f(x) = \frac{\sin(2x)}{1-x^4 + (-x)^2 + 1} = \frac{-\sin(2x)}{x^4 + x^2 + 1} = -f(x) \therefore \text{odd}$$

$$\textcircled{3} \int_{-1}^a x dx = 0 \text{ Find } a$$

$$\left[\frac{x^2}{2} \right]_{-1}^a = 0 \rightarrow \frac{a^2}{2} - \frac{(-1)^2}{2} = 0 \rightarrow \frac{a^2}{2} - \frac{1}{2} \rightarrow a = \pm 1$$

$$\textcircled{4} \text{ If } f(x) = \begin{cases} 2x, & 0 \leq x \leq 2 \\ x^2, & 2 \leq x \leq 3 \end{cases} \text{ Find } \int_0^3 f(x) \cdot dx$$

$$\int_0^3 f(x) dx = \int_0^2 f(x) dx + \int_2^3 f(x) dx$$

$$= \int_0^2 2x dx + \int_2^3 x^2 dx$$

$$= \left[\frac{2x^2}{2} \right]_0^2 + \left[\frac{x^3}{3} \right]_2^3 = (2^2 - 0) + \left(\frac{3^3}{3} - \frac{2^3}{3} \right) =$$

$$= 13 - \frac{8}{3}$$

$$= 5 \ln 6 - 5 \ln 3 = 5(\ln 6 - \ln 3) = 5 \ln \frac{6}{3} = 5 \ln 2 = \ln 2^5$$

$$= \ln 32$$

$$(3) \int \frac{2x}{1+x^2} dx = \frac{1}{2} \ln |1+x^2| + C$$

$$(4) \int \tan x dx = - \int \frac{\sin x}{\cos x} dx = - \ln |\cos x| + C$$

$$= \ln |\sec x| + C$$

$$(5) \int \frac{3x^2}{7+x^3} dx = \frac{1}{3} \ln |7+x^3| + C$$

$$(6) \int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \ln |e^x - e^{-x}| + C$$

$$(7) \int (\ln e^x + \ln e^{-x}) \cdot dx = \int (x + -x) dx$$

$$= \int 0 \cdot dx = 0 + C = C$$

$$(8) \int \frac{2 \cos(2x)}{1 + \sin(2x)} dx = \frac{1}{2} \ln |1 + \sin(2x)| + C$$

Lecture 66

ex 1 $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

$$\int \frac{g(x)}{1+(g(x))^2} \cdot dx = \tan^{-1}(g(x)) + C$$

Ex 2

① $\int \frac{1}{1+x^2} dx = \frac{1}{1} \tan^{-1}\left(\frac{x}{1}\right) + C = \tan^{-1}x + C$

② $\int \frac{1}{16+x^2} dx = \int \frac{1}{4^2+x^2} dx = \frac{1}{4} \tan^{-1}\left(\frac{x}{4}\right) + C$

③ $\int \frac{1}{5+x^2} \cdot dx = \frac{1}{5} \int \frac{1}{(\sqrt{5})^2+x^2} dx = \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$

④ $\int \frac{3}{1+9x^2} dx = \int \frac{\textcircled{3}^{g'}}{1+(\textcircled{3x})^2} = \tan^{-1}(3x) + C$
 $\downarrow$
 $g(x)$

⑤ $\int \frac{1}{1+16x^2} \cdot dx = \frac{1}{4} \int \frac{4}{1+(4x)^2} dx = \frac{1}{4} \int \frac{u}{1+u^2} \cdot dx$
 $= \frac{1}{4} \tan^{-1}(4x) + C$

⑥ $\int \frac{t}{1+t^4} dt = \frac{1}{2} \int \frac{2t}{1+(t^2)^2} dt = \frac{1}{2} \int \frac{2t}{1+(t^2)^2} dt$
 $= \frac{1}{2} \tan^{-1} t^2 + C$

$$\textcircled{7} \int \frac{x^2}{1+x^6} dx = \frac{3}{3} \int \frac{x^2}{1+(x^3)^2} dx = \frac{1}{3} \int \frac{3x^2}{1+(x^3)^2} dx$$

$$= \frac{1}{3} \tan^{-1}(t^3) + C$$

$$\textcircled{8} \int \frac{e^{2x}}{1+e^{4x}} dx = \frac{2}{2} \int \frac{e^{2x}}{1+(e^{2x})^2} dx = \frac{1}{2} \int \frac{2e^{2x}}{1+(e^{2x})^2} dx$$

$$= \frac{1}{2} \tan^{-1}(e^{2x}) + C$$

$$\textcircled{9} \int \frac{e^x}{1+e^{2x}} dx = \int \frac{e^x}{1+(e^x)^2} dx = \tan^{-1}(e^x) + C$$

$$\textcircled{10} \int \frac{t^2-1}{t^4-1} dt = \int \frac{1}{1+t^2} dt = \tan^{-1}(t) + C$$

t^2+1

Lecture 67

$$\boxed{10} \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{g'(x)}{\sqrt{1 - (g(x))^2}} dx = \sin^{-1}(g(x)) + C$$

Ex:

$$\textcircled{1} \int \frac{1}{\sqrt{9 - x^2}} dx = \int \frac{1}{\sqrt{3^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{3}\right) + C$$

$$\textcircled{2} \int_{\ln 2}^{\ln(2/\sqrt{3})} \frac{e^{-x}}{\sqrt{1 - e^{-2x}}} dx =$$

$$= - \int_{\ln 2}^{\ln(2/\sqrt{3})} \frac{e^{-x}}{\sqrt{1 - (e^{-x})^2}} dx = - \sin^{-1}(e^{-x}) \Big|_{\ln 2}^{\ln(2/\sqrt{3})}$$

$$= - \sin^{-1}(e^{-\ln(2/\sqrt{3})}) - (- \sin^{-1}(e^{-\ln 2}))$$

$$= - \sin^{-1}(e^{\ln(\frac{\sqrt{3}}{2})}) + \sin^{-1}(e^{\ln \frac{1}{2}})$$

$$= - \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right) = -\frac{\pi}{3} + \frac{\pi}{6} = -\frac{\pi}{6}$$

$$\textcircled{3} \int \frac{\sec^2 \theta}{\sqrt{1 - \tan^2 \theta}} d\theta = \int \frac{\sec^2 \theta}{\sqrt{1 - (\tan \theta)^2}} d\theta = \sin^{-1}(\tan \theta) + C$$

$$(4) \int \frac{dx}{x \sqrt{1-(\ln x)^2}} = \int \frac{\frac{1}{x}}{\sqrt{1-(\ln x)^2}} dx = \sin^{-1}(\ln x) + C$$

$$\boxed{III} \int \frac{1}{x \sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{g'(x)}{g(x) \sqrt{(g(x))^2 - 1}} dx = \sec^{-1}(g(x)) + C$$

Exo:

$$\textcircled{1} \int \frac{1}{x \sqrt{x^2 - 4}} dx = \int \frac{1}{x \sqrt{x^2 - 2^2}} dx = \frac{1}{2} \sec^{-1}\left(\frac{x}{2}\right)$$

$$\textcircled{2} \int \frac{1}{x \sqrt{x^2 - 1}} dx = \sec^{-1}(x) + C$$

Lecture 88

II] Integration by substitution ; تغيير المتكامل

⊗ Ex:

$$\textcircled{1} \int \frac{\sqrt{\tan^{-1} x}}{1+x^2} \cdot dx$$

$$= \int \frac{\sqrt{y}}{1+x^2} \cdot (1+x^2) dy$$

$$= \int \sqrt{y} \, dy = \int y^{\frac{1}{2}} \cdot dy = \frac{2}{\frac{3}{2}} y^{\frac{3}{2}} + C$$

$$= \frac{2}{3} (\tan^{-1} x)^{\frac{3}{2}} + C$$

$$\begin{aligned} \text{let } y &= \tan^{-1} x \\ \frac{dy}{dx} &= \frac{1}{1+x^2} \\ dx &= (1+x^2) dy \end{aligned}$$

$$\textcircled{2} \int_0^{\frac{\pi}{2}} \sin^2 x \cos x \, dx$$

$$\int_0^1 y^2 \cos x \cdot \frac{dy}{\cos x}$$

$$\int_0^1 y^2 \, dy = \left[\frac{y^3}{3} \right]_0^1 = \frac{1}{3} - \frac{0^3}{3} = \frac{1}{3}$$

$$\begin{aligned} \text{let } y &= \sin x \\ dy &= \cos x \, dx \\ dx &= \frac{dy}{\cos x} \end{aligned}$$

$$\text{at } x=0 \rightarrow y=0$$

$$\text{at } x=\frac{\pi}{2} \rightarrow y=1$$

$$\textcircled{3} \int_0^1 x \sqrt{1-x} \cdot dx$$

$$\int x y \cdot -2 \sqrt{1-x} \cdot dy$$

$$\text{let } y = \sqrt{1-x}$$

$$dy = \frac{-1}{2\sqrt{1-x}} \cdot dx$$

$$dx = -2\sqrt{1-x} \, dy$$

$$= \int (x)(y) (-2)(y) \cdot dy$$

$$= \int (1-y^2) (-2y^2) \cdot dy$$

$$= \int (-2y^2 + 2y^4) \cdot dy$$

$$= \left[-\frac{2}{3} y^3 + \frac{2}{5} y^5 \right]$$

$$= \left[-\frac{2}{3} (\sqrt{1-x})^3 + \frac{2}{5} (\sqrt{1-x})^5 \right]_0^1$$

$$= \left(-\frac{2}{3} (\sqrt{1-1})^3 + \frac{2}{5} (\sqrt{1-1})^5 \right) - \left(-\frac{2}{3} (\sqrt{1-0})^3 + \frac{2}{5} (\sqrt{1-0})^5 \right)$$

$$= \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$$

$$\begin{cases} y^2 = 1-x \\ y^2 - 1 = -x \\ x = 1 - y^2 \end{cases}$$

Lecture (601)

$$\text{Ex: } \int (1+\sqrt{x})^{20} dx$$

$$= \int y^{20} \cdot 2\sqrt{x} dy$$

~~$$= \int y^{20} \cdot 2\sqrt{x} dy$$~~

$$= \int 2y^{20} (y-1) dy$$

$$= \int 2y^{21} - 2y^{20} dy$$

$$= \frac{2}{22} y^{22} - \frac{2}{21} y^{21} + C$$

$$= \frac{2}{22} (1+\sqrt{x})^{22} - \frac{2}{21} (1+\sqrt{x})^{21} + C$$

$$\text{let } y = 1+\sqrt{x}$$

$$dy = \frac{1}{2\sqrt{x}} dx$$

$$2\sqrt{x} dy = dx$$

$$y-1 = \sqrt{x}$$

$$\textcircled{2} \int \frac{\sin x}{16+9\cos^2 x} \cdot dx$$

$$= \int \frac{\sin x}{16+9y^2} \cdot \frac{-1}{\sin x} = \int \frac{-1}{16+9y^2} \cdot dy$$

$$\text{let } y = \cos x$$

$$\frac{-1}{\sin x} dy = -\frac{\sin x}{\sin x} dx$$

$$dx = \frac{-1}{\sin x} dy$$

$$= -\int \frac{1}{16+9y^2} \cdot dy$$

$$= -\frac{1}{9} \int \frac{1}{(\frac{4}{3})^2 + y^2} \cdot dy$$

$$\int \frac{1}{a^2 + y^2} \cdot dy = \frac{1}{a} \tan^{-1}\left(\frac{y}{a}\right) + C$$

$$= -\frac{1}{9} \cdot \frac{1}{\frac{4}{3}} \tan^{-1}\left(\frac{y}{\frac{4}{3}}\right) + C$$

$$= \frac{-3}{(4)(9)} \tan^{-1}\left(\frac{3}{4} \cos x\right) + C = \frac{-1}{12} \tan^{-1}\left(\frac{3}{4} \cos x\right) + C$$

③ If $\int_1^5 f(x) dx = 10$ Find $\int_0^2 x f(x^2+1) \cdot dx$

$$\begin{aligned} & \int_1^5 \cancel{x} f(\cancel{x}) \cdot \frac{dy}{2\cancel{x}} \\ &= \frac{1}{2} \int_1^5 f(y) dy \\ &= \frac{1}{2} (10) = 5 \end{aligned}$$

Let $y = x^2 + 1$

$$dy = 2x dx$$

$$dx = \frac{dy}{2x}$$

at $x=0 \rightarrow y = 0^2 + 1 = 1$

at $x=2 \rightarrow y = 2^2 + 1 = 5$

Lecture 70

$$\boxed{13} \int \cosh(ax+b) dx = \frac{1}{a} \sinh(ax+b) + C$$

$$\int \sinh(ax+b) dx = \frac{1}{a} \cosh(ax+b) + C$$

$$\int \operatorname{sech}^2(ax+b) dx = \frac{1}{a} \tanh(ax+b) + C$$

$$\int \operatorname{csch}^2(ax+b) dx = -\frac{1}{a} \coth(ax+b) + C$$

$$\int \operatorname{sech}(ax+b) \tanh(ax+b) dx = -\frac{1}{a} \operatorname{sech}(ax+b) + C$$

$$\int \sinh(ax+b) \cosh(ax+b) dx = \frac{1}{a} \cosh(ax+b) + C$$

Ex:

$$\textcircled{1} \int \sinh(5x+4) dx = \frac{1}{5} \cosh(5x+4) + C$$

$$\textcircled{2} \int \operatorname{sech}(9x) \tanh(9x) dx = -\frac{1}{9} \operatorname{sech}(9x) + C$$

$$\textcircled{3} \int \operatorname{sech}^2(3-9x) dx = -\frac{1}{9} \tanh(3-9x) + C$$

$$\textcircled{4} \int (\sinh x + \cosh x) dx = \cosh x + \sinh x + C$$

$$= e^x + C$$

$$\textcircled{5} \int \frac{\operatorname{sech}(5x) \tanh(5x)}{4 + \operatorname{sech}(5x)} dx$$

~~$$\int \frac{\operatorname{sech}(5x) \tanh(5x)}{4 + \operatorname{sech}(5x)} dx$$~~



$$\text{let } y = \operatorname{sech}(5x)$$

$$dy = -5 \operatorname{sech}(5x) \tanh(5x) dx$$

$$dx = \frac{dy}{-5 \operatorname{sech}(5x) \tanh(5x)}$$

$$-5 \operatorname{sech}(5x) \tanh(5x)$$

$$= \int \frac{\operatorname{sech}(5x) \tanh(5x)}{4+y} \cdot \frac{dy}{-5 \operatorname{sech}(5x) \tanh(5x)}$$

$$= -\frac{1}{5} \int \frac{1}{4+y} \cdot dy$$

$$= -\frac{1}{5} \ln|4+y| + C$$

$$= -\frac{1}{5} \ln|4 + \operatorname{sech}(5x)| + C$$

119 $\frac{d}{dx} \int_a^b f(x) dx = 0$

$$\frac{d}{dx} \int f(x) dx = f(x)$$

Ex:

① Find $\frac{d}{dx} \int_0^u \sqrt{5+\cos x} dx = 0$

② if $y = \int \sqrt{5+\cos x}$ Find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d}{dx} \int \sqrt{5+\cos x} \cdot dx = \sqrt{5+\cos x}$$

③ If $\int f(x) \cdot dx = x^4 + x + 4$ Find $f(1)$, $f'(1)$

$$\frac{d}{dx} \int f(x) \cdot dx = \frac{d}{dx} (x^4 + x + 4)$$

$$f(x) = 4x^3 + 1$$

$$f(1) = 4(1)^3 + 1 = 5$$

$$f'(x) = 12x^2$$

$$f'(1) = 12(1)^2 = 12$$

(4) If $y = e^{2x} + \ln(x^2 + 1) + \int_0^x \frac{5x^2}{\cos x} \cdot dx$ Find $\frac{dy}{dx}$

$$\frac{dy}{dx} = 2e^{2x} + \frac{2x}{x^2 + 1} + 0$$

[15] $\int f'(x) = f(x) + C$

$$\int f''(x) = f'(x) + C$$

Ex:

(1) If $f'(x) = 5x^4 + 2x + 3$, $f(2) = 3$

$$\int f'(x) dx = \int (5x^5 + 2x + 3) dx$$

$$f(x) + C = x^5 + x^2 + 3x$$

[f(2) = 3] $f(2) + C = (2)^5 + (2)^2 + 3(2)$

$$3 + C = 32 + 4 + 6 \rightarrow \boxed{C = 39}$$

$$\therefore f(x) + 39 = x^5 + x^2 + 3x \Rightarrow$$

$$\boxed{f(x) = x^5 + x^2 + 3x - 39}$$

Find $f(1)$:

$$f(1) = (1)^5 + (1)^2 + 3(1) - 39 = -34$$

Lecture 21

[16] The Fundamental Theorem of calculus

$$\frac{d}{dx} \int_{f(x)}^{g(x)} h(t) dt = h(g(x)) g'(x) - h(f(x)) \cdot f'(x)$$

Ex:

$$\textcircled{1} \frac{d}{dx} \int_1^x \sin t \cdot dt = \sin(x) \cdot (1) - \cancel{\sin(1) \cdot (0)} = \sin x$$

$$\begin{aligned} \textcircled{2} \frac{d}{dx} \int_{2x}^{x^2} (t^2 - 1) dt &= ((x^2)^2 - 1)(2x) - ((2x)^2 - 1) \cdot 2 \\ &= 2x(x^4 - 1) - 2(4x^2 - 1) \\ &= 2x^5 - 2x - 8x^2 + 2 \end{aligned}$$

$$\begin{aligned} \textcircled{3} \frac{d}{dx} \int_0^{2x} \sqrt{1+t^4} dt &= (\sqrt{1+(2x)^4})(2) - \cancel{\sqrt{1+(0)^4} \cdot (0)} \\ &= 2\sqrt{1+16x^4} \end{aligned}$$

$$\textcircled{4} \text{ If } g(x) = x^2 \cdot \int_6^{3x} \frac{t^5}{t^2+5} dt \text{ Find } g'(2)?$$

$$\begin{aligned} g'(x) &= 2x \int_6^{3x} \frac{t^5}{t^2+5} dt + x^2 \cdot \frac{d}{dx} \int_6^{3x} \frac{t^5}{t^2+5} dx \\ &= 2x \int_6^{3x} \frac{t^5}{t^2+5} dt + x^2 \left(\frac{(3x)^5}{(3x)^2+5} (3) - \cancel{\frac{6^5}{6^2+5} (0)} \right) \end{aligned}$$

$$g'(x) = 2x \int_6^{3x} \frac{t^5}{t^2+5} dt + \frac{3x^2 (3x)^5}{(3x)^2+5}$$

$$g'(2) = 2(2) \int_6^{3(2)} \frac{t^5}{t^2+5} dt + \frac{3(2)^2 (3(2))^5}{(3(2))^2+5}$$

$$g'(2) = 12 \left(\frac{6^5}{36+5} \right)$$

⑤ If $\int_3^{2x} f(t) dt = \frac{5}{x^2+3}$ Find $f(x)$?

$$\frac{d}{dx} \int_3^{2x} f(t) dt = \frac{d}{dx} \frac{5}{(x^2+3)^2} = -5(2x)$$

$$f(2x) \cdot \frac{2}{2} - f(3)(0) = \frac{-10}{(x^2+3)^2} \cdot \frac{x}{2}$$

$$f(2x) = \frac{-5}{(x^2+3)^2}$$

$$\text{Let } y = 2x \\ x = \frac{y}{2}$$

$$f(y) = \frac{-5 \left(\frac{y}{2} \right)}{\left(\left(\frac{y}{2} \right)^2 + 3 \right)^2}$$

$$f(x) = \frac{-\frac{5}{2} x}{\left(\left(\frac{x}{2} \right)^2 + 3 \right)^2}$$

$$\textcircled{6} \quad \lim_{x \rightarrow 1} \frac{\int_2^{2x} (t+3) dt}{x-1} \quad \left(\frac{0}{0} \right) = \frac{\frac{d}{dx} \int_2^{2x} (t+3) dt}{\frac{d}{dx} (x-1)}$$

~~lim~~
~~x → 1~~

$$= \lim_{x \rightarrow 1} \frac{(2x+3)(2) - (2+3)(0)}{1}$$

$$\lim_{x \rightarrow 1} \frac{4x+6}{1} = 4(1) + 6 = 10$$

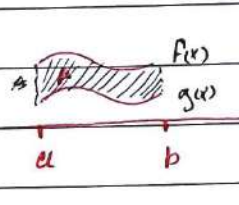
Lecture 72

Area between two curves:



~~Theorem:~~

Theorem: $A = \int_a^b (f(x) - g(x)) dx$



مثال ٢: إيجاد نقاط التقاطع بين $g \circ f$ و $g(x) = f(x)$

E x 2

① Find the area of the region ~~enclosed~~ enclosed by

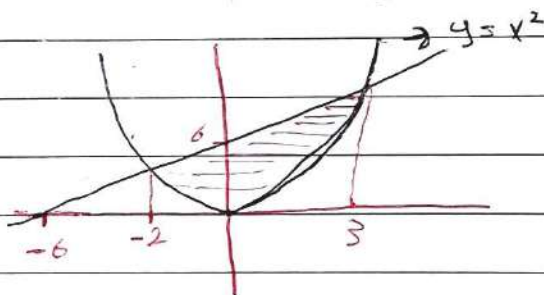
$$y = x^2, y = x + 6$$

$$x^2 = x + 6$$

$$x^2 - x - 6 = 0$$

$$(x+2)(x-3) = 0$$

$$x = -2 \quad x = 3$$



$$A = \int_{-3}^3 ((x+6) - (x^2)) dx$$

$x + 6 - x^2$

-2 0 3

مجموعه

ای افغان

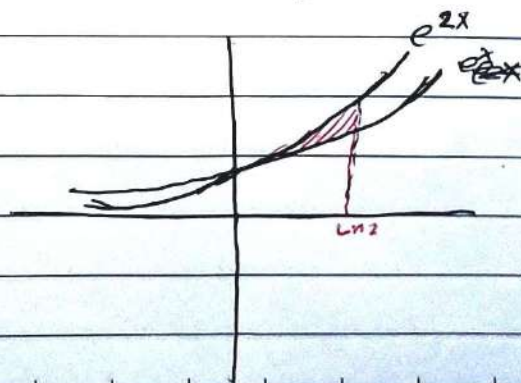
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② Find the area bounded by $y = e^x$, $y = e^{2x}$, $x \geq 0$, $x \leq \ln 2$

$$A_5 = \int_0^{\ln 2} (e^{2x} - e^x) dx$$

$$e^{2x} = e^x \rightarrow \ln e^{2x} \neq \ln e^x$$

$$2x \leq x \rightarrow x \leq 0$$



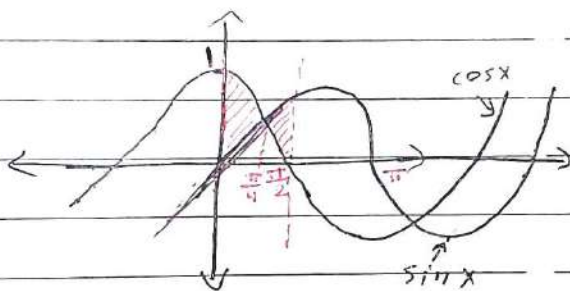
$$\int_0^{\ln 2} (e^{2x} - e^x) dx = \dots$$

$$\left[\frac{e^{2x} - e^x}{2} \right]_0^{\ln \frac{3}{2}}$$

$$e^{\ln \frac{3}{2}} = \left[\frac{3}{2} \right]$$

$$e^{2 \ln \frac{3}{2}} = e^{\ln \frac{9}{4}} = \left[\frac{9}{4} \right]$$

③ Find the area enclosed by $y = \sin x$, $y = \cos x$, $x \leq 0$, $x \leq \frac{\pi}{2}$?



$$\left[\cos x - \sin x \right]_0^{\frac{\pi}{4}} + \left[\sin x - \cos x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$\sin x = \cos x$$

$$x = \frac{\pi}{4}$$

$$A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x) dx = \dots$$