

2-10-2017

CH 13.1 Complex number and their Geometric representation

$$x^2 = -1$$

$$x = \pm \sqrt{-1}$$

$$= \pm i \text{ or } \pm j$$

- any complex number should be represented by :-

$$Z = x + iy \rightarrow \text{cartesian form}$$

$$x = \operatorname{Re}(Z)$$

$$y = \operatorname{Im}(Z)$$

$$Z = x + iy = (x, y)$$

Ex:- $Z_1 = 3 + 2i$

$$x = 3, y = 2$$

$$Z_3 = 5 \rightarrow \text{complex}$$

$$= 5 + i(0)$$

$$x = 5, y = 0$$

$$Z_2 = -3 - 4i$$

$$x = -3, y = -4$$

$$Z_4 = -5i$$

$$= 0 + (-5)i$$

$$x = 0, y = -5$$

Rule:- Two complex number are equal if real part are equal, Im part ----

Ex:- If $Z_1 = 2x + i5y$

$$Z_2 = 10 + i15 \text{ and } Z_1 = Z_2 \text{ Find } (x, y) ?$$

$$Z_1 = Z_2$$

$$2x = 10$$

$$x = 5$$

$$5y = 15 \rightarrow y = 3$$

* Summation

$$Z_1 = X_1 + iy_1$$

$$Z_2 = X_2 + iy_2$$

$$Z_1 + Z_2 = (X_1 + X_2) + i(y_1 + y_2)$$
$$\underline{X} + i\underline{Y}$$

• $(Z = X + iy)$...

* Subtraction

$$Z_1 - Z_2 = X_1 + iy_1 - (X_2 + iy_2)$$

$$= (X_1 - X_2) + i(y_1 - y_2)$$

$$= \underline{X} + i\underline{Y}$$

* Multiplication

$$Z_1 Z_2 = (X_1 + iy_1)(X_2 + iy_2)$$

$$= X_1 X_2 + iy_1 X_2 + i y_1 X_2 + i^2 y_1 y_2$$

note: $(\sqrt{-1})^2 = (i)^2 \Rightarrow i^2 = -1$

$$= (X_1 X_2 - y_1 y_2) + i(y_2 X_1 + y_1 X_2)$$
$$\underline{X} + i\underline{Y}$$

* Scalar multiplication: ~

$$Z_1 = X_1 + iy_1$$

$$c Z_1 = c(X_1 + iy_1)$$

$$= cX_1 + i c y_1$$

$$\underline{X} + i\underline{Y}$$

division: ~

note: ~ conjugate of complex number.

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2} \quad \leftarrow \quad \begin{matrix} x+iy & x-iy \\ \hline & \end{matrix}$$

$$= \frac{x_1 x_2 - iy_2 x_1 + iy_1 x_2 - i^2 y_1 y_2}{x_2^2 - iy_2 x_2 + iy_1 x_2 - i^2 y_1^2} = \frac{(x_1 x_2 + y_1 y_2) + i(y_1 x_2 - y_2 x_1)}{x_2^2 + y_2^2}$$

$$= \frac{(x_1 x_2 + y_1 y_2)}{x_2^2 + y_2^2} + i \frac{(y_1 x_2 - y_2 x_1)}{x_2^2 + y_2^2}$$

* $z_1 = 2 + 2i$ $z_2 = 1 - i$

1. $z_1 + z_2 = 3 + i$

3. $z_1 z_2 = 2 - 2i + 2i - 2i^2$
 $= 2 + 2 = 4 \rightarrow = 4 + i(0)$

2. $z_1 - z_2 = 1 + 3i$

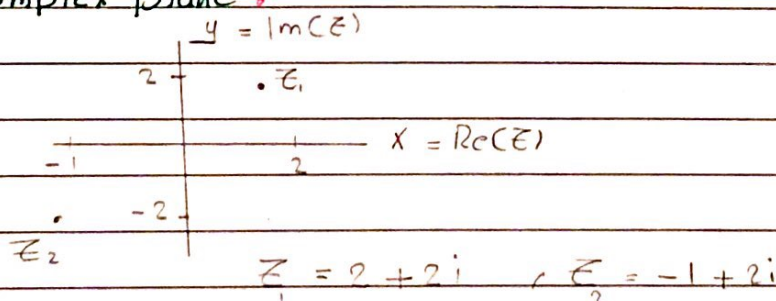
4. $\frac{z_1}{z_2} = \frac{2+2i}{1-i} \cdot \frac{1+i}{1+i}$

5. $\bar{z}_1 = 2 - 2i$

$$= \frac{2+2i+2i+4i^2}{2} = \frac{-2+4i}{2}$$

$$= -1 + 2i$$

- complex plane: ~



Rules :-

$$1. z \bar{z} = x^2 + y^2$$

$$2. x = \operatorname{Re}(z) = \frac{z + \bar{z}}{2}$$

$$3. y = \operatorname{Im}(z) = \frac{z - \bar{z}}{2}$$

$$4. (\overline{z_1 + z_2}) = \bar{z}_1 + \bar{z}_2$$

$$\frac{x+iy + x-iy}{2} - \frac{2x}{x} = x$$

$$5. (\overline{z_1 - z_2}) = \bar{z}_1 - \bar{z}_2$$

$$6. \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

$$7. \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

$$Z = X + iy = (x, y)$$

cartesian form

13.2 polar form of complex number

power, root

$$Z = X + iy = re^{i\theta}$$

cartesian polar

$$1. r = |Z| = \sqrt{x^2 + y^2}$$

magnitude, normal, modulus

$$2. \theta = \tan^{-1} \left(\frac{y}{x} \right) \begin{matrix} \nearrow \text{img} \\ \searrow \text{real} \end{matrix}$$

Ex:- let $Z = 1 + i$ find the polar form?

$$x = 1, y = 1$$

$$Z = re^{i\theta}, r = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$$

$$\theta = \tan^{-1} \left(\frac{y}{x} \right)$$

$$= \tan^{-1} \left(\frac{1}{1} \right) = \frac{\pi}{4}$$

$$Z = \sqrt{2} e^{i\frac{\pi}{4}}$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

Ex:- let $Z = \sqrt{2} e^{i\frac{\pi}{2}}$ find the cartesian form?

$$Z = re^{i\theta}$$

$$r = \sqrt{2}, \theta = \frac{\pi}{2}$$

$$x = \sqrt{2} \cos \frac{\pi}{2} = 0$$

$$y = \sqrt{2} \sin \frac{\pi}{2} = \sqrt{2}$$

$$Z = 0 + (\sqrt{2})i$$

$$Z = X + iy = (x, y)$$

cartesian form

13.2 polar form of complex number

power, root

$$Z = X + iy = re^{i\theta}$$

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Ex:- let $Z = 1 + i$ find the polar form?

$$x = 1, y = 1$$

$$Z = re^{i\theta}, r = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$$

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$$Z = \sqrt{2} e^{i\frac{\pi}{4}}$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

Ex:- let $Z = \sqrt{2} e^{i\frac{\pi}{2}}$ find the cartesian form?

$$Z = re^{i\theta}$$

$$r = \sqrt{2}, \theta = \frac{\pi}{2}$$

$$x = \sqrt{2} \cos \frac{\pi}{2} = 0$$

$$y = \sqrt{2} \sin \frac{\pi}{2} = \sqrt{2}$$

$$Z = 0 + (\sqrt{2})i$$

* Euler formula :-

$$e^{i\theta} = \cos \theta + i \sin \theta$$

Ex:- Find θ for :-

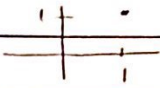
$$Z_1 = 1 + i$$

$$\text{for } Z_1 = 1 + i$$

$$Z_2 = -1 + i$$

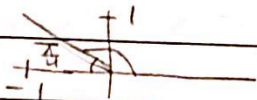
$$r = \sqrt{2}, \theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$Z_3 = 1 - i$$



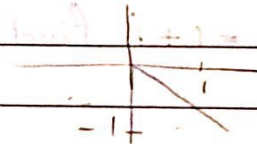
$$\text{for } Z_2 = -1 + i$$

$$r = \sqrt{2}, \theta = \tan^{-1}\left(\frac{1}{-1}\right) = \frac{4\pi}{4} - \frac{\pi}{4} = \frac{3\pi}{4}$$



$$\text{for } Z_3 = 1 - i$$

$$r = \sqrt{2}, \theta = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$$



$$\text{for } Z_4 = -1 - i$$

$$r = \sqrt{2}, \theta = \tan^{-1}\left(\frac{-1}{-1}\right) = -\frac{\pi}{4} + \frac{\pi}{4} = -\frac{3\pi}{4}$$

$$-\pi \leq \theta < \pi$$

$$\arg(z) = \theta$$

$$\text{- Principle value} = \text{Arg}(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\text{- } -\pi \leq \text{Arg}(z) \leq \pi$$

Observation

$$\text{- } \arg(z) = \text{Arg}(z) \pm 2\pi n$$

* Power: ~

$$Z = (1+i)^2 \text{ write it in simple form?}$$

$$Z = X + iy$$

$$Z = (1+i)(1+i)$$

$$= 1 + 2i + (-1)$$

$$= 2i = 0 + 2i$$

$$\text{De Moivre's } (n)$$

$$Z = (X + iy)$$

$$1. X + iy = r e^{i\theta}$$

$$2. Z = (r e^{i\theta})^n$$

$$3. Z = r^n (e^{i\theta})^n = r^n e^{i\theta n}$$

$$4. r^n e^{i\theta n} = r^n [\cos(\theta n) + i \sin(\theta n)]$$

$$= r^n \cos(\theta n) + i r^n \sin(\theta n)$$

X

i Y

$n = 1, 2, 3, 4, \dots$

*X Ex:- Find in simple form the $Z = (1+i)^{10}$?

$$1+i \rightarrow$$

$$r = \sqrt{(1)^2 + (1)^2} = \sqrt{2}, \theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$Z = (\sqrt{2} e^{i\frac{\pi}{4}})^{10} \rightarrow Z = 2^5 e^{i\frac{5\pi}{2}}$$

$$= 32 [\cos(\frac{5\pi}{2}) + i \sin(\frac{5\pi}{2})]$$

$$= 32i$$

$$\text{Ex:- } Z = (1+i)^{-10}$$

$$Z = (1+i)^{-10} = \frac{1}{(1+i)^{10}}$$

$$Z = \frac{1}{32 e^{i5\pi/2}} = \frac{1}{32} e^{-i5\pi/2} = \frac{1}{32} [\cos(-\frac{5\pi}{2}) + i \sin(-\frac{5\pi}{2})]$$

$$= \frac{-i}{32}$$

$$\frac{1}{32i} \times \frac{-i}{-i} = \frac{-i}{32}$$

* De Moivre's Formula :-

$$e^{i\theta n} = e^{i\theta n}$$

$$[e^{i\theta}]^n = e^{i(\theta n)}$$

$$[\cos(\theta) + i\sin(\theta)]^n = \cos(\theta n) + i\sin(\theta n)$$

- Root of complex number :-

$$\sqrt[n]{x+iy} = \sqrt[n]{r} \left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right) \right]$$

where n = real integer value number

$$r = \sqrt{x^2 + y^2}$$

1, 2, 3, 4, ...

$$\theta = \tan^{-1}\left(\frac{y}{x}\right), k = 0, 1, 2, 3, \dots, \boxed{n-1}$$

Ex:- Find $\sqrt[3]{1+i}$?

$$\boxed{n=3} \quad r = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

1, 2, 3, 4, ...

$$k = 0, 1, 2$$

$$, n-1 = 3-1 = \boxed{2}$$

$$k = 0, 1, 2$$

for $k=0 \rightarrow 1^{\text{st}}$ root

$$\sqrt[3]{1+i} = \sqrt[3]{r} \left[\cos \frac{\frac{\pi}{4} + 2\pi(0)}{3} + i \sin \frac{\frac{\pi}{4} + 2\pi(0)}{3} \right]$$

$$= (2)^{1/6} \left[\cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right]$$

for $k=1 \rightarrow 2^{\text{nd}}$ root

$$= \sqrt[3]{r} \left[\cos \left(\frac{\pi}{4} + 2\pi(1) \right) + i \sin \left(\frac{\pi}{4} + 2\pi(1) \right) \right]$$

$$= (2)^{1/6} \left[\cos \frac{9\pi}{12} + i \sin \frac{9\pi}{12} \right]$$

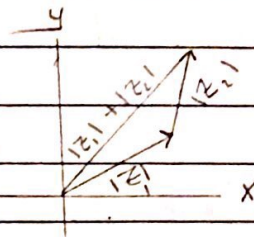
for $k=2 \rightarrow 3^{\text{rd}}$ root

$$= (2)^{1/6} \left[\cos \frac{17\pi}{12} + i \sin \frac{17\pi}{12} \right]$$

- Triangular Inequality :-

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

في المثلث (في شكله في الصورة)
الزاوية



* multiplication and division of complex number in Polar Form :-

$$z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2}$$

$$1. z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$2. \frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

$$\text{Ex:- let } z_1 = 2e^{i\frac{\pi}{4}}, \quad z_2 = 3e^{i\frac{\pi}{2}}$$

$$z_1 z_2 = 6e^{i(\frac{3\pi}{4})}, \quad \frac{z_1}{z_2} = \frac{2}{3} e^{i(\frac{\pi}{4} - \frac{\pi}{2})} = \frac{2}{3} e^{-i\pi/4}$$

Rule :-

$$1. |z_1 z_2| = |z_1| |z_2|$$

$$2. \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$$

$$3. \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

$$4. \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

Ex: - $\sqrt[4]{1}$

$$\sqrt[4]{1+0i}$$

$$n=4$$

$$r = \sqrt{1^2 + 0^2} = 1 \quad -\Theta = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

$$\begin{aligned}\sqrt[4]{1+0i} &= \sqrt[4]{1} [\cos(0 + \frac{2\pi k}{4}) + i \sin(0 + \frac{2\pi k}{4})] \\ &= \cos \frac{\pi k}{2} + i \sin \frac{\pi k}{2}, \quad k=0,1,2,3\end{aligned}$$

for $k=0$

$$k=1$$

$$k=3$$

$$\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}$$

* Root of complex number :-

$$\sqrt[n]{x+iy} = \sqrt[n]{r} \left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i \sin\left(\frac{\theta+2\pi k}{n}\right) \right]$$

$$r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right), \quad k=0, 1, 2, \dots, [n-1]$$

Ex:- $\sqrt[3]{1}$

$$\sqrt[3]{1+0i} = 1 \left[\cos\left(\frac{0+2\pi k}{3}\right) + i \sin\left(\frac{0+2\pi k}{3}\right) \right]$$

$$n=3, \quad x=1, \quad y=0$$

$$r=1, \quad \theta = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

$$k=0, 1, 2$$

for $k=0 \rightarrow 1^{\text{st}} \text{ root}$

$$= 1$$

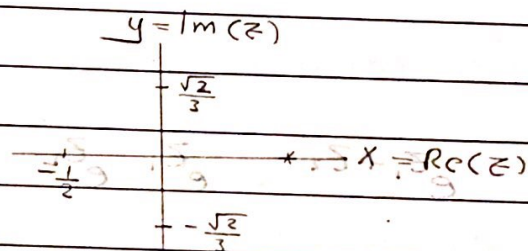
for $k=1 \rightarrow 2^{\text{nd}} \text{ root}$

$$= \frac{-1}{2} + i \frac{\sqrt{3}}{2}$$

for $k=2 \rightarrow 3^{\text{rd}} \text{ root}$

$$= \frac{-1}{2} - i \frac{\sqrt{3}}{2}$$

Complex plane :-



Rule :-

$$|\operatorname{Re}(z)| \leq |z|$$

$$|\operatorname{Im}(z)| \leq |z|$$

- Parallelogram equality :-

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$$

* note :- rotational number

$$z^{\frac{n_1}{n_2}} = \left[(z)^{n_1} \right]^{\frac{1}{n_2}}$$

Power
root

$$\begin{aligned}
 \text{Ex: } (1+i)^{2/3} &= [1+2i+i^2]^{1/3} = (2i)^{1/3} = \sqrt[3]{2i} = \sqrt[3]{0+i(2)} \\
 &\Rightarrow \sqrt[3]{0+i(2)} = \sqrt[3]{2} \left[\cos \frac{\pi}{2} + i \sin \left(\frac{\pi}{2} + 2\pi k \right) \right] \\
 ((1+i)^2)^{1/3} \quad r &= \sqrt{0^2+2^2} = \sqrt{2} \quad \theta = \tan^{-1} \left(\frac{2}{0} \right) = \frac{\pi}{2}, k=0,1,2 \\
 \text{Power} \quad \text{root} \quad \text{Polar form} &\Rightarrow \sqrt[3]{2} \left[\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right] \\
 &\Rightarrow \sqrt[3]{2} \left[\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right] \\
 &\Rightarrow \sqrt[3]{2} \left[\cos \frac{\pi}{2} + i \sin \left(\frac{\pi}{2} + 2\pi k \right) \right]
 \end{aligned}$$

13.5 Exponential function for complex number :-

$$z = e^z$$

$$z = x + iy$$

$$z = e^{x+iy} = e^x e^{iy}$$

$$= e^x [\cos y + i \sin y]$$

Euler Formula

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$z = e^x \underbrace{\cos(y)}_{\text{Re}} + i e^x \underbrace{\sin(y)}_{\text{Im}}$$

$$\text{note: } e^{z_1+z_2} = e^{z_1} e^{z_2}$$

$$\Rightarrow |z| = r = \sqrt{x^2 + y^2}$$

$$= \sqrt{(e^x \cos y)^2 + (e^x \sin y)^2}$$

$$= \sqrt{(e^x)^2 [\cos^2(y) + \sin^2(y)]}$$

$$= e^x \quad \left\{ \begin{array}{l} \cos^2(y) + \sin^2(y) = 1 \end{array} \right. \quad \text{using identity}$$

$$\theta = \tan^{-1} \left(\frac{y}{x} \right) = \tan^{-1} \left(\frac{e^x \sin y}{e^x \cos y} \right) = \tan^{-1} (\tan y)$$

$$= \tan^{-1} (\tan y)$$

$$= y$$

* principle value, $\text{Arg}(z), \theta \rightarrow y$

$$* \arg(z) = \text{Arg}(z) \pm 2\pi n$$

$$\arg(z) = y \pm 2\pi n$$

$$* (e^z)' = e^z$$

- Find $e^{\pi i}$, $e^{2\pi i}$, $e^{(\pi/2)i}$

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$$

$$e^{i2\pi} = \cos(2\pi) + i\sin(2\pi) = 1$$

$$e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$

Ex:- write $e^{1.4 - 0.6i}$ in simple form?

$$z = e^{1.4 - 0.6i}$$

$$= e^{1.4} e^{-0.6i}$$

$$= e^{1.4} [\cos(-0.6) + i\sin(-0.6)]$$

note $\rightarrow \cos(-x) = \cos(x)$

$$\sin(-x) = -\sin(x)$$

$$= e^{1.4} [\cos(0.6) - i\sin(0.6)]$$

$$= e^{1.4} \cos(0.6) - i e^{1.4} \sin(0.6)$$

$$|z| = \sqrt{x^2 + y^2}$$

$$= \sqrt{[e^{1.4} \cos(0.6)]^2 + [-e^{1.4} \sin(0.6)]^2}$$

$$= e^{1.4}$$

$$\theta = -0.6$$

$$\arg(z) = -0.6 + 2\pi n$$

$$e^z \neq 0 \quad \text{--- X-X-X ---}$$

exp is entire function

13.6 Trigonometric and hyperbolic function

$$e^{iz} = \cos(z) + i\sin(z) \quad \text{--- ① ---}$$

$$e^{i(-z)} = \cos(-z) + i\sin(-z)$$

$$= \cos(z) - i\sin(z) \quad \text{--- ② ---}$$

$$\textcircled{1} + \textcircled{2} \quad e^{iz} + e^{-iz} = 2 \cos z \quad \textcircled{1} - \textcircled{2} \quad e^{iz} - e^{-iz} = 0 + 2i \sin(z)$$

$$** \cos(z) = \frac{e^{iz} + e^{-iz}}{2} \quad ** \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\tan(z) = \frac{\sin(z)}{\cos(z)} \quad \cot(z) = \frac{1}{\tan(z)} = \frac{\cos(z)}{\sin(z)} \quad \sec(z) = \frac{1}{\cos(z)}$$

$$\csc(z) = \frac{1}{\sin(z)}$$

• Rule: ~

$$(\cos(z))' = -\sin(z) \quad \cos(z) = \cos(x) \cosh(y) - i \sin(x) \sinh(y)$$

$$(\sin(z))' = \cos(z) \quad \sin(z) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$$

$$(\tan(z))' = \sec^2(z)$$

Ex: ~ Find $(\cos(\pi + i\pi))$

$$\begin{aligned} \cos(\pi + i\pi) &= \cos(\pi) \cosh(\pi) - i \sin(\pi) \sinh(\pi) \\ &= -\cosh(\pi) + i \cdot 0 \end{aligned}$$

$$-\sin^2(z) + \cos^2(z) = 1$$

$$-\cosh(x) = 1 + \sinh(x)$$

$$-|\cos(z)|^2 = \cos^2(x) + \sinh^2(y)$$

$$-|\sin(z)|^2 = \sin^2(x) + \sinh^2(y)$$

$$-\sin(z_1) \cos(z_2) = \frac{1}{2} [\sin(z_1 + z_2) + \sin(z_1 - z_2)]$$

$$-\cos(z_1 \pm z_2) = \cos(z_1) \cos(z_2) \mp \sin(z_1) \sin(z_2)$$

$$-\sin(z_1 + z_2) = \sin(z_1) \cos(z_2) \pm \sin(z_2) \cos(z_1)$$

$$\cosh^2(z) + \sinh^2(z) = \cosh(2z)$$

* Complex Trigonometric and hyperbolic function are related

$$\cosh(iZ) = \cos(Z)$$

$$\sinh(iZ) = i \sin(Z)$$

$$\cos(iZ) = \cosh(Z)$$

$$\sin(iZ) = i \sinh(Z)$$

$$iZ = i(X+iy) \quad , \quad Z = \frac{X+iy}{1} \cdot \frac{0-i}{0-i} = \frac{0-iX+y}{0-i} = \frac{y-iX}{-i} = y + i(-X)$$

$$= iX + i^2 y$$

$$= -y + iX$$

$$\rightarrow \cosh(Z) = \cosh(X) \cos(y) + i \sinh(X) \sin(y)$$

$$\sinh(Z) = \sinh(X) \cos(y) + i \cosh(X) \sin(y)$$

$$\cosh(Z_1 + Z_2) = \cosh(Z_1) \cosh(Z_2) + \sinh(Z_1) \sinh(Z_2)$$

$$\sinh(Z_1 + Z_2) = \sinh(Z_1) \cosh(Z_2) + \cosh(Z_1) \sinh(Z_2)$$

$$\cosh^2(Z) + \sinh^2(Z) = \cosh(2Z)$$

$$\cosh^2(Z) - \sinh^2(Z) = 1$$

Ex: $\cosh(2\pi + i\pi)$

$$= \cosh(2\pi) \cos(\pi) + i \sinh(2\pi) \sin(\pi)$$

$$= -\cosh(2\pi) \text{ or } -\cos(2\pi i)$$

13.7 :- Logarithm, general power, principle value ?

$$(Z = X+iy = re^{i\theta}) \quad \ln = \ln(re^{i\theta})$$

$$\ln Z = \ln(X+iy) = \ln(re^{i\theta})$$

$$= \ln(r) + \ln e^{i\theta}$$

$$= \ln(r) + i\theta \ln e = \ln(r) + i\theta$$

$$r = \sqrt{x^2 + y^2} \quad , \quad \theta = \tan^{-1}\left(\frac{y}{x}\right) \pm 2\pi n$$

Ex:- Find $\ln(1+i)$ in simple form?

$$\ln(x+iy) = \ln(r) + i\theta$$

$$x=1, y=1 \rightarrow r=\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) \pm 2\pi n = \frac{\pi}{4} \pm 2\pi n$$

$$\ln(1+i) = \ln(\sqrt{2}) + i\left[\frac{\pi}{4} \pm 2\pi n\right]$$

Ex:- Find $\ln(3-4i)$ in simple form?

$$\ln(3-4i) = \ln(r) + i\theta$$

$$r = \sqrt{9+16} = \sqrt{25} = 5$$

$$\theta = \tan^{-1}\left(\frac{-4}{3}\right) \pm 2\pi n = -0.93 \pm 2\pi n$$

$$\ln(5) + i[-0.93 \pm 2\pi n]$$

* Principle value $\ln(z)$ called $L_n(z)$

$$L_n(z) = \ln(z) + i \operatorname{Arg}(z) \dots \text{--- (1)}$$

$$\ln(z) = L_n(z) \pm 2\pi ni \dots \text{--- (2)} \quad \begin{matrix} n=0 \\ \hline \ln(z) = L_n(z) \end{matrix}$$

Ex:- $L_n(3-4i) = \ln(5) + i[-0.93]$

Ex:- $L_n(1+i) = \ln(\sqrt{2}) + i\left[\frac{\pi}{4}\right]$

1. $\ln(z_1 z_2) = \ln(z_1) + \ln(z_2)$

2. $\ln\left(\frac{z_1}{z_2}\right) = \ln(z_1) - \ln(z_2)$

3. $[\ln(z)]' = \frac{1}{z}$

* General power :-

$$(z)^c = c - \text{real}$$

- +ve, -ve

- Complex number

- rotational

$$(z)^c = e^{\ln(z)c} = e^{c \ln(z)}$$

$$\ln(z) = \ln(r) + i\theta$$

$$= e^{c[\ln(r) + i\theta]} = e^{c \ln(r) + i c \theta} = e^{x_3 + i y_3}$$

$$e^{x_3 + i y_3} = e^{x_3} [\cos(y_3) + i \sin(y_3)]$$

$$= e^{x_3} \cos(y_3) + i e^{x_3} \sin(y_3)$$

Real
Im

Ex:- Find $(i)^i$ in simple form?

$$(i)^i = e^{\ln(i)i} = e^{i \ln(i)}$$

$$\ln(i) = \ln(0 + i(1)) = \ln(r) + i\theta$$

$$r = \sqrt{0^2 + 1^2} = 1, \theta = \tan^{-1}\left(\frac{1}{0}\right) \pm 2\pi n$$

$$= \frac{\pi}{2} \pm 2\pi n$$

$$\ln(i) = \ln(1) + i\left(\frac{\pi}{2} \pm 2\pi n\right)$$

$$= i\left(\frac{\pi}{2} \pm 2\pi n\right)$$

$$= e^{i\left(\frac{\pi}{2} \pm 2\pi n\right)} = e^{-\frac{\pi}{2} \pm 2\pi n}$$

Ex:- Find $(1+i)^{2-i}$

$$(1+i)^{2-i} = e^{\ln(1+i)(2-i)} = e^{(2-i)\ln(1+i)}$$

$$\ln(1+i) = \ln(\sqrt{2}) + i\left(\frac{\pi}{4} \pm 2\pi n\right)$$

$$(2-i)(\ln(\sqrt{2}) + i(\frac{\pi}{4} \pm 2\pi n))$$

$$(z_1)^{z_2} = e^{z_2 \ln(z_1)}$$

$$(x_1 + i y_1)^{(x_2 + i y_2)} = ?$$

Solution:-

$$\frac{(x_2 + iy_2)}{(x_1 + iy_1)} = e^{\frac{\ln(x_2 + iy_2)}{(x_2 + iy_2) \ln(x_1 + iy_1)}} = e^{\frac{\ln(x_2 + iy_2)}{(x_2 + iy_2) \ln(x_1 + iy_1)}}$$

$$\ln(x_1 + iy_1) = \ln(\sqrt{x_1^2 + y_1^2}) + i(\tan^{-1}(\frac{y_1}{x_1}) \pm 2\pi n)$$

$$= x_3 + iy_3 = (x_2 + iy_2) [\ln \sqrt{x_1^2 + y_1^2} + i(\tan^{-1}(\frac{y_1}{x_1}) \pm 2\pi n)]$$

$$\rightarrow e^{x_3 + iy_3} = e^{x_3} e^{iy_3}$$

$$= e^{x_3} [\cos(y_3) + i \sin(y_3)]$$

$$= \underbrace{e^{x_3} \cos(y_3)}_{\text{Re}} + i \underbrace{e^{x_3} \sin(y_3)}_{\text{Im}}$$

Chapter matrix, vector, addition and scalar multiplication

matrix :- Is a rectangular array of numbers or function which we will enclose in brackets.

element entry $\rightarrow$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0.2 & 0.1 \end{bmatrix}$, $\begin{bmatrix} \cos x & \sin x \\ 2 & 0.3 \end{bmatrix}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{array}{l} \text{row} = 3 \\ \text{column} = 3 \end{array}$$

size of $A = \text{row} \times \text{column}$
 3×3

* Column vector

$$V_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

* row vector

$$V_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$$

* linear system : $Ax = b$

$$Ax = b$$

$$4x_1 + 6x_2 + 9x_3 = 6$$

$$6x_1 + 0x_2 - 2x_3 = 20$$

$$5x_1 - 8x_2 + x_3 = 10$$

$$\begin{bmatrix} 4 & 6 & 9 \\ 6 & 0 & -2 \\ 5 & -8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 10 \end{bmatrix}$$

* Augmenting matrix $A = \tilde{A} = [A | b]$

$$\tilde{A} = \left[\begin{array}{ccc|c} 4 & 6 & 9 & 6 \\ 6 & 0 & -2 & 20 \\ 5 & -8 & 1 & 10 \end{array} \right] \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{1}{2} \\ -1 \end{bmatrix}$$

* For matrix $A = [a_{ij}]$ has $m \times n$ row and column

size of $A = m \times n$
 row $\rightarrow$ column

ex: - $A = 3 \times 3$

* Square matrix :- Is matrix that has rows = column

* main diagonal elements : A

$$\rightarrow a_{11}, a_{22}, a_{33} \leftarrow \text{main diagonal}$$

* Addition of matrix :-

rule :- equality of matrix

Two matrix $A = [a_{jk}]$ and $B = [b_{jk}]$ are equal iff

1. have same size

2. corresponding element are equal

If $A \neq B$ then A and B are different iff :-

1. do not have same size

2. and corresponding element are not equal

Ex:- let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2 \times 2}$ $B = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}_{2 \times 2}$

If $A = B$ find $a_{11}, a_{12}, a_{21}, a_{22}$?

$a_{11} = 4, a_{12} = 0, a_{21} = 3, a_{22} = -1$

Ex:- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2 \times 2}$, $B = \begin{bmatrix} 2 & 2 \\ 2 & 4 \\ 5 & 6 \end{bmatrix}_{3 \times 2}$

$A = B$? A and B are different

Def:- The sum of two matrix $A = [a_{jk}]$ and $B = [b_{jk}]$ of same size is written as $A + B = [a_{jk} + b_{jk}]$ and obtained by adding the corresponding element.

Note:- matrix with different size can not be added.

ex:- $A = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix}_{2 \times 3}$, $B = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}_{2 \times 3}$
Find $A + B = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}_{2 \times 3}$, $A - B = \begin{bmatrix} -9 & 7 & 3 \\ -3 & 0 & 2 \end{bmatrix}_{2 \times 3}$

ex:- let $A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}_{2 \times 2}$, $B = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & 3 \end{bmatrix}_{2 \times 3}$

Find $A + B$ = can not be added

* Def: scalar multiplication

The product of any $m \times n$ matrix $A = [a_{ij}]$ and any scalar (C) written as CA and is $m \times n$ matrix $CA = [ca_{ij}]$ obtained by multiplying each element by C .

Ex: If $A = \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}_{2 \times 2}$ Find $2A$?

$$2A = \begin{bmatrix} 4 & 6 \\ -4 & 2 \end{bmatrix}_{2 \times 2} \quad \frac{-1}{2} A = \begin{bmatrix} 1 & \frac{3}{2} \\ -1 & \frac{1}{2} \end{bmatrix}_{2 \times 2}$$

$$-1A = \begin{bmatrix} -2 & -3 \\ 2 & -1 \end{bmatrix}_{2 \times 2}$$

Addition rules:-

1. $A + B = B + A$

2. $(A + B) + C = A + (B + C) = A + B + C$

3. $A + \underline{0}_{\text{matrix}} = A$

4. $A + (-A) = \underline{0}_{\text{matrix}}$

3. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \underline{0} \Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Rule of scalar multiplication:-

1. $C(A + B) = CA + CB$

2. $(C + k)A = CA + kA$

3. $C(kA) = (ck)A$

4. $1A = A$

- matrix multiplication :-

Def:- The product $C=AB$ of an $m \times n$ matrix $A=[a_{jk}]$ times an $n \times p$ matrix $B=[b_{jk}]$ is defined iff $n=r$ and the $m \times p$ matrix $C=[c_{jk}]$ with entries

$$c_{jk} = \sum_{l=1}^n a_{jl} b_{lk}$$

$j = 1, 2, \dots, m$
 $k = 1, 2, \dots, p$

$$\begin{matrix} AB & = & C \\ \begin{matrix} m \times n & n \times p \\ \text{Equal} \end{matrix} & & m \times p \end{matrix}$$

Ex:- let $A = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix}$

3×3 3×4

Find AB ?

$$\begin{matrix} AB & = & C \\ \begin{matrix} 3 \times 3 & 3 \times 4 \\ \text{equal} \end{matrix} & & 3 \times 4 \end{matrix} \quad C = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & 14 & 6 \\ -9 & 4 & -37 & -29 \end{bmatrix}$$

$$C_{11} = (3)(2) + (5)(5) + (-1)(9) = 22$$

$$C_{12} = (3)(-2) + (5)(0) + (-1)(-4) = -2$$

$$C_{13} = (-6)(1) + (-3)(8) + (3)(11) = -18$$

Ex:- let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ Find AB ? $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$$AB = C = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) \end{bmatrix}$$

Ex:- 1. $\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 22 \\ 43 \end{bmatrix}$

2×2 2×1 2×1

2. $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 1 \\ 6 & 12 & 2 \\ 12 & 24 & 4 \end{bmatrix}$

3×1 1×3 3×3

3. $\begin{bmatrix} 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 19 \end{bmatrix}$

1×3 3×1 1×1

note :- $AB \neq BA$

rule :- 1. $(kA)(B) = k(AB)$

2. $A(BC) = (AB)C$

3. $(A+B)C = AC + BC$

4. $C(A+B) = CA + CB$

* product in terms of row and column vector :-

$$\text{let } A_{3 \times 3} = \begin{bmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{bmatrix} \quad AB_{\substack{3 \times 3 \quad 3 \times 4 \\ \text{equal}}} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 & a_1 b_4 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 & a_2 b_4 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 & a_3 b_4 \end{bmatrix}$$

$$B_{3 \times 4} = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ | & | & | & | \\ | & | & | & | \\ | & | & | & | \end{bmatrix}$$

- parallel processing :-

$$AB = A \begin{bmatrix} b_1 & b_2 & b_3 & b_n \\ | & | & | & | \\ | & | & | & | \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & Ab_3 & Ab_n \\ | & | & | & | \\ | & | & | & | \\ | & | & | & | \end{bmatrix}$$

$$\text{Ex:- } AB = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 7 \\ -1 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 11 & 4 & 34 \\ -17 & 8 & -23 \end{bmatrix}$$

$$1. Ab_1 = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ -17 \end{bmatrix} \quad 3. \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 34 \\ -23 \end{bmatrix}$$

$$2. Ab_2 = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \quad \therefore AB = \begin{bmatrix} 11 & 4 & 34 \\ -17 & 8 & -23 \end{bmatrix}$$

* Trans position :-

Def:- The transpose of $m \times n$ matrix $A = [a_{ij}]$ is $n \times m$ matrix A^T "A transpose" that has first row of A as its first column and second row of A as its second column and so on.

Ex:- let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ Find A^T ?

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Ex:- let $A = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}$ Find A^T ?

$$A^T = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}$$

Ex:- let $A = [6 \ 2 \ 3]$ Find A^T ?

$$A^T = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$$

rules:-

1. $(A^T)^T = A$

3. $(CA)^T = CA^T$

2. $(A+B)^T = A^T + B^T$

4. $(AB)^T = B^T A^T$

$\neq A^T B^T$

* Square matrix $\rightarrow A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2 \times 2}$

* symmetric matrix iff $A^T = A$

* skew-symmetric matrix iff

$$A^T = -A$$

* upper triangular matrix

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

* lower triangular matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 8 & -1 & 0 \\ 7 & 6 & 3 \end{bmatrix}$$

* diagonal matrix

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

* scalar matrix

$$S = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$$

* Unit matrix, Identity matrix

$$I_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AI - IA = A$$

$$T_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

7.3 linear system of equation

Gauss-elimination

→ let linear system with m -equation and n -unknowns

$$\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \dots + a_{2n}x_n &= b_2 \end{aligned}$$

$$a_{m1}x_1 + \dots + a_{mn}x_n = b_m$$

1. linear system $\rightarrow x_1, x_2, \dots, x_n$

2. For L.S with (b_j) are Zero

The system is homogeneous

3. For L.S with (b_j) at least one is not Zero

("non-homogeneous")

any linear system can be written as $Ax = b$

$$\begin{matrix} 3 \times 3 \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \end{matrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

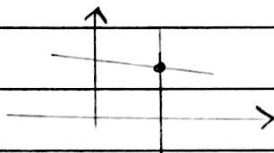
$$A \quad x = b$$

→ Augmented matrix A

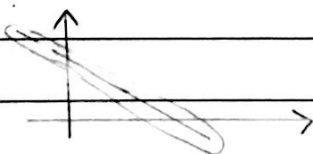
$$\tilde{A} = [A | b]$$

* For $Ax = b$ has three possible solution:-

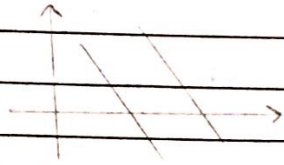
1. Precisely one solution if lines intersect



2. Infinitely many solution if the lines coincide



3. no - solution if the lines are parallel



x. Gauss - elimination and back substitution

$$2x_1 + 5x_2 = 2$$

$$+ 13x_2 = -26$$

$$x_2 = -2$$

$$x_1 = 6$$

1. $AX = b$

$$\begin{bmatrix} 2 & 5 \\ 0 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -26 \end{bmatrix}$$

2. $\tilde{A} = [A \mid b]$

$$= \left[\begin{array}{cc|c} 2 & 5 & 2 \\ 0 & 13 & -26 \end{array} \right] \downarrow$$

Gauss-elimination

$$x_2 = \frac{-26}{13} = -2 \quad \text{back}$$

$$x_1 = 6 \quad \text{substitution}$$

Ex:- $2x_1 + 5x_2 = 2$

$$-4x_1 + 3x_2 = -30$$

Find x_1, x_2 ?

$$2 \text{ [eq (1)]} + \text{eq (2)}$$

$$4x_1 + 10x_2 = 4$$

$$+ 4x_1 + 3x_2 = -30$$

$$0 + 13x_2 = -26$$

$$\rightarrow x_2 = -2$$

$$x_1 = 6$$

Using Gauss - elimination

1. $Ax = b$

$$\begin{bmatrix} 2 & 5 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -30 \end{bmatrix}$$

2. $\tilde{A} = [A | b]$

$$\tilde{A} = \left[\begin{array}{cc|c} 2 & 5 & 2 \\ -4 & 3 & -30 \end{array} \right] \xrightarrow{\text{Pivot}}$$

$$\left[\begin{array}{cc|c} 2 & 5 & 2 \\ 0 & 13 & -26 \end{array} \right] \sim 2R_1 + R_2$$

We eliminate x_1 ?

$$13x_2 = -26$$

$$x_2 = -2, \quad x_1 = \frac{2 - 5x_2}{2} = 6$$

Ex:- with one - solution

$$x_1 - x_2 + x_3 = 0$$

$$-x_1 + x_2 - x_3 = 0$$

$$10x_2 + 25x_3 = 90$$

$$20x_1 + 10x_2 = 80$$

Find x_1, x_2, x_3 ?

1. $Ax = b$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 10 & 25 \\ 20 & 10 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 90 \\ 80 \end{bmatrix}$$

2. $\tilde{A} = [A | b]$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array} \right] \xrightarrow{\text{pivot}}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array} \right] \xrightarrow{R_1 + R_2} \xrightarrow{-20R_1 + R_4}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\text{Pivot}}$$

eliminate x_2

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{3R_2 + R_3}$$

back substitution

$$-95x_3 = -190 \quad x_3 = 2$$

$$0x_1 + 10x_2 + 25x_3 = 90 \quad x_2 = 4$$

$$x_1 - x_2 + x_3 = 0 \quad x_1 = 0$$

* elementary row operation of matrix

1. Inter change of two rows.
 2. addition of constant multiple of row to another row.
 3. multiplication of row by non-zero constant.
- has no-effect on A or solution

Theorem :- Row equivalent system has same solution.

Ex:-

$$\textcircled{A} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\textcircled{B} \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_3 \\ b_1 \end{bmatrix}$$

[A + B has same solution]

Note :-

1. Over determined L.S if it has ~~same~~ more equation than Unknowns.
2. determined L.S if it has equal number of equation and Unknowns.
3. Under determined L.S if it has fewer equation than Unknowns.
4. Consistent L.S if it has one solution or least. ایک یا زائد
5. In Consistent L.S if it has no solution.

Ex: Find with infinitely many solution

$$3x_1 + 1.2x_2 + 2x_3 - 5x_4 = 8$$

$$0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 = 2.7$$

$$1.2x_1 - 0.3x_2 - 0.3x_3 + 2.4x_4 = 2.1$$

solution $\rightarrow$ لا يوجد حل فريد

① $Ax = b$ (over determining)

$$\begin{bmatrix} 3 & 1.2 & 2 & -5 \\ 0.6 & 1.5 & 1.5 & -5.4 \\ 1.2 & -0.3 & -0.3 & 2.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 2.7 \\ 2.1 \end{bmatrix}$$

② $\tilde{A} = [A | b]$

$$\tilde{A} = \left[\begin{array}{cccc|c} 3 & 1.2 & 2 & -5 & 8 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array} \right] \rightarrow \text{pivot}$$

③ eliminate x_1

$$\left[\begin{array}{cccc|c} 3 & 1.2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array} \right] \begin{array}{l} \rightarrow R_2 - 0.2R_1 \\ \rightarrow R_3 - 0.4R_1 \end{array}$$

$$3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$$

$$1.1x_2 + 1.1x_3 - 4.4x_4 = 1.1$$

Back substitution

لا يوجد حل فريد

$$x_3 = x_4 = 0 \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = 2 - x_4$$

$$x_3 = \text{arbitrary}$$

$$x_2 = 1 - x_3 + 4x_4$$

$$x_4 = \text{arbitrary}$$

Infinitely many solution

② no-solution

$$3x_1 + 2x_2 + x_3 = 3$$

$$2x_1 + x_2 + x_3 = 0$$

$$6x_1 + 2x_2 + 4x_3 = 6$$

solution $\rightarrow$ find x_1, x_2, x_3 s.t.
 Determine x_1, x_2, x_3

① $Ax = b$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 6 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$$

③ eliminate x_1

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -2 \\ 0 & -2 & 2 & 0 \end{array} \right] \rightarrow R_2 - \frac{2}{3}R_1, R_3 - 2R_1$$

② $\tilde{A} = [A | b]$

$$= \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array} \right] \rightarrow \text{Pivot}$$

④ eliminate x_2

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -2 \\ 0 & 0 & 0 & 12 \end{array} \right] \rightarrow R_3 - 6R_2$$

$$3x_1 + 2x_2 + x_3 = 3$$

$$-\frac{1}{2}x_2 + \frac{1}{2}x_3 = -2$$

$$0 = 12 \text{ False statement}$$

No - solution $\neq$

* Row Echelon Form and information from it: ~

$\rightarrow$ For L.S $\rightarrow Ax = b$

$$\tilde{A} = [A | b]$$

a. after Gauss - elimination [Rif]

low ~

$$[R, f] = \left[\begin{array}{ccc|c} r_{11} & r_{12} & r_{1n} & f_1 \\ 0 & r_{22} & r_{2n} & f_2 \\ & & r_{nn} & f_n \\ & & & f_{n+1} \\ & & & f_{n+2} \\ & & & f_m \end{array} \right]$$

1. $r_{ii} \neq 0$

2. $r \leq m$

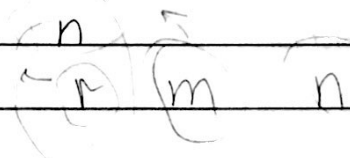
3. all element in the triangle and Rectangular are zero.

4. $r = \text{rank}(R) = \text{rank}(A)$

A. no - solution: ~

if $r < m$ and at least one number

$f_{n+1}, f_{n+2}, \dots, f_m$ is not zero



B. Unique one solution

if $r = n$ and all numbers from $r+1$ to m are zero

$$\left[\begin{array}{cc|c} 2 & 5 & 2 \\ 0 & 13 & -26 \end{array} \right]$$

$$r_1 = 13 = r_2$$

$n = n \rightarrow R_1 = R_2 = -26$ has one solution.

C. Infinitely many solution

if $r < m$ and all number from $r+1$ to m are zero

$$\left[\begin{array}{ccccc} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$r = 2 \quad 2 < 3$$

$$m = 3, \quad R_1 = R_2 = 1.1$$

7.4:- linear Independent, Rank of matrix, vector space.

Theorem:- let $a_1, a_2, a_3, \dots, a_m$ are m -vector

$c_1, c_2, c_3, \dots, c_m$ are m -scalar

then $c_1 a_1 + c_2 a_2 + c_3 a_3 + \dots + c_m a_m = 0$ — eq (1)

1. If eq (1) holds for all c_i are zero then system is linearly Independent.

2. If eq (1) holds for at least one c_i is not zero then system is linearly dependent.

$$a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$a_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = 0$$

$$c_1 + 2c_2 = 0$$

$$c_1 = -2c_2$$

$$2c_1 + 4c_2 = 0$$

$$c_2 = 1$$

not zero $\rightarrow$ linearly dependent

$$3c_1 + 6c_2 = 0$$

Ex:- $a_1 = [3 \ 0 \ 22]$

$a_2 = [-6 \ 42 \ 24 \ 54]$

$a_3 = [21 \ -12 \ 0 \ -15]$

are a_1, a_2, a_3 L.D or L.I.D ?

$$C_1 [3 \ 0 \ 22] + C_2 [-6 \ 42 \ 24 \ 54] + C_3 [21 \ -12 \ 0 \ -15] = 0$$

$$3C_1 - 6C_2 + 21C_3 = 0$$

$$0C_1 + 42C_2 - 12C_3 = 0$$

$$2C_1 + 24C_2 + 0C_3 = 0$$

$$2C_1 + 54C_2 - 15C_3 = 0$$

$$C_1 = 6, C_2 = -\frac{1}{2}, C_3 = -1$$

a_1, a_2, a_3 are linearly dependent.

1-11-2017

ملاحظة من المحاضر: إذا لم تكن متأكد من الإجابة، فاستخدم اختبار رتبة المصفوفة.

Rank of matrix :-

def :- rank of matrix $A = \text{Rank}(A)$ is the maximum number of linearly independent row vector of A .

Ex:- let $A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} \begin{matrix} \rightarrow a_1 \\ \rightarrow a_2 \\ \rightarrow a_3 \end{matrix}$

Find rank(A) ?

1. a_1, a_2

$$C_1 a_1 + C_2 a_2 = 0$$

$$C_1 \begin{bmatrix} 3 \\ -6 \\ 21 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 42 \\ -21 \\ 24 \end{bmatrix} = 0$$

$$3C_1 + (-6)C_2 = 0$$

$$0C_1 + 42C_2 = 0$$

$$2C_1 + 24C_2 = 0$$

$$2C_1 + 54C_2 = 0$$

$$C_1 = C_2 = 0$$

a_1, a_2 are linearly Independent

2. a_1, a_3

$$C_1 a_1 + C_3 a_3 = 0$$

$$3C_1 + 21C_3 = 0$$

$$0C_1 + 21C_3 = 0$$

$$2C_1 + 0C_3 = 0$$

$$2C_1 + (-15)C_3 = 0$$

$$C_1 = C_3 = 0$$

a_1, a_3 are linearly Indep

3. a_2, a_3

$$C_2 a_2 + C_3 a_3 = 0$$

$$-6C_2 + 21C_3 = 0$$

$$42C_2 - 21C_3 = 0$$

$$21C_2 + 0C_3 = 0$$

4. a_1, a_2, a_3

$$C_1 a_1 + C_2 a_2 + C_3 a_3 = 0$$

$$C_1 = 6$$

$$C_2 = -\frac{1}{2}$$

$$C_3 = -1$$

(rank is equal to no. of zeros eliminated)

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank} = \text{no. of non-zero rows} = 2 \quad \left(\begin{array}{l} \text{rank of matrix is equal to} \\ \text{no. of non-zero rows} \end{array} \right)$$

note: $\text{rank}(A) = 0$ iff $A = 0$

* Row-equivalent matrix :-

1. matrix A_1 is row equivalent to matrix A_2 if A_1 can be obtained from A_2 by elementary row-operation.
2. Row-equivalent matrix has same rank.

Theorem :-

Consider p -vector that each has n -component then these vectors are linearly independent if the matrix formed with these vectors as row vectors has rank p . However these vectors are linearly dependent if it has rank less than p .

$$\text{Ex:- } p_1 = \begin{bmatrix} - & - & - & - \end{bmatrix}$$

$$p_2 = \begin{bmatrix} - & - & - & - \end{bmatrix}$$

$$p_3 = \begin{bmatrix} - & - & - & - \end{bmatrix}$$

$$p = 3 ; n = 4$$

$$\text{rank}(\text{formed matrix}) = p \quad \text{L. indep}$$

$$\text{rank}(\text{formed matrix}) < p \quad \text{L. dep}$$

* rank of matrix :-

Is the max number of L. indep column vector of A .

$$\text{Rank}(A) = \text{Rank}(A^T)$$

* Consider p -vectors each has n -component if $n < p$ then these vector are linearly dep.

Ex: - $p_1 = [1 \ 2]$, $p_2 = [5 \ 6]$, $p_3 = [9 \ 10]$

$p_4 = [3 \ 4]$, $p_5 = [7 \ 8]$ (two element fixed)

$p = 5$, $n = 2$, $n < p \Rightarrow$ L. dep
 $2 < 5$

* Vector space

$$\text{let } V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & & v_n \\ | & | & & | \end{bmatrix}$$

1. dimension of V = the max. number of linearly independent vectors in V .
 (rank = dimension)
2. basis of V = sub-set of V with max possible number of linearly indep vectors.
3. span of V = sub-set of V with max possible number of linearly dep vectors.

Theorem: - The vector space $\mathbb{R}^n$ consisting of all vectors with n -component (where n is real number) has dimension n .

- The row-space and the column space of matrix A have the same dimension equal to $\text{rank}(A)$.

row space and column space

Theorem: - $\text{Rank}(A) + \text{nullity}(A) = \text{number of columns}$
 $= \text{number of unknowns}.$

note: - ① row space of A = the span of rows vectors of A

② column = = = = = column = = =

Theorem:-

$$Ax = b$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{array}{l} m \text{ - equation} \\ n \text{ - unknown} \end{array} \rightarrow \text{1. Degree of freedom}$$

$$\tilde{A} = [A \mid b]$$

1. solution exist:- linearly system has solution

$$\text{iff } \text{Rank}(A) = \text{Rank}(\tilde{A})$$

solution infinitely one ∞

2. Uniqueness:- L.S has one solution

$$\text{iff } \text{Rank}(A) = \text{Rank}(\tilde{A}) = n \text{ } \rightarrow \text{p.d.f.}$$

3. Infinitely many solution

$$\text{iff } \text{Rank}(A) < n$$

4. No - solution

$$\text{iff } \text{Rank}(A) \neq \text{Rank}(\tilde{A})$$

7.6 second and third order determinants

$$\text{let } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$D = \det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

* Cramer rule :-

For $Ax=b$ let A 2×2

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = b_1a_{22} - a_{12}b_2$$

$$\text{Ex:- } 4x_1 + 3x_2 = 12$$

$$2x_1 + 5x_2 = -8$$

Find x_1, x_2 ?

$$1. Ax=b$$

$$\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \end{bmatrix}$$

$$x_1 = \frac{D_1}{D} = \frac{\begin{vmatrix} 12 & 3 \\ -8 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = 6$$

$$x_2 = \frac{D_2}{D} = \frac{\begin{vmatrix} 4 & 12 \\ 2 & -8 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = -4$$

* Third order determinants

$$Ax=b$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$D = \det(A) \quad 3 \times 3 \text{ upto } 3 \times 3$$

$$a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D}$$

$$D_1 = \begin{vmatrix} b_1 & a_{11} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ b_3 & a_{32} & a_{31} \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

7.7 determinat + Crumer rules :-

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

* The minore matrix of AM :-

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

$$m_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$m_{12} = \begin{vmatrix} a_{11} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$m_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

- The cofactor matrix of AC :-

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$\rightarrow C = \begin{bmatrix} m_{11} & -m_{12} & m_{13} \\ -m_{21} & m_{22} & -m_{23} \\ m_{31} & -m_{32} & m_{33} \end{bmatrix}$$

$$C_{jk} = (-1)^{j+k} m_{jk}$$

$$[A \rightarrow M \rightarrow C]$$

- Theorem:-
- Inter change of two rows multiplies the value of determinat by (-1)
 - addition of multiple of row to another row dose not alter the value of determinat.
 - multiplication of row by none zero constant (C) multiply the value of determinat by (C)

Rule:-

$$\det(CA) = C^n \det A$$

$C = \text{scalar}$

$n = \text{number of row in } A$

$A = \text{matrix}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{Ex: } A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = -2$$

$$2A = \begin{vmatrix} 2 & 4 \\ 6 & 8 \end{vmatrix} = (2)(8) - (4)(6) = 16 - 24 = -8$$

$$\det(2A) = 2 \det(A) = (4)(-2) = -8$$

Evaluation of determinant by reduction to triangular form
"Gauss - elimination"

$$D = \begin{vmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ 3 & 8 & 9 & 1 \end{vmatrix} \xrightarrow{\text{Pivot}} \text{Using G-E} = \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 2 & 6 & -1 \\ 0 & 8 & 3 & 16 \end{vmatrix} \begin{matrix} \rightarrow R_2 - 2R_1 \rightarrow \text{Pivot} \\ \rightarrow R_4 + 1.5R_1 \end{matrix}$$

$$\begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & -11.4 & 29.2 \end{vmatrix} \begin{matrix} \rightarrow \text{Pivot} \\ \rightarrow R_3 - 0.4R_2 \\ \rightarrow R_4 - 1.6R_2 \end{matrix} \rightarrow \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & 0 & 47.25 \end{vmatrix} \begin{matrix} \rightarrow \text{Pivot} \\ \rightarrow R_4 + 4.75R_3 \end{matrix}$$

$$D = (2)(5)(2.4)(47.25) = 1134$$

قواعد ← row و column

d. Transpose leave the value of determinant unchanged :-

$$\det(A) = \det(A^T)$$

المتعكس

c. Zero row or Zero Column render the value of determinant to Zero.

d. Proportional rows or columns render the value of determinant to Zero.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Theorem: Rank in terms of determinant For $(m \times n)$ matrix $A = [a_{jk}]$

1. A has rank $(R) \geq 1$ iff A has $r \times r$ sub matrix with non-zero determinant.

2. The determinant of any square sub matrix with more than r -rows contained in A has value equal to Zero.

3. if $m=n$ an $(n \times n)$ matrix A has rank (n) iff $\det(A) \neq 0$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

2x2 sub matrix :-

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \neq 0$$

$$\text{Rank}(A) = r = 2$$

iff $A_{3 \times 3}$

$$\text{Rank}(A) = 2, 1$$

$$\det(A) = 0$$

2 ol sub matrix
3x3 rank 2
تفسيره ان rank 2
فان D = 0

* Cramer rules :-

$$Ax = b$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{n1} & a_{nn} \end{bmatrix}$$

D_n is the determinant

obtained from D by replacing

in D the $r \times r$.

column by column with element

$$x = \begin{bmatrix} x_1 \\ x_n \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_n \end{bmatrix}$$

$$x = \frac{D_1}{D}, x_2 = \frac{D_2}{D} \dots x_n = \frac{D_n}{D}$$

$$D = \det(A)$$

7.8 Invers of matrix « Gauss - Jordan - elimination »

→ For square matrix $n \times n$ A its inverse

$$A^{-1} \rightarrow A A^{-1} = A^{-1} A = I$$

Note: 1. If A has inverse then A called non-singular matrix.

2. $\Rightarrow$ no inverse $\Rightarrow$ singular $\Rightarrow$

For system $AX=b$ we can find X using A^{-1} ?

$$A^{-1} A X = A^{-1} b$$

$$I X = A^{-1} b \rightarrow X = A^{-1} b$$

Theorem: Existence of inverse:

The inverse A^{-1} of an $n \times n$ matrix A exist

iff ① $\text{rank}(A) = n$

② $\det(A) \neq 0$

③ A is non-singular

iff $\text{rank}(A) < n$

then 1. A is singular

2. A is no singular

Finding inverse by Gauss-Jordan-elimination

let $A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$ Find A^{-1} ?

$$\tilde{A} = [A | I] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array} \right] \rightarrow \text{Pivot}$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 1 \end{array} \right] \rightarrow R_2 + 3R_1 \rightarrow \text{Pivot}$$

$$\rightarrow R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -3 & -4 & -1 & 1 \end{array} \right] \rightarrow R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3.5 & 1.5 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 & -0.2 \end{array} \right] \begin{array}{l} \rightarrow -R_1 \\ \rightarrow 0.5R_2 \\ \rightarrow 0.2R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0.6 & 0.4 & -0.4 & 0 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 & 0 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 & -0.2 \end{array} \right] \begin{array}{l} \rightarrow R_1 + 2R_3 \\ \rightarrow R_2 - 3.5R_3 \\ \rightarrow \text{Pivot} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 & 0 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 & 0 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 & -0.2 \end{array} \right] \rightarrow R_1 + R_2$$

$$\tilde{A} = [I | A^{-1}]$$

$$A^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

Theorem: ~ Inverse Using Determinant

$$A^{-1} = \frac{1}{|A|} C^T$$

$$A \rightarrow M \rightarrow C \rightarrow C^T$$

$$\text{let } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Ex: ~ let $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ Find A^{-1} ?

$$A^{-1} = \frac{1}{|A|} C^T = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$$

$$\text{let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{Find } A^{-1} ?$$

$$A^{-1} = \frac{1}{|A|} C^T$$

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{jk} = (-1)^{j+k} M_{jk}$$

$$C = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}$$

$$C^T = \begin{bmatrix} M_{11} & -M_{21} & M_{31} \\ -M_{12} & M_{22} & -M_{32} \\ M_{13} & -M_{23} & M_{33} \end{bmatrix}$$

Ex:- Find A^{-1} ?

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} C^T$$

$$|A| = -1(-4-3) - 1(12+1) + 2(9-1) = 10$$

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13$$

$$C_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8 ; C_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2$$

$$C_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2 ; C_{23} = -\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2$$

$$C_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3 ; C_{32} = -\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = 7$$

$$C_{33} = \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = 2 ; C = \begin{bmatrix} -7 & -13 & 8 \\ 2 & -2 & 2 \\ 3 & 7 & -2 \end{bmatrix}$$

$$C^T = \begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix} ; A^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

Theorem:- For diagonal matrix ?

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} ; A^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & 0 \\ 0 & \frac{1}{a_{22}} & 0 \\ 0 & 0 & \frac{1}{a_{33}} \end{bmatrix}$$

$$\text{Ex:- } A = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rules:-

1. $(AC)^{-1} = C^{-1}A^{-1}$

2. $(A^{-1})^{-1} = A$

3. $(A^T)^{-1} = (A^{-1})^T$

$(A^T)^{-1} = (A^{-1})^T$

* Un-usual Properties of matrix :-

1. $AB \neq BA$

2. $AB = 0$

it does not mean $A=0$ or $B=0$

let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

$AB = 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3. $AC = AD$

it does not mean $C=D$

Theorem :- Cancellation laws

let A, B, C be $n \times n$ matrix then

a. If $\text{rank}(A) = n$ $AB = AC$, then $B = C$

b. " " " " $AB = 0$, then $B = 0$

c. if $AB = 0$ and $A \neq 0$, $B \neq 0$ then
 $\text{rank}(A) < n$; $\text{rank}(B) < n$

d. if A is singular, then AB, BA are singular

• inverse هو العكس

Theorem :- $\det(AB) = \det(BA) = \det A \det B$

* Adjoint matrix :-

1. $\text{Adj}(A) = [C_{ij} \text{ of } (A)]^T = C^T$

$A^{-1} = \frac{1}{|A|} C^T$; $A^{-1} = \frac{1}{|A|} \text{adj}(A)$

2. $(A) [\text{adj}(A)] = [\text{adj}(A)](A) = \det(A) I$
scalar

chapter 8

8.1 The matrix Eigen value Problem

let $A = [a_{ij}]$ is $n \times n$ square matrix then matrix Eigen Value Problem is $Ax = \lambda X$

where $A = \text{matrix} - \text{square}$

$\lambda = \text{Eigen value}$, $X = \text{Eigen vector}$

note: $X = 0$ is solution with no-interest we are looking for $X \neq 0$

* The spectrum of A is the set of all Eigen value of A that consist of at least one eigen value and at most n -eigen value.

$$A = \begin{bmatrix} & \\ & \\ & \end{bmatrix}_{n \times n}$$

$$\lambda_1, \dots, \lambda_n$$

* How to find Eigen value

$$\text{for } Ax = \lambda x$$

$$Ax - \lambda x = 0$$

$$[A - \lambda I]x = 0$$

$$1. \det(A - \lambda I) = 0$$

characteristic equation

from $\det(A - \lambda I) = 0$ we can find λ 's

2. by finding λ 's $\equiv$ eigen value

we can then substitute in $(A - \lambda I)x = 0$

to find X 's $\equiv$ eigen vectors

Ex:- let $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$ find eigen value + eigen vectors?

$$|A - \lambda I| = 0 \rightarrow \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}$$

$$\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$(-5-\lambda)(-2-\lambda) - (4) = 0$$

$$\lambda^2 + 7\lambda + 6 = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = -6$$

eigen vector eigen value

1. For $\lambda_1 = -1$ the corresponding eigen vector X is

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$AX = b$$

$$\hat{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

2. For $\lambda_2 = -6$ the X is

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\hat{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

* normalization of $X = \frac{X}{|X|}$

$$|X| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Ex:- Find λ 's, X 's?

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

1. $\det(A - \lambda I) = 0$ main diag is λ upari

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

$$\rightarrow C = 2 - \lambda \quad \left| \begin{array}{ccc|ccc} 1-\lambda & 6 & -2 & 2 & -6 & +(-3) \\ -2 & 0-\lambda & & -1 & 0-\lambda & \\ 2 & 1-\lambda & & -1 & -2 & \end{array} \right|$$

$$= -\lambda^3 - \lambda^2 + 21\lambda + 45 = 0 \rightarrow \lambda = 5, -3, -3$$

For $\lambda = 5$ the X is in

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by Gauss-elimination:-

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \text{--- Pivot}$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & -\frac{24}{7} & -\frac{48}{7} \\ 0 & -\frac{16}{7} & -\frac{32}{7} \end{bmatrix} \rightarrow R_2 + \frac{2}{7}R_1 \text{--- Pivot}$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & -\frac{24}{7} & -\frac{48}{7} \\ 0 & 0 & 0 \end{bmatrix} \rightarrow R_3 - \frac{2}{3}R_2$$

$$-7x_1 + 2x_2 - 3x_3 = 0 \rightarrow -\frac{24}{7}x_2 - \frac{48}{7}x_3 = 0$$

$$x_2 = -1, x_1 = 1, x_3 = 2$$

$$\hat{X} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

For $\lambda = -3$ the X is in -

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using G-E:-

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \rightarrow \text{Pivot}$$

$$\rightarrow x_1 + 2x_2 - 3x_3 = 0 \quad \hat{X} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow R_2 - 2R_1$$

$$\rightarrow R_3 + R_1$$

Ex: show that $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ is an eigen vector

Cross Producting for eigen value

2. For A

$$A = \begin{bmatrix} 4 & -1 & -1 \\ 0 & -1 & 3 \\ 1 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 4 & -1 & -1 \\ 0 & -1 & 3 \\ -1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \stackrel{?}{=} 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$3 \times 3 \quad 3 \times 1$

$$AX = \lambda X$$

$$\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

3×1

Ex: Find λ 's, X 's?

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$1. \det(A - \lambda I) = 0 \sim \begin{vmatrix} 0 - \lambda & 1 \\ -1 & 0 - \lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0 \sim \lambda = +i, -i$$

$\sim$ For $\lambda = i$ the X is:-

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{using G-E :- } \hat{X} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$\sim$ For $\lambda = -i$ the cross producting X 's is:-

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{using G-E :- } \hat{X} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Theorem:- A^T and A has the same Eigen value

$$A^T = \begin{bmatrix} 0 & -1 \\ i & 0 \end{bmatrix}$$

$$1. \det(A - \lambda I) = 0$$

$$\begin{vmatrix} 0 - \lambda & 1 \\ 1 & 0 - \lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda = i, -i$$

8.5 symmetric, skew symmetric and orthogonal matrix :-

Def:- a real square matrix $A = [a_{ij}]$ is called

1. symmetric iff $A^T = A$
2. skew symmetric iff $A^T = -A$
3. orthogonal iff $A^T = A^{-1}$

$$A^{-1} = \frac{1}{|A|} C^T$$

Ex:- 1. symmetric

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix} \leadsto A^T = A$$

2. skew symmetric

$$A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix} \leadsto A^T = -A$$

3. orthogonal

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \leadsto A^T = A^{-1}$$

* For any real square matrix A

1. symmetric matrix $(R) = \frac{1}{2} (A + A^T)$
2. skew symmetric $(S) = \frac{1}{2} [A - A^T]$

Ex:- $A = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix}$ Find S, R ?

$$R = \frac{1}{2} \left[\begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix} + \begin{bmatrix} 9 & 2 & 3 \\ 5 & 3 & 4 \\ 2 & -8 & 3 \end{bmatrix} \right] = \begin{bmatrix} 9 & 3.5 & 2.5 \\ 3.5 & 3 & -2 \\ 3.5 & -2 & 3 \end{bmatrix}$$

$$S = \frac{1}{2} [A - A^T] = \begin{bmatrix} 0 & 1.5 & 1.5 \\ -1.5 & 0 & -6 \\ 1.5 & 6 & 0 \end{bmatrix}$$

* Theorem :-

1. The eigen value of symmetric matrix are real. ≤ 5
2. The eigen value of skew symmetric matrix are pure imaginary or zero. $(25i, -25i, 0)$

Ex:- Find λ 's for A :-

$$A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix} \rightarrow \text{skew-symmetric}$$

$$1. \det(A - \lambda I) = 0$$

$$\begin{vmatrix} 0-\lambda & 9 & -12 \\ -9 & 0-\lambda & 20 \\ 12 & -20 & 0-\lambda \end{vmatrix} = 0$$

$$-\lambda^3 - 625\lambda = 0$$

$$-\lambda [\lambda^2 + 625] = 0 \rightarrow \lambda = 0, \lambda = -25i, \lambda = 25i$$

* Orthogonal matrix :-

Ex:

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$A^T = A^{-1} \rightarrow A^T = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$A^{-1} = \frac{1}{1} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

- Theorem :-
1. The determinantal of orthogonal matrix is 1, -1
 2. The eigen value of orthogonal matrix A are real or complex conjugate in pairs and have absolute value of 1.

$$z = x + iy$$

$$\bar{z} = x - iy$$

$$|z| = \sqrt{x^2 + y^2} = 1$$

Ex:- Find det + λ 's for A ?

$$A = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$$

$$\det(A) = 1$$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} \frac{2}{3} - \lambda & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{5}{3} - \lambda & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} - \lambda \end{vmatrix} = 0 \rightarrow -\lambda^3 + 2\lambda^2 + \frac{2}{3}\lambda - 1 = 0$$

$$\lambda = \frac{-1}{3}, \frac{5 \pm \sqrt{11}}{6}$$

real complex - conj

$$= \sqrt{\left(\frac{-1}{3}\right)^2 + \left(\frac{\sqrt{11}}{6}\right)^2} = \sqrt{\frac{25}{36} + \frac{11}{36}} = \sqrt{\frac{36}{36}} = 1$$

Theorem:-

① For vector a, b

$$a = [a_1 \ a_2 \ \dots \ a_n]$$

$$b = [b_1 \ b_2 \ \dots \ b_n]$$

$$a \cdot b = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

dot-product

② $\|a\| = \text{norm} = \text{length} = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$

eigen vector $\rightarrow$ use $\rightarrow$ find eigen value $\rightarrow$

8.4:- Eigen-bases - diagonalization

Theorem:- if an $n \times n$ matrix A has n -distinct eigen value

then A has basis of eigen vector $x_1, x_2, \dots, x_n$ for $\mathbb{R}^n$

Ex:- Find λ, X 's for A ?

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \quad 1. \det(A - \lambda I) = 0$$

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0 \rightarrow (5-\lambda)^2 - 9 = 0$$

$$\lambda = 2, 8$$

For $\lambda = 2$ the X 's is:-

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \hat{X} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For $\lambda = 8$ the X 's is

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \hat{X} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

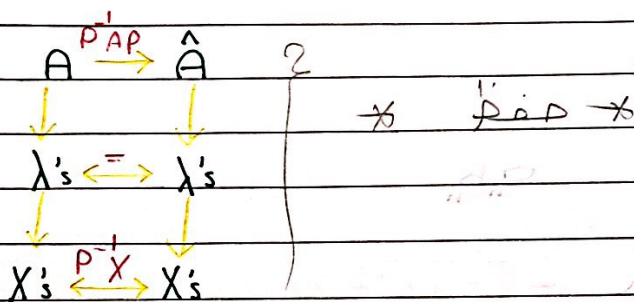
- similar matrix, similarity transformation :-

Def:- let $n \times n$ matrix $\hat{A}$ is called similar to $n \times n$ matrix A iff $\hat{A} = P^{-1}AP \neq P^{-1}PA$

where P is non-singular + P is $n \times n$ matrix. Def ① if $\hat{A}$ is similar

Def:- 1. if $\hat{A}$ is similar to A then $\hat{A}$ has the same eigen value as A . λ 's = λ 's eigen value

2. if X is an eigen vector of A then $y = P^{-1}X$ is the eigen vector of $\hat{A}$ corresponding to same eigen value :-



Ex:- $A = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 13 \\ 14 \end{bmatrix}$

Find λ 's, X 's for $A, \hat{A}$?

→ for A ?

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 6-\lambda & -3 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)(-1-\lambda) - (-12) = 0 \rightarrow \lambda = 2, 3$$

for $\lambda = 2$ the X 's is :-

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 4 & -3 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \hat{X} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

for $\lambda = 3$ the X 's is

$$\begin{bmatrix} 3 & -3 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \hat{X} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\hat{A} = P^{-1}AP$$

$$P^{-1} = \frac{1}{43} \begin{bmatrix} 11 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$1. \det(\hat{A} - \lambda I) = 0$$

$$\begin{vmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0 \rightarrow (3-\lambda)(2-\lambda) = 0$$

$$\lambda = 2, 3$$

$\rightarrow$ For $\lambda = 2$ the y is:-

$$(\hat{A} - \lambda I)y = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\hat{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\rightarrow$ For $\lambda_2 = 3$ the y is

$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \hat{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

خطوة 9: إيجاد المتجهات الذاتية (خطوات 1 و 2 و 3 و 4 و 5 و 6 و 7 و 8)

المتجهات الذاتية

المتجهات

$$y_1 = P^{-1}x_1 = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y_2 = P^{-1}x_2 = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Diagonalization of matrix :-

If an $n \times n$ matrix A has eigen vectors

then $D = X^{-1} A X$

$D \rightarrow$ diagonal matrix

$$D = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$X =$ eigen vector of matrix A as column vector

$$D = X^{-1} A X$$

$$D^2 = DD = X^{-1} A X \underbrace{X^{-1} A X}_{D}$$

$$X^{-1} A \underbrace{I}_1 A X$$

$$X^{-1} A A X$$

$$X^{-1} A^2 X$$

$$D^m = X^{-1} A^m X$$

Ex:- find D for A :-

$$A = \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}$$

Solution:-

$$D = X^{-1} A X$$

$$1. \det(A - \lambda I) = 0$$

$$\begin{vmatrix} 7.3 - \lambda & 0.2 & -3.7 \\ -11.5 & 1 - \lambda & 5.5 \\ 17.7 & 1.8 & -9.3 - \lambda \end{vmatrix} = 0$$

$$= (7.3 - \lambda) \begin{vmatrix} 1 - \lambda & 5.5 \\ 1.8 & -9.3 - \lambda \end{vmatrix} - 0.2 \begin{vmatrix} -11.5 & 5.5 \\ 17.7 & -9.3 - \lambda \end{vmatrix} + (-3.7) \begin{vmatrix} -11.5 & 1 - \lambda \\ 17.7 & 1.8 \end{vmatrix}$$

$$= -\lambda^3 - \lambda^2 - 12\lambda = 0$$

$$-\lambda(\lambda - 3)(\lambda + 4) = 0 \rightarrow \lambda = 0, 3, -4$$

$$\begin{vmatrix} -11.5 & 1 - \lambda \\ 17.7 & 1.8 \end{vmatrix}$$

For $\lambda = 5$ the X_3 is:

$$(A - \lambda I)X = 0 \rightarrow \hat{X} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$$

For $\lambda = 0$ the X_3 is:

$$(A - \lambda I)X = 0 \rightarrow \hat{X} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

For $\lambda = -4$ the X_3 is:

$$(A - \lambda I)X = 0 \rightarrow \hat{X} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

$$X = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \rightarrow X^{-1} = \frac{1}{|X|} C^T$$

$$X^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

$$D = X^{-1} A X$$

$$= \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix} * \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1 & 5.5 \\ 12.7 & 1.8 & -9.3 \end{bmatrix} * \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

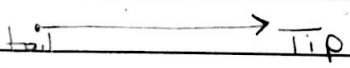
For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Trace of $A = a_{11} + a_{22} + a_{33}$

Chapter 9 :-

9.1 Vectors in 2-space + 3-space

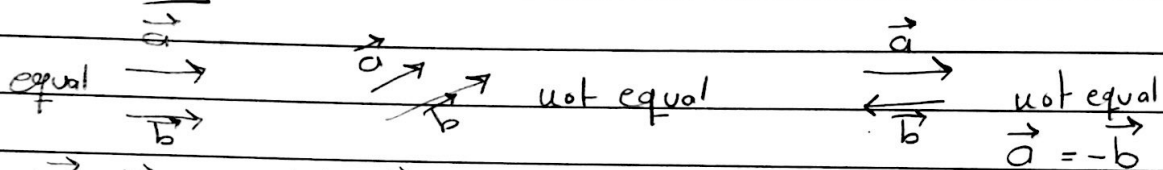
1. Scalar :- Quantity that is determined by its magnitude like
2. Vector :- Quantity that is determined by its magnitude and direction like

* For vector $\vec{a} = \vec{a}$ 
Initial Point Terminal Point

$$|\vec{a}| = \text{length} = \text{norm} = \text{magnitude} = \text{Euclidean norm}$$

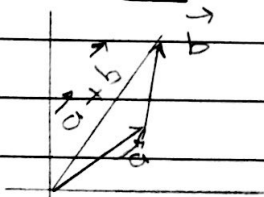
If $|\vec{a}| = 1$ $\vec{a}$ is unit-vector

Def:- Two vectors $\vec{a}, \vec{b}$ are equal $\vec{a} = \vec{b}$ if they have same length and same direction.



$$|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$$

Triangle inequality



* Parallel gram equality

$$|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = 2[|\vec{a}|^2 + |\vec{b}|^2]$$

note:- $\mathbf{i} \cdot \mathbf{i} = [1, 0, 0] \cdot [1, 0, 0] = 1$

$$1. \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$

$$2. \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{i} = \mathbf{i} \cdot \mathbf{k} = 0$$

* Vector addition, scalar Multiplication

$$\vec{a} = [a_1, a_2, a_3]$$

$$\vec{b} = [b_1, b_2, b_3]$$

$$\vec{a} + \vec{b} = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$$

$$\vec{a} - \vec{b} = [a_1 - b_1, a_2 - b_2, a_3 - b_3]$$

* properties of vector addition :-

$$1. \vec{a} + \vec{b} = \vec{b} + \vec{a}$$

$$2. (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$

$$3. \vec{a} + (-\vec{a}) = \vec{0} \text{ (Zero vector } (0, 0, 0))$$

$$4. \vec{a} + (c\vec{a}) = \vec{a}$$

* scalar Multiplication :-

$$\vec{a} = [a_1, a_2, a_3]$$

c = scalar

$$|c\vec{a}| = |c| |\vec{a}|$$

* properties :-

$$1. c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$$

$$2. (c+k)\vec{a} = c\vec{a} + k\vec{a}$$

$$3. c(k\vec{a}) = (ck)\vec{a}$$

$$4. 1\vec{a} = \vec{a}$$

Zero scalar

Zero vector

$$5. 0\vec{a} = \vec{0} (0, 0, 0)$$

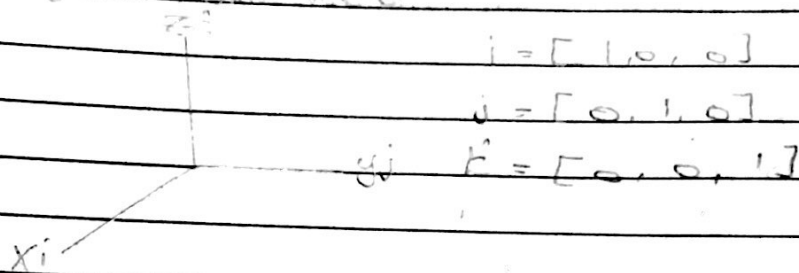
$$6. -1\vec{a} = -\vec{a}$$

$$\text{Ex:- let } \vec{a} = [4, 0, 1], \vec{b} = [2, -5, \frac{1}{3}]$$

$$\text{Find } -\vec{a} ? \quad -\vec{a} = [-4, 0, -1]$$

$$7\vec{a} = [28, 0, 7], \quad \vec{a} + \vec{b} = [6, -5, \frac{4}{3}]$$

$$2(\vec{a} - \vec{b}) = [4, 10, \frac{4}{3}]$$

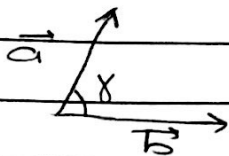


any vector $\vec{a}$

$$\vec{a} = [a_1, a_2, a_3] \equiv \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

9.2 Inner Product (dot product)

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta \rightarrow \text{Scalar}$$



If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ then $\vec{a} \cdot \vec{b} = 0$

$$\vec{a} = [a_1, a_2, a_3]$$

$$\vec{b} = [b_1, b_2, b_3]$$

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Theorem: $\vec{a}$ and $\vec{b}$ are orthogonal or perpendicular

iff $\theta = 90^\circ$ or $\frac{\pi}{2}$

$$\text{so } \vec{a} \cdot \vec{b} = 0$$



$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

Ex: let $\vec{a} = [1, 2, 0]$

$\vec{b} = [3, -2, 1]$

Find θ between $\vec{a}, \vec{b}$?

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{3 + -4 + 0}{\sqrt{(1)^2 + (2)^2 + (0)^2} \sqrt{3^2 + (-2)^2 + 1^2}} = \frac{-1}{\sqrt{5} \sqrt{14}}$$

$$\theta = 96.865^\circ$$

* Properties :-

$$1. (q\vec{a} + r\vec{b}) \cdot \vec{c} = q\vec{a} \cdot \vec{c} + r\vec{b} \cdot \vec{c}$$

$$2. \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

$$3. \vec{a} \cdot \vec{a} \geq 0$$

$$4. \vec{a} \cdot \vec{a} = 0 \text{ iff } \vec{a} = \vec{0}$$

$$5. (\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$$

6. Cauchy - schwarz :-

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}| |\vec{b}|$$

* Component of vectors :-

For $\vec{a}$ has initial point $P(x_1, y_1, z_1)$ and terminal point $Q(x_2, y_2, z_2)$

$$a_1 = x_2 - x_1, \quad \vec{a} = [a_1, a_2, a_3]$$

$$a_2 = y_2 - y_1, \quad |\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$a_3 = z_2 - z_1$$

Ex: - $\vec{a}$

$P(4, 0, 2)$

$Q(6, -1, 2)$

اذا اعدل انا تغييرا فهذا البعد ...

بدايتها من ادم ...

ولا يطلع على الجرد ان لب ادم باكب ...

$$\vec{a} = [2, -1, 0]$$

$$|\vec{a}| = \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5}$$

* position vector :- $\vec{r}$ of point $A(x, y, z)$ is vector with origin $(0, 0, 0)$ as its initial point and A as terminal point.

$$(0, 0, 0) \rightarrow (x, y, z)$$

* Zero-vector $\vec{0}$ has zero length + no-direction

$$\vec{a} = [a_1, a_2, a_3]$$

$$\vec{b} = [b_1, b_2, b_3]$$

$$\vec{0} = \vec{0}$$

$$\text{i.e. } a_1 = b_1, a_2 = b_2, a_3 = b_3$$